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FRACTION-FREE COMPUTATION OF DETERMINANTS

DEANNA LEGGETT AND JOHN PERRY (FACULTY ADVISOR)

Abstract. Here we examine Dodgson's Method, a fraction-free methods for computing determinants. In some cases, this method fails due to division by zero. We propose a workaround for Dodgson's Method that ensures it also can be used to compute the determinant of any integer matrix.

1. Introduction: Fraction-free?

It is a fact of linear algebra that the determinant of a matrix of integers is an integer. Thus any fractions introduced while computing its determinant will be eliminated at some step in the algorithm. Even if the final step of such methods yields the correct integer for the determinant of A, the fractions introduced along the way make the method more complex. This is true both if the calculations are done by hand or if they are carried out by a computer program. Thus we need a *fraction-free method*, that is, a method of computing determinants such that any divisions that are introduced are exact [4, p. 262].

Cofactor expansion is one of the most common fraction-free methods and is generally taught in elementary linear algebra courses. However, it becomes cumbersome if the dimension of the matrix is nontrivial. Bareiss' Algorithm is based on row reduction but can also be proven using Sylvester's Indentity ([2, 4]). Here we will focus on third fraction-free method: Dodgson's Method.

2. Dodgson's Method

In each iteration of Dodgson's Method, create a new matrix whose entries are contiguous two-by-two determinants made up of the entries of the matrix created in the previous iteration. We then divide each of the entries of the new matrix by the corresponding entry in the interior of the matrix created two iterations before. In this way, each iteration yields a matrix whose dimension is one less than that of the matrix from the previous iteration. For this reason, Dodgson's Method, like Bareiss' Algorithm, is often called a "condensation method."

Example 1. Find $\det(A)$ using Dodgson's Method, where

$$A = \begin{pmatrix} 1 & -2 & 1 & 2 \\ -1 & 4 & -2 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{pmatrix}.$$

Let $A^{(4)} = A$. The first iteration of Dodgson's Method yields

$$A^{(3)} = \begin{pmatrix} \begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 4 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 3 & 3 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 5 \\ -15 & 18 & -11 \\ 9 & -9 & -11 \end{pmatrix}.$$

Since there is no $A^{(5)}$, there is no division involved in the first iteration of Dodgson's Method. However, in the second iteration, we divide the determinant of each 2×2 by the corresponding entry in the interior of

$A^{(4)}$:

$$A^{(2)} = \begin{pmatrix} \dfrac{\begin{vmatrix} 2 & 0 \\ -15 & 18 \end{vmatrix}}{4} & \dfrac{\begin{vmatrix} 0 & 5 \\ 18 & -11 \end{vmatrix}}{-2} \\ \dfrac{\begin{vmatrix} -15 & 18 \\ 9 & -9 \end{vmatrix}}{3} & \dfrac{\begin{vmatrix} 18 & -11 \\ -9 & -11 \end{vmatrix}}{3} \end{pmatrix} = \begin{pmatrix} 9 & 45 \\ -9 & -99 \end{pmatrix}.$$

Likewise,

$$A^{(1)} = \frac{\begin{vmatrix} 9 & 45 \\ -9 & -99 \end{vmatrix}}{18} = \left(\frac{-486}{18} \right) = (-27).$$

Thus $\det(A) = -27$.

The algorithm is based on a theorem of Jacobi [3, p. 48].

Theorem 2 (Dodgson's Condensation Theorem). *Let A be an $n \times n$ matrix. After k successful condensations, Dodgson's method produces the matrix*

$$A^{(n-k)} = \begin{pmatrix} |A_{1\ldots k+1,1\ldots k+1}| & |A_{1\ldots k+1,2\ldots k+2}| & \cdots & |A_{1\ldots k+1,n-k\ldots n}| \\ |A_{2\ldots k+2,1\ldots k+1}| & |A_{2\ldots k+2,2\ldots k+2}| & \cdots & |A_{2\ldots k+2,n-k\ldots n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-k\ldots n,1\ldots k+1}| & |A_{n-k\ldots n,2\ldots k+2}| & \cdots & |A_{n-k\ldots n,n-k\ldots n}| \end{pmatrix}$$

whose entries are the determinants of all $(k+1) \times (k+1)$ contiguous submatrices of A [3, p. 8].

