Master in Statistics and Operations Research
Master Thesis

A Capacitated Commodity Trading Model with Market Power

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Abstract

This Master Thesis proposes an analytical model of commodity trading in the presence of market power, which is new to the literature. We establish the rational behavior of traders between two markets and derive price spread dynamics based on the trading volumes. We consider two commodity markets that quote different prices for the same product, and a trader capable of purchasing in one market to resell in the other, within a trading capacity. For simplicity, we assume that the inventory present between the two markets is constant, i.e., the amount stored is fixed (and is directly linked to the trading capacity). We first establish the structure of the optimal trading policy. This policy determines, given a realization of the current price spread between the two markets, how much (if any) volume must be bought/sold from/at the first market and sold/bought at/from the second market, so that the long-run average profit is maximized. When optimizing, the trader takes into account its impact on future spreads. We then describe the parameters of the optimal policy and we apply our model to kerosene prices in New York and Los Angeles, and illustrate the insights derived.

Contents

Introduction

Commodity trading moves a colossal amount of volume and money worldwide. This Master thesis considers the bidirectional problem of a commodity trader which may purchase in one market and resell in the other (or vice versa) depending on the spot price at the two markets.

We consider the trader to be *capacitated*; in other words, we have operational restrictions (e.g. logistics) in the volume he can move between the markets. We also take into account the influence of the trading activities in the spot prices as the *spread* (difference in price) is reduced when trading takes place (spreads tend to close over time). This is technically known as *market power*.

This Thesis has several objectives:

- The first objective is to build a model that characterizes the problem taking into account the capacity restrictions and market power. We will need to understand and model the price spread dynamics.

- Our primary objective is to find and characterize the optimal trading policy for the trader's problem. This policy must determine, given a realization of the current price spread between the two markets, how much (if any) volume must be bought/sold from/at the first market and sold/bought at/from the second market, so that the long-run average profit is maximized.

- The topic of this Thesis lays between two different study areas. The first one is finance and economics, the second is operations management and operations research. We will be particulary careful about introducing the basic concepts and technical vocabulary of the first area as they can be new for someone who does not work in it. In some sense, we hope that this Thesis will be a bridge between the two areas as it combines the ideas and methodologies of both.

- We will test our theoretical model and apply its insights to a real example. With this we hope to obtain a valuable feedback.

- Without avoiding scientific precision we pretend to make the intuition beneath our results as clear as possible so that a trader who reads this Thesis does not find difficulties in understanding what (and why) we claim he should do to optimize his benefits.

Since November 2006, I have been working as the research assistant of professor Victor Martínez-de-Albéniz from the Production, Technology and Operations Management Department at IESE Business School. The main result of our collaboration is a co-authorized paper *A Capacitated Commodity Trading Model with Market Power* submitted to *Management Science* December 2007. In this Master Thesis I will expose the results and conclusions we present in the paper as well as the methodology and techniques we have used.

The Thesis is organized as follows. In Chapter 1, we make a short introduction to commodity trading and we review the previous literature and how this Thesis is related to it. Chapter 2 describes in detail the model with its assumptions and notation. Then, in Chapter 3 we present the results for different market situations (monopoly, oligopoly), together with the optimality equations that characterize the optimal policy, and a numerical sensitivity analysis. In Chapter 4 we show an application of our model to kerosene prices and discuss future research directions. Finally, we summarize, discuss and present the conclusions.

Chapter 1

Commodity Trading

1.1 About Commodities

Etymologically, the word commodity came into use in English in the 15th century, derived from the French word *commodité*, similar in meaning to "convenience" in terms of quality of services. The Latin root is *commoditas*, referring variously to the appropriate measure of something; a fitting state, time or condition; a good quality; efficaciousness or propriety; and advantage, or benefit. The German equivalent is *die Ware*, i.e., wares or goods offered for sale. The French equivalent is *produit de base* like energy, goods, or industrial raw materials. The Spanish translation is *mercancía* and *mercaderia* in Catalan.

In the original and simplified sense, commodities were things of value, of uniform quality, that were produced in large quantities by many different producers. The items from each different producer were considered equivalent. In a broader sense one can think a commodity to be anything for which there is demand, but which is supplied without qualitative differentiation across a given market.

Examples include not only minerals, metals and agricultural products such as iron ore, aluminum, silver, gold, sugar, coffee, rice, wheat, but also energy sources such as coal, oil, and natural or liquified gas, and even intermediary or manufactured products such as chemicals or generic drugs.

Nowadays, commodities have become the building block of a large part of our economy. They are relatively easily traded and can be physically delivered anywhere in the world. The volume of commodity trading in the world is huge: more than 91 million 60kg coffee bags were traded in 2006, more than 84 million barrels of oil are daily consumed... and in some cases in clear expansion (the natural gas sector grows 2% per year, liquified natural gas (LNG) grows 10%). Many less-developed countries depend heavily on the exports of a small number of primary commodities. There are big firms with huge departments devoted exclusively to commodity trading (e.g. Glencore, LouisDreyfus).

Commodities are traded in very active markets, such as the Chicago Board of Trade (CBOT), the New York Mercantile Exchange (NYMEX), or the London Metal Exchange (LME), to cite a few. Prices are determined by the market, rather than by the large suppliers or large buyers, with the exception perhaps of the Organization of the Petroleum Exporting Countries (OPEC).

Spot prices at commodity markets exhibit several salient features: they are highly auto-correlated, and extremely volatile with rare but violent explosions in price. Figure 1.1 is an example of the time series plot of a commodity (coffee) with high volatility and spikes in price. Deaton and Laroque [8] explained the volatility by inventory holding dynamics: where speculators carry inventory, the price remains stable, but when they are out-of-stock, the price fluctuates wildly. Table 4.1 summarizes some of the statistical facts about eight primary commodities over the period November 1999-January 2008 (4-month and one-year a-c are the first order autocorrelation coefficients when considering the data in trimesters or annually. The Kurtosis is calculated with respect to that of a Normal Distribution).
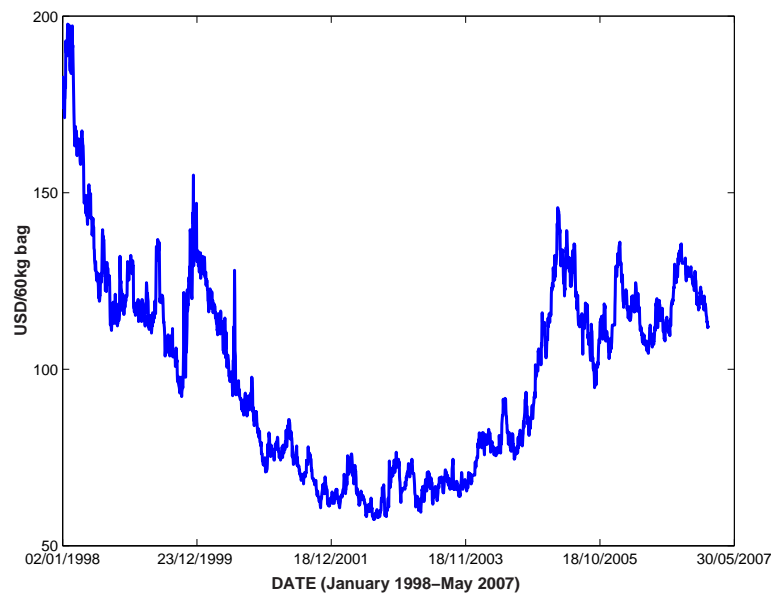

Figure 1.1: Daily coffee prices

It is commonly assumed that the price of a commodity is the same in any point in the world, as occurs for stocks or bonds. This is not so, as the price in each location is adjusted for logistics costs and local market conditions (local balance of supply and demand) among others. For example, a coffee producer with several factories in Europe will pay different prices for deliveries in each one of the factories. One would assume that these differences are very small, or at least stable over time. This is true in the long-run but in some cases price spreads, i.e., price difference between two locations, can be significant in the short term (we will examine a particular example in Chapter 4).

Table 1.1: Commodity price facts November 1999-January 2008. Source: Reuters.

	4-month a-c	One-year a-c	St.Dev	Skewness	Kurtosis
Rice	0.93	0.61	4.92	0.55	0.06
Tea	0.79	0.51	18.45	0.48	0.65
Wheat	0.87	0.56	73.85	1.77	8.55
Cotton	0.79	0.59	17.53	0.02	0.23
Sugar	0.83	0.24	66.85	3.16	12.67
Coffee	0.82	0.40	31.14	0.76	0.95
Cocoa	0.91	0.67	16.42	0.47	-0.38
Copper	0.96	0.85	97.7	2.70	7.36

In such markets, there is a good opportunity for commodity traders to enter the market, and, in the short-run, make a profit. In stock and bond markets, these constitute arbitrage opportunities that disappear as soon as they appear, these markets are considered to be arbitrage-free (Baxter [1] defines an *arbitrage opportunity* as the making of a guaranteed risk-free profit with a trade o series of trades in the market). For commodities, however, it takes time for traders to close the price spreads to its average levels. In order to understand the market, it is important to elucidate how a trader should behave, according to its own trading capacity. Trading capacity may consist of shipping from one market to the other, which may be costly, for example. We must of course consider that the trading activity has an impact on the price levels in each market, i.e., the price at a location increases if a trader buys from it, and decreases if he sells to it: each trader has market power.

1.2   Literature Review

There is significant research on commodity prices and commodity trading, with at least two different literature streams: the first one, from economics and finance, and the second one, from operations management.

The first research stream studies the properties of commodity price evolution in established markets. One of the first important papers in this area is Deaton and Laroque[8], which show that, as said before, commodity prices have high autocorrelation and variability. They develop a *rational expectations model*[1] for commodity prices that explains these features. They base their model on supply/demand forces and in particular on the existence of competitive speculators that hold inventory of the commodity, with which they expect to buy low and sell high. This behavior results in commodity prices normally staying within some limited range; here, speculators will build inventory when the price is low, and deplete it when it is high. As a result, the speculators act as regulators of the commodity prices. Sometimes, however, when supply and demand shocks push the price high, the speculators may run out of inventory, and hence spikes in prices will occur. Deaton and Laroque [9] extend this work and find that speculation only is not enough to explain the high positive autocorrelation in observed commodity prices and that part of it is caused by the underlying processes of supply and demand.

Williams and Wright [28], Chambers and Bailey [4] and Routledge et al. [20] also use a competitive rational expectations model to explain the properties of commodity prices in markets. They all consider the existence of *convenience yields*, which are defined in Hull[17] as a measure of the benefits from owning a commodity asset versus holding a long futures contract on the asset. A *futures contract* is a contract that obligates the holder to buy or sell an asset at a predetermined delivery price during a specified future time period. A convenience yield is generally positive since carrying inventory allows the owner to make profits from trading opportunities that may arise, for instance from temporary local shortages; on the other hand, holding a futures contract does not allow it. As an example, an oil refiner is unlikely to regard a futures contract on crude oil as equivalent to crude oil held in inventory: the crude oil in inventory can be an input to the refining process whereas a futures contract cannot be used for this purpose. The convenience yield typically reflects the market's expectations concerning the future availability of the commodity. As noted in Hull [17], the greater the chances that shortages will occur, the higher the convenience yield. One can describe spot prices evolution by modeling convenience yields and spot prices as separate stochastic processes, possibly correlated, see Gibson and Schwartz [12]. Schwartz [21] develops three variations of a mean-reverting stochastic model driven by one, two or three factors taking into account mean reversion of

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[1]The theory of *rational expectations* was first proposed by John F. Muth of Indiana University in the early sixties. He used the term to describe the many economic situations in which the outcome depends partly upon what people expect to happen. It is based on the standard economic assumption that people behave in ways that maximize their utility (their enjoyment of life) or profits. As an example, the price of an agricultural commodity, for example, depends on how many acres farmers plant, which in turn depends on the price that farmers expect to realize when they harvest and sell their crops

commodity prices, convenience yields and stochastic interest rates. Schwartz and Smith [22] provide a two-factor stochastic spot price model determined by the combination of short-term and long-term factors, that allow volatility in both terms. This two-factor model is equivalent to the stochastic convenience yield model developed in Gibson and Schwartz [12]. These models are empirically validated for copper, gold and crude oil. Our work does not directly model mean-reverting spot price processes or convenience yields, as in these papers, but focuses instead on modeling price spreads.

Routledge et al. [19] consider the correlation between prices of different commodities within broad families (e.g., natural gas and electricity in energy). They model the substitutability of these commodities and find existence of equilibrium price processes, where rational agents convert one commodity into the other. In particular, they focus on natural gas and electricity, and analyze the spread (price difference) between these two commodities. This is known in the industry as *the spark spread*, and is used extensively by commodity traders. The industry also uses *the dark spread*, i.e., the spread between coal and electricity prices. Reuters (the world's largest international multimedia news agency) provides to its users calculators for these two types of spreads. In our work, we focus on price spreads between different geographical locations, although the work could be extended to spreads between different commodities, as discussed later. We take a similar approach, and specifically consider the individual operational actions of rational traders, which allows us to describe in closed-form the agents' actions.

A second stream of literature on commodities exists in operations management. The papers in this group typically focus on the management of inventory of commodities, in the presence of price uncertainty, with buy/sell decisions in a single market. There is extensive literature on inventory management models, see for example Zipkin [30]. Goel and Gutierrez [13][14][15], apply these type of models specifically into commodities. These papers try to incorporate the information given by the convenience yield in the inventory and buy/sell decisions, e.g., Caldentey et al. [3] for copper mining operations in Chile, where they use the stochastic process in Schwartz [21] to model copper spot prices. In this sense, they try to combine finance and operations models.

In this group, Golabi [16] models the prices of the commodity in future periods as random variables with known distribution functions. Assuming constant demand, he proves that a sequence of critical price levels at a given period determines the optimal ordering strategy. Wang [27] proves that a myopic inventory policy is optimal for a multi-period model with stochastic demand and decreasing prices. Secomandi [23] considers optimal commodity trading and provides a much more detailed view of the operations involved in trading. He focuses on *storage assets*, i.e. storage facilities or contracts that ensure that one will have the inventory at a pre-determined time. His model is based on inventory and well-behaved flow constraints. He shows that the optimal policy is, depending on the region, to buy and withdraw, to do nothing or to sell and inject. This type of policy is used for contract valuation in Secomandi [23] and Wang et al. [26].

Our work combines the ideas of price equilibrium, from economics and finance, together with a more detailed view of operations. We are specifically interested in optimizing

the inventory management policy, given that the trading activity may influence the price spread process.

Chapter 2

The Model

2.1 Two Markets

Let us consider the trading of a single commodity in only two locations, market A and market B. Each one of these markets has its own local dynamics, and as a result the price at which the commodity trades in each place may be different. As said, we name these differences in price as price spreads or simply *spreads*.

Trading may also take place between A and B. In particular, traders may choose to buy in one place and ship (send) the commodity to the other. Of course, doing so, the traders incur trading costs. We will focus on this type of trading between two locations.

The **first assumption** of our model is that there are no lead-times when shipping the commodity from one market to the other. Implicitly, we are assuming that at any given time, what is bought in one location is sold at the other. Although in reality shipping of physical commodities cannot be done immediately, our first assumption simplifies the problem and makes it more tractable.

