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Recommended Citation

Reutskiy, S. Y., Chen, C., Tian, H. Y. (2008). A Boundary Meshless Method Using Chebyshev Interpolation and Trigonometric Basis Function for Solving Heat Conduction Problems. *International Journal for Numerical Methods in Engineering, 74*(10), 1621-1644.
Available at: https://aquila.usm.edu/fac_pubs/1513

NME 2230
pp: 1–24 (col.fig.: Nil)
PROD. TYPE: COM
ED: Vijaya
PAGN: Veena – SCAN: Monica

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng (2007)
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.2230

A boundary meshless method using Chebyshev interpolation and trigonometric basis function for solving heat conduction problems

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SUMMARY

A boundary meshless method has been developed to solve the heat conduction equations through the use of a newly established two-stage approximation scheme and a trigonometric series expansion scheme to approximate the particular solution and fundamental solution, respectively. As a result, no fundamental solution is required and the closed form of approximate particular solution is easy to obtain. The effectiveness of the proposed computational scheme is demonstrated by several examples in 2D and 3D. We also compare our proposed method with the finite-difference method and the other meshless method showed in Šarler and Vertnik (*Comput. Math. Appl.* 2006; **51**:1269–1282). Excellent numerical results have been observed. Copyright © 2007 John Wiley & Sons, Ltd.

Received 8 February 2007; Revised 23 July 2007; Accepted 28 September 2007

KEY WORDS: method of fundamental solutions; particular solution; Chebyshev interpolation; C-expansion; diffusion equations

1. INTRODUCTION

During the past decade, meshless methods have attracted great attention in the area of scientific computing. Various types of numerical techniques for solving science and engineering problems without domain discretization have been developed. One of the common goals of developing meshless methods is to solve a given set of partial differential equations (PDEs) with minimum human and computational costs. Hence, other than the accuracy and efficiency, the simplicity of the implementation of the developed meshless algorithm is also of great importance.

Among all the proposed meshless methods, Trefftz-type methods have been vigorously re-investigated in recent years. A special type of Trefftz method is the method of fundamental solutions

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(MFSs) which has been extended to solve various elliptic and time-dependent problems [1–3]. However, the fundamental solution of a given differential equation is not always available. The ill conditioning of the MFS and the location of source points are also issues to be resolved. As a result, despite the effectiveness of the MFS, its applicability is somehow limited. To alleviate these difficulties, we introduce the method of approximate fundamental solutions (MAFSs) [4] in which the trial function is approximated by the truncated trigonometric series. In this way, the fundamental solution of the given partial differential operator is not required. For non-homogeneous equation, getting a closed form particular solution is not a trivial task [2, 3]. Recently, a novel and effective numerical technique for approximating the particular solutions of a new class of differential equations has been developed [5]. It combines the advantages of polynomial interpolation and trigonometric approximation to the source function so that the closed form approximate particular solution can be easily and accurately obtained. This technique includes two major steps: (i) approximating the source function using Chebyshev polynomials, (ii) Chebyshev interpolants are further approximated by a *C-Expansion* approximation scheme, a trigonometric-based scheme. One of the advantages of this technique is that the closed form of approximate particular solution can be easily obtained. This approach is also highly accurate due to the spectral convergence of Chebyshev interpolation. We would like to note that such a two-stage approximation scheme is not necessary from the point of view of pure function approximation. However, our ultimate goal is to develop an approximation scheme so that the approximate particular solution can be evaluated efficiently for a more general class of differential operators. Furthermore, it is interesting that an approximate fundamental solution can be obtained in a way similar to the derivation of particular solutions using the same trigonometric basis functions [4, 5]. Encouraged by the success of these novel approaches for solving elliptic problems [5], we extend these techniques to solve transient heat conduction problems.

There are various numerical approaches to solving heat conduction problems. The common approaches are (i) time–space separation [6]; (ii) Laplace transform or Fourier transform to remove the time dependence [7, 8]; (iii) time difference scheme [9, 10]. There are advantages and disadvantages in each approach. Apparently, the time difference scheme is the most popular approach being applied for solving time-dependent problems. In this paper, we will focus on this approach.

We consider the following initial boundary value problem

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = L[u(\mathbf{x},t)] + f(\mathbf{x},t), \quad \mathbf{x}\in\Omega\subset\mathbb{R}^d, \ \ t>0, \ \ d=1,2,3 \tag{1}$$

$$B[u(\mathbf{x},t)] = g(\mathbf{x},t), \quad \mathbf{x}\in\partial\Omega \tag{2}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega \tag{3}$$

where L is a time-independent linear differential operator, B is a boundary operator, and Ω is a simply connected domain bounded by a simple closed curve $\partial\Omega$.

Finite difference in time transforms (1)–(3) to a sequence of elliptic equations. Using the likewise Crank–Nicholson (C–N) scheme with the second order approximation in time

$$\frac{u^{j+1}(\mathbf{x}) - u^j(\mathbf{x})}{\Delta t} = \frac{1}{2}[L[u^{j+1}(\mathbf{x})] + L[u^j(\mathbf{x})]] + f^{j+1/2}(\mathbf{x})$$

DOI: 10.1002/nme

we obtain a sequence of inhomogeneous equations

$$(L-p)[u^{j+1}(\mathbf{x})] = -(L+p)[u^{j}(\mathbf{x})] - 2f^{j+1/2}(\mathbf{x}), \quad \mathbf{x}\in\Omega, \quad j=0,1,2,\ldots \tag{4}$$

$$B[u^{j+1}(\mathbf{x})] = g^{j+1}(\mathbf{x}), \quad \mathbf{x}\in\partial\Omega \tag{5}$$

where $u^j(\mathbf{x})=u(\mathbf{x},t^j)$, $f^{j+1/2}(\mathbf{x})=f(\mathbf{x},(t^j+t^{j+1})/2)$, $g^{j+1}(\mathbf{x})=g(\mathbf{x},t^{j+1})$, $t^j=j\Delta t$, Δt is a time step, and $p=2/\Delta t$. The boundary value problem (4)–(5) can be solved by the method of particular solution in which an effective way of evaluating an approximate particular solution is crucial. We refer the readers to the references in [2, 3] for further details for the evaluation of particular solutions. In this paper, we employ the newly established Chebyshev interpolation and *C-Expansion* approximation scheme, which has been recently published in this journal, to evaluate the approximate particular solution [5].

For a homogeneous equation, many boundary meshless methods can be applied [2, 11]. According to the MAFS, an approximate solution of (4) at the $j+1$th time step can be expressed in the form

$$u^{j+1}(\mathbf{x}) = u_p^{j+1}(\mathbf{x}) + \sum_{k=1}^{K} a_k^{j+1}\Psi_k(\mathbf{x}) \tag{6}$$

where $u_p^{j+1}(\mathbf{x})$ is a particular solution at the $j+1$th time layer, and $\Psi_k(\mathbf{x})$, $k=1,\ldots,K$ are the approximate fundamental solutions. Note that, if differential operators of the form

$$L = \sum_{k_1,k_2=0}^{l} A_{k_1,k_2}\frac{\partial^{2k_1+2k_2}}{\partial x_1^{2k_1}\partial x_2^{2k_2}}, \quad A_{k_1,k_2}=\text{const.} \tag{7}$$

are considered and if the right-hand side of (4) is approximated by the trigonometric series, e.g.

$$-(L+p)[u^j(\mathbf{x})] - 2f^{j+1/2}(\mathbf{x}) \simeq \sum_{n=1}^{M}\sum_{m=1}^{M} H_{n,m}^{(j)} \sin\left(n\pi\frac{x+1}{2}\right)\sin\left(m\pi\frac{y+1}{2}\right) \tag{8}$$

then the particular solution u_p^{j+1} can be written in the analytic form.

In the MAFS, the trial functions $\Psi_k(\mathbf{x})$ in (6) satisfy $(L-p)[\Psi_k]=I(x,y,\xi_k,\eta_k)$, where $I(x,y,\xi_k,\eta_k)$ is a 2D delta-shaped function, in the infinite domain. The detailed formulation of the MAFS will be given in the next section.

The organization of this paper is as follows. In Section 2, we briefly introduce the basis functions of the MAFS and provide three regularization methods for the formulation of MAFS. In Section 3, a finite-difference time-stepping scheme is employed to reduce the given heat conduction problem to a sequence of modified Helmholtz equations. In Section 4, the method of particular solution has been employed to solve the modified Helmholtz equation for each time step. To demonstrate the effectiveness of the proposed approach in this paper, numerical examples of heat conduction problems in regular and irregular domains in 2D and 3D are given in Section 5.

DOI: 10.1002/nme

2. BASIS FUNCTIONS OF THE MAFS

In this section, we briefly introduce the formulation of trial function using MAFs. The Laplace operator can be written as the sum of two 1D operators

$$\nabla^2 = -l^{(x)} - l^{(y)}, \quad l^{(x)} = -\frac{\partial^2}{\partial x^2}, \quad l^{(y)} = -\frac{\partial^2}{\partial y^2}$$

We use minus sign before the operators in order to obtain a positive spectra.