Like Bareiss' Algorithm, Dodgson's Method also encounters division by zero for some matrices. However, swapping rows of an intermediate matrix in Dodgson's Method and continuing does not yield the determinant of the original matrix.

Example 3. Find $\det(A)$ using Dodgson's Method, where

$$A = \begin{pmatrix} 1 & -4 & 1 & 2 & 1 \\ -1 & 4 & 4 & 1 & 0 \\ 3 & 3 & 3 & 4 & -2 \\ 2 & 5 & 2 & -1 & 4 \\ 4 & 1 & 3 & 2 & 1 \end{pmatrix}.$$

Since there are no zeros in the interior of $A^{(5)}$, we apply Dodgson's Method as usual.

$$A^{(5)} = \begin{pmatrix} 1 & -4 & 1 & 2 & 1 \\ -1 & 4 & 4 & 1 & 0 \\ 3 & 3 & 3 & 4 & -2 \\ 2 & 5 & 2 & -1 & 4 \\ 4 & 1 & 3 & 2 & 1 \end{pmatrix} \Longrightarrow A^{(4)} = \begin{pmatrix} 0 & -20 & -7 & 1 \\ -15 & 0 & 13 & -2 \\ 9 & -9 & -11 & 14 \\ -18 & 13 & 7 & -9 \end{pmatrix}.$$

Notice the zero that is introduced in $a_{22}^{(4)}$. Since this is an interior element of $A^{(4)}$, we will encounter division by zero when trying to find $a_{22}^{(2)}$. We cannot swap the rows of $A^{(4)}$; this gives an incorrect answer. Thus Dodgson's Method fails for this matrix.

Dodgson proposed a workaround by swapping rows of the *original* matrix but this does not work in all cases. Neither does the solution of [3]. This leads us to ask the question "Is there a way to change Dodgson's Method such that it can be used to find the determinant of any integer matrix?"

3. New Method: Modify Interior Row

The Modify Interior Row method uses the original Dodgson's Method until division by zero occurs. We then use Theorem 2 to determine the "problem submatrix" B; that is, the submatrix of A whose interior has a determinant of zero. Next we add a strategic multiple of row one of B to row two of B and use Dodgson's Method to find the determinant of B. If we encounter division by zero while calculating the determinant of B, then we add a strategic multiple of the last row of B to row two of B and recalculate the determinant of

Algorithm 1

algorithm *Dodgson's with Modify Interior Row*
  inputs
    $M \epsilon \mathbb{Z}^{n \times n}$
  outputs
    $\det(M)$
  do
    Let $C = 0$
    Let $A = M$
  while (number of columns in A)> 1 **do**
    Let m be the number of rows in A
    Let D be an $(m-1) \times (m-1)$ matrix of zeros
    for $i \in \{1, \ldots, m-1\}$ **do**
      for $j \in \{1, \ldots, m-1\}$ **do**
        Let $b_{ij} = a_{ij} \cdot a_{i+1,j+1} - a_{i+1,j} \cdot a_{i,j+1}$
      if $C \neq 0$ **then**
        for $i \in \{1, \ldots, m-1\}$ **do**
          for $j \in \{1, \ldots, m-1\}$ **do**
            if $c_{i+1,j+1} \neq 0$ **then**
              Let $d_{ij} = \frac{d_{ij}}{c_{i+1,j+1}}$
            else
              Let $l = n - (m-1)$
              Let B be the submatrix of M whose interior corresponds to $c_{i+1,j+1}$
              Add to row 2 of B a strategic multiple of row 1 of B
              Use Dodgson's Method to find $\det(B)$
              if Dodgson's Method fails for B **then**
                Add to row 2 of B a strategic multiple of row n of B
                if Dodgson's Method fails for B **then**
                  Let $\det(B) = 0$
              Let $d_{ij} = \det(B)$
      Let $C = A$
      Let $A = D$
    return A

B. We will show that if Dodson's Method fails a third time, the determinant of B is zero. We then continue with Dodgson's Method to find the determinant of A. See Algorithm 1.

