

Multiplicity and Poincaré series for mixed multiplier ideals

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Let X be a complex surface with at most a rational singularity at a point $O \in X$ and $\mathfrak{m} = \mathfrak{m}_{X,O}$ be the maximal ideal of the local ring $\mathcal{O}_{X,O}$ at O. Given a tuple of \mathfrak{m} -primary ideals $\mathfrak{a} := (\mathfrak{a}_1, ..., \mathfrak{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ we will consider a common *log-resolution*, that is a birational morphism $\pi : X' \to X$ such that X' is smooth, $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for some effective Cartier divisors F_i , i = 1, ..., r and $\sum_{i=1}^r F_i + E$ is a divisor with simple normal crossings where $E = Exc(\pi)$ is the exceptional locus. Actually, the divisors F_i are supported on the exceptional locus since the ideals are \mathfrak{m} -primary.

We define the mixed multiplier ideal at a point $\mathbf{c} := (c_1, .., c_r) \in \mathbb{R}^r_{\geq 0}$ as ¹

$$\mathcal{J}(\mathbf{a}^{\mathbf{c}}) := \mathcal{J}(\mathfrak{a}_{1}^{c_{1}} \cdots \mathfrak{a}_{r}^{c_{r}}) = \pi_{*}\mathcal{O}_{X'}([K_{\pi} - c_{1}F_{1} - \cdots - c_{r}F_{r}])$$

where $\lceil \cdot \rceil$ denotes the round-up and $K_{\pi} = \sum_{i=1}^{s} k_j E_j$ is the relative canonical divisor, a Q-divisor on X' supported on the exceptional locus E which is characterized by the property $(K_{\pi} + E_i) \cdot E_i = -2$ for every exceptional component E_i , $i = 1, \ldots, s$.

Associated to any point $\boldsymbol{c} \in \mathbb{R}^{r}_{\geq 0}$, we consider the *region* of \boldsymbol{c} as:

$$\mathcal{R}_{\mathbf{a}}\left(\mathbf{c}
ight) = \left\{\mathbf{c}' \in \mathbb{R}^{r}_{\geqslant 0} \mid \mathcal{J}\left(\mathbf{a}^{\mathbf{c}'}
ight) \supseteq \mathcal{J}\left(\mathbf{a}^{\mathbf{c}}
ight)
ight\} \,.$$

The boundary of the region $\mathcal{R}_{\mathfrak{a}}(c)$ is what we call the *jumping wall* associated to c. From now on we will denote by $\mathbf{JW}_{\mathfrak{a}}$ the set of jumping walls of \mathfrak{a} .

1 Multiplicities of jumping points

We define the multiplicity attached to a point $\boldsymbol{c} \in \mathbb{R}_{\geq 0}^{r}$ as the codimension of $\mathcal{J}(\boldsymbol{a}^{c})$ in $\mathcal{J}(\boldsymbol{a}^{(1-\varepsilon)c})$ for $\varepsilon > 0$ small enough, i.e.

$$m(\boldsymbol{c}) := \dim_{\mathbb{C}} rac{\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{(1-arepsilon)}
ight)}{\mathcal{J}\left(\boldsymbol{\mathfrak{a}}^{c}
ight)}.$$

One can compute these multiplicities using the following result:

¹By an abuse of notation, we will also denote $\mathcal{J}(\mathbf{a}^{\mathbf{c}})$ its stalk at O so we will omit the word "sheaf" if no confusion arises.