The **second** important **assumption** is considering constant linear trading costs. This is a strong assumption, since we are saying that the costs are independent of the price of the commodity. We could have considered other types of costs. As an example: if the trading cost is a fraction of the dollar volume of trade, the cost would be a constant times the spot price at the buying market times the trading quantity. While this is numerically solvable, the analysis becomes intractable. However, assuming linear costs has been made before in the literature, e.g., in Goel and Gutierrez [14].

Our basic model describes a pipeline, where no intermediate storage is allowed. In other words, a trader simply opens a faucet so that the commodity flows from A to B or vice versa. We consider in addition that the pipeline capacity is given by the maximum achievable flow, in units per period.

We denote:

- q^{AB} the trading capacity from A to B, (per period)

- q^{BA} the trading capacity the one from B to A.

- u_t^{AB} trading quantity from A to B at time t

- u_t^{BA} trading quantity from B to A at time t

- c^{AB} the cost of moving one unit from A to B

- c^{BA} the cost of moving one unit from B to A.

The capacity constraint implies that $0 \leq u_t^{AB} \leq q^{AB}$ and $0 \leq u_t^{BA} \leq q^{BA}$ and the total trading cost incurred is equal to $c^{AB} u_t^{AB} + c^{BA} u_t^{BA}$.
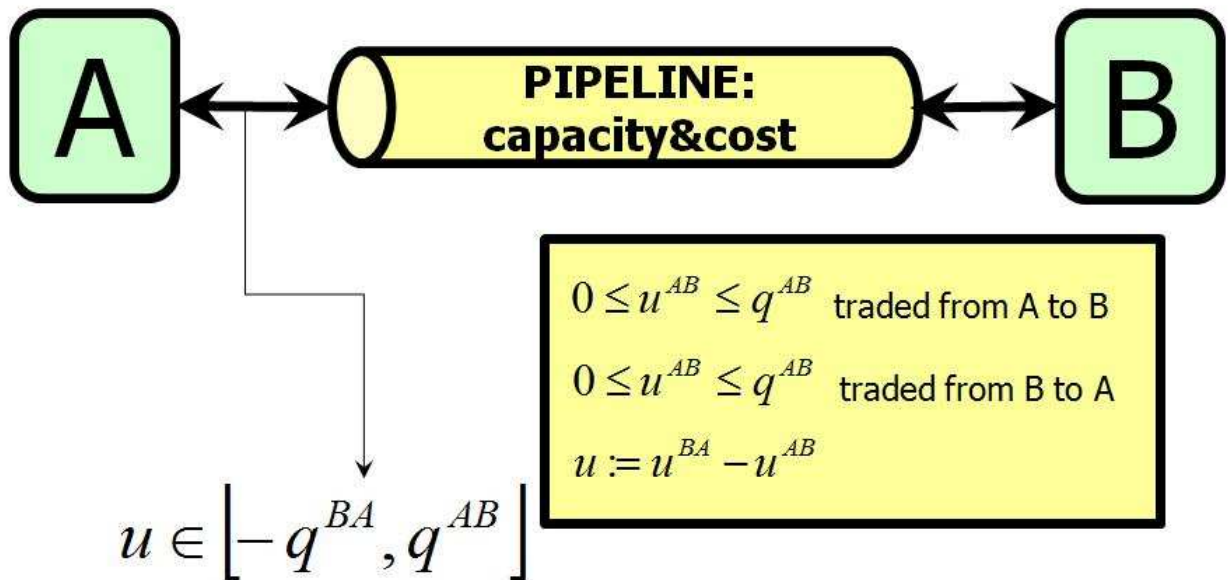

Figure 2.1: Summary of model features. The trader can ship u^{AB} units of commodity from A to B and u^{BA} units from B to A.

The model could also handle a shipping process that does not involve a pipeline, as long as the inventory level in each part of the route remains constant. For instance, consider a set of trucks or ships that on each day can move towards A or towards B (all together). The quantity originating in A and arriving in B will be identical, and will be constrained by the maximum speed in the system.

2.1.1   Trading between two markets

Any trading policy, if decided rationally, should obviously depend on the prices quoted in A and B. For example, when $P_t^B - P_t^A > c^{AB}$, it is profitable to trade from A to B. Clearly, if the current trading activity does not influence the future prices, the trader would try to maximize the present profit from trading at t, i.e.,

$$\max_{u^{AB}, u^{BA} \geq 0} (P_t^B - P_t^A)(u_t^{AB} - u_t^{BA}) - c^{AB} u_t^{AB} - c^{BA} u_t^{BA}.$$

In this case, it is easy to see that the optimal policy is such that either $u_t^{AB} = 0$ or $u_t^{BA} = 0$ and

$$u_t := u_t^{AB} - u_t^{BA} = \begin{cases} q^{AB} & \text{if } P_t^B - P_t^A \geq c^{AB} \\ 0 & \text{if } c^{AB} \geq P_t^B - P_t^A \geq -c^{BA} \\ -q^{BA} & \text{otherwise.} \end{cases}$$

We define the *net trading quantity* at time t as $u_t := u_t^{AB} - u_t^{BA}$. With this notation, the capacity constraints become $-q^{BA} \leq u_t \leq q^{AB}$. Notice that u_t *determines the trading policy.*

In other words, we will trade at maximum capacity when the price spread is bigger than the trading cost. As noted before, if P_t^A and P_t^B are independent of the trading quantities, this trading policy is optimal, as shown in Secomandi [24].

It is a basic economical fact that in a financial market (for a constant supply) when demand increases (many traders willing to buy) prices increase and when demand decreases prices go down. Then, if many traders follow the policy described before, the price at the cheaper market would rise and the price at the other market would go down, so the spread would tend to close over time. Hence, it is clear that the trading volume must impact the prices in the future. If this was not the case, we would see large spreads between locations, and more importantly, that these spreads may not close over time.

As a result, if the trading quantities do influence future prices, it is no longer clear what type of policy is now optimal for the trader. This requires modeling the relationship between global trading policies and the price spread process. This is done in the next section.

2.2   The Price Processes

We model time as a continuous variable, $t \geq 0$, and we denote the price of the commodity in markets A and B as P_t^A and P_t^B respectively. Using the same notation as in the previous section, $v_t := v_t^{AB} - v_t^{BA}$ denotes what the other traders in the market trade from A to B. Of course, this is positive when the quantity flows from A to B and negative otherwise.

Using *Game Theory* nomenclature we could see the traders in the market as players in a game. We will use the two words (trader/player) indistinctively.

We model the price processes at A and B as correlated random walk, i.e.,

$$\begin{pmatrix} dP_t^A \\ dP_t^B \end{pmatrix} = \begin{pmatrix} \alpha(P_t^A, P_t^B) + \beta_A(u_t + v_t) \\ \alpha(P_t^A, P_t^B) - \beta_B(u_t + v_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{A1} & \sigma_{A2} \\ \sigma_{B1} & \sigma_{B2} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} \qquad (2.1)$$

where W_t^1 and W_t^2 are independent Wiener processes, α can be any exogenous function, and $\beta_A, \beta_B, \sigma_{...} \geq 0$.

With this formulation, it is easy to see that the trade between A and B has a direct impact on the prices process, which captures the players' market power. Specifically, the *total net trade* from A to B at t, i.e., $u_t + v_t$, if positive, increases the price at A and decreases it at B. This is qualitatively intuitive. Furthermore, we model this dependency as *linear*. This is a strong assumption, which is related to having linear price-quantity demand curves.

Notice also that the proposed model could exhibit mean-reversion properties, *as long as the mean-reverting adjustment is the same in both markets.* For example, one could let $\alpha(P_t^A, P_t^B) = \kappa \left[\mu - (\theta^A P_t^A + \theta^B P_t^B) \right]$, where μ would be the long-term mean price, defined as a weighted average of prices at A and B.

Finally, in contrast with models in finance, e.g., Schwartz [21], we consider linear random walks, instead of geometric ones. This assumption guarantees tractability of the trading problem. However, since geometric Brownian motions might be more appropriate, we explore an alternative model in Chapter 4.

As said, the profit captured by the trader depends on the price spread. We define $G_t := P_t^B - P_t^A$ the price spread at time t between A and B.

From Equation (2.1), we have that $dG_t = -(\beta_A + \beta_B)(u_t + v_t)dt + (\sigma_{B2} - \sigma_{A2})dW_t^2 - (\sigma_{A1} - \sigma_{B1})dW_t^1$. Thus, G_t is a stochastic process that can be expressed as

$$dG_t = -\beta(u_t + v_t)dt + \sigma dW_t, \qquad (2.2)$$

where $\beta = \beta_A + \beta_B$, $\sigma^2 = (\sigma_{B2} - \sigma_{A2})^2 + (\sigma_{A1} - \sigma_{B1})^2$ and W_t a Wiener process.

In the remainder of the Thesis, we assume that *the traded quantity v_t (related to the rest of the market) only depends on G_t.* This assumption guarantees tractability. As a result, since both revenue and cost depend only on G_t, it follows that, at optimality, the trader's policy u_t should only be a function of the spread G_t.[1]

Hence, considering that u_t and v_t are functions of G_t only, i.e., $u_t = u(G_t)$ and $v_t = v(G_t)$,

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[1]In a game setting, this would be a Nash equilibrium, since the best-response to a trading policy that depends only on G_t is a trading policy that depends only on G_t. This is true when each trader in the market has operating costs $c^{ij} u_t^{ij}$.

the random variable G, satisfies the stochastic differential equation

$$dG_t = A(G)dW_t + B(G)dt \qquad (2.3)$$

with $A(G) = \sigma$ and $B(G) = -\beta(u(G) + v(G))$.

Now, if for a certain $\epsilon > 0$, $\lim\limits_{G \to -\infty} B \geq \epsilon$ and $\lim\limits_{G \to +\infty} B \leq -\epsilon$, G has a stationary probability density function (p.d.f.), denoted $f(G)$, which satisfies

$$\frac{1}{2}A^2(G)\frac{\partial^2 f}{\partial G^2} - B(G)\frac{\partial f}{\partial G} = 0. \qquad (2.4)$$

This is known as the Kolmogorov equation, and is a backward parabolic partial differential equation.

We give a quick justification of how we reach equation 2.4. As we are interested in the long term average profits we do not consider the transients solutions of the Kolmogorov equation, that involve $\dfrac{\partial f}{\partial t}$. We can consider the backward problem. The probability that the spread is G at time $t + dt$, denoted as $f(G, t + dt)$, depends only on where the spread was at time t, as we deal with a Markovian process. So we have

$$f(G, t + dt) = \mathbb{E}_\phi\Big[f(G - B(G)dt - A(G)\sqrt{dt}\epsilon, t)\Big]$$

where $\phi(\epsilon)$ is a Gaussian distribution ($\mathbb{E}_\phi[\epsilon] = 0$, $\mathbb{E}_\phi[\epsilon^2] = 1$, and $\int_{-\infty}^{\infty}\phi(\epsilon)d\epsilon = 1$). When the p.d.f. is stationary, a second order Taylor approximation yields

$$
\begin{aligned}
f(G) = \ & f(G, t + dt) = \int_{-\infty}^{\infty} f(G - B(G)dt - A(G)\sqrt{dt}\epsilon, t)\phi(\epsilon)d\epsilon \\
= \ & \int_{-\infty}^{\infty}\Big[f(G) - \frac{\partial f}{\partial G}[B(G)dt + A(G)\sqrt{dt}\epsilon] + \frac{1}{2}\frac{\partial^2 f}{\partial G^2}A(G)^2 dt\epsilon^2\Big]\phi(\epsilon)d\epsilon \\
= \ & \int_{-\infty}^{\infty}\big[f(G) - \frac{\partial f}{\partial G}B(G)dt\big]\phi(\epsilon)d\epsilon - \int_{-\infty}^{\infty}\frac{\partial f}{\partial G}A(G)\sqrt{dt}\epsilon\phi(\epsilon)d\epsilon + \int_{-\infty}^{\infty}\frac{1}{2}\frac{\partial^2 f}{\partial G^2}A(G)^2 dt\epsilon^2\phi(\epsilon)d\epsilon \\
= \ & f(G) - \frac{\partial f}{\partial G}B(G)dt + \frac{1}{2}\frac{\partial^2 f}{\partial G^2}A(G)^2 dt.
\end{aligned}
$$

This yields Equation (2.4). For more details on the methodology, see Wilmott [29].

In our case, Equation (2.4) can be written as

$$\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial G^2} + \beta(u(G) + v(G))\frac{\partial f}{\partial G} = 0. \qquad (2.5)$$

Thus, the net trading capacity $u(G)$ has a direct influence on the p.d.f. of the spread (we denote it by $f_u(G)$ to make that dependency apparent). The only variable that depends on the trader decision is $u(G)$, so it acts as a control for Equation 2.5. As an illustration, we show in Figure 2.2 the stationary p.d.f. for two controls u, where $v \equiv 0$.
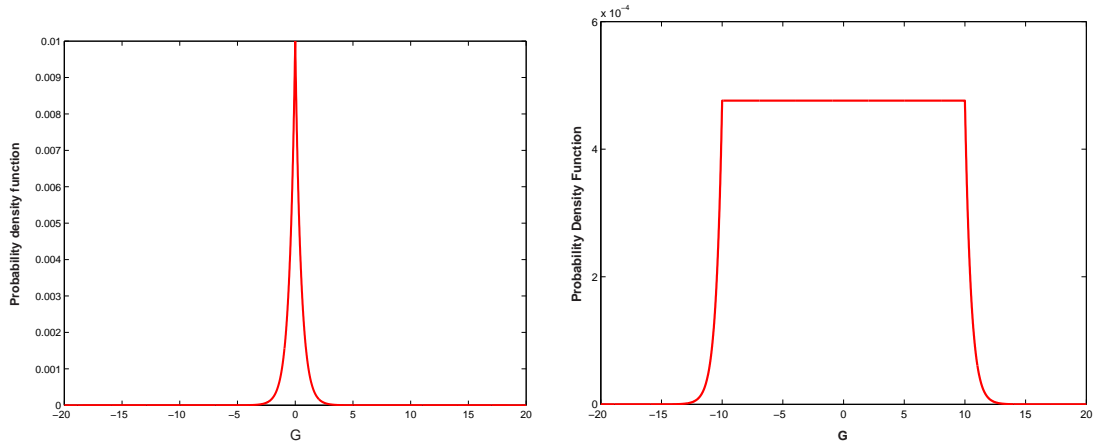
Figure 2.2: Stationary p.d.f. of G, with $v \equiv 0$, $\beta = 1$ and $\sigma = 1$, where, in the figure on the left, $u = -2$ for $G < 0$ and $u = +2$ for $G \geq 0$; and, in the figure on the right, $u = -2$ for $G < -10$, $u = 0$ for $-10 \leq G < 10$ and $u = +2$ for $G \geq 10$.

For simplicity, we restrict our attention to having both u and v piecewise constant functions. The model can incorporate non-piecewise linear controls, but the analysis becomes significantly more complex. However, this restriction is quite reasonable as the market is likely to react with fixed capacity to each spread interval. Now, if $u(G) + v(G)$ is piecewise constant, Equation (2.5) can be rewritten[2] as

$$u(G) = -\frac{\sigma^2}{2\beta} \frac{f'_u(G)}{f_u(G)} - v(G). \tag{2.6}$$

This explicitly links the traded quantity u to the price spread distribution f_u.

