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Induced matter: Curved N-manifolds encapsulated in Riemann-flat N+1 dimensional space

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Abstract

Liko and Wesson have recently introduced a new 5-dimensional induced matter solution of the Einstein equations, a negative curvature Robertson-Walker space embedded in a Riemann-flat 5-dimensional manifold. We show that this solution is a special case of a more general theorem prescribing the structure of certain N+1 dimensional Riemann-flat spaces which are all solutions of the Einstein equations. These solutions encapsulate N-dimensional curved manifolds. Such spaces are said to "induce matter" in the sub-manifolds by virtue of their geometric structure alone. We prove that the N-manifold can be *any* maximally symmetric space.

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The concept of “induced matter”, was originally introduced by Wesson [1, 2]. While investigating 5D Kaluza-Klein theory, he recognized that a curved 4-space could be embedded in a Ricci-flat ($R_{AB} = 0$; $A, B, ... \in \{0, 1, 2, 3, 4\}$) 5-space. This is a reflection of the Campbell-Magaard theorem [3] which, applied to 5D, states that it is always possible to embed a curved 4D manifold in a 5D Ricci-flat space. Seahra and Wesson [4] provide an overview and rigorous proof of the Campbell-Magaard theorem with applications to higher dimensions. Wesson takes “induced matter” to mean that the left-hand geometric side extra terms of the flat 5D Ricci-tensor provide the source terms in the 4D curved Ricci-tensor of the embedded space. A “weak” version of this concept utilizing an embedding of the Friedmann-Roberston-Walker(FRW) 4-space in a Minkowski 5-space has been used to visualize the big bang sectionally [5]. Here, the 5-space is Riemann-flat ($R_{ABCD} = 0$) since it is Minkowski. There is no physics in the 4D subspace, except with reference to the original FRW coordinates. This simply provides a Euclidean embedding diagram.

More recently, Liko and Wesson have introduced a new 5D, Riemann-flat solution [6] which they found could “encapsulate” a 4D curved FRW space. We use the term “encapsulate” as distinct from embed since in this 5-space, the coordinates are not Minkowski. The 4D subspace is itself curved in the same 5D coordinates. It is true that a flattening transformation can be found to a 5D Minkowski space. However, this would simply produce another embedding diagram. The physics seems to lie in the encapsulating 5D metric. We shall use the term ”induced matter” to include a Riemann-flat 5D manifold encapsulating a curved 4D subspace. The Liko-Wesson induced matter solution goes on to describe an apparently inflationary universe as a negative curvature FRW space embedded in a special 5D universe. The RW space undergoes accelerated expansion subject to a repulsive “dark energy” ($P = -\rho$). We will show in this paper that the Liko-Wesson solution is a special case of a more general class of maximally symmetric sub-manifolds embedded in Riemann-flat space. A detailed discussion of maximally symmetric sub-manifolds based on Poincarè metrics and their consequences can be found in ref.[7]. For convenience, we repeat some critical definitions and calculations.

Consider the Riemann manifold defined by

$$dS^2 = \tilde{g}_{ij} dx^i dx^j. \tag{1}$$

This space is said to be *maximally symmetric* if and only if it has constant *sectional curvature* $\kappa = \kappa(i,j)$, for any $1 \le i \ne j \le N$. In the plane spanned by the basis vectors $(\hat{e}_i,\, \hat{e}_j)$ the sectional curvature is defined by,

$$\kappa(i,j) = \tilde{g}^{ii} R^j_{iji} \qquad (i,j \text{ not summed}). \tag{2}$$

For a maximally symmetric space $R^j_{iji} = \kappa \tilde{g}_{ii}$, $j \ne i$. For such a space,

$$R_{ii} = -\kappa(N-1)\tilde{g}_{ii}. \tag{3}$$

Theorem: Let $\tilde{g}_{ij}$ represent a maximally symmetric space of sectional curvature κ. *The metric*

$$dS^2 = d\tau^2 - D\tau^2 \tilde{g}_{ij} dx^i dx^j, \quad i,j \ldots, \in \{1,2,\ldots,N\}\ , \tag{4}$$

is Riemann-flat whenever $D = -\kappa$.