We start the construction of the MAFS basis functions with the formal Fourier series for Dirac's delta function. It is well known that the eigenfunctions

$$\varphi_n(x) = \sin(\lambda_n(x+1)), \quad \lambda_n = \frac{n\pi}{2}, \quad n = 1, 2, \ldots \tag{9}$$

are the solutions of the following *Sturm–Liouville* problem on the interval $[-1, 1]$:

$$l^{(x)}\varphi = \lambda^2\varphi, \quad \varphi(-1) = \varphi(1) = 0 \tag{10}$$

The eigenfunctions $\varphi_n(x)$ form an orthogonal system on $[-1, 1]$ with the scalar product

$$\int_{-1}^{1} \varphi_n(x)\varphi_m(x)\,\mathrm{d}x = \delta_{n,m} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

Thus, Dirac's delta function can be formally expressed as follows:

$$\delta(x-\xi) = \sum_{n=1}^{\infty} \varphi_n(\xi)\varphi_n(x) \tag{11}$$

Note that this series diverges at any point in the interval $[-1, 1]$. With various kinds of regularization techniques, a smooth delta-shaped function, $I(x, \xi)$, can be constructed through the formal series expansion (11); i.e. the regularized delta-shaped functions have the form

$$I(x, \xi) = \sum_{n=1}^{M} r_n(M, \gamma)\varphi_n(\xi)\varphi_n(x) \tag{12}$$

Note that $r_n(M, \gamma)$ is the regularization factor that can be obtained by the following regularization techniques:

1. The Lanczos regularization technique:

$$r_n(M, \gamma) = [\sigma_n(M)]^{\gamma}, \quad \sigma_n(M) = \frac{\sin[v(n, M)]}{v(n, M)}, \quad v(n, M) = \frac{n\pi}{M+1} \tag{13}$$

   where $\sigma_n(M)$ are called the Lanczos sigma factors that are used to overcome Gibb's phenomenon in the Fourier series expansion of non-smooth functions [12]. This technique was employed in [4, 13] for solving stationary and time-dependent problems. The parameters M and γ should be taken in coupling. In all the calculations presented in this paper, we use $\gamma = 4, 6, 8, 12, 14, 16, 18$ for $M = 10, 20, 30, 40, 50, 80, 100$. This choice of the regularization parameter is found to be close to the optimal one.

DOI: 10.1002/nme

2. The Riesz regularization technique:

$$r_n(M,\gamma)=\left(1-\frac{\lambda_n^2}{\lambda_{M+1}^2}\right)^{\gamma} \tag{14}$$

This was proposed in [14] for solving elliptic PDEs with scattered data in irregular domains.

3. The Abel regularization technique: Let us consider the following equation:

$$\frac{\partial w(t,x,\xi)}{\partial t}=\frac{\partial^2 w(t,x,\xi)}{\partial x^2} \tag{15}$$

with the initial distribution

$$w(0,x,\xi)=\delta(x-\xi)=\sum_{n=1}^{\infty}\varphi_n(\xi)\varphi_n(x) \tag{16}$$

We consider diffusion of the initial delta distribution. Let us look for a solution in the same form of the series

$$w(t,x,\xi)=\sum_{n=1}^{\infty}w_n(t)\varphi_n(\xi)\varphi_n(x) \tag{17}$$

From (15), we obtain

$$w_n(t)=\exp(-\lambda_n^2 t)$$

The time t plays the role of the regularizing parameter. We have

$$w(t,x,\xi)\to 0 \quad \text{when } t\to\infty$$

for all x,ξ. We set the regularizing coefficients in the following way:

$$r_n(\alpha)=\exp(-\alpha\lambda_n^2) \tag{18}$$

i.e. α is the time moment in which we consider $w(t,x,\xi)$. This summation is also known as a heat-kernel regularized sum or a generalized Dirichlet series [15].

In the practical applications we use the truncated series

$$I(x,\xi)=\sum_{n=1}^{M}r_n(\alpha)\varphi_n(\xi)\varphi_n(x) \tag{19}$$

where $r_{M+1}(\alpha)=\varepsilon$ is a small prescribed value. In all the numerical results presented in this paper, we use $\alpha=0.005$–0.01 for $M=30$, $\alpha=0.001$–0.005 for $M=50$, and $\alpha=0.0012$–0.0015 for $M=100$.

The graph of 1D smooth approximations $I(x,\xi)$ of Dirac's delta function using the Lanczos regularization techniques is shown in Figure 1. The graphs of $I(x,\xi)$ using the Riesz and Abel regularizations are similar to that in Figure 1. Note that we place here the graphics of the scaled values $I(x,\xi)/I(\xi,\xi)$. As shown in the figure, $I(x,\xi)$ can approximate Dirac's delta function $\delta(x-\xi)$ as closely as we want by properly choosing the regularization factors. One important distinction between these two functions is that $I(x,\xi)\in C^{\infty}$ while $\delta(x-\xi)$ is not a differentiable function.

DOI: 10.1002/nme

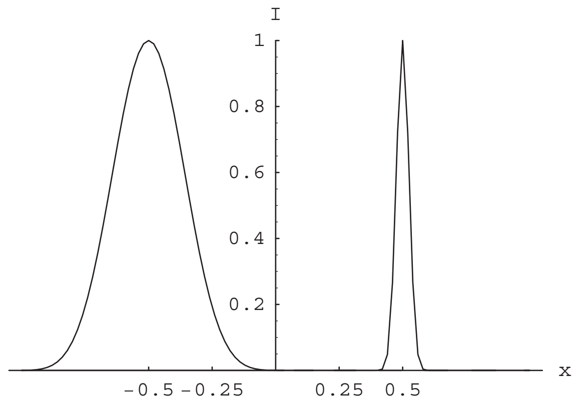

Figure 1. The Lanczos regularization. Approximations of the delta-function with $M=20$, $l=6$ (left) and $M=200$, $l=18$ (right).

The 2D delta-shaped functions can be obtained through the tensor product of the 1D ones; i.e.

$$I(x,y,\xi,\eta)=I(x,\xi)I(y,\eta)=\sum_{n,m=1}^{M} c_{n,m}(\xi,\eta)\varphi_n(x)\varphi_m(y) \tag{20}$$

where the coefficients $c_{n,m}(\xi,\eta)$ depend on the regularizing technique used.

In Figure 2, we plot the graphs of 2D delta-shaped basis functions $I(x,y,\xi,\eta)$ using Abel's regularization technique with $M=20$, $\alpha=0.01$ and $M=200$, $\alpha=0.0003$, respectively. They are infinitely differentiable and are not 'identical' to zero in any interval. However, by visual observation, $I(x,y,\xi,\eta)$ differs from zero only inside some neighborhood of the center point (ξ,η). In a way, $I(x,y,\xi,\eta)$ can be characterized as 'approximate locally supported functions'.

Regardless of the type of regularization technique, all the approximations $I(x,y,\xi,\eta)$ have the form of a truncated series over $\varphi_n(x)\varphi_m(y)$. The approximate fundamental solution $\Psi(x,y,\xi,\eta)$ of a given PDE can be obtained by using $I(x,y;\xi,\eta)$ as the forcing term. For example, for the modified Helmholtz equation, we have

$$(\nabla^2-p)\Psi(x,y,\xi,\eta)=I(x,y,\xi,\eta) \tag{21}$$

Technically, $\Psi(x,y,\xi,\eta)$ in (21) is not only an approximate fundamental solution, but also a particular solution. Since $I(x,y,\xi,\eta)$ is a linear combination of trigonometric functions, the particular solution $\Psi(x,y,\xi,\eta)$ in (21) has to be a linear combination of trigonometric functions also. Hence, $\Psi(x,y,\xi,\eta)$ has to be in the following form:

$$\Psi(x,y,\xi,\eta)=\sum_{m,n=1}^{M} D_{n,m}(\xi,\eta)\varphi_n(x)\varphi_m(y) \tag{22}$$

where $D_{n,m}$ are to be determined. Substituting (20) and (22) in (21), by the method of undetermined coefficients, we have

$$D_{n,m}(\xi,\eta)=-\frac{c_{n,m}(\xi,\eta)}{\lambda_n^2+\lambda_m^2+p} \tag{23}$$

DOI: 10.1002/nme

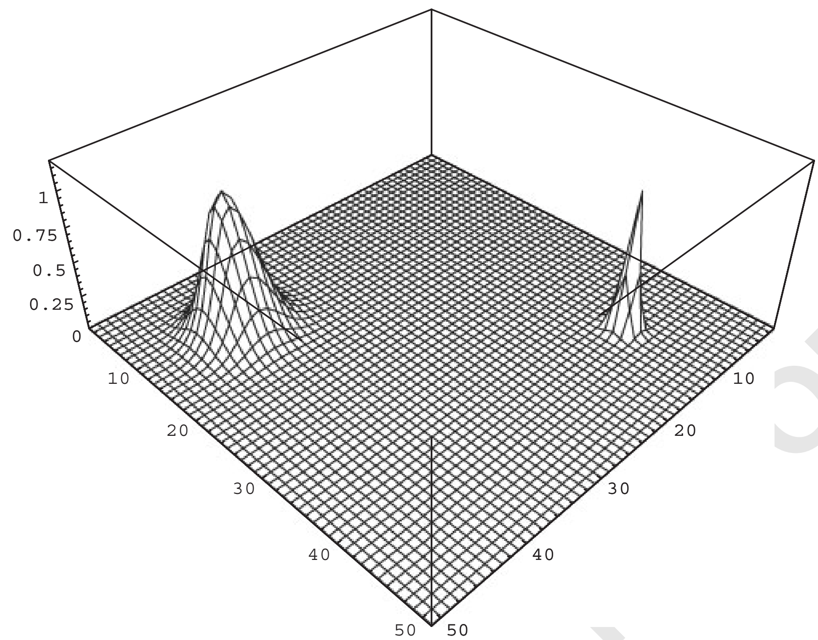

Figure 2. The graphs of 2D delta-shaped functions by Abel's regularization technique with $M = 20, \alpha = 0.01$ and $M = 200, \alpha = 0.0003$, respectively.