2.3   Optimization of Trading Policies

Given a realization of the price spread at time t, G_t, and given capacities q^{AB}, q^{BA}, we are interested in finding the optimal operating policy u that maximizes the trader's long-run average profits. The optimal policy provides, given a price spread between the two markets, where and how much product to buy and sell. There are two components in the profit: the revenue, $G_t u_t$, and the trading cost, $c^{AB} u_t^+ + c^{BA} u_t^-$, where $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$. In other words,

$$J(u) := \mathbb{E}_u \left[G_t u_t - c^{AB} u_t^+ - c^{BA} u_t^- \right] \tag{2.7}$$

where the expectation is taken with respect to f_u, the stationary distribution of G_t. Hence, the problem in (2.7) can be equivalently expressed as

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[2] As $f(G)$ is a density function, the constant of integration must be 0.

$$J^* = \sup_u J(u) = \int_{-\infty}^{\infty} \left(Gu(G) - c^{AB}u^+(G) - c^{BA}u^-(G) \right) f_u(G)dG$$

$$\text{subject to} \quad u(G) = -\frac{\sigma^2}{2\beta}\frac{f_u'(G)}{f_u(G)} - v(G) \tag{2.8}$$

$$-q^{BA} \le u(G) \le q^{AB}$$

$$f_u \text{ a p.d.f.}$$

where $f_u(G)$ is the spread distribution that depends on our control. This formulation has the advantage of being quite compact.

Alternatively, we could have considered a discounted profit model, where the control would again depend only on G_t. However, by examining the alternative formulation,

$$J(u, G_0) = \mathbb{E}\left[\int_0^{\infty} \left(G_t u(G_t) - c^{AB}u^+(G_t) - c^{BA}u^-(G_t) \right) e^{-rt}dt \right], \tag{2.9}$$

one can observe that here we would need to consider the transient distribution of G_t, given that at time $t = 0$, G_0 is the spread.

A discrete version of the problem would raise similar complications:

$$J(G_0) = \max_{-q^{BA} \le u \le q^{AB}} \left\{ G_0 u - c^{AB}u^+ - c^{BA}u^- + \alpha\mathbb{E}_{\tilde{W}}\left[J\left(G_0 - \beta(u + v(G_0))\Delta t + \sigma\sqrt{\Delta t}\tilde{W} \right) \right] \right\} \tag{2.10}$$

The problem posed by maximizing $J(u, G_0)$ in Equation (2.9) can be solved using optimal control. Interestingly, as we show in §4.2.1, the structure of the results that we derive for our basic problem setting, i.e., focusing on average profits, Equation (2.8), hold for this alternative formulation.

2.4   Summary

In this chapter we have presented the model and characterized the optimization problem to be solved. This can be summarized in the following key points:

- We have describe what type of trading between two locations we consider, and modeled it as a **pipeline** between A and B. The capacity of the pipeline is its maximum flow and no intermediate storage nor lead-times are considered. We have discussed why we consider **linear trading costs**.

- We have modeled the spread dynamics, G_t, as a stochastic process that depends on the net trading quantity, $u_t + v_t$ and that satisfies $dG_t = -\beta(u_t + v_t)dt + \sigma dW_t$. With this expression we capture the impact of trading quantities in prices, i.e, **market power**.

- We have explicitly linked our trading quantity (**control** u) to the spread probability density function, f_u, through the following equation: $u(G) = -\frac{\sigma^2}{2\beta}\frac{f_u'(G)}{f_u(G)} - v(G)$

- We have presented the trader's optimization problem (2.8) that gives the optimal trading policy when our aim is to maximize the long-run average profits.

Chapter 3

Optimal Trading Policies

Once related the trading quantity to the price spread distribution, we are interested in finding the optimal policy u that maximizes the trader's problem (2.8). This policy must determine, given a realization of the current price spread, G_t, how much (if any) volume must be bought/sold from/at the A market and sold/bought at/from the B market, so that the long-run average profit is maximized.

We first analyze the case of *monopoly*, where there are no other traders between A and B (i.e., $v \equiv 0$) and then solve the general case.

3.1 Monopoly: $v \equiv 0$

We consider in this section an extreme case of our problem: the trader has no competitors trading between A and B. We call this a monopoly. Note that this does not mean that the trader is a price-setter in A or B, but rather that he is the only player in the link between A and B. In this case, Equation (2.6) can be written as

$$u(G) = -\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)}$$

Intuitively, one would think that this is the situation in which the trader can make highest profits. The next propositions show that, in fact, it is optimal to trade as little as possible, unless the price spread is really large, in order to maximize the average trading profit.

Proposition 1. *If*

(i) $c^{AB} = c^{BA} = 0$ *and*

(ii) $v(G) = 0 \; \forall G,$

then any policy is optimal in (2.8) and $J^* = \dfrac{\sigma^2}{2\beta}$.

Proof.

$$\mathbb{E}\Big\{Gu(G)\Big\} = \int_{-\infty}^{\infty} Gu(G)f(G)dG = \int_{-\infty}^{\infty} G\left(-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)}\right)f(G)dG = \frac{\sigma^2}{2\beta}\int_{-\infty}^{\infty} -Gf'(G)dG = \frac{\sigma^2}{2\beta}$$

after integrating by parts $Gf'(G)$ and remembering that $\displaystyle\int_{-\infty}^{\infty} f(G)dG = 1$ as f is a p.d.f.

∎

This result is somehow surprising. Indeed, in the short term, it would be better to instantly gain, at time t, $G_t q^{AB}$ if $G_t > 0$ or $-G_t q^{BA}$ if $G_t < 0$. However, by doing this every time, the trader would quickly modify the spread p.d.f. and push G_t close to zero, which would reduce future profits. Alternatively, the trader could delay the trade until $|G_t|$ is large, in which case, he would obtain higher profits, but in fewer occasions. The proposition shows that both strategies will actually yield the same average profit in the long term when the trading cost is zero. In other words, the overall average benefit does not depend on the concrete feasible policy: it is the same to trade often, even with small spreads, or to delay the trade until spreads are large.

In contrast, the optimal policy does matter when there are trading costs.

Proposition 2. *If*

    (i) $c^{AB}, c^{BA} > 0$ and

    (ii) $v(G) = 0 \ \forall G$,

then the optimal policy for our trading problem (2.8) is the limit, when $M \longrightarrow \infty$, of

$$u(G) = \begin{cases} -q^{BA} & \text{if } G \leq M \\ 0 & \text{if } -M < G \leq M \\ q^{AB} & \text{if } G > M \end{cases}$$

and yields, at the limit, $J^* = \dfrac{\sigma^2}{2\beta}$.

Proof. On the one hand we have that the expected long-term average gains are:

$$\mathbb{E}\left\{Gu(G) - c^{AB}u^+(G) - c^{BA}u^-(G)\right\}$$

$$= \int_{-\infty}^{\infty}\left(Gu(G) - c^{AB}u^+(G) - c^{BA}u^-(G)\right)f(G)dG$$

$$= \int_{-\infty}^{\infty}\left(G\left[-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)}\right] - c^{AB}\left[-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)}\right]^+ - c^{BA}\left[-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)}\right]^-\right)f(G)dG$$

$$= \frac{\sigma^2}{2\beta} - c^{AB}\int_{-\infty}^{\infty}\left[-\frac{\sigma^2}{2\beta}f'(G)\right]^+ dG - c^{BA}\int_{-\infty}^{\infty}\left[-\frac{\sigma^2}{2\beta}f'(G)\right]^- dG$$

$$\leq \frac{\sigma^2}{2\beta}.$$

On the other hand, the profits obtained with the policy given by the proposition are:

$$\int_{-\infty}^{\infty}Gu(G)f(G)dG - c^{AB}\int_{M}^{\infty}q^{AB}f(G)dG - c^{BA}\int_{-\infty}^{-M}q^{BA}f(G)dG$$

$$= \frac{\sigma^2}{2\beta} - c^{AB}q^{AB}\int_{M}^{\infty}f(G)dG - c^{BA}q^{BA}\int_{-\infty}^{-M}f(G)dG.$$

Now, at the limit for $M \to \infty$, this value is $\frac{\sigma^2}{2\beta}$.

Indeed, we have $f(G) = f(-M)e^{\frac{2\beta q^{BA}(G+M)}{\sigma^2}}$ for $G < -M$, $f(G) = f(-M) = f(M)$ for $-M < G < M$ and $f(G) = f(M)e^{\frac{-2\beta q^{AB}(G-M)}{\sigma^2}}$ for $G > M$. Of course, $f(-M) = f(M)$ is a constant such that f integrates to one. As a result, $\lim_{M\to\infty}\int_{M}^{\infty}f = \lim_{M\to\infty}\int_{-\infty}^{-M}f = 0$. This implies that, regardless of the policy $u(G)$, the maximum long term profit we can possibly obtain is $\frac{\sigma^2}{2\beta}$. This is achieved as the limit of the proposed policy. ∎

Proposition 2 incorporates cost. Proposition 1 implied that when $v \equiv 0$, and $c^{AB} = c^{BA} = 0$, then the average long-run profit is independent of the trading quantity. When the costs are positive, the policy of *not trading anything* therefore yields the same average profit, and is optimal. Since such a policy is actually not feasible (the p.d.f. is not well-defined), we show that the same objective can be achieved as the limit of feasible policies.

3.2   Oligopoly: $v(G) \neq 0$

Economically, an oligopoly is a market form in which a market or industry is dominated by a small number of traders.

We have so far assumed that we, as traders, were the only player with the ability to trade and make profits from the price spread. We consider below a market with several competitors. Competition might change the structure of the optimal policy because the distribution now also depends on how the competitors influence the market. We have denoted this influence as $v(G)$, where $v(G)$ is the total net quantity traded from A to B, by all the competitors. In that case, the control satisfies, from Equation (2.6),

$$u(G) = -\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G).$$

When there are no trading costs, i.e., $c^{AB} = c^{BA} = 0$, the objective function can be written as

$$\int_{-\infty}^{\infty} Gu(G)f(G)dG = \int_{-\infty}^{\infty} G\left(-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G)\right)f(G)dG$$
$$= \frac{\sigma^2}{2\beta} + \int_{-\infty}^{\infty} -Gv(G)f(G)dG$$

after integrating by parts $Gf'(G)$. Hence, the maximization problem is equivalent

$$\begin{aligned}
\sup_{f} \quad & \int_{-\infty}^{\infty} -Gv(G)f(G)dG \\
\text{subject to} \quad & -q^{BA} \leq -\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G) \leq q^{AB} \\
& f \text{ a p.d.f.}
\end{aligned} \tag{3.1}$$

This reformulation has the advantage that the trading volume u is no longer present. We can thus optimize directly taking f as the decision variable. The optimization problem can be fully solved, with the support of a technical lemma (1) which we show for the general case. In short, we find necessary conditions that characterize the optimal function f. Since the p.d.f. and the trading quantity $u(G)$ are directly related, this is equivalent to finding the optimal trading policy.

As for the monopoly case we begin with a proposition that characterizes the optimal trading policy when there are no trading costs.

Proposition 3. *If*

(i) $c^{AB} = c^{BA} = 0$ *and*

(ii) $v(G)$ *is non-decreasing,* $\lim_{G \to +\infty} v(G) > 0$ *and* $\lim_{G \to -\infty} v(G) < 0$,

then there exists m such that the optimal policy for our trading problem (2.8) is

$$u(G) = \begin{cases} -q^{BA} & \text{if } G \leq m \\ q^{AB} & \text{if } G > m. \end{cases}$$

Proof. The proof of Proposition 3 is a particular case of the proof of Theorem 1 ∎

Notice that the proposition requires a regularity assumption on v, in order to have a well-defined optimal policy, and avoid asymptotically optimal solutions, as in Proposition 2.

Under such condition, when there are no costs, we find that the optimal trading policy is to trade as much as possible from one market to the other, with the trading direction depending on whether the spread G is higher or lower that a threshold m. We show the result by establishing that, in problem (3.1), one of the two constraints on $\dfrac{f'}{f}$ is always binding.

We next generalize the preceding results to the general case, with $c^{AB}, c^{BA} > 0$. In that case, similarly to Equation (3.1), the trading problem (2.8) is equivalent to

$$\sup_{f} \int_{-\infty}^{\infty} \left[-Gv(G) - c^{BA}\left(-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G) \right)^{-} - c^{AB}\left(-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G) \right)^{+} \right] f(G)dG$$

$$\text{subject to} \quad -q^{BA} \leq -\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G) \leq q^{AB}$$

$$f \text{ a p.d.f.}$$

$$(3.2)$$

Solving the problem above yields the following theorem. This is the main result of our work as it gives the optimal trading policy for the general case.

Theorem 1. *If*

(i) $c^{AB}, c^{BA} > 0$ *and*

(ii) $v(G)$ *is non-decreasing,* $\lim_{G \to +\infty} v(G) > 0$ *and* $\lim_{G \to -\infty} v(G) < 0$,

then there exists m^{BA}, m^{AB}, *such that* $m^{BA} \leq m^{AB}$, *and the optimal policy for our trading problem (2.8) is*

$$u(G) = \begin{cases} -q^{BA} & \text{if } G \leq m^{BA} \\ 0 & \text{if } m^{BA} < G \leq m^{AB} \\ q^{AB} & \text{if } G > m^{AB}. \end{cases}$$

In order to prove Theorem 1, we first need to prove the following lemma.