But what do we mean by "a strategic multiple" of these rows?

Definition 4. Let $\mathbf{v}_i$ and $\mathbf{v}_j$ be rows i and j, respectively, of an $n \times n$ matrix A, and let $b \in \mathbb{Z}$. Then $b \cdot \mathbf{v}_j$ is a *strategic multiple* of $\mathbf{v}_j$ if the row vector $b \cdot \mathbf{v}_j + \mathbf{v}_i$ does not introduce any zeros in columns $2, 3, \ldots, n-1$ that were not already zero in $\mathbf{v}_i$. That is, if $a_{ik} \neq 0$ then $b \cdot a_{jk} + a_{ik} \neq 0$ for $k = 2, 3, \ldots, n-1$.

Thus this method can be used to find the determinant of any integer matrix. It also has two other major advantages: the reuse of calculations and the simplicity of the required calculations.

Example 5. Find $\det(A)$ using the modified Dodgson's Method, where

$$A = \begin{pmatrix} 1 & -4 & 1 & 2 & 1 \\ -1 & 4 & 4 & 1 & 0 \\ 3 & 3 & 3 & 4 & -2 \\ 2 & 5 & 2 & -1 & 4 \\ 4 & 1 & 3 & 2 & 1 \end{pmatrix}.$$

Recall from Example 3 that

$$A^{(4)} = \begin{pmatrix} 0 & -20 & -7 & 1 \\ -15 & 0 & 13 & -2 \\ 9 & -9 & -11 & 14 \\ -18 & 13 & 7 & -9 \end{pmatrix},$$

which contains a zero at $a_{22}^{(4)}$. Since this is an interior element of $A^{(4)}$, we will have to apply Modify Interior Row. As indicated in Algorithm 1, we need to recalculate only the determinant of the submatrix whose interior has a determinant of zero. The remainder of the calculations will be carried out using the traditional Dodgson's Method:

$$A^{(3)} = \begin{pmatrix} \frac{300}{4} & \frac{-260}{4} & \frac{27}{1} \\ \frac{135}{3} & \frac{117}{3} & \frac{160}{4} \\ \frac{-45}{9} & \frac{80}{2} & \frac{1}{-1} \end{pmatrix} = \begin{pmatrix} 75 & -65 & 27 \\ 45 & 39 & 40 \\ -9 & 40 & -1 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} \frac{0}{0} & \frac{-3653}{13} \\ \frac{2152}{-9} & \frac{-1639}{-11} \end{pmatrix} = \begin{pmatrix} ? & -281 \\ -239 & 149 \end{pmatrix}.$$

By Theorem 2, we know that the determinant of the upper-right 4×4 submatrix of A is equal to $a_{11}^{(2)}$. That is,

$$a_{11}^{(2)} = \begin{vmatrix} 1 & -4 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{vmatrix}$$

To find this determinant, we add a strategic multiple of row one to row two. Here it suffices to subtract the first row from the second. We can now use Dodgson's Method to calculate the determinant of the new submatrix:

$$a_{11}^{(2)} \begin{vmatrix} 1 & -4 & 1 & 2 \\ -1 & 4 & 4 & 1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 1 & 2 \\ -2 & 8 & 3 & -1 \\ 3 & 3 & 3 & 4 \\ 2 & 5 & 2 & -1 \end{vmatrix} = 245.$$

By substituting this value into $A^{(2)}$, we have

$$A^{(2)} = \begin{pmatrix} 245 & -281 \\ -239 & 149 \end{pmatrix} \Longrightarrow A^{(1)} = \left(\frac{(245) \cdot (149) - (-281) \cdot (-239)}{39} \right) = (-786).$$

In fact, $\det(A) = -786$.