The basis functions of the MAFS for the general linear differential operators such as (7) can be obtained in a similar way.

3. FINITE-DIFFERENCE TIME-STEPPING ALGORITHM

As mentioned above, the trigonometric functions $\varphi_n(x)$ and their products form a natural basis for the problems we considered. Through this section we consider the following 2D heat conduction equation:

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} = \chi \nabla^2 u(\mathbf{x}, t) + f(\mathbf{x}, t) \tag{24}$$

where χ is a constant.

Let us recall C–N scheme used in the finite-difference approximation of the parabolic equations. In particular, C–N scheme for the heat equation in one spatial dimension $\partial_t u = \chi \partial_{xx} u$ can be expressed in the form

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{\chi}{2}[D_{xx}[u^{n+1}] + D_{xx}[u^n]]$$

DOI: 10.1002/nme

where D_{xx} denotes the finite-difference approximation of the second derivative. In the case $\partial_t u = \chi(\partial_{xx}u + \partial_{yy}u)$ with two spatial dimensions, the C–N scheme can be written in a similar form

$$\frac{u^{n+1}-u^n}{\Delta t} = \frac{\chi}{2}[D_{xx,yy}[u^{n+1}] + D_{xx,yy}[u^n]]$$

where $D_{xx,yy}$ denotes the well-known finite-difference approximation of the Laplacian.

To approximate Equation (1), we follow a similar format as above

$$\frac{u^{n+1}-u^n}{\Delta t} = \frac{1}{2}[L_{xx,yy}[u^{n+1}] + L_{xx,yy}[u^n]] + f(\mathbf{x}, t^{n+1/2})$$

which approximates the PDE with second order in time at the time moment $t^{n+1/2}$. However, $L_{xx,yy}$ here denotes the spectral approximation instead of the finite-difference approximation for the space operator L in (1). Hence, strictly speaking, we do not use the C–N approximation scheme but the likewise C–N scheme that has the same second-order approximation in time and the average of the space operators on u^{n+1} and u^n. When $L = \chi\nabla^2$, using the likewise C–N scheme, we obtain

$$\frac{u^{n+1}-u^n}{\Delta t} = \frac{\chi}{2}(\nabla^2 u^{n+1} + \nabla^2 u^n) + f(\mathbf{x}, t^{n+1/2}) \tag{25}$$

i.e.

$$\nabla^2 u^{n+1} - \frac{2}{\chi\Delta t}u^{n+1} = -\nabla^2 u^n - \frac{2}{\chi\Delta t}u^n - \frac{2}{\chi}f(\mathbf{x}, t^{n+1/2}) \tag{26}$$

The inhomogeneous Helmholtz equation (26) can be rewritten as

$$\nabla^2 u^{n+1} - pu^{n+1} = h^n \tag{27}$$

where

$$p = \frac{2}{\chi\Delta t}, \quad h^n = -\nabla^2 u^n - pu^n - \frac{2}{\chi}f(\mathbf{x}, t^{n+1/2}) \tag{28}$$

We note that the evaluation of $\nabla^2 u^n$ in (26) at each time step is tedious and it may subject to the loss of accuracy due to the difficulty in evaluating the second derivative. To avoid these difficulties, we modify the above numerical scheme by applying the algorithm proposed by Ramachandran and Balakrishnan [16]. By introducing the intermediate variable

$$\widetilde{u} = \tfrac{1}{2}(u^{n+1} + u^n) \tag{29}$$

(26) can be re-casted in the following form:

$$\frac{2\widetilde{u} - 2u^n}{\Delta t} = \chi\nabla^2\widetilde{u} + f(\mathbf{x}, t^{n+1/2}) \tag{30}$$

or

$$\nabla^2\widetilde{u} - \frac{2}{\chi\Delta t}\widetilde{u} = -\frac{2}{\chi\Delta t}u^n - \frac{1}{\chi}f(\mathbf{x}, t^{n+1/2}) \tag{31}$$

The values of u^{n+1} can then be obtained from (29) after $\widetilde{u}$ is computed from (31). We employ both approaches (26) and (31) in the section of numerical results and observe little difference between them. In both cases, we need to evaluate the approximate particular solution.

DOI: 10.1002/nme

4. THE METHOD OF PARTICULAR SOLUTIONS

We would like to start this section with a brief review of the two-stage approximation scheme introduced in [5]. Here, we focus on the bivariate case since higher dimensional cases can be dealt with by a similar scheme. Let us consider a function $g(x,y)$. The main idea of this approximation scheme is to perform Chebyshev interpolation to $g(x,y)$ first and then to further approximate the Chebyshev polynomials by the C-expansion.

The Chebyshev interpolant using Gauss–Lobatto nodes [17–19] for rectangular $[a,b]\times[c,d]$ takes the form

$$\tilde{g}(x,y)=\sum_{i=0}^{N_x}\sum_{j=0}^{N_y} a_{ij}T_i\left(\frac{2x-b-a}{b-a}\right)T_j\left(\frac{2y-d-c}{d-c}\right) \tag{32}$$

where

$$a_{ij}=\frac{4}{N_xN_y\bar{c}_i\bar{c}_j}\sum_{p=0}^{N_x}\sum_{q=0}^{N_y}\frac{f(x_p,y_q)}{\bar{c}_p\bar{c}_q}\cos\left(\frac{\pi pi}{N_x}\right)\cos\left(\frac{\pi qj}{N_y}\right)$$

and $\bar{c}_0=\bar{c}_{N_x}=\bar{c}_{N_y}=2$, $\bar{c}_i=1$, $1\leqslant i\leqslant N_x-1$, and $\bar{c}_j=1$, $1\leqslant j\leqslant N_y-1$. Note that N_x and N_y are the numbers of Gauss–Lobatto nodes in the x and y directions, respectively. It is well known that the use of Gauss–Lobatto nodes will ensure the spectral convergence for the Chebyshev interpolation.

The above Chebyshev interpolation is followed by the C-expansion procedure. Instead of using the multi-dimensional generalization of the C-expansion procedure, we use 1D C-expansion since every term in the right-hand side of (32) is a product of 1D functions. For each 1D function $T_i((2x-b-a)/(b-a))$, its C-expansion is in the form

$$T_i\left(\frac{2x-b-a}{b-a}\right)\simeq\sum_{m=1}^{M} t_{i,m}\varphi_m(x)$$

where φ_m are given in (9) and $t_{i,m}$ are the expansion coefficients. Thus, the C-expansion for $T_i((2x-b-a)/(b-a))T_i((2y-d-c)/(d-c))$ is given by

$$T_i\left(\frac{2x-b-a}{b-a}\right)T_j\left(\frac{2y-d-c}{d-c}\right)\simeq\sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{M}} T_{m_1,m_2}^{i,j}\varphi_{m_1}(x)\varphi_{m_2}(y)$$

where $\mathbf{m}=(m_1,m_2)$, $\mathbf{M}=(M_1,M_2)$, $\mathbf{1}=(1,1)$, $T_{m_1,m_2}^{i,j}=t_{i,m_1}t_{j,m_2}$.