Lemma 1. *Given functions $v(G)$ and $h(G) := -Gv(G)$, and an optimal solution $f(G)$ of the maximization problem (3.2)*

If there is $[g_0, g_0 + \delta]$ such that $\dfrac{-\sigma^2}{2\beta} \dfrac{f'}{f} > -q^{BA} + v(G)$ then we must have

$$\frac{\int_{g_0}^{\infty} \left(h - c^{AB} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^+ - c^{BA} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^- \right) f + \frac{\sigma^2}{2\beta} f(g_0) \left(c^{AB} \mathbf{1}_{\left(\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \geq 0 \right)} - c^{BA} \mathbf{1}_{\left(\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \leq 0 \right)} \right)}{\int_{g_0}^{\infty} f}$$

$$\leq \frac{\int_{-\infty}^{\infty} \left(h - c^{AB} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^+ - c^{BA} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^- \right) f}{\int_{-\infty}^{\infty} f}.$$

If there is $[g_0, g_0 + \delta]$ such that $\dfrac{-\sigma^2}{2\beta} \dfrac{f'}{f} < q^{AB} + v(G)$ then we must have

$$\frac{\int_{g_0}^{\infty} \left(h - c^{AB} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^+ - c^{BA} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^- \right) f + \frac{\sigma^2}{2\beta} f(g_0) \left(c^{AB} \mathbf{1}_{\left(\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \geq 0 \right)} - c^{BA} \mathbf{1}_{\left(\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \leq 0 \right)} \right)}{\int_{g_0}^{\infty} f}$$

$$\geq \frac{\int_{-\infty}^{\infty} \left(h - c^{AB} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^+ - c^{BA} \left[\frac{-\sigma^2}{2\beta} \frac{f'}{f} - v \right]^- \right) f}{\int_{-\infty}^{\infty} f}.$$

Proof. We consider an optimal f that satisfies (3.2) such that, within an interval $[g_0, g_0 + \delta]$, $\dfrac{-\sigma^2}{2\beta} \dfrac{f'}{f} > -q^{BA} + v(G)$, or in other words, $\dfrac{\sigma^2}{2\beta} \dfrac{f'}{f} < q^{BA} - v(G)$. Define $\tilde{u}(G) = u(G) + \epsilon$ for $G \in [g_0, g_0 + \delta]$, and equal to $u(G)$ otherwise. The corresponding $\tilde{f}$ satisfies

$$\frac{\tilde{f}'(G)}{\tilde{f}(G)} = \begin{cases} \dfrac{f'}{f} + \epsilon & \text{if } G \in [g_0, g_0 + \delta] \\ \dfrac{f'}{f} & \text{otherwise} \end{cases} \tag{3.3}$$

and hence

$$\tilde{f}(G) = \begin{cases} \dfrac{f}{\int_{-\infty}^{g_0} f + \int_{g_0}^{g_0+\delta} e^{\epsilon(G-g_0)} f(G) dG + e^{\epsilon\delta} \int_{g_0+\delta}^{\infty} f} & \text{if } G < g_0 \\[2ex] \dfrac{f e^{\epsilon(G-g_0)}}{\int_{-\infty}^{g_0} f + \int_{g_0}^{g_0+\delta} e^{\epsilon(G-g_0)} f(G) dG + e^{\epsilon\delta} \int_{g_0+\delta}^{\infty} f} & \text{if } G \in [g_0, g_0 + \delta] \\[2ex] \dfrac{f e^{\delta\epsilon}}{\int_{-\infty}^{g_0} f + \int_{g_0}^{g_0+\delta} e^{\epsilon(G-g_0)} f(G) dG + e^{\epsilon\delta} \int_{g_0+\delta}^{\infty} f} & \text{if } G > g_0 + \delta \end{cases} \tag{3.4}$$

In $G \in [g_0, g_0 + \delta]$, for small $\epsilon > 0$,

$$\left[\frac{-\sigma^2}{2\beta} \frac{\tilde{f}'(G)}{\tilde{f}(G)} - v(G) \right]^+ = \left[\frac{-\sigma^2}{2\beta} \frac{f'(G)}{f(G)} - v(G) \right]^+ - \frac{\sigma^2 \epsilon}{2\beta} \mathbf{1}_{\left(\frac{-\sigma^2}{2\beta} \frac{f'(G)}{f(G)} - v(G) \geq 0 \right)}$$

and

$$\left[\frac{-\sigma^2}{2\beta}\frac{\tilde{f}'(G)}{\tilde{f}(G)} - v(G)\right]^- = \left[\frac{-\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G)\right]^- + \frac{\sigma^2\epsilon}{2\beta}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'(G)}{f(G)} - v(G) \leq 0\right)}$$

where $\mathbf{1}_Z = 1$ if condition Z is satisfied and 0 otherwise. The objective using policy $\tilde{u}$ can be written as

$$J_{\epsilon,\delta}(G) = \int_{-\infty}^{\infty} \left(h - c^{AB}\left[\frac{-\sigma^2}{2\beta}\frac{\tilde{f}'(G)}{\tilde{f}(G)} - v(G)\right]^+ - c^{BA}\left[\frac{-\sigma^2}{2\beta}\frac{\tilde{f}'(G)}{\tilde{f}(G)} - v(G)\right]^-\right)\tilde{f}(G)dG$$

$$= \frac{\int_{-\infty}^{g_0} Hf + e^{\epsilon\delta}\int_{g_0}^{\infty} Hf + \int_{g_0}^{g_0+\delta}\frac{\sigma^2\epsilon}{2\beta}\left(c^{AB}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \geq 0\right)} - c^{BA}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \leq 0\right)}\right)fe^{\epsilon(G-g_0)}dG}{\int_{-\infty}^{g_0} f + \int_{g_0}^{g_0+\delta} e^{\epsilon(G-g_0)}fdG + e^{\epsilon\delta}\int_{g_0+\delta}^{\infty} f}$$

where $H = \left(h - c^{AB}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^+ - c^{BA}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^-\right)$ to simplify notation.

A Taylor approximation with respect to ϵ and δ (hence considering that $\delta^2 \ll \delta$), yields that

$$J_{\epsilon,\delta}(G) = \frac{\int_{-\infty}^{\infty} Hf + \epsilon\delta\left\{\int_{g_0}^{\infty} Hf + \frac{\sigma^2}{2\beta}f(g_0)\left(c^{AB}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \geq 0\right)} - c^{BA}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \leq 0\right)}\right)\right\}}{\int_{-\infty}^{\infty} f + \epsilon\delta\int_{g_0}^{\infty} f} + o(\epsilon\delta)$$

where $\lim_{x \to 0}\frac{o(x)}{x} = 0$.

Since f is optimal, we have $J_{\epsilon,\delta} \leq J$, for ϵ, δ small, which yields after some algebra, that

$$\frac{\int_{g_0}^{\infty} Hf + \frac{\sigma^2}{2\beta}f(g_0)\left(c^{AB}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \geq 0\right)} - c^{BA}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \leq 0\right)}\right)}{\int_{g_0}^{\infty} f} \leq \frac{\int_{-\infty}^{\infty} Hf}{\int_{-\infty}^{\infty} f}.$$

The same argument applied with $\frac{-\sigma^2}{2\beta}\frac{f'}{f} < q^{AB} + v(G)$ yields the second part of the lemma. $\blacksquare$

We are now ready to proof the theorem.

Proof. Denote W the right-hand side of the expression in Lemma 1, i.e.,

$$W := \frac{\int_{-\infty}^{\infty}\left(h - c^{BA}\left[-\frac{\sigma^2}{2\beta}\frac{f'}{f} - v\right]^- - c^{AB}\left[-\frac{\sigma^2}{2\beta}\frac{f'}{f} - v\right]^+\right)fdG}{\int_{-\infty}^{\infty} f} < 0,$$

from assumption (ii) of the Theorem. Consider f the optimal solution of problem (3.2), and $u = -\dfrac{\sigma^2}{2\beta}\dfrac{f'}{f} - v$. We claim that

(i) $u(G) = q^{AB}$ when $G \to \infty$;

(ii) $u(G) = q^{AB}$ for $G > m^{AB}$ for some m^{AB};

(iii) $u(G) = 0$ for $m^{BA} < G < m^{AB}$ for some $m^{BA} \leq m^{AB}$; and

(iv) $u(G) = -q^{BA}$ for $G < m^{BA}$.

We start with (i). If $g_0 \to \infty$ and $u(G) < q^{AB}$, Lemma 1 provides a necessary condition,

$$\frac{\int_{g_0}^{\infty}\left(h - c^{AB}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^+ - c^{BA}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^-\right)f + \frac{\sigma^2}{2\beta}f(g_0)\left(c^{AB}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \geq 0\right)} - c^{BA}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \leq 0\right)}\right)}{\int_{g_0}^{\infty} f} \geq W.$$

The left-hand side expression of the necessary condition is bounded from above by

$$\frac{\int_{g_0}^{\infty} hf + \frac{\sigma^2}{2\beta}c^{AB}f(g_0)}{\int_{g_0}^{\infty} f} = \frac{\int_{g_0}^{\infty}\left(-Gv(G) - \frac{\sigma^2}{2\beta}c^{AB}\frac{f'(G)}{f(G)}\right)f(G)dG}{\int_{g_0}^{\infty} f}$$

$$\leq \frac{\int_{g_0}^{\infty}\left(-Gv(G) + c^{AB}(q^{AB} + v(G))\right)f(G)dG}{\int_{g_0}^{\infty} f}$$

$$= c^{AB}q^{AB} + \frac{\int_{g_0}^{\infty} -(G - c^{AB})v(G)f(G)dG}{\int_{g_0}^{\infty} f}$$

Since, for $g_0 > c^{AB}$, $-(G - c^{AB})v(G)$ is decreasing (because $v(G)$ is positive when G is very large), the right-hand side tends to $-\infty$ as $g_0 \to \infty$. Hence this contradicts the necessary condition from the lemma. As a result, for G large enough $u(G) = q^{AB}$, and that implies (i).

Given (i), $u(G) = q^{AB} > 0$ for large G implies for large g_0,

$$\frac{\int_{g_0}^{\infty}\left(h - c^{AB}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^+ - c^{BA}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^-\right)f + \frac{\sigma^2}{2\beta}f(g_0)\left(c^{AB}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \geq 0\right)} - c^{BA}\mathbf{1}_{\left(\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v \leq 0\right)}\right)}{\int_{g_0}^{\infty} f}$$

$$= \frac{\int_{g_0}^{\infty}\left(-Gv(G) - c^{AB}q^{AB}\right)f(G)dG + \frac{\sigma^2}{2\beta}c^{AB}f(g_0)}{\int_{g_0}^{\infty} f}$$

$$= \frac{\int_{g_0}^{\infty} -(G - c^{AB})v(G)f(G)dG}{\int_{g_0}^{\infty} f}$$

where again we transformed $f(g_0)$ into $\int_{g_0}^{\infty}(-f'/f) \times f$, and used Equation (2.6). As before, this is continuous and hence it is optimal to set $u(g_0) = q^{AB}$ for $g_0 > m^{AB}$, where m^{AB} is such that

$$\frac{\int_{m^{AB}}^{\infty}(-Gv(G) - c^{AB}q^{AB})f(G)dG + \frac{\sigma^2}{2\beta}c^{AB}f(m^{AB})}{\int_{m^{AB}}^{\infty} f} = \frac{\int_{m^{AB}}^{\infty} -(G - c^{AB})v(G)f(G)dG}{\int_{m^{AB}}^{\infty} f} = W$$

This proves (ii). For $g_0 = m^{AB} - \epsilon$, we have that

$$\frac{\int_{g_0}^{\infty}\left(h - c^{AB}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^+ - c^{BA}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^-\right)f + \frac{\sigma^2}{2\beta}c^{AB}f(g_0)}{\int_{g_0}^{\infty} f} = W + \epsilon' > W,$$

for $\epsilon' > 0$ very small. Is is clear that $u(g_0) > 0$ cannot be optimal, as the necessary condition for $u < q^{AB}$ is not satisfied. Similarly, $u(g_0) < 0$ cannot be optimal, since

$$\frac{\int_{g_0}^{\infty}\left(h - c^{AB}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^+ - c^{BA}\left[\frac{-\sigma^2}{2\beta}\frac{f'}{f} - v\right]^-\right)f - \frac{\sigma^2}{2\beta}c^{BA}f(g_0)}{\int_{g_0}^{\infty} f} = W + \epsilon' - 2\frac{\sigma^2}{2\beta}c^{BA}\frac{f(g_0)}{\int_{g_0}^{\infty} f} < W.$$

Hence, it is optimal to set $u(g_0) = 0$. This is true while

$$W > \frac{\int_{g_0}^{m^{AB}} -Gv(G)f(G)dG + \int_{m^{AB}}^{\infty}(-Gv(G) - c^{AB}q^{AB})f(G)dG - \frac{\sigma^2}{2\beta}c^{BA}f(g_0)}{\int_{g_0}^{\infty} f}$$

$$= \frac{\int_{g_0}^{\infty} -(G + c^{BA})v(G)f(G)dG + \int_{m^{AB}}^{\infty} -(c^{AB} + c^{BA})q^{AB}f(G)dG}{\int_{g_0}^{\infty} f}$$

and

$$W < \frac{\int_{g_0}^{m^{AB}} -Gv(G)f(G)dG + \int_{m^{AB}}^{\infty}(-Gv(G) - c^{AB}q^{AB})f(G)dG + \frac{\sigma^2}{2\beta}c^{AB}f(g_0)}{\int_{g_0}^{\infty} f}$$

$$= \frac{\int_{g_0}^{\infty} -(G - c^{AB})v(G)f(G)dG}{\int_{g_0}^{\infty} f}.$$

These equations hold until the first one becomes an equality (since the right-hand side of the second equation is decreasing if equal to W, and the right-hand side of the first equation also decreases). As a result, they are satisfied for $g_0 > m^{BA}$, which proves (iii). m^{BA} is uniquely defined by

$$\frac{\int_{m^{BA}}^{m^{AB}} -Gv(G)f(G)dG + \int_{m^{AB}}^{\infty}(-Gv(G) - c^{AB}q^{AB})f(G)dG - \frac{\sigma^2}{2\beta}c^{BA}f(m^{BA})}{\int_{m^{BA}}^{\infty} f} = W.$$

Finally, for $G < m^{BA}$, the optimal decision is to set $u(G) = -q^{BA}$, as the necessary condition for $u < -q^{BA}$ is not satisfied. ∎
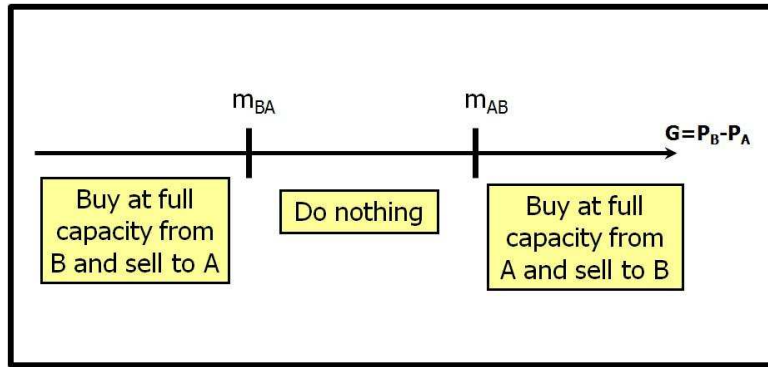
Figure 3.1: Structure of the optimal trading policy

In essence, the optimal trading policy is determined by three regions: $[-\infty, m^{BA}]$ where it is optimal to move as much product as possible from market B to the market A; $(m^{AB}, \infty]$, where the same is true from A to B, the opposite direction; and $(m^{BA}, m^{AB}]$ where it is optimal to do nothing, as one can see in Figure 3.1.

This is the same structure as in the case with no market power. In our notation, with no market power, $m^{BA} = -c^{BA}$ and $m^{AB} = c_{AB}$. Hence, market power changes the optimal trading policy by delaying the point where the traders begin to ship product from one market to the other.

The theorem thus generalizes Propositions 1, 2 and 3. The proof of the theorem is interesting, as it uses necessary conditions to determine the thresholds m^{BA}, m^{AB}.

3.3   The Optimal Trading Thresholds

In this section, we establish how to compute the optimal trading policy, or equivalently, the spread p.d.f. We show as well some properties of the problem depending on the function $v(G)$.

