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Integral and Nonnegativity Preserving Approximations of Functions

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Abstract

In this paper we consider the problem of approximating a function by continuous piecewise linear functions that preserve the integral and nonnegativity of the original function.

Key words: Integral preserving, Positivity preserving

1 Introduction

The problem of approximating a function by piecewise polynomials is central in many branches of mathematics. In this paper we consider the following problem: given a finite uniform partition of the unit interval $I = [0, 1]$ or the unit square $I \times I = [0, 1] \times [0, 1]$, find a continuous piecewise linear function that is integral and nonnegativity preserving for every integrable function. This problem has applications in, e.g., the numerical analysis of Markov operators in stochastic analysis and Frobenius-Perron operators in ergodic theory [2]. For example, the famous Ulam conjecture [6], [5] is related to integral and nonnegativity preserving approximations via piecewise constant functions.

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In the next section we give two results for L^1 spaces. Then in Sections 3 and 4 we concentrate on the context of $L^1(I)$ and $L^1(I \times I)$, respectively.

2 Some Averaging Operators on $L^1(X)$

Let $(X, \mathcal{A}, P)$ be a probability space and let $\psi_0, \psi_1, \ldots, \psi_m$ be nonnegative $\mathcal{A}$-measurable functions on X such that $\psi_0 + \psi_1 + \cdots + \psi_m = 1$. Assume that $\psi_0, \psi_1, \ldots, \psi_m$ are linearly independent in $L^1(X)$ and let Ψ_m denote the linear span of $\psi_0, \psi_1, \ldots, \psi_m$ in $L^1(X)$.

Let T be a continuous linear operator from $L^1(X)$ to Ψ_m. Given $f \in L^1(X)$ and $g \in L^\infty(X)$, define $\langle f, g\rangle = \int_X fg\, dP$. Since the dual of $L^1(X)$ is $L^\infty(X)$, there exist $w_0, , w_1, \ldots, w_m \in L^\infty(X)$ such that

$$T(f) = \sum_{i=0}^{m} \langle f, w_i\rangle\, \psi_i \quad \text{for each} \quad f \in L^1(X).$$

T is called *nonnegative* if T maps nonnegative functions to nonnegative functions. We say that T preserves integrals if $\int_X T(f)\, dP = \int_X f\, dP$ for each $f \in L^1(X)$. We say that T is an *averaging operator* from $L^1(X)$ to Ψ_m if $T(1) = 1$ and if T is nonnegative and preserves integrals.

Theorem 1 *Let $\psi_0, \psi_1, \ldots, \psi_m$ and Ψ_m be as above. Let $w_0, w_1, \ldots, w_m \in L^\infty(X)$ and define $T : L^1(X) \to \Psi_m$ by*

$$T(f) = \sum_{i=0}^{m} \langle f, w_i\rangle\, \psi_i \quad \textit{for each} \quad f \in L^1(X).$$

(1) $T(1) = 1$ if and only if $\langle 1, w_i\rangle = 1$ for $i = 0, 1, \ldots, m$.
(2) T is nonnegative if $w_i \geq 0$ a.e. for $i = 0, 1, \ldots, m$.
(3) T preserves integrals if and only if $\sum_{i=0}^{m} \langle \psi_i, 1\rangle\, w_i = 1$ a.e.

PROOF. (1): Suppose $T(1) = 1$. Then $\sum_{i=0}^{m} \psi_i = 1 = T(1) = \sum_{i=0}^{m} \langle 1, w_i\rangle\, \psi_i$. Hence, $\langle 1, w_i\rangle = 1$ for $i = 0, 1, \ldots, m$. The converse is clear. (2): Clearly T is nonnegative if $w_i \geq 0$ a.e. for $i = 0, 1, \ldots, m$. (3): If $f \in L^1(X)$, then

$$\langle T(f), 1\rangle = \left\langle \sum_{i=0}^{m} \langle f, w_i\rangle\, \psi_i, 1 \right\rangle = \sum_{i=0}^{m} \langle f, w_i\rangle \langle \psi_i, 1\rangle = \left\langle f, \sum_{i=0}^{m} \langle \psi_i, 1\rangle\, w_i \right\rangle.$$