By combining approximations for all terms $a_{ij}T_iT_j$ in the right-hand side of (32), the two-stage approximation for the given function $g(x,y)$ gives the following form:

$$g(x,y)\simeq\sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{M}} G_{m_1,m_2}\varphi_{m_1}(x)\varphi_{m_2}(y)$$

where

$$G_{m_1,m_2}=\sum_{i=0}^{N_x}\sum_{j=0}^{N_y} T_{m_1,m_2}^{i,j} \tag{33}$$

DOI: 10.1002/nme

Hence, by the above two-stage approximation process, u^n in (28) can be approximated by the truncated trigonometric series:

$$u^n(\mathbf{x}) \simeq \sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{M}} U^n_{m_1,m_2}\varphi_{m_1}(x_1)\varphi_{m_2}(x_2) \tag{34}$$

Similarly, $f(\mathbf{x}, t^{n+1/2})$ in (28) can be approximated as follows:

$$f(\mathbf{x}, t^{n+1/2}) \simeq \sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{M}} F^n_{m_1,m_2}\varphi_{m_1}(x_1)\varphi_{m_2}(x_2) \tag{35}$$

where the coefficients $U^n_{m_1,m_2}$ and $F^n_{m_1,m_2}$ in (34) and (35) are to be determined by (33). As a result, the right-hand side of (27) can be expressed in the same form

$$h^n(\mathbf{x}) \simeq \sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{M}} H^n_{m_1,m_2}\varphi_{m_1}(x_1)\varphi_{m_2}(x_2) \tag{36}$$

where

$$H^n_{m_1,m_2} = (\lambda^2_{m_1} + \lambda^2_{m_2} - p)U^n_{m_1,m_2} - \frac{2}{\chi}F^n_{m_1,m_2} \tag{37}$$

and $\{\lambda_{m_1}, \lambda_{m_2}\}$ are given in (9).

Using the method of particular solutions [2], we can split the solution of (27) into the following form:

$$u^{n+1} = u^{n+1}_p + u^{n+1}_h$$

where u^{n+1}_p is a particular solution that does not necessarily satisfy the boundary condition (2) and u^{n+1}_h is the corresponding homogeneous solution. Since we approximate the right-hand side $h^n(\mathbf{x})$ of (27) by the truncated series, it is easy to find u^{n+1}_p in the analytic form

$$u^{n+1}_p(\mathbf{x}) = \sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{M}} U^{n+1}_{p,m_1,m_2}\varphi_{m_1}(x_1)\varphi_{m_2}(x_2) \tag{38}$$

where

$$U^{n+1}_{p,m_1,m_2} = -\frac{H^n_{m_1,m_2}}{\lambda^2_{m_1} + \lambda^2_{m_2} + p} \tag{39}$$

The homogeneous solution u^{n+1}_h satisfies the corresponding homogeneous Helmholtz equation

$$\nabla^2 u^{n+1}_h - p u^{n+1}_h = 0, \quad \mathbf{x} \in \Omega \tag{40}$$

$$B[u^{n+1}_h(\mathbf{x})] = g^{n+1}(\mathbf{x}) - B[u^{n+1}_p(\mathbf{x})], \quad \mathbf{x} \in \partial\Omega \tag{41}$$

where $g^{n+1}(\mathbf{x}) = g(\mathbf{x}, t^{n+1})$. Note that u^{n+1}_h can be approximated by the linear combination of the approximate fundamental solutions $\Psi(\mathbf{x}, \boldsymbol{\xi}_k)$ in (22), i.e.

$$u^{n+1}_h \simeq \sum_{k=1}^{K} q_k \Psi(\mathbf{x}, \boldsymbol{\xi}_k) \tag{42}$$

DOI: 10.1002/nme

where q_k are coefficients to be determined and the source points ξ_k are placed outside the solution domain Ω. By fitting the boundary condition in (41) using collocation method, q_k can be easily determined [2]. It is interesting that the particular solution and the fundamental solution have the same basis functions. After q_k are obtained, the solution at each time step can be obtained as follows:

$$u^{n+1}(\mathbf{x}) \simeq \sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{M}} U^{n+1}_{m_1,m_2} \varphi_{m_1}(x_1)\varphi_{m_2}(x_2) \tag{43}$$

where

$$U^{n+1}_{m_1,m_2} = U^{n+1}_{p,m_1,m_2} + \sum_{k=1}^{K} q_k^{n+1} D^k_{m_1,m_2} \tag{44}$$

$$D^k_{m_1,m_2}(\xi,\eta) = -\frac{c_{m_1,m_2}(\xi_k,\eta_k)}{\lambda^2_{m_1}+\lambda^2_{m_2}+p} \tag{45}$$

Starting from the initial condition by the two-stage interpolation scheme, we carry out the proposed calculations (34)–(45) at each time step.

5. NUMERICAL RESULTS

5.1. 2D cases

From our numerical experiments, the Riesz regularization technique (14) leads to the divergence of the solution. Hence, throughout this section we use Lanczos and Abel regularizing techniques only. In all the calculations presented in this section the source points are distributed on a fictitious boundary which is a circle with its center at $(0,0)$ and radius $R_s = 0.99$. The number of the collocation points on $\partial\Omega$ is taken as twice as many as that of the source points. We test our algorithm on problems with known exact solutions u_{ex}. We also compare some of the problems with the traditional finite difference method and meshless method in [20]. To validate the performance of the proposed algorithm, the mean square root (MSR) errors

$$\text{MSR} = \sqrt{\frac{1}{N_e}\sum_{i=1}^{N_e}[u(x_i,y_i,t)-u_{\text{ex}}(x_i,y_i,t)]^2}$$

are computed using a uniform 9×9 mesh for the square domain, where N_e is the total number of nodes on the mesh. In the case of irregular domain the test points are obtained using RNUF pseudorandom number generator from the Microsoft IMSL Library.

Since approximations (34)–(36) vanish on the boundary of the square $[-1,1]\times[-1,1]$ due to the zero boundary condition (10) of $\varphi_n(x)$, in the following we consider problems defined either on the square $[-0.5,0.5]\times[-0.5,0.5]$ or in an irregular region that is strictly inside the square $[-0.5,0.5]\times[-0.5,0.5]$ and bounded away from $[-1,1]\times[-1,1]$. If this is not the case originally, appropriate translation and scaling operations are required. In the following examples, the numbers of Chebyshev's polynomials for x and y directions are denoted, respectively, by N_x and N_y, the

DOI: 10.1002/nme

number of trigonometric harmonics in the C-expansion is denoted by M, and the number of source points for the MAFS is denoted by K.

Example 1
Let us consider the following diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = \nabla^2 u(x,y,t) + f(x,y,t), \quad (x,y)\in\Omega, \quad t>0 \tag{46}$$

where $\Omega=[-0.5,0.5]\times[-0.5,0.5]$. The initial condition, the Dirichlet boundary condition, and $f(x,y,t)$ are chosen such that the exact solution is

$$u_{\text{ex}}(x,y,t)=\sin[(x+0.5)(y+0.5)]\cos t$$

The numerical results with Lanczos regularization scheme for different time steps Δt are shown in Table I. The numerical results in the first four columns are obtained using $N_x=N_y=25$, $M=30$, $K=30$. Initially, the errors improved when Δt decreased. This is consistent with the well-known C–N scheme. This means that the error in the approximation of the PDE was dominated by these parameters. However, for $\Delta t\leqslant\frac{1}{16}$ the error does not improve with the further reduction of Δt. This implies that the error in the approximation of the PDE becomes the non-dominating one and the error in solution is caused by other reasons. To further reduce the error, we need to improve the approximation of the forcing term using Chebyshev and C-expansion scheme [5]. The results in the last column of Table I are obtained using $N_x=N_y=30$, $M=50$, and $K=50$.

The same problem is solved using the Abel regularization. We choose the same parameters N_x, N_y, M and K as in Table I. Some results are presented in Table II. The numerical results in the first four columns are obtained using $\alpha=0.005$ with $M=30$ and in the last column using $\alpha=0.002$ with $M=70$.

Table I. The MSR errors in Example 1 using the Lanczos regularization in a squared domain.

t	$\Delta t=\frac{1}{4}$	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{32}$	$\Delta t=\frac{1}{32}$
1	1.8×10^{-5}	2.1×10^{-5}	5.9×10^{-6}	7.6×10^{-6}	3.6×10^{-7}
2	6.3×10^{-5}	1.8×10^{-5}	8.0×10^{-6}	8.7×10^{-6}	8.8×10^{-7}
5	5.5×10^{-5}	1.4×10^{-5}	5.7×10^{-6}	6.8×10^{-6}	7.4×10^{-7}
10	5.3×10^{-5}	2.3×10^{-5}	1.7×10^{-5}	1.0×10^{-5}	7.8×10^{-7}

Table II. The MSR errors and CPU time in Example 1 using the Abel regularization in a squared domain.

t	$\Delta t=\frac{1}{4}$	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{32}$	$\Delta t=\frac{1}{32}$
1	1.8×10^{-5}	9.8×10^{-6}	2.4×10^{-6}	7.6×10^{-6}	3.6×10^{-7}
2	6.3×10^{-5}	1.6×10^{-5}	4.1×10^{-6}	8.7×10^{-6}	8.8×10^{-7}
5	5.5×10^{-5}	1.2×10^{-5}	3.6×10^{-6}	6.8×10^{-6}	7.4×10^{-7}
10	5.3×10^{-5}	1.5×10^{-5}	6.5×10^{-6}	1.0×10^{-5}	7.8×10^{-7}
CPU	7.4	14.3	28.0	55.6	137.0