Interestingly, the optimal policy derived in Theorem 1 is given by three optimality equations. Knowing that the optimal policy is given by two thresholds m^{BA}, m^{AB}, since

$$-\frac{\sigma^2}{2\beta}\frac{f'(G)}{f(G)} = \begin{cases} -q^{BA} + v(G) & \text{if } G \leq m^{BA} \\ v(G) & \text{if } m^{BA} < G \leq m^{AB} \\ q^{AB} + v(G) & \text{if } G > m^{AB} \end{cases}$$

let

$$\varphi(G) = \begin{cases} e^{\int_{m^{BA}}^{G}(-v(t)+q^{BA})\frac{2\beta}{\sigma^2}dt} & \text{if } G \leq m^{BA} \\ e^{\int_{m^{BA}}^{G}-v(t)\frac{2\beta}{\sigma^2}dt} & \text{if } m^{BA} < G \leq m^{AB} \\ e^{\int_{m^{BA}}^{m^{AB}}-v(t)\frac{2\beta}{\sigma^2}dt}e^{\int_{m^{AB}}^{G}(-v(t)-q^{AB})\frac{2\beta}{\sigma^2}dt} & \text{if } G > m^{AB}. \end{cases} \quad (3.5)$$

Hence, we have that $f = \dfrac{\varphi}{\int_{-\infty}^{\infty}\varphi}$. We can express the objective function as

$$W := \left(\frac{1}{\int_{-\infty}^{\infty}\varphi dG}\right)\left(\begin{array}{c} \int_{-\infty}^{m^{BA}}\left(-Gv(G)-c^{BA}q^{BA}\right)\varphi(G)dG \\ + \int_{m^{BA}}^{m^{AB}}-Gv(G)\varphi(G)dG \\ + \int_{m^{AB}}^{\infty}\left(-Gv(G)-c^{AB}q^{AB}\right)\varphi(G)dG \end{array}\right). \quad (3.6)$$

Maximizing W with respect to m^{BA} and m^{AB} yields the following optimality conditions.

Proposition 4. *Consider u be the optimal control as defined in Theorem (1). m^{BA}, m^{AB} and W are jointly characterized by Equation (3.6) together with*

$$W = \frac{\int_{m^{AB}}^{\infty}-(G-c^{AB})v(G)\varphi(G)dG}{\int_{m^{AB}}^{\infty}\varphi}, \quad (3.7)$$

$$W = \frac{\int_{-\infty}^{m^{BA}}-(G+c^{BA})v(G)\varphi(G)dG}{\int_{-\infty}^{m^{BA}}\varphi}. \quad (3.8)$$

Proof. W is the value of the trader's problem for the optimal policy described at Theorem 1, and the equation for m^{AB} is Equation (3.7), given in the proof of Theorem 1. In that same proof, m^{BA} is defined by

$$\frac{\int_{m^{BA}}^{m^{AB}}-Gv(G)\varphi(G)dG + \int_{m^{AB}}^{\infty}(-Gv-c^{AB}q^{AB})\varphi(G)dG - \frac{\sigma^2}{2\beta}c^{BA}\varphi(m^{BA})}{\int_{m^{BA}}^{\infty}\varphi} = W.$$

Now taking in account the expression for W, the following algebra yields Equation (3.8).

$$\left(\int_{-\infty}^{\infty}\varphi\right)\left(W - \frac{\int_{-\infty}^{m^{BA}}(-Gv-c^{BA}q^{BA})\varphi}{\int_{-\infty}^{\infty}\varphi}\right) - \frac{\sigma^2}{2\beta}c^{BA}\varphi(m^{BA}) = \left(\int_{m^{BA}}^{\infty}\varphi\right)W$$

$$\Rightarrow \left(\int_{-\infty}^{\infty}\varphi - \int_{m^{BA}}^{\infty}\varphi\right)W - \int_{-\infty}^{m^{BA}}(-Gv-c^{BA}q^{BA})\varphi - \frac{\sigma^2}{2\beta}c^{BA}\varphi(m^{BA}) = 0$$

$$\Rightarrow \frac{\int_{-\infty}^{m^{BA}}(-Gv-c^{BA}q^{BA})\varphi dG + \frac{\sigma^2}{2\beta}c^{BA}\varphi(m^{BA})}{\int_{-\infty}^{m^{BA}}\varphi} = W.$$

Rewriting $\varphi(m^{BA})$ as $\int_{-\infty}^{m^{BA}} (f'/f) \times f$, and using Equation (2.6) yields

$$\frac{\int_{-\infty}^{m^{BA}} (-Gv - c^{BA}q^{BA})\varphi dG + \frac{\sigma^2}{2\beta}c^{BA}\varphi(m^{BA})}{\int_{-\infty}^{m^{BA}} \varphi} = \frac{\int_{-\infty}^{m^{BA}} (-Gv - c^{BA}q^{BA} - c^{BA}(-q^{BA} + v))\varphi dG}{\int_{-\infty}^{m^{BA}} \varphi}$$

and hence Equation (3.8). ∎

With the equations given by Proposition 4, we can find numerically the value of the parameters and hence the optimal function $f(G)$.

In addition, under some properties of $v(G)$, the equations allow us to derive some properties on the optimal thresholds.

Proposition 5. *If $c^{AB} = c^{BA}$, $q^{AB} = q^{BA}$ and $v(G)$ is anti-symmetrical, i.e., $v(-G) = -v(G)$ for all G, then $m^{BA} = -m^{AB}$.*

Proof. We denote $c^{AB} = c^{BA} = c$, $q^{AB} = q^{BA} = q$. Let us consider the following change of variables: $G = -s$, $\tilde{m}^{BA} = -m^{AB}$, $\tilde{m}^{AB} = -m^{BA}$ and $\tilde{\varphi}(s) = \varphi(-s)$ where m^{BA} and m^{AB} are given by Equations (3.8) and (3.7) in Proposition 4. We claim that if $v(-G) = -v(G)$ then $\tilde{m}^{BA}, \tilde{m}^{AB}$ and $\tilde{W} = W$ also solve the optimality equations of Proposition 4.

Indeed, we have that

$$
\begin{aligned}
\tilde{W} &= \frac{\int_{-\infty}^{\tilde{m}^{BA}} (-Gv(G) - cq)\tilde{\varphi}(G)dG + \int_{\tilde{m}^{BA}}^{\tilde{m}^{AB}} -Gv(G)\tilde{\varphi}(G)dG + \int_{\tilde{m}^{AB}}^{\infty} (-Gv(G) - cq)\tilde{\varphi}(G)dG}{\int_{-\infty}^{\infty} \tilde{\varphi}(G)dG} \\
&= \frac{\int_{+\infty}^{-\tilde{m}^{BA}} (sv(-s) - cq)\tilde{\varphi}(-s)(-ds) + \int_{-\tilde{m}^{BA}}^{-\tilde{m}^{AB}} sv(-s)(-ds)\tilde{\varphi}(-s)(-ds) + \int_{-\tilde{m}^{AB}}^{-\infty} (sv(-s) - cq)\tilde{\varphi}(-s)(-ds)}{\int_{+\infty}^{-\infty} \tilde{\varphi}(-s)(-ds)} \\
&= \frac{\int_{m^{AB}}^{\infty} (-sv(s) - cq)\varphi(s)ds + \int_{m^{BA}}^{m^{AB}} -sv(s)ds\varphi(s)ds + \int_{-\infty}^{m^{BA}} (-sv(s) - cq)\varphi(s)ds}{\int_{-\infty}^{\infty} \varphi(s)ds} \\
&= W,
\end{aligned}
$$

where we made the change of variables $s = -G$ and used the fact that v is anti-symmetrical. It is now clear that applying the change of variables,

$$\frac{\int_{-\infty}^{\tilde{m}^{BA}} -(G + c)v(G)\tilde{\varphi}(G)dG}{\int_{-\infty}^{\tilde{m}^{BA}} \tilde{\varphi}(G)dG} = \frac{\int_{m^{AB}}^{\infty} -(G - c)v(G)\varphi(G)dG}{\int_{m^{AB}}^{\infty} \varphi(G)dG} = W = \tilde{W},$$

i.e., satisfies Equation (3.8). Similarly, it satisfies Equation (3.7) as well. In conclusion, $\tilde{m}^{BA}, \tilde{m}^{AB}$ and $\tilde{W}$ are also solutions of the optimality equations and hence, since the solutions are unique, $m^{BA} = -m^{AB}$. ∎

Furthermore, for the extreme case where the trader's capacity is very small (very limited market power) m^{BA} and m^{AB} are closer to the trading costs.

Proposition 6. *If $q^{AB}, q^{BA} \to 0$, then $m^{BA} \to -c^{BA}$ and $m^{AB} \to c^{AB}$.*

Proof. When $q^{AB} = q^{BA} = 0$, then $\dfrac{\sigma^2}{2\beta} \dfrac{\varphi'}{\varphi} = -v$ and hence

$$
W = \frac{\int_{-\infty}^{\infty} -Gv\varphi dG}{\int_{-\infty}^{\infty} \varphi dG} = \frac{\int_{-\infty}^{\infty} G\left(\frac{\sigma^2}{2\beta}\frac{\varphi'}{\varphi}\right)\varphi dG}{\int_{-\infty}^{\infty} \varphi dG} = -\frac{\sigma^2}{2\beta}
$$

On the other hand, (3.8) is now

$$
\begin{aligned}
W &= \frac{\int_{-\infty}^{m^{BA}} -(G+c^{BA})v(G)\varphi(G)dG}{\int_{-\infty}^{m^{BA}} \varphi dG} \\
&= \frac{\int_{-\infty}^{m^{BA}} (G+c^{BA})\frac{\sigma^2}{2\beta}\varphi'(G)dG}{\int_{-\infty}^{m^{BA}} \varphi dG} \\
&= \frac{\frac{\sigma^2}{2\beta}[(G+c^{BA})\varphi]_{-\infty}^{m^{BA}} - \frac{\sigma^2}{2\beta}\int_{-\infty}^{m^{BA}} \varphi}{\int_{-\infty}^{m^{BA}} \varphi dG} \\
&= -\frac{\sigma^2}{2\beta} + \frac{\sigma^2}{2\beta}\frac{(c^{BA}+m^{BA})\varphi(m^{BA})}{\int_{-\infty}^{m^{BA}} \varphi}
\end{aligned}
$$

Therefore, as the optimal solution is unique, we have that $m^{BA} + c^{BA} = 0$ and hence $m^{BA} = -c^{BA}$. Similarly $m^{AB} = c^{AB}$ when $q^{AB} = q^{BA} = 0$.

When $q^{AB}, q^{BA} > 0$ but very small, we know that (m^{BA}, m^{AB}) are uniquely defined by

$$
\begin{aligned}
0 &= \frac{\int_{m^{AB}}^{\infty} -(G-c^{AB})v(G)\varphi(G)dG}{\int_{m^{AB}}^{\infty} \varphi} \\
&\quad - \frac{\int_{-\infty}^{m^{BA}}\left(-Gv(G)-c^{BA}q^{BA}\right)\varphi(G)dG + \int_{m^{BA}}^{m^{AB}} -Gv(G)\varphi(G)dG + \int_{m^{AB}}^{\infty}\left(-Gv(G)-c^{AB}q^{AB}\right)\varphi(G)dG}{\int_{-\infty}^{\infty} \varphi dG} \\
0 &= \frac{\int_{-\infty}^{m^{BA}} -(G+c^{BA})v(G)\varphi(G)dG}{\int_{-\infty}^{m^{BA}} \varphi} \\
&\quad - \frac{\int_{-\infty}^{m^{BA}}\left(-Gv(G)-c^{BA}q^{BA}\right)\varphi(G)dG + \int_{m^{BA}}^{m^{AB}} -Gv(G)\varphi(G)dG + \int_{m^{AB}}^{\infty}\left(-Gv(G)-c^{AB}q^{AB}\right)\varphi(G)dG}{\int_{-\infty}^{\infty} \varphi dG}.
\end{aligned}
$$

This is a system of two equations $\psi_1(m^{BA}, m^{AB}, q^{BA}, q^{AB}) = 0$ and $\psi_2(m^{BA}, m^{AB}, q^{BA}, q^{AB}) = 0$ with two unknowns, and two parameters q^{BA}, q^{AB}. Since ψ is continuously differentiable, then (m^{BA}, m^{AB}) is continuous in (q^{BA}, q^{AB}) around $(0,0)$. We thus have that, for small

q^{BA}, q^{AB}, the deviations around $m^{BA} = -c^{BA}$ and $m^{AB} = c^{AB}$ are respectively $m^{BA} + c^{BA}$ and $m^{AB} - c^{AB}$, and hence

$$\frac{d\psi}{dq^{BA}}q^{BA} + \frac{d\psi}{dq^{AB}}q^{AB} + \frac{d\psi}{dm^{BA}}(m^{BA} + c^{BA}) + \frac{d\psi}{dm^{AB}}(m^{AB} - c^{AB}) + o(q^{BA}) + o(q^{AB}) = 0,$$
(3.9)

where $o(x)/x \to 0$ when $x \to 0$.

Interestingly, at $q^{AB} = 0$,

$$\frac{1}{\varphi}\frac{\partial\varphi}{\partial m^{BA}} = \frac{2\beta}{\sigma^2}v(-c^{BA}) \text{ and } \frac{1}{\varphi}\frac{\partial\varphi}{\partial m^{AB}} = 0.$$

Also,

$$\frac{1}{\varphi}\frac{\partial\varphi}{\partial q^{BA}} = -\frac{2\beta}{\sigma^2}(-c^{BA} - G)^+ \text{ and } \frac{1}{\varphi}\frac{\partial\varphi}{\partial q^{AB}} = -\frac{2\beta}{\sigma^2}(G - c^{AB})^+.$$

Hence, at $q^{AB} = q^{BA} = 0$, we have that

$$\frac{1}{\psi_1}\frac{d\psi_1}{dm^{BA}} = \frac{-\varphi(-c^{BA})}{\int_{-\infty}^{-c^{BA}}\varphi} \text{ and } \frac{1}{\psi_1}\frac{d\psi_1}{dm^{AB}} = 0,$$

$$\frac{1}{\psi_2}\frac{d\psi_2}{dm^{AB}} = \frac{\varphi(c^{AB})}{\int_{c^{AB}}^{\infty}\varphi} \text{ and } \frac{1}{\psi_2}\frac{d\psi_2}{dm^{BA}} = 0.$$

In addition, using that $W = -\frac{\sigma^2}{2\beta}$,

$$\frac{1}{\psi_1}\frac{d\psi_1}{dq^{BA}} = \left(\frac{2\beta}{\sigma^2}\right)\begin{bmatrix} \frac{2\beta}{\sigma^2}\frac{\int_{-\infty}^{-c^{BA}}(G+c^{BA})^2 v(G)\varphi(G)dG}{\int_{-\infty}^{-c^{BA}}\varphi} + \frac{\int_{-\infty}^{-c^{BA}}-(G+c^{BA})\varphi(G)dG}{\int_{-\infty}^{-c^{BA}}\varphi} \\ -c^{BA}\frac{\int_{-\infty}^{-c^{BA}}\varphi}{\int_{-\infty}^{\infty}\varphi} - \frac{2\beta}{\sigma^2}\frac{\int_{-\infty}^{-c^{BA}}G(G+c^{BA})v(G)\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} - \frac{\int_{-\infty}^{-c^{BA}}-(G+c^{BA})\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} \end{bmatrix},$$

$$\frac{1}{\psi_1}\frac{d\psi_1}{dq^{AB}} = \left(\frac{2\beta}{\sigma^2}\right)\begin{bmatrix} -c^{AB}\frac{\int_{c^{AB}}^{\infty}\varphi}{\int_{-\infty}^{\infty}\varphi} + \frac{2\beta}{\sigma^2}\frac{\int_{c^{AB}}^{\infty}G(G-c^{AB})v(G)\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} - \frac{\int_{c^{AB}}^{\infty}(G-c^{AB})\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} \end{bmatrix};$$

and

$$\frac{1}{\psi_2}\frac{d\psi_2}{dq^{BA}} = \left(\frac{2\beta}{\sigma^2}\right)\begin{bmatrix} -c^{BA}\frac{\int_{-\infty}^{-c^{BA}}\varphi}{\int_{-\infty}^{\infty}\varphi} - \frac{2\beta}{\sigma^2}\frac{\int_{-\infty}^{-c^{BA}}G(G+c^{BA})v(G)\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} - \frac{\int_{-\infty}^{-c^{BA}}-(G+c^{BA})\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} \end{bmatrix},$$

$$\frac{1}{\psi_2}\frac{d\psi_2}{dq^{AB}} = \left(\frac{2\beta}{\sigma^2}\right)\begin{bmatrix} -\frac{2\beta}{\sigma^2}\frac{\int_{c^{AB}}^{\infty}(G-c^{AB})^2 v(G)\varphi(G)dG}{\int_{c^{AB}}^{\infty}\varphi} + \frac{\int_{c^{AB}}^{\infty}(G-c^{AB})\varphi(G)dG}{\int_{c^{AB}}^{\infty}\varphi} \\ -c^{AB}\frac{\int_{c^{AB}}^{\infty}\varphi}{\int_{-\infty}^{\infty}\varphi} + \frac{2\beta}{\sigma^2}\frac{\int_{c^{AB}}^{\infty}G(G-c^{AB})v(G)\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} - \frac{\int_{c^{AB}}^{\infty}(G-c^{AB})\varphi(G)dG}{\int_{-\infty}^{\infty}\varphi} \end{bmatrix}.$$