Hence, $\sum_{i=0}^{m} \langle \psi_i, 1\rangle\, w_i = 1$ a.e. if and only if $\langle T(f), 1\rangle = \langle f, 1\rangle$ for each f in $L^1(X)$. $\square$

Note. Let $X = I$ with the Lebesgue measure. Suppose $\psi_0(x) = 1 - x/2$ and $\psi_1(x) = x/2$. If we choose $w_0 = 1$ and $w_1 = -1$, then $Tf(x) = \langle f, 1\rangle (1 - x)$ and so T is nonnegative. Thus T is nonnegative does not imply that $w_i \geq 0$ for all i.

Theorem 2 *Let $\psi_0, \psi_1, \ldots, \psi_m$ and Ψ_m be as above. Let $V_0, V_1, \ldots, V_m \in \mathcal{A}$ such that $P(V_i) > 0$ for each $i = 0, 1, \ldots, m$. Define $Q : L^1(X) \to \Psi_m$ by*

$$Q(f) = \sum_{i=0}^{m} \left\langle f, \frac{1}{P(V_i)}\chi_{V_i} \right\rangle \psi_i \quad \textit{for each} \quad f \in L^1(X).$$

Assume $P(V_i \cap V_j) = 0$ if $i \neq j$. Then Q is an averaging operator from $L^1(X)$ to Ψ_m if and only if $\langle \psi_k, 1\rangle = P(V_k)$ for each $k = 0, 1, \ldots, m$.

PROOF. Set $w_i = \frac{1}{P(V_i)}\chi_{V_i}$ for $i = 0, 1, \ldots, m$. Assume Q is an averaging operator from $L^1(X)$ to Ψ_m. By Theorem 1(3), $\sum_{i=0}^{m} \langle \psi_i, 1\rangle w_i = 1$ a.e. Thus, for $k = 0, 1, \ldots, m$, we have

$$P(V_k) = \langle \chi_{V_k}, 1\rangle = \sum_{i=0}^{m} \langle \chi_{V_k}, w_i\rangle \langle \psi_i, 1\rangle = \langle \psi_k, 1\rangle .$$

Assume $\langle \psi_k, 1\rangle = P(V_k)$ for each $k = 0, 1, \ldots, m$. Then

$$1 = \langle 1, 1\rangle = \left\langle \sum_{k=0}^{m} \psi_k, 1 \right\rangle = \sum_{k=0}^{m} \langle \psi_k, 1\rangle = \sum_{k=0}^{m} P(V_k).$$

Since $P(V_i \cap V_j) = 0$ for $i \neq j$, it follows that $\sum_{k=0}^{m} \chi_{V_k} = 1$ a.e. and so $\sum_{i=0}^{m} \langle \psi_i, 1\rangle w_i = \sum_{i=0}^{m} \chi_{V_i} = 1$ a.e. Thus Q preserves integrals by Theorem 1(3). $\square$

3 Some Averaging Operators on $L^1(I)$

Divide $I = [0, 1]$ into n equal subintervals $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Let $h = 1/n = m(I_i)$, where m is the Lebesgue measure. Let Φ_n denote the space of all continuous piecewise linear functions associated with the partition $0 = x_0 < x_1 < \cdots < x_n = 1$. Let φ_i be the unique function in Φ_n such that φ_i is 1 at the node x_i and 0 at all other node points. The $(n+1)$ nodal functions $\{\varphi_i\}_{i=0}^{n}$ form a canonical basis for Φ_n.