DOI: 10.1002/nme

Table III. The MSR errors and CPU time in Example 1 using C–N approximation scheme of the FDM in a squared domain.

t	$\Delta t=\frac{1}{4}$	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{32}$	$\Delta t=\frac{1}{32}$
1	1.3×10^{-5}	4.6×10^{-6}	1.6×10^{-6}	8.0×10^{-7}	4.0×10^{-7}
2	5.0×10^{-5}	1.2×10^{-5}	3.3×10^{-6}	1.1×10^{-6}	8.6×10^{-7}
5	4.1×10^{-5}	1.0×10^{-5}	2.7×10^{-6}	8.5×10^{-7}	7.0×10^{-7}
10	4.4×10^{-5}	1.3×10^{-5}	3.5×10^{-6}	1.5×10^{-6}	8.9×10^{-7}
CPU	0.31	0.64	1.22	2.43	36.35

Table IV. The MSR errors and CPU time in Example 1 using the Abel regularization in a squared domain.

t	$\Delta t=\frac{1}{4}$	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{32}$	$\Delta t=\frac{1}{32}$
1	1.4×10^{-3}	4.6×10^{-4}	2.3×10^{-4}	1.1×10^{-4}	4.8×10^{-5}
2	5.4×10^{-3}	8.1×10^{-4}	1.7×10^{-4}	7.9×10^{-5}	3.6×10^{-5}
5	1.0×10^{-3}	3.8×10^{-4}	1.1×10^{-4}	5.5×10^{-5}	2.5×10^{-5}
10	5.4×10^{-3}	1.2×10^{-3}	3.5×10^{-4}	1.8×10^{-4}	7.5×10^{-5}
CPU	19.1	35.9	69.7	137.3	170.2

Table V. The MSR errors and CPU time in Example 1 using C–N approximation scheme of the FDM in a squared domain.

t	21×21 mesh	41×41 mesh	61×61 mesh	81×81 mesh
1	1.3×10^{-2}	3.5×10^{-3}	1.6×10^{-3}	9.1×10^{-4}
2	1.1×10^{-2}	2.7×10^{-3}	1.2×10^{-3}	6.9×10^{-4}
5	7.1×10^{-3}	1.8×10^{-3}	8.2×10^{-4}	4.7×10^{-4}
10	2.1×10^{-2}	5.4×10^{-3}	2.5×10^{-3}	1.4×10^{-3}
CPU	2.1	28.5	152.0	450.0

We compare our method with the classical finite difference method (FDM) using C–N scheme for this problem on $\Omega=[0,1]\times[0,1]$ in which the corresponding exact solution is $u_{\text{ex}}=\sin(xy)\cos(t)$. The numerical results in the first four columns of Table III are obtained using 21×21 uniform mesh while the results in the last column are obtained using 41×41 uniform mesh. By comparing the results shown in Tables II and III, it is evident that the FDM is more efficient than our method. However, the situation changes when we consider the problem with more complex exact solution which has more oscillations:

$$u_{\text{ex}}(x,y,t)=\sin(20xy)\cos(t)$$

on $\Omega=[0,1]\times[0,1]$ by the FDM, which corresponds to the exact solution

$$u_{\text{ex}}(x,y,t)=\sin[20(x+0.5)(y+0.5)]\cos(t)$$

DOI: 10.1002/nme

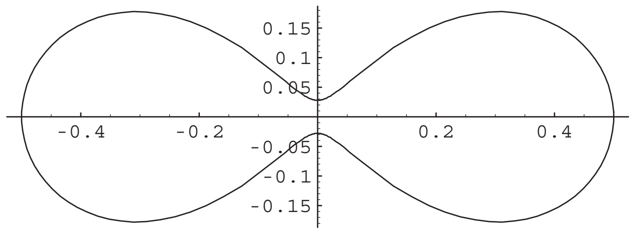

Figure 3. Oval of Cassini.

Table VI. The MSR errors in Example 2 using the Lanczos regularization in the domain of oval of Cassini.

t	$\Delta t=\frac{1}{4}$	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{16}$
1	4.6×10^{-6}	2.8×10^{-7}	2.2×10^{-7}	2.5×10^{-7}
2	1.3×10^{-5}	2.2×10^{-6}	5.3×10^{-7}	5.6×10^{-7}
5	5.7×10^{-6}	1.7×10^{-6}	4.5×10^{-7}	4.3×10^{-7}
10	7.2×10^{-6}	1.8×10^{-6}	3.9×10^{-6}	4.3×10^{-7}

on $\Omega=[-0.5,0.5]\times[-0.5,0.5]$ by our method. The computational results are shown in Tables IV and V. We observe that our proposed method is better than FDM. The results shown in the first four columns of Table IV are obtained using $N_x=N_y=30$, $M=70$, $K=50$, $\alpha=0.002$, and the results shown in the last column correspond to $N_x=N_y=30$, $M=100$, $K=50$, $\alpha=0.0012$. In Table V, the results are obtained using FDM with $\Delta t=\frac{1}{32}$ and the best results are obtained using 81×81 mesh which can be easily achieved using our proposed method with fairly large time step $\Delta t=\frac{1}{8}$ as shown in the second column of Table IV. Furthermore, the CPU times are 450 and 35.9 s for FDM and our proposed method, respectively. The CPU time by FDM increases rapidly with the growth of the mesh size. Hence, we conclude that our proposed method is more effective than FDM for the cases of complicated solution. For a regular domain such as square, FDM may have the advantage for solving smooth problems. However, for irregular domains or high-dimensional problems, our proposed meshless approach has the clear advantage due to the difficulty of meshing the domain using FDM.

Example 2

In this example, we consider the same problem as in the previous example for an irregular-shaped domain as depicted in Figure 3 which is represented by the following parametric equation:

$$x(t)=R(t)\cos(t),\quad y(t)=R(t)\sin(t)$$

where

$$R(t)=c^2\cos(2t)+\sqrt{a^4-c^4\sin^2(2t)},\quad 0\leqslant t\leqslant 2\pi$$

Here we set $c=0.353$, $a=\sqrt{0.25-c^2}$. The numerical results in Table VI are obtained using Lanczos regularization scheme with $N_x=N_y=25$, $M=30$, $K=30$ in the first three columns of the table and $N_x=N_y=30$, $M=50$, $K=30$ in the last column.

DOI: 10.1002/nme

Table VII. The MSR errors in Example 2 using the Abel regularization in the domain of oval of Cassini.

t	$\Delta t=\frac{1}{4}$	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{32}$
1	4.7×10^{-6}	4.2×10^{-7}	3.2×10^{-7}	8.3×10^{-8}
2	1.3×10^{-5}	2.2×10^{-6}	5.8×10^{-7}	1.8×10^{-7}
5	5.7×10^{-6}	1.8×10^{-6}	4.5×10^{-7}	1.6×10^{-7}
10	7.4×10^{-6}	5.5×10^{-6}	5.4×10^{-7}	1.3×10^{-6}

Similar to the last example, the numerical results obtained using Abel's regularization technique are shown in Table VII. The results in the first three columns are obtained using $M=30$ and $K=20$. The results in the last column correspond to $M=50$ and $K=25$. The regularization parameter is taken as $\alpha=0.005$ in all the calculations placed in this table. The collocation points are distributed uniformly, in terms of angle, on the boundary.

For the irregular-shaped domain, it is a major task to meshing the domain using FDM. Our proposed method is meshless and the solution procedure for the previous example with square domain and the current one with irregular domain is the same.

To further demonstrate the effectiveness of the proposed algorithm, in the following we show two examples with more complicated forcing term.

Example 3
Let us consider the problem with the same diffusion equation (46) in the domain $\Omega=[-0.5,0.5]\times[-0.5,0.5]$ with the initial and Dirichlet boundary conditions and $f(x,y,t)$ being artificially imposed in such a way that the exact solution is

$$u_{\text{ex}}(x,y,t)=f_1(x,y)\cos t \tag{47}$$

where

$$f_1(x,y)=\frac{3}{4}\exp\left(-\frac{(9x+2.5)^2+(9y+2.5)^2}{4}\right)+\frac{3}{4}\exp\left(-\frac{(9x+5.5)^2+(9y+5.5)^2}{49}\right)$$
$$+\frac{1}{2}\exp\left(-\frac{(9x-2.5)^2+(9y+1.5)^2}{4}\right)-\frac{1}{5}\exp(-(9x+0.5)^2-(9y-2.5)^2) \tag{48}$$

Note that $f_1(x,y)$ is a re-scaled Franke's function (see Figure 4) which is widely used as a benchmark problem for surface reconstruction [21]. The function $f_1(x,y)$ in (48) was originally defined in the unit square.

To obtain the numerical results, we couple the Chebyshev and C-expansion approximation with the Lanczos regularization scheme and let $N_x=N_y=45$, $M=100$, $K=100$. The numerical results with different time steps Δt are shown in Table VIII.