As a result, we can express Equation (3.9) as

$$m^{BA} = -c^{BA} - \left(\frac{\psi_1}{\frac{d\psi_1}{dm^{BA}}}\right)\left(\frac{1}{\psi_1}\frac{d\psi_1}{dq^{BA}}q^{BA} + \frac{1}{\psi_1}\frac{d\psi_1}{dq^{AB}}q^{AB}\right) + o(q^{BA}) + o(q^{AB})$$

and

$$m^{AB} = c^{AB} - \left(\frac{\psi_2}{\frac{d\psi_2}{dm^{AB}}} \right) \left(\frac{1}{\psi_2} \frac{d\psi_2}{dq^{BA}} q^{BA} + \frac{1}{\psi_2} \frac{d\psi_2}{dq^{AB}} q^{AB} \right) + o(q^{BA}) + o(q^{AB}).$$

∎

This result shows that our model is robust and that in the limit (when $q^{AB} = q^{BA} = 0$), our optimal policy is the same as that of a rational trader with no market power. In addition, one is able to obtain first-order approximations of m^{BA} and m^{AB} when the capacities q^{AB}, q^{BA} are small.

Proposition 6 allows us to derive some interesting conclusions in a "perfect" market, where all traders are small. In that situation, each trader is going to start trading when the spread is beyond its transaction cost. As a result, there is a direct relationship between traders' costs, installed capacities and the market spread long-run distribution. We use this assumption later on, in §4.1, in order to estimate the amount of capacity present in the market based on the price spread distribution.

3.4   Sensitivity analysis

In this section we show some properties of the model with respect to small changes in the parameters.

In a first attempt we tried to do it analytically using the Envelope Theorem. The envelope theorem is a general principle describing how the value of an optimization problem changes as the parameters of the problem change. [1] Notwithstanding, the analytical expressions for the partial derivatives are untractable.

We decided then to do the sensitivity analysis numerically with MATLAB (one can find the scripts with the code in Appendix A). For this purpose, we have taken two different approaches. We have implemented the explicit analytical function defined in §3.3 and compared the results with explicit simulations of concrete policies that satisfy $dG_t = -\beta(u_t + v(G_t))dt + \sigma dW_t$. The results found in both cases are similar, so we can say that our model is robust. Figure 3.3 is an example of this: the simulated p.d.f. (with 10,000

---

[1]Roughly: consider an arbitrary maximization problem where the objective function, f, depends on a parameter (a):

$$J(a) = \max_x f(x, a)$$

where the function J(a) gives the maximized value of f as a function of a. Let $x(a)$ be the value of x that solves the maximization problem in terms of paramater a. The envelope theorem tells us how J(a) changes as a changes, manely:

$$\frac{dJ(a)}{da} = \frac{\partial f(x^*, a)}{\partial a}\Big|_{x^*=x(a)}$$

data points) "converges" to the analytical p.d.f. Figure 3.2 shows the result of a simulated policy. Since both approaches yield similar results, we have used below the analytical implementation because the computations are faster than with simulations.
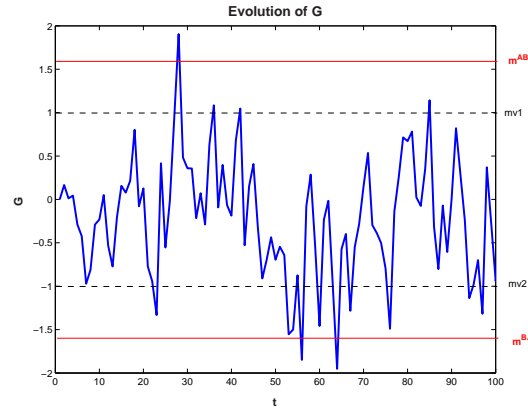

Figure 3.2: Simulating a policy. Parameters: $c^{AB} = c^{BA} = 0$, $q^{AB} = q^{BA} = 1$, $v(G) = 1$ when $G \geq 1$, $v(G) = -1$ when $G \leq -1$ and $v(G) = 0$ otherwise
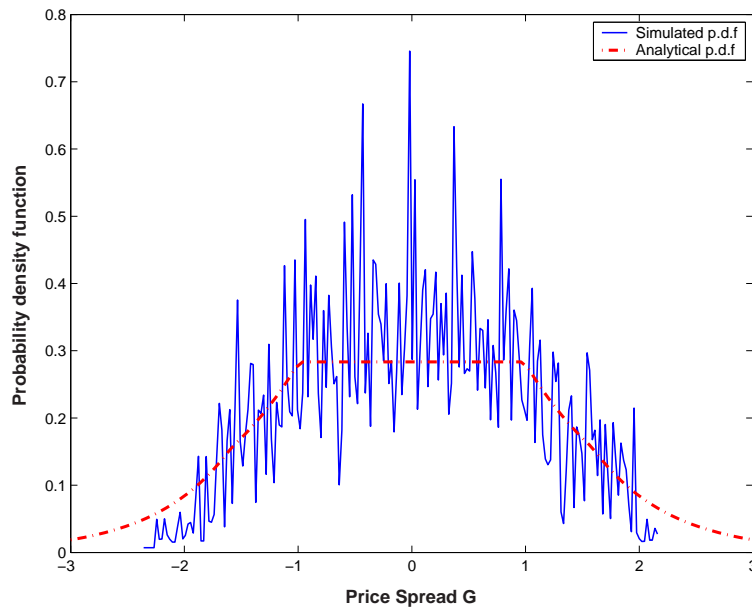

Figure 3.3: Comparison between the analytical pdf (red) and the simulated pdf (blue).

For given $c^{BA}, c^{AB}, q^{BA}, q^{AB}, \beta, \sigma, v(\cdot)$, the optimal thresholds m^{BA}, m^{AB} are found, through an exhaustive search, even though it would be computationally more efficient just to solve Equations (3.6), (3.8) and (3.7). We also compute the average profit J^* defined in (2.7). In Figure 3.4 we show the value of J^* for a concrete trading policy, with and without trading costs.
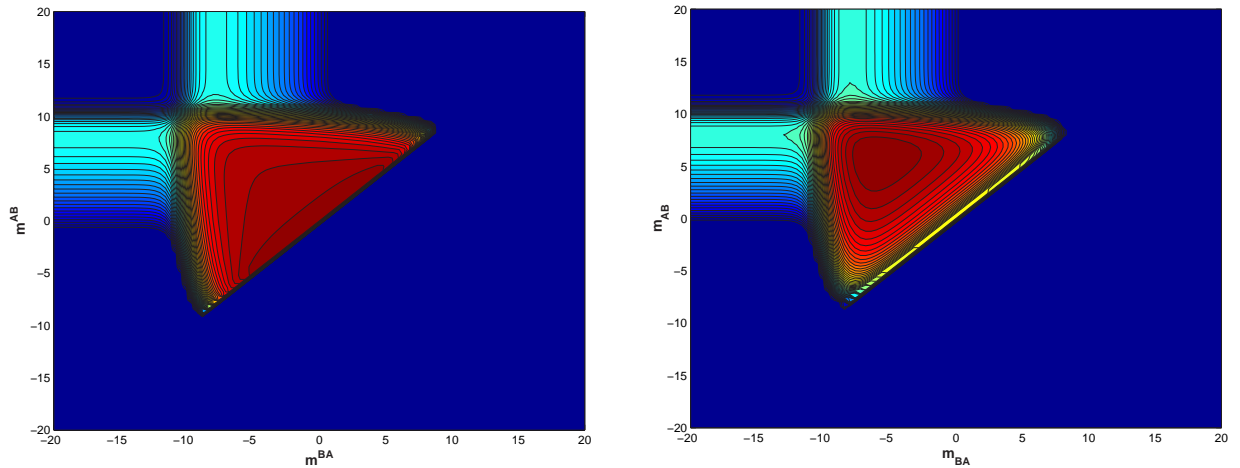
Figure 3.4: Long-term average benefits. Left: no trading costs. Right: positive costs

Without lost of generality we have assumed that $\frac{\sigma^2}{2\beta} = 1$. This is only a scaling assumption. In addition, we have considered two main cases:

- *the symmetric trader*: $c^{BA} = c^{BA} = c, q^{BA} = q^{AB} = q$ and v(G) antisymmetric;

- *the uni-directional trader*: $c^{BA} = \infty, c^{AB} = c$ and $q^{BA} = 0, q^{AB} = q$.

For both scenarios, we numerically show the impact of cost (c) and capacity (q) on the optimal average profit $J^* = \frac{\sigma^2}{2\beta} - W$, and on the thresholds m^{BA}, m^{AB}. Note that for the symmetric trader scenario, we can apply Proposition 5 and hence $m^{BA} = -m^{AB}$.

We observe in Figures 3.5 and 3.6 that, as one would expect, the trader's average profit decreases to zero with cost. Any capacity is actually useless for $c \geq 5$, since, as the competitor starts trading $|v| = 1$ for $|G_t| \geq 1$, the spread never goes above c. The threshold at which the trader begins to operate is actually very close to the cost, for the entire range.

In addition, the trader's profit is concave in the capacity, implying decreasing marginal returns on the capacity. The marginal value of capacity is very small for $q \geq 2$, since more capacity tends to reduce the price spreads. As the capacity grows, the threshold also increases, and can be quite high compared to the cost, e.g., for the symmetric trader, when $q = 5, c = 1$ and $m^{AB} \approx 1.6$.

Interestingly, comparing Figures 3.5 and 3.6, one can see that the symmetric trader makes slightly more than twice the profit for the unidirectional trader. The optimal trading thresholds are very similar in both cases. This suggests that, in the case of a symmetric
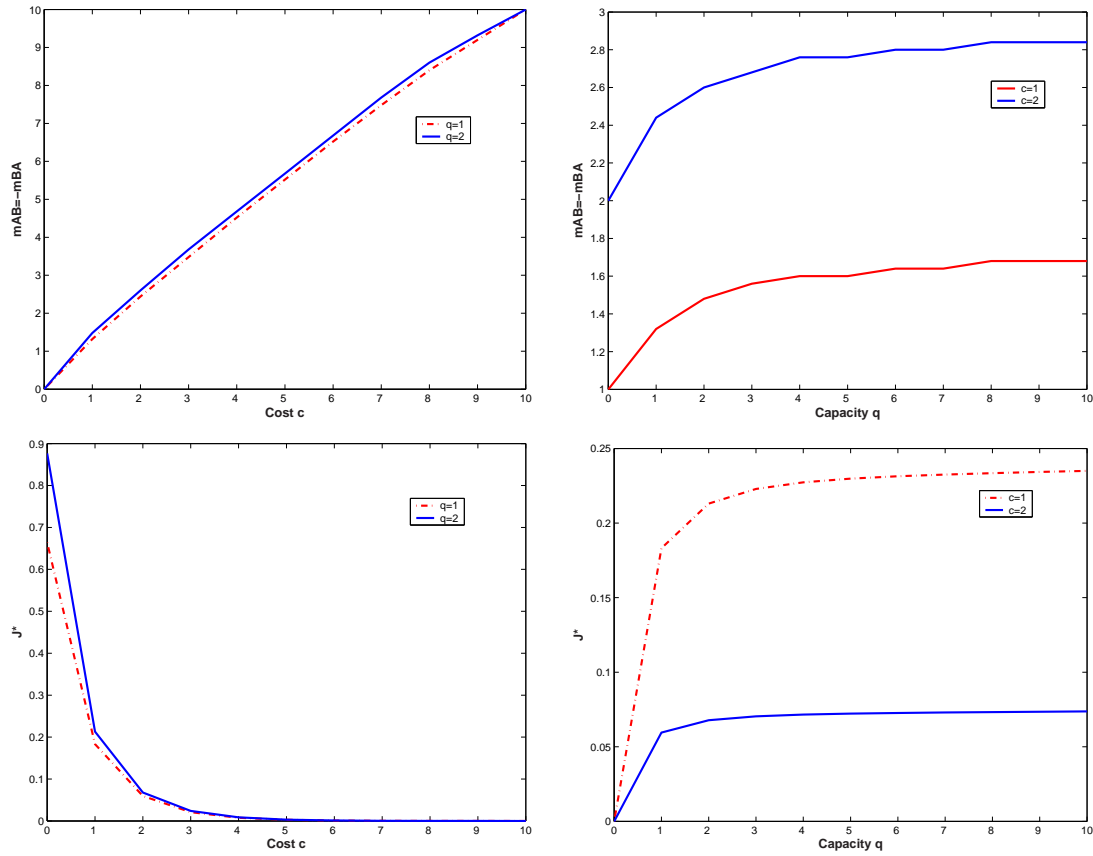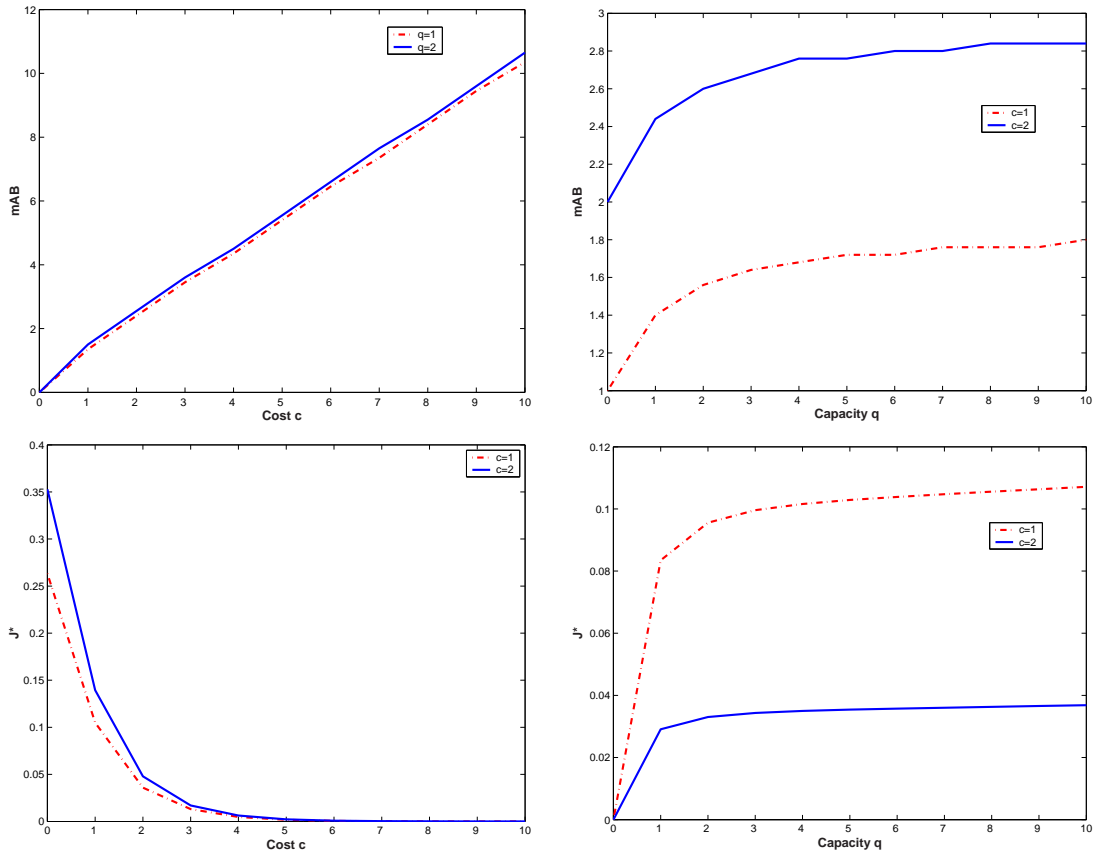
Figure 3.5: **The symmetric trader.** The top figures depict the changes of the optimal m^{BA}, m^{AB} with changes in c (left) and q (right). The bottom figures show the corresponding changes in average profit J^*, again with changes in c (left) and q (right). In these figures, we consider that $v(G) = 1$ when $G \geq 1$, $v(G) = -1$ when $G \leq -1$ and $v(G) = 0$ otherwise.

trader, the thresholds could be computed independently, i.e., find m^{AB} using $c^{AB}, q^{AB} > 0$ and $q^{BA} = 0$, and similarly for m^{BA}. This policy would be close to optimal.