Let T be a continuous linear operator from $L^1(I)$ to Φ_n. There exist $w_i \in L^\infty(I)$ for $i = 0, 1, \ldots, n$ such that

$$T(f) = \sum_{i=0}^{n} \langle f, w_i\rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).$$

Theorem 3 *Let $w_0, w_1, \ldots, w_n \in L^\infty(I)$ and define $T : L^1(I) \to \Phi_n$ by*

$$T(f) = \sum_{i=0}^{n} \langle f, w_i \rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).$$

(1) $T(1) = 1$ if and only if $\langle 1, w_i \rangle = 1$ for $i = 0, 1, \ldots, n$.
(2) T is nonnegative if and only if $w_i \geq 0$ a.e. for $i = 0, 1, \ldots, n$.
(3) T preserves integrals if and only if $w_0 + 2\sum_{i=1}^{n-1} w_i + w_n = 2n$ a.e., or equivalently, $\frac{1}{n}\sum_{i=1}^{n}(w_{i-1} + w_i)/2 = 1$ a.e.

PROOF. Parts (1) and (3) follow from Theorem 1. In part (3), we need to use $\langle \varphi_0, 1 \rangle = \langle \varphi_n, 1 \rangle = 1/2n$ and $\langle \varphi_i, 1 \rangle = 1/n$ for $1 \leq i \leq n-1$. Clearly T is nonnegative if $w_i \geq 0$ a.e. for $i = 0, 1, \ldots, n$. Suppose T is nonnegative. Let $A_i = \{x : w_i(x) < 0\}$. Then

$$0 \leq T(\chi_{A_i})(x_i) = \sum_{j=0}^{n} \langle \chi_{A_i}, w_j \rangle \varphi_j(x_i) = \langle \chi_{A_i}, w_i \rangle .$$

Hence $m(A_i) = 0$ and so $w_i \geq 0$ a.e. for $i = 0, 1, \ldots, n$. □

Note. Let T be defined as in Theorem 3. If $T(1) = 1$ and if T preserves integrals, then T need not be nonnegative even for the case $n = 1$. Simply take $w_0 = 3\chi_{[0,1/2]} - \chi_{[1/2,1]}$ and $w_1 = 3\chi_{[1/2,1]} - \chi_{[0,1/2]}$.

Let S_i be the closed support of φ_i and let V_i be a closed subinterval of S_i such that $m(V_i) > 0$ for $i = 0, 1, \ldots, n$. Define $Q_n : L^1(I) \to \Phi_n$ by

$$Q_n(f) = \sum_{i=0}^{n} \left\langle f, \frac{1}{m(V_i)}\chi_{V_i} \right\rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).$$

Then Q_n satisfies the conditions in (1) and (2) of Theorem 1. We wish to find $V_0, V_1, \ldots, V_n$ such that Q_n is an averaging operator from $L^1(I)$ to Φ_n.

Example 4 *Set $w_i = \frac{1}{m(S_i)}\chi_{S_i}$ for $i = 0, 1, \ldots, n$. Define $\alpha_n : L^1(I) \to \Phi_n$ by*

$$\alpha_n(f) = \sum_{i=0}^{n} \langle f, w_i \rangle \varphi_i \quad \text{for each} \quad f \in L^1(I).$$

Using Theorem 3, it is easy to check that α_n is an averaging operator from $L^1(I)$ to Φ_n. Clearly $w_i \geq 0$ and $\langle 1, w_i \rangle = 1$ for $i = 0, 1, \ldots, n$. Also, $w_0 + 2\sum_{i=1}^{n-1} w_i + w_n = 2n$ except at the points $\{x_1, \ldots, x_{n-1}\}$.

Note. α_n was first constructed in [1] to calculate fixed densities of Frobenius-Perron operators associated with chaotic interval mappings.

Example 5 *Let* $W_0 = [0, h/2]$, $W_n = [1-h/2, 1]$ *and* $W_i = [x_i - h/2, x_i + h/2]$ *for* $i = 1, \ldots, n-1$. *Set* $w_i = \frac{1}{m(W_i)}\chi_{W_i}$ *for* $i = 0, 1, \ldots, n$. *Define* $\beta_n : L^1(I) \to \Phi_n$ *by*

$$\beta_n(f) = \sum_{i=0}^{n} \langle f, w_i \rangle \varphi_i \quad \textit{for each} \quad f \in L^1(I).$$

Using Theorem 3, it is easy to check that β_n *is an averaging operator from* $L^1(I)$ *to* Φ_n. *Clearly* $w_i \geq 0$ *and* $\langle 1, w_i \rangle = 1$ *for* $i = 0, 1, \ldots, n$. *Also,* $w_0 + 2\sum_{i=1}^{n-1} w_i + w_n = 2n$ *except at the points* $\{x_0 + h/2, \ldots, x_{n-1} + h/2\}$.