Proof: Consider the metric

$$dS^2 = d\tau^2 - f(\tau)^2 \tilde{g}_{ij} dx^i dx^j, \tag{5}$$

where $\tilde{g}_{ij}$ denotes a maximally symmetric space. We compute the independent components of the curvature tensor (the overprime denotes differentiation in τ):

$$\begin{aligned}
R^i{}_{0j0} &= -\frac{f''}{f}\delta^i{}_j \\
R^0{}_{i0j} &= -ff''\tilde{g}_{ij} \\
R^k{}_{ikj} &= \tilde{R}_{ij} - (N-1)f'^2\tilde{g}_{ij} \\
&= -\kappa(N-1)\tilde{g}_{ij} - (N-1)f'^2\tilde{g}_{ij} \\
&= -(N-1)(f'^2+\kappa)\tilde{g}_{ij},
\end{aligned} \tag{6}$$

where we have made use of the result Eq.(3). It is evident from Eq.(6) that the space will be Riemann-flat if and only if $f'' = 0$ and $f'^2 + \kappa = 0$. Let

$f(\tau) = \sqrt{D}\,\tau$. Then $f'' = 0$ and $f'^2 - D = 0$. It follows that $D = -\kappa$ and the proof is complete.

Liko and Wesson [6] introduce the line element (with overall sign of dS^2 reversed from ours),

$$dS^2 = d\tau^2 - \frac{\tau^2}{L^2}\Big[dt^2 - L^2 \sinh^2(\frac{t}{L}) d\sigma_3^2\Big], \tag{7}$$

where,

$$d\sigma_3^2 = \Big(1 + \frac{kr^2}{4}\Big)^{-2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2), \tag{8}$$

is the Robertson-Walker 3-space with $k = -1$.

Define coordinates $x^A = \{\tau, r, \theta, \phi, t\},\ A \in \{0, 1, 2, 3, 4\}$.

We can then identify in Eq.(4),

$$\tilde{g}_{ii} = \{-f, -fr^2, -fr^2 \sin^2\theta, 1\}, \qquad f = L^2 \sinh^2(t/L)\Big(1 + \frac{kr^2}{4}\Big)^{-2}, \tag{9}$$

and $D = 1/L^2$. Eq.(7) is thus of the form (4) and will satisfy the theorem provided that the sectional curvature of the 4-space is $\kappa = -1/L^2$. Direct evaluation of the sectional curvature for two typical cases (by symmetry, the remaining cases are identical) results in,

$$\begin{aligned} \kappa(1,2) &= \tilde{g}^{11} R^2_{121} = -\frac{1}{L^2} \\ \kappa(4,2) &= \tilde{g}^{44} R^2_{424} = -\frac{1}{L^2} \end{aligned}$$

which show that the conditions (6) are met. That is, the 4-space has constant sectional curvature which then results in a Riemann-flat 5-space. We have thus shown that the new metric solution Eq.(7) introduced by Liko and Wesson is a special case of our more general theorem Eq.(4) which allows the N-space to be any maximally symmetric manifold.

References

[1] P.S. Wesson, Gen. Relativ. Gravit. **22**, 707 (1990).

[2] P.S. Wesson, Astrophysical Journal **394**, 19 (1992).

[3] J. Campbell, *A course on differential geometry*, Claredon, Oxford, (1926).

[4] Sanjeev S. Seahra and Paul Wesson Class. Quant. Grav. **20**, 1321 (2003).

[5] Sanjeev S. Seahra and Paul Wesson, Class. Quant. Grav. **19**, 1139 (2002).

[6] Tomáš Liko and Paul S. Wesson, J. Math. Phys. **46**, 062504-1 (2005).

[7] Lawrence R. Mead, Harry I. Ringermacher and Sungwook Lee, submitted to the Journal of Mathematical Physics.