Figure 3.6: **The uni-directional trader.** The top figures depict the changes of the optimal m^{AB} with changes in c (left) and q (right). The bottom figures show the corresponding changes in average profit J^*, again with changes in c (left) and q (right). In these figures, we consider that $v(G) = 1$ when $G \geq 1$, $v(G) = -1$ when $G \leq -1$ and $v(G) = 0$ otherwise.

3.5   Summary

In this chapter we have presented many results and shown important insights of our model that can be summed up in the following key points:

- For the **monopoly case**: Proposition 1 shows that when there are no trading costs the average long-run profit is independent of the trading quantity. This is not true when the costs are positive as shown in Proposition 2.

- When solving the trader's problem we have been able to reformulate (2.8) into an optimization problem (3.2) with f as decision variable. Since the p.d.f f and the trading quantity $u(G)$ are directly linked; this is equivalent to finding the optimal trading policy.

- For the **oligopoly case**: Theorem 1 characterizes the optimal trading policy, $u(G)$,

which is determined by two thresholds m^{AB} and m^{BA}. The trader must ship at full capacity from market B to market A when the spread is below m^{BA}, the same but in the opposite direction when the spread is above m^{AB}, and do nothing elsewhere.

- We have analyzed the **impact of market power** in the structure of the optimal trading policy: it delays the point where the traders begin to ship product from one market to the other.

- Proposition 4 gives the **closed-form equations** which characterizes the thresholds m^{AB} and m^{BA}.

- Proposition 5 shows that $m^{AB} = -m^{BA}$ when v(G) is anti-symmetrical. Proposition 6 allows us to derive some interesting conclusions when all traders are small ($q \approx 0$). In that situation, each trader is going to start trading when the spread is beyond its transaction cost. Hence, there is a direct **relationship** between traders' costs, installed capacities and the market spread long-run distribution.

- In §3.4 we illustrate how the **changes in cost and capacity** affect the average long-run profits and the thresholds described by the optimal policy. As expected, revenues decrease with cost and are concave with capacity, as more capacity tends to reduce the price spreads..

Chapter 4

Applications and Extensions

4.1   Application to Jet-Fuel (Kerosene) Prices

In this section, we analyze the data of jet-fuel (kerosene) spot prices in two locations: New York Harbor (NY) and Los Angeles (LA). We have taken the data from the Energy Information Administration from the U.S.Goverment, see http://www.eia.doe.gov. This is an illustrative example as Kerosene is a product with no qualitative differences, i.e, as the planes are the same worldwide the jet-fuel must be the same everywhere.

We show in Figure 4.1 the evolution of kerosene jet-fuel spot prices, over more than fifteen years. As can be seen in the left figure, the evolution of the price in these two locations is very similar. This implies that in the long-run the price spread is small. However, when examining, in the right figure, the price differences over the period, it is surprising to see short-run spreads of tens of cents of a dollar. Another salient feature is that the spreads do not close quickly, in some cases lasting more than one month.

Our objective here is to apply our model to the trading between these two locations. Trading, as we use it, can be done in several forms. For example, one can imagine a dedicated trader that buys kerosene in one market, loads it in a tank truck (which hold from 4000 to 9000 gallons), and ships it to the other market. This is an expensive way of trading, and would yield per-gallon costs of the order of 10 USD cents. The capacity available for this would probably be very large. One can also think of an airline that flies the NY-LA route and refuels its planes every day in both locations. By buying a bit more in the cheaper market and a bit less in the more expensive market, this airline is effectively trading between the two markets. The capacity here would be the number of planes of the airline in this route. This form of trading is also expensive: a B767-400 consumes 7400 liters per hour, weights 200 tons when fully loaded, which implies that, shipping one gallon between NY and LA consumes about 0.2 gallons, around 40 USD cents per gallon traded at 2007 kerosene prices. Our model can be used by this trader to optimize his actions. We analyze below the *outcome* of the actions of the traders: the spread price process.
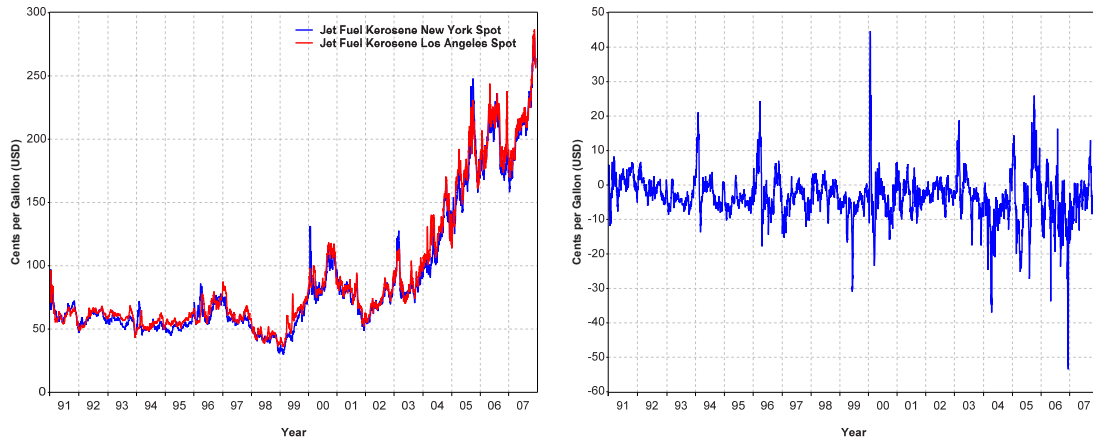
Figure 4.1: Plot of kerosene jet-fuel daily spot prices on the left (in USD cents per Gallon) in New York and Los Angeles, and the corresponding spread (price difference) $G_t = P_t^{NY} - P_t^{LA}$.

The series reflects the spot price in USD cents per gallon from January 1991 to November 2007. We are using prices on markets that open from Monday to Friday, although in a few cases we do not have data over the entire 5-days week, due to local holidays. As a result, when calculating the spread series, we have removed the days where information of the price of one or both locations was missing. These facts (week-ends, holidays and missing data) generate some complications in our study. In fact, when calculating auto-correlations, i.e., correlations across successive days, we have to take into account this phenomenon. In other words, although not having information of the price spread in week-ends, trading on a pipeline or other shipping systems (e.g., trucks) can occur, moving product from one location to the other. So as a result, the system has three times more capacity between Friday and Monday than during the week. Thus, we index the series using natural days, i.e., the time t is indexed from Monday to Sunday.

We have calculated the spread $G_t = P_t^{NY} - P_t^{LA}$. Table 4.1 provides the basic statistics summary of the series.

Notice that the maximum and minimum value of the spread are highly significant as well as the high variance. These facts are interesting for a player trading in these markets.

It is also interesting to examine the auto-correlation diagram, shown in Figure 4.2. Using the time series terminology, the figure is equivalent to the autocorrelation function (ACF) but taking in account the missing data. In other words, when calculating the autocorrelation coefficients (see [5] for more details) at lag k the terms where G_t and/or G_{t+k} are missing are not considered. The slow exponential decay in the ACF is a feature that, for example, autoregressive models (AR) exhibit, and so does ours. In fact, if the data was modeled with a simple AR(1) process, then we would have a spread process as follows: $G_t - \mu = \alpha(G_{t-1} - \mu) + \sigma\epsilon_t$ where ϵ_t is a white noise, with mean zero and standard deviation one. This is in some way an alternative to Equation (2.2). High auto-correlation

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Table 4.1: Basic statistics of $G_t = P_t^{NY} - P_t^{LA}$

Number of observations (market days)	4238
Minimum	-53.50
Maximum	44.70
First quartile	-6.17
Third quartile	-0.25
Mean	-3.33
Median	-3.25
Variance	41.12
Standard deviation	6.42
Skewness	-0.20
Kurtosis	8.28

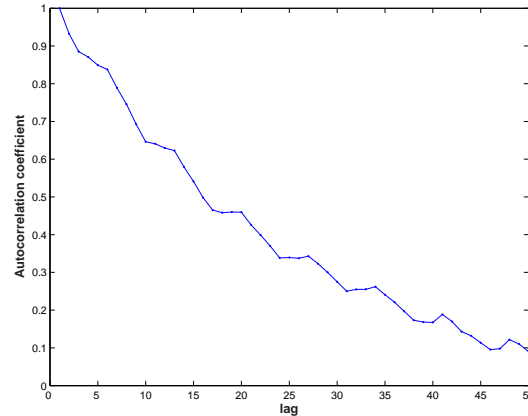in the spread could be a sign of "inefficiency" of the market, in other words, financial opportunities for traders.

Figure 4.2: The autocorrelations for the jet kerosene data taking in account the missing information. The lag is measured in days.

More importantly, we depict the spread distribution of the series. Figure (4.3) shows both the p.d.f. and its logarithm. In our model, see Equation (2.6), the slope of the logarithm of the p.d.f., f'/f is equal to $-2\beta(u(G) + v(G))/\sigma^2$. This can be calculated from the figure. In other words, we can estimate $-2\beta(u(G) + v(G))/\sigma^2$ from the data. In a sense, this provides an aggregate measure for the intensity with which the market brings an off-average spread back to the average. Interestingly, this measure can be well approximated through a piecewise constant function, as follows.

$$\frac{f'}{f}(G) = -\frac{2\beta(u(G) + v(G))}{\sigma^2} = \begin{cases} 0.29 & \text{for } G \leq -5.5 \\ 0.00 & \text{for } -5.5 < G \leq -1.5 \\ -0.31 & \text{for } G > -1.5 \end{cases}$$
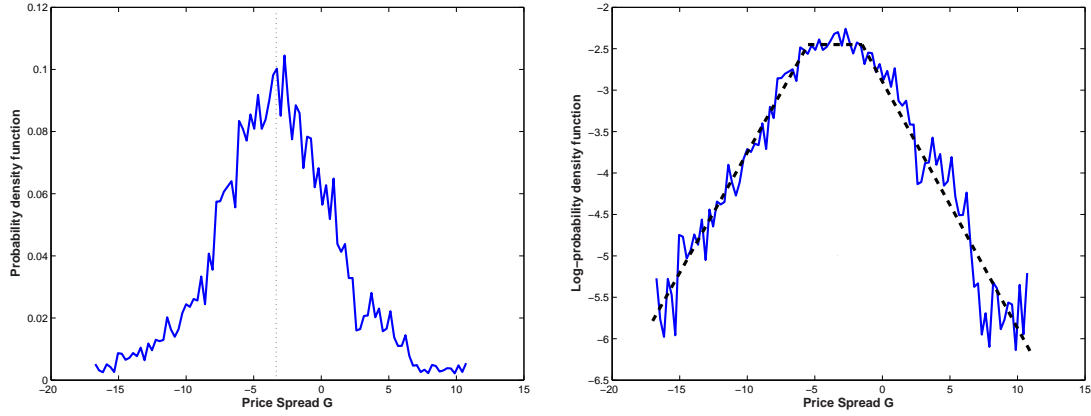
Figure 4.3: On the left, the shape of the p.d.f. for the data of jet kerosene is similar to what we could have expected when applying our trading policies. On the right, we show that the linear approximation (dashed line) fits the logarithm of the p.d.f. well. The vertical line reflects the mean -3.33 USD cents of the data. Finally, we plot the distribution for a data range of plus/minus two standard deviations.

The approximation fits the data well, as shown in Figure 4.3. Interestingly, it would be more accurate than trying to model the spread series as an AR(1) process. Indeed, for an AR(1) process, Equation (2.4) would yield to $\frac{f''}{f'} = \frac{-2(1-\alpha)G}{\sigma^2}$. Hence the logarithm of the p.d.f. (right-hand side in Figure 4.3) would exhibit a much larger "convexity" in G, an approximation that does not fit the data as well as our model.

If all traders are small compared to the market (which seems a reasonable assumption), then our model would provide a way to estimate how much total trading capacity $u(G) + v(G)$ is available at each cost level G. This approach can be used to calibrate some of the parameters of our model. Some more detailed estimates can be obtained by directly expressing all the price variations, $\Delta G = G_{t+1} - G_t$, when $G_t \in [G - \delta, G + \delta]$, as the sum of a constant plus a white noise. This has the advantage of estimating directly $-\beta(u(G) + v(G))$ (the constant) and σ (the standard deviation of the noise). However, it turns out that the estimates are very noisy, and only look consistent when we aggregate them significantly (large δ), which is somehow similar to the estimation performed on f'/f.

4.2   Extensions

In this section we consider different types of model we could have used to solve the problem, their differences with our model, advantages and disadvantages.

4.2.1   Discounted Profit Maximization

One of the strong assumptions of our model is that the trader cares about long-term average profit. This assumption does simplify the analysis, as it allows treating the objective in an explicit way, and, as a result, computing closed-form thresholds for the optimal policy.

In section §2.3, we have presented an alternative way for the optimization of the trading policies: the discounted profit maximization case. We use optimal control theory[1] to solve the problem. We use the notation and nomenclature from Bertsekas [2].