The numerical results shown in Table IX are obtained using Abel's regularization scheme with $\alpha=0.0012$, $M=100$, and $K=100$. As we can see, the numerical results in both approaches are excellent.

DOI: 10.1002/nme

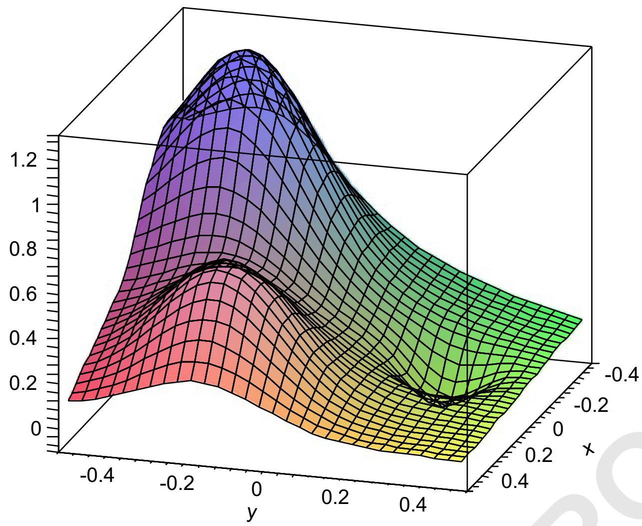

Figure 4. Franke's function $f_1(x, y)$.

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Table VIII. The MSR errors in Example 3 using the Lanczos regularization in the squared domain.

t	$\Delta t = \frac{1}{8}$	$\Delta t = \frac{1}{16}$	$\Delta t = \frac{1}{32}$
1	1.6×10^{-4}	4.3×10^{-5}	1.1×10^{-5}
2	1.4×10^{-4}	3.3×10^{-5}	8.2×10^{-6}
5	9.0×10^{-5}	2.2×10^{-5}	6.4×10^{-6}
10	2.7×10^{-4}	6.7×10^{-5}	3.0×10^{-4}

Table IX. The MSR errors in Example 3 using the Abel regularization in the squared domain.

t	$\Delta t = \frac{1}{8}$	$\Delta t = \frac{1}{16}$	$\Delta t = \frac{1}{32}$
1	1.6×10^{-4}	4.5×10^{-5}	1.4×10^{-5}
2	1.4×10^{-4}	3.4×10^{-5}	8.6×10^{-6}
5	9.0×10^{-5}	2.3×10^{-5}	2.0×10^{-6}
10	2.7×10^{-4}	6.7×10^{-5}	4.5×10^{-4}

Example 4

In this example we consider the same diffusion equation (46) in the domain $\Omega=[-0.5, 0.5]\times[-0.5, 0.5]$. The initial condition, the Dirichlet boundary condition, and $f(x, y, t)$ are imposed such that the exact solution is

$$u_{\rm ex}(x, y, t) = f_2(x, y)\cos t \tag{49}$$

DOI: 10.1002/nme

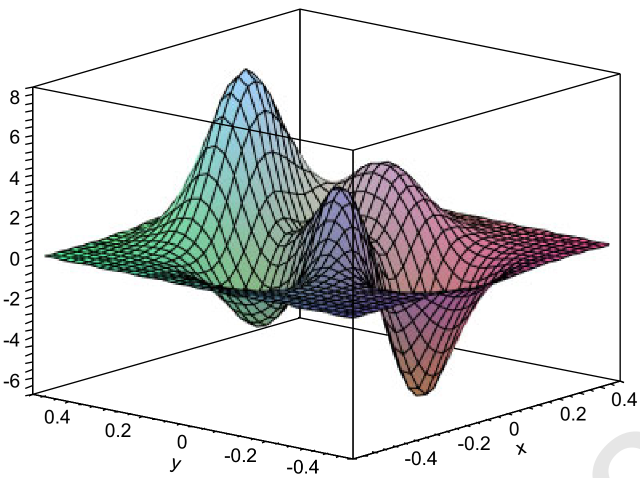

Figure 5. The PEAK's function $f_2(x, y)$.

Table X. The MSR errors in Example 4 using the Lanczos regularization in the squared domain.

t	$\Delta t = \frac{1}{8}$	$\Delta t = \frac{1}{16}$	$\Delta t = \frac{1}{32}$
1	1.8×10^{-3}	5.5×10^{-4}	1.5×10^{-4}
2	2.0×10^{-3}	4.2×10^{-4}	1.1×10^{-4}
5	1.1×10^{-3}	2.8×10^{-4}	7.2×10^{-4}
10	3.5×10^{-3}	8.7×10^{-4}	1.6×10^{-3}

where

$$f_2(x, y) = 3(1-6x)^2 \exp(-36x^2 - (6y+1)^2) - 10\left(\frac{6x}{5} - (6x)^3 - (6y)^5\right) \times \exp(-36(x^2+y^2)) - \frac{1}{3}\exp(-(6x+1)^2 - 36y^2) \tag{50}$$

is the re-scaled PEAKS function (see Figure 5) from MATLAB [22]. To evaluate the particular solutions, we use the Chebyshev and C-expansion approximation with Lanczos regularization scheme and let $N_x = N_y = 40$, $M = 100$, and $K = 100$. The numerical results with different time steps Δt are shown in Table X. Using Abel's regularization scheme with the parameters $\alpha = 0.0012$, $M = 100$, and $K = 100$, we obtain the numerical results as shown in Table XI.

We note that the FDM scheme cannot give any reasonable solution for the problems in Examples 3 and 4.

The previous examples have shown the effectiveness of our proposed method in solving problems with complicated domain or forcing term. In the next example, we will demonstrate how our method can be applied to problems with different types of boundary conditions. By examining

DOI: 10.1002/nme

Table XI. The MSR errors in Example 4 using the Abel regularization in the squared domain.

t	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{32}$
1	1.8×10^{-3}	5.5×10^{-4}	1.4×10^{-4}
2	2.0×10^{-3}	4.2×10^{-4}	1.1×10^{-4}
5	1.1×10^{-3}	2.8×10^{-4}	7.1×10^{-5}
10	3.5×10^{-3}	8.8×10^{-4}	1.2×10^{-3}

Table XII. The MSR errors in Example (5) using the Abel regularization in a squared domain with mixed boundary conditions.

t	$\Delta t=\frac{1}{4}$	$\Delta t=\frac{1}{8}$	$\Delta t=\frac{1}{16}$	$\Delta t=\frac{1}{32}$	$\Delta t=\frac{1}{32}$
1	9.4×10^{-5}	3.8×10^{-5}	4.9×10^{-5}	8.2×10^{-6}	1.8×10^{-6}
2	1.3×10^{-4}	7.0×10^{-5}	2.3×10^{-5}	9.3×10^{-6}	6.1×10^{-6}
5	1.2×10^{-4}	4.0×10^{-5}	7.2×10^{-6}	8.3×10^{-6}	4.7×10^{-6}
10	1.3×10^{-4}	3.1×10^{-5}	1.1×10^{-5}	1.8×10^{-5}	6.6×10^{-6}

Equations (40)–(42), we obtain a collocation system with the matrix $B[\Psi(\mathbf{x}_i,\xi_k)]$, i.e. the entries of the collocation matrix are obtained by applying the boundary operator B to the approximate fundamental solutions $\Psi(\mathbf{x},\xi_k)$, and then evaluating at the boundary collocation points $\mathbf{x}_i$. For example, the entries of the collocation matrix corresponding to Neumann boundary condition is $\partial\Psi(\mathbf{x}_i,\xi_k)/\partial n$.

Example 5

We consider the same diffusion equation in Example 1 with mixed boundary conditions in the domain $\Omega=[-0.5,0.5]\times[-0.5,0.5]$. The mixed boundary conditions are

$$u(-0.5,y,t)=g_1(y,t),\quad -0.5\leqslant y\leqslant 0.5,\quad t>0$$

$$u(0.5,y,t)=g_2(y,t),\quad -0.5\leqslant y\leqslant 0.5,\quad t>0$$

$$u(x,-0.5,t)+\frac{\partial u}{\partial n}(x,-0.5,t)=g_3(x,t),\quad -0.5\leqslant x\leqslant 0.5,\quad t>0$$

$$u(x,0.5,t)+\frac{\partial u}{\partial n}(x,0.5,t)=g_4(x,t),\quad -0.5\leqslant x\leqslant 0.5,\quad t>0$$

The boundary data $g_i, i=1,2,3,4$, the source function $f(x,y,t)$, and the initial condition $u(x,y,0)$ are given according to the exact solution $u_{\text{ex}}(x,y,t)=\sin[(x+0.5)(y+0.5)]\cos t$. The results using Abel regularization with various Δt are given in Table XII. The results in the first four columns are obtained using $N_x=N_y=25$, $M=50$, $K=30$, $\alpha=0.003$ and in the last column using $N_x=N_y=25$, $M=50$, $K=50$, $\alpha=0.003$. Again, the results in both tables are extremely accurate.