To prove that Algorithm 1 terminates correctly, we first need the following proposition and lemma.

Proposition 6. [1, p. 287] *If A is an $n \times n$ matrix, then the following are equivalent.*

(a) $\det(A) \neq 0$.
(b) The column vectors of A are linearly independent.
(c) The row vectors of A are linearly independent.

Lemma 7. *Let A be an $n \times n$ matrix whose interior has a determinant of zero. Suppose adding a constant multiple of row 1 of A to row 2 of A still yields a zero for the determinant of the interior. Suppose the same is true when a constant multiple of row n of A is added to row 2 of A. Then* $\det(A) = 0$.

Proof. Let A be an $n \times n$ matrix whose interior has a determinant of zero, and, for each $i = 1, 2, 3, \ldots n$, let $\mathbf{r}_i = A_{i,2\ldots n-1}$. By Proposition 6, the row vectors of the interior of A, $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_{n-1}$, are linearly dependent. Suppose that the determinant of the submatrix created by adding a constant multiple b of $\mathbf{r}_1$ to $\mathbf{r}_2$ is zero. That is,

$$\det \begin{pmatrix} \mathbf{r}_2 + b\mathbf{r}_1 \\ \mathbf{r}_3 \\ \vdots \\ \mathbf{r}_{n-1} \end{pmatrix} = 0.$$

This implies that $\mathbf{r}_2 + b\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_4, \ldots, \mathbf{r}_{n-1}$ are linearly dependent (by 6). Then, by definition of linear dependence, there exist constants $c_3, c_4, \ldots, c_{n-1}$ such that

$$\mathbf{r}_2 + b\mathbf{r}_1 = c_3\mathbf{r}_3 + c_4\mathbf{r}_4 + \ldots + c_{n-1}\mathbf{r}_{n-1}.$$

Solving for $\mathbf{r}_1$ yields

$$\begin{aligned} \mathbf{r}_1 &= \frac{1}{b}\left(-\mathbf{r}_2 + c_3\mathbf{r}_3 + c_4\mathbf{r}_4 + \ldots + c_{n-1}\mathbf{r}_{n-1}\right) \\ &= -\frac{1}{b}\mathbf{r}_2 + \frac{c_3}{b}\mathbf{r}_3 + \frac{c_4}{b}\mathbf{r}_4 + \ldots + \frac{c_{n-1}}{b}\mathbf{r}_{n-1}. \end{aligned}$$

Since $\mathbf{r}_1$ can be rewritten as a sum of constant multiples of $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_{n-1}$, the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1}$ are linearly dependent. Likewise, suppose that

$$\det\begin{pmatrix} \mathbf{r}_2 + d\mathbf{r}_n \\ \mathbf{r}_3 \\ \vdots \\ \mathbf{r}_{n-1} \end{pmatrix} = 0.$$

Then an argument similar to the one above can be used to show that

$$\mathbf{r}_n = -\frac{1}{d}\mathbf{r}_2 + \frac{c'_3}{d}\mathbf{r}_3 + \frac{c'_4}{d}\mathbf{r}_4 + \ldots + \frac{c'_{n-1}}{d}\mathbf{r}_{n-1}.$$

Then $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$ are also linearly dependent.