Theorem 1 Let $\mathbf{a} \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of \mathfrak{m} -primary ideals, $\mathbf{c} \in \mathbb{R}^r_{>0}$ a point and $H_{\mathbf{c}}$ the reduced divisor defined as $H_{\mathbf{c}} = \lceil K_{\pi} - (1 - \varepsilon)c_1F_1 - \cdots - (1 - \varepsilon)c_rF_r \rceil - \lceil K_{\pi} - c_1F_1 - \cdots - c_rF_r \rceil$ for a sufficiently small $\varepsilon > 0$. Then,

 $m(\mathbf{c}) = \left(\left\lceil K_{\pi} - c_1 F_1 - \dots - c_r F_r \right\rceil + H_{\mathbf{c}} \right) \cdot H_{\mathbf{c}} + \# \left\{ \text{connected components of } H_{\mathbf{c}} \right\}.$

2 Poincaré series of mixed multiplier ideals

Given a m-primary ideal $\mathfrak{a} \subseteq \mathcal{O}_{X,O}$, Galindo and Montserrat in 2010 introduced its *Poincaré* series as

$$P_{\mathfrak{a}}(t) = \sum_{c \in \mathbb{R}_{>0}} m(c) \ t^c.$$

For a tuple of m-primary ideals $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ we are going to give a generalization of this series by considering a sequence of mixed multiplier ideals indexed by points in a ray $L : \mathbf{c}_0 + \mu \mathbf{u}$ in the positive orthant $\mathbb{R}^r_{>0}$ with a vector $\mathbf{u} = (u_1, \ldots, u_r) \in \mathbb{Z}^r_{\geq 0}, \mathbf{u} \neq \mathbf{0}$ and $\mathbf{c}_0 \in \mathbb{Q}^r_{\geq 0}$. Here we are considering, for simplicity, a point \mathbf{c}_0 belonging to a coordinate hyperplane but not necessarily being the origin and $\mu \in \mathbb{R}_{>0}$. Namely, we consider the sequence of mixed multiplier ideals

$$\mathcal{J}\left(\mathbf{a}^{\mathbf{c}_{0}}\right) \supseteq \mathcal{J}\left(\mathbf{a}^{\mathbf{c}_{1}}\right) \supseteq \mathcal{J}\left(\mathbf{a}^{\mathbf{c}_{2}}\right) \supseteq \cdots \supseteq \mathcal{J}\left(\mathbf{a}^{\mathbf{c}_{i}}\right) \supseteq \cdots$$

where $\{c_i\}_{i>0} = L \cap \mathbf{JW}_{\mathfrak{a}}$ or equivalently $\{c_i\}_{i>0}$ is the set of jumping points of this sequence. Then we define the *Poincaré series of* \mathfrak{a} *alongside the ray* L as

$$P_{\mathbf{a}}(\underline{t};L) = \sum_{\mathbf{c}\in L} m(\mathbf{c}) \ \underline{t}^{\mathbf{c}}$$

where $\underline{t}^{\boldsymbol{c}} := t_1^{c_1} \cdots t_r^{c_r}$.

Theorem 2 Let $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_r) \subseteq (\mathcal{O}_{X,O})^r$ be a tuple of \mathfrak{m} -primary ideals and let $L : \mathbf{c}_0 + \mu \mathbf{u}$ be a ray in the positive orthant $\mathbb{R}^r_{\geq 0}$ with $\mathbf{u} \in \mathbb{Z}_{\geq 0}$, $\mathbf{u} \neq \mathbf{0}$. The Poincaré series of \mathbf{a} alongside Lcan be expressed as

$$P_{\mathbf{a}}(\underline{t};L) = \underline{t}^{\mathbf{c}_0} \sum_{\mu \in [0,1)} \left(\frac{m(\mathbf{c}_0 + \mu \mathbf{u})}{1 - \underline{t}^{\mathbf{u}}} + \rho_{\mathbf{c}_0 + \mu \mathbf{u},\mathbf{u}} \frac{\underline{t}^{\mathbf{u}}}{(1 - \underline{t}^{\mathbf{u}})^2} \right) \underline{t}^{\mu \mathbf{u}}.$$

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References

[1] M. Alberich-Carramiñana, J. Àlvarez Montaner, F. Dachs-Cadefau and V. González-Alonso, Multiplicities of jumping points for mixed multiplier ideals, To appear.