Note. It has been shown [3] that $\beta_n f$ is a better approximation to $f \in L^1(I)$ than $\alpha_n f$.

Let $V_0, \ldots, V_n$ and Q_n be as above. If Q_n is integral preserving and if E is a subinterval of $[x_k, x_{k+1}]$ and $0 \leq k < n$, then

$$m(E) = \langle \chi_E, 1 \rangle = \langle Q_n(\chi_E), 1 \rangle = \sum_{i=0}^{n} \left\langle \chi_E, \frac{1}{m(V_i)}\chi_{V_i} \right\rangle \langle \varphi_i, 1 \rangle$$

and so

$$m(E) = \frac{m(E \cap V_k)}{m(V_k)} \frac{m(S_k)}{2} + \frac{m(E \cap V_{k+1})}{m(V_{k+1})} \frac{m(S_{k+1})}{2}. \tag{1}$$

If Q_n is an averaging operator from $L^1(I)$ to Φ_n, then we will show that either $Q_n = \alpha_n$ or $Q_n = \beta_n$.

Lemma 6 *Let* $V_0, \ldots, V_n$ *and* Q_n *be as above. Assume* Q_n *is an averaging operator from* $L^1(I)$ *to* Φ_n. *If* $m(V_i \cap V_{i+1}) = 0$ *for* $i = 0, 1, \ldots, n-1$, *then* $Q_n = \beta_n$.

PROOF. Assume $m(V_i \cap V_{i+1}) = 0$ for $i = 0, 1, \ldots, n-1$. By Theorem 2, it follows that $m(V_k) = \langle \varphi_k, 1 \rangle$ for $0 \leq k \leq n$. Thus, $m(V_0) = m(V_n) = h/2$ and $m(V_i) = h$ for $i = 1, \ldots, n-1$. It follows that $V_i = W_i$ for $i = 1, \ldots, n$ where $W_0, W_1, \ldots, W_n$ are as in Example 2. Hence, $Q_n = \beta_n$. $\square$

Lemma 7 *Let* $V_0, \ldots, V_n$ *and* Q_n *be as above. Assume* Q_n *is an averaging operator from* $L^1(I)$ *to* Φ_n. *If* $0 \leq k < n$ *and if* $m(V_k \cap V_{k+1}) > 0$, *then* $V_k = S_k$ *and* $V_{k+1} = S_{k+1}$.

PROOF. Let $0 \le k < n$ and assume $m(V_k \cap V_{k+1}) > 0$. Applying equation (1) with $E = V_k \cap V_{k+1}$, we see that

$$m(V_k \cap V_{k+1}) = \frac{m(V_k \cap V_{k+1})}{m(V_k)} \frac{m(S_k)}{2} + \frac{m(V_k \cap V_{k+1})}{m(V_{k+1})} \frac{m(S_{k+1})}{2}.$$

It follows that $2 = m(S_k)/m(V_k) + m(S_{k+1})/m(V_{k+1})$ and so $V_k = S_k$ and $V_{k+1} = S_{k+1}$. □

Lemma 8 *Let $V_0, \ldots, V_n$ and Q_n be as above and assume $n > 1$. Assume Q_n is an averaging operator from $L^1(I)$ to Φ_n. If $V_k = S_k$ for some $0 < k < n$, then $Q_n = \alpha_n$.*

PROOF. Let $0 < j < n$ and assume $V_j = S_j$. Applying equation (1) with $E = [x_j, x_{j+1}]$, we see that

$$m([x_j, x_{j+1}]) = \frac{m([x_j, x_{j+1}])}{2} + \frac{m([x_j, x_{j+1}] \cap V_{j+1})}{m(V_{j+1})} \frac{m(S_{j+1})}{2}.$$