Create a new matrix by swapping columns 1 and $n-1$ of A; then swapping rows 1 and $n-1$ of the new matrix yields:

$$A' = \begin{pmatrix} a_{n-1,n-1} & a_{n-1,2} & \cdots & a_{n-1,n-2} & a_{n-1,1} & a_{n-1n} \\ a_{2,n-1} & a_{22} & \cdots & a_{2,n-2} & a_{21} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,n-1} & a_{n-2,2} & \cdots & a_{n-2,n-2} & a_{n-2,1} & a_{n-2,n} \\ a_{1,n-1} & a_{12} & \cdots & a_{1,n-2} & a_{11} & a_{1n} \\ a_{n,n-1} & a_{n2} & \cdots & a_{n,n-2} & a_{n1} & a_{nn} \end{pmatrix}.$$

Since $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1}$ and $\mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$ are linearly dependent, we can use row reduction to create zero entries in the first $n-2$ columns of the last two rows of this matrix. This yields the matrix

$$A'' = \begin{pmatrix} a_{n-1,n-1} & a_{n-1,2} & \cdots & a_{n-1,n-2} & a_{n-1,1} & a_{n-1n} \\ a_{2,n-1} & a_{22} & \cdots & a_{2,n-2} & a_{21} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,n-1} & a_{n-2,2} & \cdots & a_{n-2,n-2} & a_{n-2,1} & a_{n-2,n} \\ 0 & 0 & \cdots & 0 & a''_{11} & a''_{1n} \\ 0 & 0 & \cdots & 0 & a''_{n1} & a''_{nn} \end{pmatrix}.$$

Let D be the $(n-2) \times (n-2)$ matrix in the upper left corner of A'' and let B be the 2×2 matrix in the bottom right corner of A''. Then A'' has the form

$$A'' = \begin{pmatrix} D & * \\ 0 & B \end{pmatrix}.$$

From linear algebra, we know that the determinant of matrices having this form is

$$\det(A'') = \det(D) \cdot \det(B).$$

Notice that, after $n-3$ row swaps and $n-3$ column swaps, D is the interior of A. So the determinant of D is $(-1)^{2(n-3)}$ times the determinant of the interior of A, which is zero, so $\det(D) = 0$. Thus

$$\begin{aligned} \det(A'') &= \det(D) \cdot \det(B) \\ &= 0 \cdot \det(B) \\ &= 0. \end{aligned}$$

Since A'' was created from matrix A using a few row and column swaps,

$$\det(A) = (-1)^{\alpha} \det(A'') = (-1)^{\alpha} \cdot 0 = 0.$$

By substitution, $\det(A) = 0$. $\square$

Theorem 8. *Algorithm 1 terminates correctly.*

Proof. Recall that we have already proven that Dodgson's method terminates correctly. Thus we need only show that adding the steps required by Dodgson's Method: Modify Interior Row does not change the determinant given by the original Dodgson's Method.

By the Condensation Theorem (Theorem 2),

$$a_{ij}^{(k)} = \left| A_{i\ldots i+n-k, j\ldots j+n-k} \right|.$$

Let

$$B = A_{i\ldots i+n-k, j\ldots j+n-k}.$$

By the Condensation Theorem, if Dodgson's Method failed at $a_{ij}^{(k)}$ because of division by zero, then the determinant of the interior of B is zero. Suppose we create a new matrix $\tilde{B}$ by adding to row two of B a strategic multiple of row one of B. The determinant of $\tilde{B}$ is the same of B. Thus

$$a_{ij}^{(k)} = \det(B) = \det\left(\tilde{B}\right),$$

which allows us to calculate $a_{ij}^{(k)}$ and continue with Dodgson's Method to find $\det(A)$.

Suppose that we also encounter division by zero when using Dodgson's Method to calculate $\det\left(\tilde{B}\right)$. Create a third matrix $\check{B}$ by adding a strategic multiple of the last row of B to the second row of B. The determinant of $\det\left(\check{B}\right) = \det(B) = a_{ij}^{(k)}$, so we may continue with Dodgson's Method to find $\det(A)$. Suppose that, instead of being able to calculate $\det\left(\check{B}\right)$, we again encounter division by zero. By Lemma 7, we have

$$\det(B) = 0,$$

so $a_{ij}^{(k)} = 0$.

In any case, we can find $a_{ij}^{(k)}$ and continue with Dodgson's Method to find $\det(A)$. Since Dodgson's Method terminates correctly, Modify Interior Row terminates correctly. $\square$

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