DOI: 10.1002/nme

Example 6

We consider the following diffusion equation:

$$\frac{\partial u}{\partial t}=\chi\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\quad (x,y)\in D,\ \ t>0 \tag{51}$$

with initial and boundary conditions

$$u(x,y,0)=1,\quad (x,y)\in D$$

$$u(x,y,t)=0,\quad (x,y)\in\partial D,\ \ t>0$$

where $D=[-0.2,0.2]\times[-0.2,0.2]$, and $\chi=5.8\times10^{-7}$. Note that the solution is not continuous at $t=0$.

To apply our method, we need to transform the domain D to the standard domain $\Omega=[-0.5,0.5]\times[-0.5,0.5]$. After the domain transformation, we have

$$\begin{aligned}\frac{\partial u}{\partial t}&=\frac{\chi}{0.16}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\quad (x,y)\in\Omega,\ \ t>0\\ u(x,y,0)&=1,\quad (x,y)\in\Omega\\ u(x,y,t)&=0,\quad (x,y)\in\partial\Omega,\ \ t>0\end{aligned} \tag{52}$$

The exact solution is given by [23] as follows:

$$u_{\mathrm{ex}}(x,y,t)=\frac{16}{\pi^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}L_{n,m}\cos[(2n+1)\pi x]\cos[(2m+1)\pi y]\exp(-D_{n,m}t)$$

where

$$L_{n,m}=\frac{(-1)^{n+m}}{(2n+1)(2m+1)}\quad\text{and}\quad D_{n,m}=\frac{\chi}{0.16}\frac{\pi^2}{4}\left[\frac{(2n+1)^2}{0.25}+\frac{(2m+1)^2}{0.25}\right]$$

In Table XIII, we present some numerical results using Lanczos regularization technique. In this table, the results in the last column are obtained using (31). It should be noted that the calculations are performed with very large values of the parameter p as shown in (28). Note that χ in (28) should be replaced by $\chi/0.16$; i.e. $p=0.32/(\chi\Delta t)$. Thus, $p=110\,344$ for $\Delta t=5$ and $p=551\,724$ for $\Delta t=1$. In all the calculations in this example, the number of Chebyshev's polynomials in each axis direction is $N_x=N_y=5$; the number of the sources is $K=50$. The numerical results in Table XIV are obtained using Abel's regularization with $\alpha=0.0055$, $M=50$.

Example 7

In this example we consider the problem given by Šarler and Vertnik [20]; i.e. the diffusion equation (51) in $D=[0,1]\times[0,1]$ with $\chi=1$, and initial and boundary conditions

$$u(x,y,0)=1,\quad (x,y)\in D \tag{53}$$

$$u(x,1,t)=0,\quad 0\leqslant x\leqslant 1,\ \ t>0 \tag{54}$$

$$u(1,y,t)=0,\quad 0\leqslant y\leqslant 1,\ \ t>0 \tag{55}$$

DOI: 10.1002/nme

Table XIII. The MSR errors in Example 6 using the Lanczos regularization.

	$\Delta t=5$			$\Delta t=1$	
t	$M=30$	$M=40$	$M=50$	$M=50$	$M=50$, Reference [16]
1000	3.0×10^{-3}	1.1×10^{-3}	4.3×10^{-4}	3.2×10^{-4}	3.3×10^{-4}
2000	1.0×10^{-3}	3.3×10^{-4}	2.7×10^{-4}	6.4×10^{-5}	4.1×10^{-5}
5000	2.3×10^{-4}	1.7×10^{-4}	1.4×10^{-4}	4.7×10^{-5}	2.0×10^{-5}
10000	1.3×10^{-4}	9.8×10^{-5}	8.3×10^{-5}	2.8×10^{-5}	1.2×10^{-5}

Table XIV. The MSR errors in Example 6 using the Abel regularization.

t	$\Delta t=5, M=50$	$\Delta t=1, M=50$	$\Delta t=0.5, M=50$
1000	7.8×10^{-3}	7.8×10^{-3}	7.8×10^{-3}
2000	4.6×10^{-3}	4.6×10^{-3}	4.6×10^{-3}
5000	7.7×10^{-4}	7.7×10^{-4}	7.6×10^{-4}
10000	7.1×10^{-5}	3.7×10^{-5}	3.0×10^{-5}

$$\frac{\partial}{\partial x}u(0,y,t)=0, \quad 0\leqslant y\leqslant 1, \quad t>0 \tag{56}$$

$$\frac{\partial}{\partial y}u(x,0,t)=0, \quad 0\leqslant x\leqslant 1, \quad t>0 \tag{57}$$

The analytical solution [23] of the above problem is given by

$$u_{\rm ex}(x,y,t)=W(x,t)W(y,t)$$

where

$$W(x,t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\left(\frac{2n-1}{2}\pi x\right)\exp\left[-\left(\frac{2n-1}{2}\pi\right)^2 t\right] \tag{58}$$

We would like to indicate that there is a misprint for $W(x,t)$ in Equation (35) in [20].

We apply the algorithm described in Example 6 using the Lanczos regularization. The numerical results obtained in Table XV use the following parameters: $N_x=N_y=5$, $M=50$, $K=50$, $\Delta t=10^{-5}$. To compare our results with the results in Reference [20], we define the average absolute error $\|u-u_{\rm ex}\|_{\rm avg}$ and the maximal absolute error $\|u-u_{\rm ex}\|_\infty$ as in Reference [20] which are

$$\|u-u_{\rm ex}\|_\infty=\max_{1\leqslant i\leqslant N_t}|u(t,x_i,y_i)-u_{\rm ex}(t,x_i,y_i)|$$

$$\|u-u_{\rm ex}\|_{\rm avg}=\frac{1}{N_t}\sum_{i=1}^{N_t}|u(t,x_i,y_i)-u_{\rm ex}(t,x_i,y_i)|$$

The best results using $\Delta t=10^{-5}$ and 101×101 RBFs obtained in [20] are shown in Table XVI (Table 12 in [20]). Comparing the results in Tables XV and XVI, we observe that our results are slightly more accurate using only $K=50$. Furthermore, in [20] the authors indicated that their

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Table XV. $\|u-u_{\text{ex}}\|_{\text{avg}}$ and $\|u-u_{\text{ex}}\|_{\infty}$ in Example 7 using the Lanczos regularization in a squared domain.

	$\Delta t=10^{-5}$		$\Delta t=10^{-4}$		$\Delta t=10^{-3}$	
t	$\|u-u_{\text{ex}}\|_{\text{avg}}$	$\|u-u_{\text{ex}}\|_{\infty}$	$\|u-u_{\text{ex}}\|_{\text{avg}}$	$\|u-u_{\text{ex}}\|_{\infty}$	$\|u-u_{\text{ex}}\|_{\text{avg}}$	$\|u-u_{\text{ex}}\|_{\infty}$
0.001	2.9×10^{-4}	3.8×10^{-3}	7.9×10^{-4}	9.7×10^{-3}	—	—
0.01	4.3×10^{-5}	2.0×10^{-4}	4.3×10^{-4}	1.8×10^{-3}	4.4×10^{-3}	1.8×10^{-2}
0.1	1.7×10^{-5}	7.9×10^{-5}	1.1×10^{-4}	2.1×10^{-4}	1.1×10^{-3}	1.9×10^{-3}
1	3.3×10^{-6}	1.1×10^{-6}	2.1×10^{-6}	5.9×10^{-6}	1.2×10^{-5}	3.1×10^{-5}
2	—	—	—	—	9.5×10^{-8}	2.5×10^{-7}
5	—	—	—	—	4.3×10^{-14}	1.1×10^{-13}

Table XVI. $\|u-u_{\text{ex}}\|_{\text{avg}}$ and $\|u-u_{\text{ex}}\|_{\infty}$ in Example 7 by Šarler and Vertnik [20].

	$\Delta t=10^{-5}$	
t	$\|u-u_{\text{ex}}\|_{\text{avg}}$	$\|u-u_{\text{ex}}\|_{\infty}$
0.001	2.352×10^{-4}	2.809×10^{-3}
0.01	9.371×10^{-5}	3.523×10^{-4}
0.1	9.243×10^{-5}	1.582×10^{-4}
1	8.324×10^{-6}	2.066×10^{-5}

method diverges for the larger time step $\Delta t=10^{-3}$ because the explicit approach is unstable and thus requires very small time step in the time marching scheme (see page 1280 in [20]). The method presented in our paper is implicit and stable for $\Delta t=10^{-3}$ as shown in Table XV. On the other hand, the method proposed in [20] is a local method that has the advantage for solving complicated large-scale problems. Both methods have their own merits for solving different types of problems.

5.2. 3D cases

The method described in the previous section can be easily extended to 3D problems using the approximations in the form

$$\sum_{\mathbf{m}=1}^{\mathbf{M}} U_{m_1,m_2,m_3}\varphi_{m_1}(x_1)\varphi_{m_2}(x_2)\varphi_{m_3}(x_3)$$

for the approximation of the delta function, the MAFS trial functions, and the solution on each time step.