Hence, $m(V_j \cap V_{j+1}) \ge m([x_j, x_{j+1}] \cap V_{j+1}) > 0$. By Lemma 7, $V_{j+1} = S_{j+1}$. By a similar argument, we see that $V_{j-1} = S_{j-1}$. Thus if $V_k = S_k$ for some $0 < k < n$, then $V_i = S_i$ for $i = 0, 1, \ldots, n$ and so $Q_n = \alpha_n$. □

Theorem 9 *Assume Q_n is an averaging operator from $L^1(I)$ to Φ_n. Then either $Q_n = \alpha_n$ or $Q_n = \beta_n$.*

PROOF. Suppose $Q_n \ne \beta_n$. By Lemma 6, we may choose k such that $m(V_k \cap V_{k+1}) > 0$ and such that $0 \le k < n$. By Lemma 7, we have $V_k = S_k$ and $V_{k+1} = S_{k+1}$. If $n = 1$, then $V_0 = S_0$ and $V_1 = S_1$ and so $Q_n = \alpha_n$. Suppose $n > 1$. By Lemma 8, we have $Q_n = \alpha_n$ since either $0 < k < n$ and $V_k = S_k$ or $0 < k + 1 < n$ and $V_{k+1} = S_{k+1}$. □

4 Some Averaging Operators on $L^1(I \times I)$

We use the standard *Kuhn triangulation* of the domain $I \times I$. Divide the square $I \times I$ into n^2 equal sub-squares $I_i \times I_j = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ with area $h^2 = 1/n^2$. Then divide each $I_i \times I_j$ into two simplicies

$$\text{co}\{(x_{i-1}, y_{j-1}), (x_{i-1}, y_j), (x_i, y_j)\}, \quad \text{co}\{(x_{i-1}, y_{j-1}), (x_i, y_{j-1}), (x_i, y_j)\},$$

where co A denotes the convex hull of the set A. Thus, we obtain a triangulation T_h of $I \times I$ into a family of $2n^2$ triangles and each triangle has area $h^2/2$.

Let Δ_h be the space of continuous piecewise linear functions associated with the triangulation T_h. Let φ_{ij} be the unique function in Δ_h such that φ_{ij} is 1 at the node (x_i, y_j) and 0 at all the other nodes of T_h. The $(n+1)^2$ *nodal functions* $\{\varphi_{ij}\}_{i,j=0}^n$ form a canonical basis for Δ_h and $\sum_{i=0}^n \sum_{j=0}^n \varphi_{ij} = 1$.

Let T be a continuous linear operator from $L^1(I \times I)$ to Δ_h. There exist $w_{ij} \in L^\infty(I \times I)$ for $0 \leq i, j \leq n$ such that

$$T(f) = \sum_{i=0}^n \sum_{j=0}^n \langle f, w_{ij} \rangle \varphi_{ij} \quad \text{for each} \quad f \in L^1(I \times I).$$

Again Theorem 1(2) can be strengthened. As before one can show that if T is nonnegative then $w_{ij} \geq 0$ a.e. for $0 \leq i, j \leq n$. Besides, like Theorem 3 (3), T preserves integrals if and only if

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \frac{2w_{i-1,j-1} + w_{i,j-1} + w_{i-1,j} + 2w_{i,j}}{6} = 1 \text{ a.e..}$$

Let S_{ij} be the closed support of φ_{ij} and let V_{ij} be a closed convex subset of S_{ij} such that $m(V_{ij}) > 0$ for $0 \leq i, j \leq n$. Define $Q_h : L^1(I \times I) \to \Delta_h$ by

$$Q_h(f) = \sum_{i=0}^n \sum_{j=0}^n \left\langle f, \frac{1}{m(V_{ij})} \chi_{V_{ij}} \right\rangle \varphi_{ij} \quad \text{for each} \quad f \in L^1(I \times I).$$

Then Q_h satisfies the conditions in (1) and (2) of Theorem 1. We wish to find $\{V_{ij}\}_{i,j=0}^n$ such that Q_h is an averaging operator from $L^1(I \times I)$ to Δ_h, that is, Q_h satisfies the condition (3) in Theorem 1.

Example 10 *Set* $w_{ij} = \frac{1}{m(S_{ij})} \chi_{S_{ij}}$ *for* $0 \leq i, j \leq n$. *Define* $\alpha_h : L^1(I \times I) \to \Delta_h$ *by*

$$\alpha_h(f) = \sum_{i=0}^n \sum_{j=0}^n \langle f, w_{ij} \rangle \varphi_{ij} \quad \textit{for each} \quad f \in L^1(I \times I).$$

Using Theorem 1, it is easy to check that α_h *is an averaging operator from* $L^1(I \times I)$ *to* Δ_h.

Note. The numerical scheme α_h was developed in [4] to compute absolutely continuous invariant measures associated with two dimensional transformations.

Now the question is whether we can construct an averaging operator Q_h such that $m(V_{ij} \cap V_{kl}) = 0$ whenever $(i, j) \neq (k, l)$. Because of Theorem 2, all boils down to finding $\{V_{ij}\}_{i,j=0}^n$ such that $\langle \varphi_{ij}, 1 \rangle = m(V_{ij})$ for each $0 \leq i, j \leq n$. The answer is yes, but we first show that a most intuitive construction of

$\{V_{ij}\}_{i,j=0}^{n}$ fails. Let

$$V_{ij} = (I \times I) \cap \left(\left[x_i - \frac{h}{2}, x_i + \frac{h}{2} \right] \right) \times \left(\left[y_j - \frac{h}{2}, y_j + \frac{h}{2} \right] \right), \quad 0 \le i, j \le n.$$

But the corresponding Q_h fails to be integral preserving. It fails at the four corner nodes. For example, $m(V_{nn}) = h^2/4$, but $\langle \varphi_{nn}, 1 \rangle = h^2/3$. Hence by Theorem 2, Q_h is not an averaging operator.

It turns out that a correct approach is to use a centroid of each triangle in T_h. We construct W_{ij} as the convex hull of the centroids of the triangles in S_{ij}. The construction of W_{ij} is shown in Figure 1.

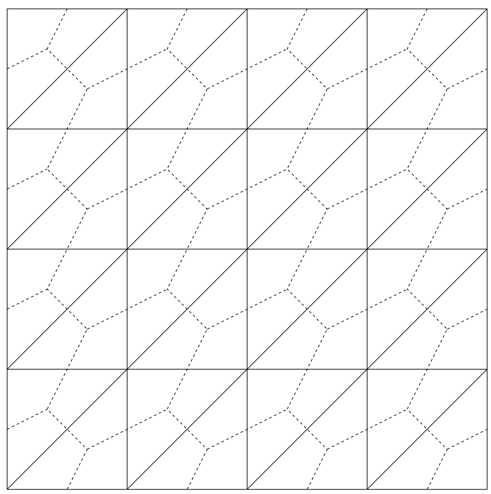

Fig. 1. Partitioning of the Unit Square for $n = 4$

Example 11 *Set* $w_{ij} = \frac{1}{m(W_{ij})} \chi_{W_{ij}}$ *for* $0 \le i, j \le n$*. Define* $\beta_h : L^1(I \times I) \to \Delta_h$ *by*

$$\beta_h(f) = \sum_{i=0}^{n} \sum_{j=0}^{n} \langle f, w_{ij} \rangle \, \varphi_{ij} \quad \textit{for each} \quad f \in L^1(I \times I).$$

Now we prove that β_h *is an averaging operator from* $L^1(I \times I)$ *to* Δ_h*.*

Note. From the theoretical analysis in [3] and the fact that each W_{ij} is a subset of S_{ij} with much smaller area, one can see that the numerical method based on β_h has a better convergence property than α_n in the computation of two dimensional absolutely continuous invariant measures; see [2] for more details on approximations of invariant measures.

PROOF. By Theorem 2, it suffices to show that $\langle \varphi_{ij}, 1 \rangle = m(W_{ij})$ for $0 \le i, j \le n$. There are four cases to consider.