Example 8

We consider the following diffusion equation in 3D:

$$\begin{aligned} \frac{1}{\chi}\frac{\partial u}{\partial t} &= \nabla^2 u + \frac{a^2-r^2}{a^2}, \quad (x,y,z)\in\Omega, \quad t>0 \\ u(x,y,z,0) &= 0, \quad (x,y,z)\in\Omega \\ u(x,y,z,t) &= 0, \quad (x,y,z)\in\partial\Omega, \quad t>0 \end{aligned} \tag{59}$$

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Table XVII. The MSR errors in Example 8 using the Lanzcos regularization.

t	$u_{\text{ex}}(0,0,0,t)$	$M=15, N=300$	$M=20, N=400$	$M=25, N=500$
0.01	8.4×10^{-3}	5.7×10^{-5}	1.5×10^{-6}	5.4×10^{-7}
0.05	2.3×10^{-2}	5.0×10^{-5}	2.5×10^{-6}	5.2×10^{-7}
0.1	2.7×10^{-2}	1.7×10^{-5}	2.5×10^{-6}	2.0×10^{-7}
0.2	2.8×10^{-2}	4.7×10^{-6}	2.5×10^{-6}	1.0×10^{-7}
0.3	2.8×10^{-2}	4.3×10^{-6}	2.5×10^{-6}	9.9×10^{-8}

Table XVIII. The MSR errors in Example 8 using the Abel regularization.

t	$\text{u}_{\text{ex}}(0,0,0,t)$	$M=15, N=300$	$M=20, N=400$	$M=25, N=500$
0.01	8.4×10^{-3}	1.7×10^{-4}	6.7×10^{-5}	9.2×10^{-6}
0.05	2.3×10^{-2}	6.3×10^{-4}	6.0×10^{-5}	1.7×10^{-5}
0.1	2.7×10^{-2}	7.3×10^{-4}	3.0×10^{-5}	1.9×10^{-5}
0.2	2.8×10^{-2}	7.4×10^{-4}	1.9×10^{-5}	2.0×10^{-5}
0.3	2.8×10^{-2}	7.4×10^{-4}	1.9×10^{-5}	2.0×10^{-5}

where $\Omega\cup\partial\Omega=\{(x,y,z): x^2+y^2+z^2\leqslant a^2\}$, $r^2=x^2+y^2+z^2$, and χ is the diffusion coefficient.

The exact solution is given by [23]

$$u_{\text{ex}}=\frac{(a^2-r^2)(7a^2-3r^2)}{60a^2}-\frac{12a^3}{r\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^5}\sin\frac{n\pi r}{a}\exp\left(-\frac{\chi n^2\pi^2 t}{a^2}\right)$$

In the following numerical computation, we consider $a=0.5$, $\chi=1$. To demonstrate that our proposed approach can be easily extended to 3D problems, we choose to solve this problem in the Cartesian coordinates without considering the special property of its spherical symmetry.

For all the numerical results obtained in the example, the number of source points K is equal to the number of collocation points N. These points are randomly distributed using the subroutine RNSPH from the IMSL Library. The source points are placed on the sphere with radius $R_s=0.95$. In all the calculations performed in this example the number of Chebyshev's polynomials in each axis direction is $N_x=N_y=N_z=5$. A total of 50 test points inside the sphere are randomly selected.

Table XVII contains the MSR solution errors on a set of test points that are distributed uniformly inside the sphere. The results are obtained by the Lanzcos regularization technique. We used the time step $\Delta t=0.01$ for $M=15$, 20 and $\Delta t=0.001$ for $M=25$.

In Table XVIII, we show some of the numerical results using Abel's regularization technique. We use the time step $\Delta t=0.01$ for $M=15$, 20 and $\Delta t=0.001$ for $M=25$. The regularization parameter is $\alpha=0.02$ for $M=15$ and $\alpha=0.01$ for $M=20$, 25. In this example, numerical results obtained using Lanzcos regularization scheme seems superior than those from Abel's regularization scheme. Both schemes produce excellent results.

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6. CONCLUDING REMARKS

The numerical technique presented in this paper can be classified as a boundary meshless method. The Chebyshev polynomial and trigonometric basis functions for the evaluation of approximate particular solution [5] are coupled with the approximate fundamental solution for finding the corresponding homogeneous solution [4]. We extend these novel approaches to solve the heat conduction problems.

In the past, the fundamental solution has been used as the trial function for the approximation of homogeneous solution and the radial basis functions have been widely used as the trial function for the approximation of particular solution. One special feature presented in this paper is that we apply the same trial function for the approximation of fundamental solution and particular solution. Since the particular solution and approximate fundamental solution are easy to derive using the proposed trial function, the same numerical scheme can be extended to a large class of linear PDEs. The proposed method is highly accurate. Hence, we expect other types of linear or non-linear time-dependent problems such as wave equations, Burger equation, convection–diffusion equations, etc. can be solved effectively using the proposed technique. Further work in extending our approach beyond the heat conduction problems is currently under investigation.

One of the challenges of the proposed approach is the optimal choice of the various parameters. The excellent numerical results in this paper merit further investigation in this respect.

REFERENCES

1. Fairweather G, Karageorghis A. The method of fundamental solutions for elliptic boundary value problems. *Advances in Computational Mathematics* 1998; **9**:69–95.
2. Golberg MA, Chen CS. The method of fundamental solutions for potential, Helmholtz and diffusion problems. In *Boundary Integral Methods—Numerical and Mathematical Aspects*, Golberg MA (ed.). Computational Mechanics Publications, 1998; 103–176.
3. Cho HA, Golberg MA, Muleshkov AS, Li X. Trefftz methods for time dependent partial differential equations. *CMC* 2004; **1**:1–37.
4. Reutskiy S. A boundary method of Trefftz type with approximate trial functions. *Engineering Analysis with Boundary Elements* 2002; **26**(4):341–353.
5. Reutskiy S, Chen CS. Approximation of multivariate functions and evaluation of particular solutions using Chebyshev polynomial and trigonometric basis functions. *International Journal for Numerical Methods in Engineering* 2006; **67**:1811–1829.
6. Young DL, Tsai CC, Fan CM. Direct approach to solve nonhomogeneous diffusion problems using fundamental solutions and dual reciprocity methods. *Journal of the Chinese Institute of Engineers* 2004; **27**:597–609.
7. Zhu S, Satravaha P, Lu X. Solving linear diffusion equations with the dual reciprocity method in Laplace space. *Engineering Analysis with Boundary Elements* 1995; **13**(1):1–10.
8. Chen CS, Rashed YF, Golberg MA. A mesh free method for linear diffusion equations. *Numerical Heat Transfer, Part B* 1998; **33**:469–486.
9. Li X, Ho CH, Chen CS. Computational test of approximation of functions and their derivatives by radial basis functions. *Neural, Parallel and Scientific Computations* 2002; **10**:25–46.
10. Ingber M, Chen CS, Tanski JA. A mesh free approach using radial basis functions and parallel domain decomposition for solving three dimensional diffusion equations. *International Journal of Numerical Methods in Engineering* 2004; **60**:2183–2201.
11. Chen W. Symmetric boundary knot method. *Engineering Analysis with Boundary Elements* 2002; **26**:489–494.
12. Lanczos C. *Applied Analysis*. Prentice-Hall: Englewood Cliffs, NJ, 1956.
13. Reutskiy S. A Trefftz type method for time-dependent problems. *Engineering Analysis with Boundary Elements* 2004; **28**:13–21.
14. Reutskiy S. A boundary method of Trefftz type for PDEs with scattered data. *Engineering Analysis with Boundary Elements* 2005; **29**:713–724.

DOI: 10.1002/nme

15. Available from: http://www.answers.com/topic/zeta-function-regularization.
16. Ramachandran PA, Balakrishnan K. Radial basis functions as approximate particular solutions: review of recent progress. *Engineering Analysis with Boundary Elements* 2000; **24**:575–582.
17. Boyd JP. *Chebyshev and Fourier Spectral Methods*. Dover: New York, 2001.
18. Bernardi C, Maday Y. Spectral methods. In *Handbook of Numerical Analysis*, vol. V, Ciarlet PG, Lions J-L (eds). 1997; 209–485.
19. Canuto C, Hussaini MY, Quarteroni A, Zang TA. *Spectral Methods in Fluid Dynamics*. Springer: New York, 1988.
20. Šarler B, Vertnik R. Meshfree explicit local radial basis function collocation method for diffusion problems. *Computers and Mathematics with Applications* 2006; **51**:1269–1282.
21. Franke R. Scattered data interpolation: test of some methods. *Mathematics of Computation* 1982; **48**:181–200.
22. MathWorks: Developers of MATLAB and Simulink. Available from: http://www.mathworks.com.
23. Carslaw HS, Jaeger JC. *Conduction of Heat in Solids*. Oxford University Press: London, 1995.

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