Case 1 $(i, j) = (0, 0)$ or $(i, j) = (n, n)$.
We consider the case $(i, j) = (n, n)$. Notice that $\langle \varphi_{nn}, 1 \rangle = h^2/3$. From Figure 2 we see that W_{nn} is a pentagon $ABCEF$ and it is made of the square $ABDF$ of dimension $h/2$ by $h/2$ and two congruent triangles BCD and DEF whose base and height are $h/2$ and $(h/2 - h/3)$, respectively. Hence

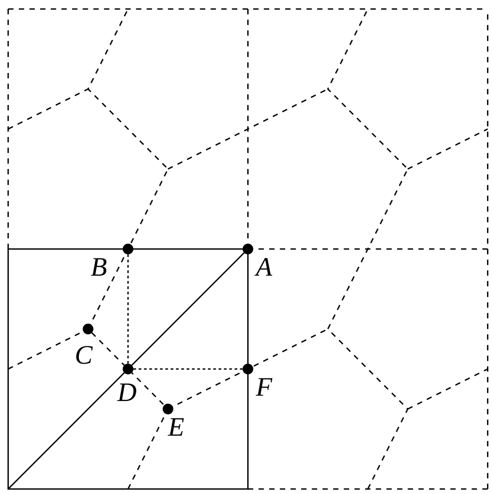

Fig. 2. Upper Right Corner Case

$$m(W_{nn}) = \frac{h}{2} \cdot \frac{h}{2} + 2 \cdot \frac{1}{2} \cdot \frac{h}{2} \left(\frac{h}{2} - \frac{h}{3} \right) = \frac{h^2}{3}.$$

Case 2 $(i, j) = (0, n)$ or $(i, j) = (n, 0)$.
We consider the case $(i, j) = (0, n)$. Notice that $\langle \varphi_{0n}, 1 \rangle = h^2/6$. As in Case 1 one can show that

$$m(W_{0n}) = \frac{h}{3} \cdot \frac{h}{3} + 2 \cdot \frac{1}{2} \cdot \left(\frac{h}{2} - \frac{h}{3} \right) = \frac{h^2}{6}.$$

Case 3 $1 \le i, j \le n - 1$ (Interior Nodes).
Notice in this case that $\langle \varphi_{ij}, 1 \rangle = h^2$. From Figure 3, we see that W_{ij} is a hexagon $ABCDEF$ and it is made of the parallelogram $BCEF$, whose base is h and height is $2h/3$, and two congruent triangles ABF and CDE whose base and height are h and $h/3$, respectively. Thus

$$m(W_{ij}) = h \cdot \frac{2}{3}h + 2 \cdot \frac{1}{2} \cdot h \cdot \frac{h}{3} = h^2.$$

Case 4 All other cases (Boundary Nodes except Four Corner Nodes).

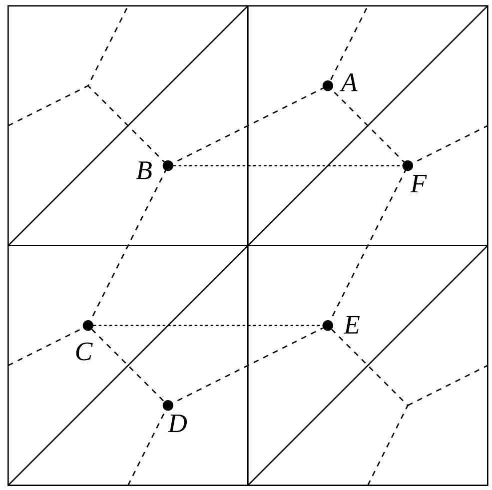

Fig. 3. Interior Case

Notice that in this case $\langle \varphi_{i,j}, 1 \rangle = h^2/2$. As in Case 3 one can verify that

$$m(W_{ij}) = h \cdot \frac{h}{3} + \frac{1}{2} \cdot h \cdot \frac{1}{3} h = \frac{h^2}{2}.$$

So by Theorem 2, β_h is an averaging operator. $\square$

Note. It is an open question whether if Q_h is an averaging operator from $L^1(I \times I)$ to Δ_h then $Q_h = \alpha_h$ or $Q_h = \beta_h$.

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