Denoting as $J(G_0)$ the profit-to-go (expected objective function value), function of the current price spread G_0, the maximization problem can be written as

$$\max_{-q^{BA} \leq u_t \leq q^{AB}} J(G_0) = \mathbb{E}\left\{ \int_0^\infty e^{-rt}\left(G_t u_t - c^{BA} u_t^- - c^{AB} u_t^+ \right) dt \right\} \tag{4.1}$$

where r is the instantaneous discount factor, and such that $dG_t = -\beta(u_t + v(G_t))dt + \sigma dW_t$

The optimality conditions can be captured through the Hamilton-Jacobi-Bellman (HJB) equation. First, the optimal control $u^*(G)$ is given by

$$\max_{-q^{BA} \leq u \leq q^{AB}} \left(Gu - c^{BA} u^- - c^{AB} u^+ \right) - \beta \frac{dJ}{dG} u.$$

This implies that $u^*(G) = -q^{BA}, 0, q^{AB}$. This is the same structure we have found when solving 2.8. Hence, a bang-bang policy is optimal in the discounted profit case as well.

In addition, if $G - \beta \dfrac{dJ}{dG}$ is increasing when equal to $-c^{BA}, 0, c^{AB}$, then there exists m^{BA}, m^{AB} such that $u^*(G) = -q^{BA}$ when $G \leq m^{BA}$; $u^*(G) = q^{AB}$ when $G \geq m^{AB}$; and finally, $u^*(G) = 0$ when $m^{BA} \leq G \leq m^{AB}$. In that case, the HJB equation satisfies:

$$
\begin{array}{lll}
\frac{\sigma^2}{2}\frac{d^2J}{dG^2} - \beta(-q^{BA} + v)\frac{dJ}{dG} - rJ - (G - c^{BA})q^{BA} = 0 & \text{when} & G \leq m^{BA}, \\
\frac{\sigma^2}{2}\frac{d^2J}{dG^2} - \beta v\frac{dJ}{dG} - rJ = 0 & \text{when} & m^{BA} \leq G \leq m^{AB} \\
\frac{\sigma^2}{2}\frac{d^2J}{dG^2} - \beta(q^{AB} + v)\frac{dJ}{dG} - rJ + (G - c^{AB})q^{AB} = 0 & \text{when} & G \geq m^{AB}
\end{array}
$$

---

[1]The optimal control can be derived using Pontryagin's maximum principle (a necessary condition), or by solving the Hamilton-Jacobi-Bellman equation (a sufficient condition).

4.2.2 Alternative Price Processes, Cost Structures and Competitor Actions

In this Master Thesis we present a simple analytical model that gives intuitive trading policies. One of the main advantages in our model is to be able to recast a problem with two stochastic processes, P_t^A and P_t^B, into a problem with a single stochastic process $G_t = P_t^B - P_t^A$. This can be done under two main assumptions:

- that the stochastic process dG_t can be expressed only as a function of the spread G_t (for instance, that the actions of the competitors v_t only depend on G_t, not on (P_t^A, P_t^B));

- and that the cost function of the trader is linear, which implies that it is independent of (P_t^A, P_t^B).

When any of these two assumptions is not satisfied, then the approach developed in this Thesis is significantly more complicated. We provide guidelines on how to deal with these more general models. The generic formulation for the price processes, replacing Equation (2.1), is:

$$dP_t^A = \left[\alpha^A(P_t^A, P_t^B) + \beta^A(P_t^A, P_t^B)\Big(u_t + v_t(P_t^A, P_t^B)\Big)\right]dt + \sigma^A(P_t^A, P_t^B)dW_t^A$$
$$dP_t^B = \left[\alpha^B(P_t^A, P_t^B) - \beta^B(P_t^A, P_t^B)\Big(u_t + v_t(P_t^A, P_t^B)\Big)\right]dt + \sigma^B(P_t^A, P_t^B)dW_t^B \tag{4.2}$$

A particular case of these is the geometric Brownian motion, where $\alpha^i = 0$, $\beta^i = P_t^i\bar{\beta}^i$, and $\sigma^i = P_t^i\bar{\sigma}^i$, with $\bar{\beta}^i, \bar{\sigma}^i$ constants.

The objective function thus becomes

$$J(u) = \max_u \mathbb{E}\left\{(P_t^B - P_t^A)u - c\Big(P_t^A, P_t^B, u\Big)\right\}$$
$$= \max_u \int\int_{-\infty}^{+\infty}\left[(p^B - p^A)u(p^A, p^B) - c\Big(p^A, p^B, u(p^A, p^B)\Big)\right]f_u(p^A, p^B)dp^A dp^B,$$

where f_u satisfies a partial differential equation similar to (2.4), but in two variables. In this case, no closed form solution can be easily computed, and f_u should be found numerically.

4.2.3 Trading in a Network

As said, the model developed in this Thesis considers a capacitated trader between two pre-determined locations, market A and market B. This is a great simplification as, in reality, trading opportunities are possible between many potential origins and destinations.

So, if we want to extend our model to three or more locations we are implicitly defining a network of n nodes (a connected graph), each node representing a market. Figure 4.4 illustrates how the graph for the kerosene prices would look like. For each arc (i, j) of the graph, we define the trading quantities between market i and market j as u^{ij}, the existing trading capacity as q^{ij} and the trading unit cost c^{ij}.

We must also redefine the price process evolution from Equation (2.1) as

$$ dP_t^i = \left[\alpha(\mathbf{P}_t) + \sum_{j \neq i} \beta^{ij}(u_t^{ij} + v_t^{ij}) - \sum_{j \neq i} \beta^{ji}(u_t^{ji} + v_t^{ji}) \right] dt + \sigma^i dW_t^i \text{ for all } i \in \{1, \ldots, n\} $$

(4.3)

where $\mathbf{P}_t = (P_t^1, \ldots, P_t^n)$, and W_t^i are (possibly correlated) Wiener processes.

Another option when extending our basic model to a network is to integrate the inventory aspect of the trade. In the model we have assumed that the quantity bought at market i is equal to the one sold at market j. This is reasonable if we are thinking of pipelines. However, if the shipping system is a serial network of warehouses, then one could consider the possibility of storing inventory somewhere in that network.

With this new formulation, similar to §4.2.2, it is impossible to transform the problem into a single-dimension state space (the price spread). The analysis consequently becomes intractable, and numerical methods must be used.
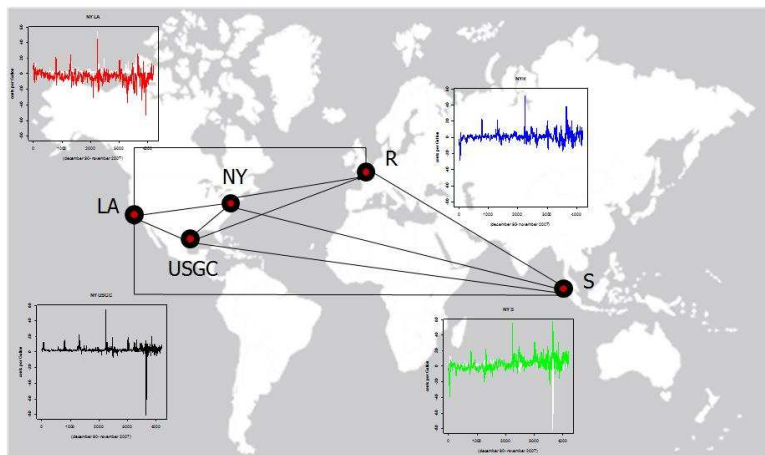
Figure 4.4: Network for the kerosene example. The main markets for the jet-kerosene are New York (NY), Los Angeles (LA), United States Gulf Coast (USGC), Rotterdam (R) and Singapore(S). We also show, close to each location, the spread with respect to the NY price

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4.3   Summary

In this chapter we have seen an application of our model to real market data and discussed alternative models and future extensions. This can be summarized in the following key points:

- We have shown that for the jet-fuel kerosene prices the short-term spreads are significant and that sometimes they take over a month to close. These features are a sign of "arbitrage" opportunity for a trader.

- We can estimate $-2\beta(u(G) + v(G))/\sigma^2$ directly from the data. This provides an aggregate measure for the **intensity** with which the market brings an off-average spread back to the average.

- We have shown that the structure of the optimal trading policy in the discounted profit case is the same as the one we had in Theorem 1.

- We provide guidelines on how to deal with the general models.

- We consider and discuss the problems when trying to extend our model to a network or integrating the inventory aspect of trade.

Conclusions

In this Master Thesis we have presented a model which analyzes the optimal trading strategy for capacitated traders in commodity markets who want to maximize their long-run average profits. We have focused on two markets only, A and B, where the prices evolve following a correlated random walk, and are influenced by the trader's actions, what we call market power.

Perhaps the biggest advantage in our model, presented in Chapter 2, is to be able to transform the trader's decision, based on the prices at both markets (P_t^A, P_t^B), into one that only depends on the price spread $G_t = P_t^B - P_t^A$. With this we are able to compute closed-form expressions for the optimal policy.

In Chapter 3, we have characterized this optimal trading policy, which is described by two thresholds $m^{BA} \leq m^{AB}$ such that it is optimal to trade as much as possible (at capacity) from B to A when $G_t \leq m^{BA}$, do nothing when $m^{BA} < G_t \leq m^{AB}$, and again trade as much as possible, from A to B this time, when $G_t > m^{AB}$.

This is the sort of policy that is intuitively optimal when the trader has no impact on future prices. We show that the structure remains the same even when the trader moves the market. Interestingly, the thresholds are very close to the trading costs c^{BA}, c^{AB} when the trading capacity is small, confirming that when the trader is small in the market, then he can safely assume that he has no market power (although his trading volume has an impact on the long-run distribution of prices). On the other hand, when the trading capacity is large, the thresholds can significantly deviate from the trading costs, as shown in §3.4.

From a traders point of view the model can be used for many different purposes. At an operational level, it allows the trader to specify when and how much to trade. At a more tactical and strategic level, the trader can predict the price spread distribution by looking at the trading capacity used for each spread level, and evaluate the impact of trading capacity additions. The specific application to kerosene spreads in Chapter 4 is just an example of what can be achieved with the model, and many alternative applications are possible, in other industries.

Summarizing, we have:

- Simple analytical model which captures the impact on prices of the trading actions of a capacitated trader.

- Intuitive and closed-form expressions for the optimal policy based on the spread and characterized by two thresholds.

- The model is useful for both operational and capacity decisions

We have interpreted the markets A and B as *locations* where an identical commodity is traded. The trading capacity represents the shipping capacity per period available, such as a pipeline, a system of trucks, etc. Interestingly, there is an alternative interpretation of our results: A and B could represent *two different commodities* priced in the same location. There, the trading capacity would represent the transformation capacity per period. Consider for example P_t^A being the price of natural gas, and P_t^B the price of electricity, in the same market. A power generation company could consider its generation capacity as trading capacity from A to B. Our model would dictate the generation policy for this company, and the distribution of the price spread (called the *spark spread*) could be found. Thus, our model is able to describe analytically what the trader/generator should do. Another example would be a refinery. Crude oil is a mix of different hydrocarbon compounds, and the proportion of the various hydrocarbons in the mix varies according to the origin of the crude oil. For a refinery operator, technically it is feasible to produce any refined product from any type of crude (A,B).

Finally, in §4.2, we formulate several extensions to more general situations, in particular where the price processes follow other dynamics, e.g., geometric Brownian motion, or the number of markets is larger than two. The trader optimization problem can there be solved numerically. Another extension of our basic model is to integrate the inventory aspect of the trade. We have considered a shipping system where the quantity bought at A is equal to the one sold at B. This is a fair assumption if we focus on a pipeline. However, if the shipping system is a serial network of warehouses, then one could consider the possibility of storing inventory somewhere in that network. The analysis becomes significantly more complex. These extensions can be further explored, and constitute a promising direction of future research.

Appendix

In this appendix we show the MATLAB code used in the numerical sensitivity analysis in §3.4. The main function is CAMfunction that gives as result the value $NPV = J^*$ and the two thresholds m^{BA}, m^{AB} depending on ten parameters:

1. mode1: when =1 we use the analytical model; when=2 we use simulation.

2. mode2: when =1 we compute the symmetric trader case; when=2 we compute the uni-directional trader case

3. c:=value of the trading costs

4. q:=value of the capacity

5. SimN:=number of simulations to be done

6. xn:= number of points

7. mv0:=lower threshold for the competence

8. mv1:=upper threshold for the competence

9. qv0:=competitors capacity from B to A

10. qv1:=competitors capacity from A to B

SENSITIVITY ANALYSIS

```
function[NPV,mAB]=CAMfunction(mode1,mode2,c,q,SimN,xn,mv0,mv1,qv0,qv1)
```

```
xmin=-20; lower limit
xmax=20; upper limit
```

```
xstep=(xmax-xmin)/(xn-1);
xrange=xmin+xstep*(0:xn-1);
m0=xmin;
```

```
N=zeros(xn,xn);
epsilon=zeros(1,SimN);

for k=1:SimN
    epsilon(k)=2*rand-1; random perturbation
end


u=zeros(1,xn); our control
v=zeros(1,xn); market power of the competitors


obj=zeros(1,xn);
fbest=ones(1,xn);
objbest=0;
m0best=xmin;
m1best=xmax;


for i=1:xn
    if xrange(i)<mv0
        v(i)=qv0;
    elseif xrange(i)>mv1
        v(i)=qv1;
    end


end


for i1=1:xn
    m1=xrange(i1);
    for i0=1:i1
        m0=xrange(i0);


        u=zeros(1,xn);
        if(mode2==1)
            for k=1:xn
                if xrange(k)<=m0
                    u(k)=-q;
                elseif xrange(k)>=m1
                    u(k)=q;
                end
            end
        end
```

```matlab
    if(mode2==2)
        for k=1:xn
            if xrange(k)<=m0
                u(k)=0;
            elseif xrange(k)>=m1
                u(k)=q;
            end
        end
    end


ANALYTICAL
    if(mode1==1)
        f=ones(1,xn);
        imid=floor(xn/2);
        for k=imid-1:-1:1
            f(k)=f(k+1)*exp((u(k+1)+v(k+1)+u(k)+v(k))/2*xstep);
        end
        for k=imid+1:xn
            f(k)=f(k-1)*exp(-(u(k-1)+v(k-1)+u(k)+v(k))/2*xstep);
        end


        sumf=sum(f);
        f=f/sumf;


        for k=2:(xn-1)
            obj(k)=(xrange(k)*u(k)-c*abs(u(k)))*(f(k+1)+f(k-1))/2;
        end


        objf=sum(obj);
        N(i0,i1)=objf;
    end


SIMULATED
    if(mode1==2)


        G=0;
        objf=0;
        Gplot=[G];
```

```
for k=1:SimN


    if(mode2==1)
        if (G<m0)
            uG=-q;
        elseif (G>m1)
            uG=q;
        else
            uG=0;
        end


    if (G<mv0)
            vG=qv0;
        elseif (G>mv1)
            vG=qv1;
        else
            vG=0;
        end
    end


    if(mode2==2)
        if (G<m0)
            uG=0;
        elseif (G>m1)
             uG=q;
        else
            uG=0;
        end


        if (G<mv0)
            vG=qv0;
        elseif (G>mv1)
            vG=qv1;
        else
            vG=0;
        end
    end


    G=G-(uG+vG)+epsilon(k);
    objf=objf+G*uG-c*abs(uG);
    Gplot=[Gplot G];
```

```
            end


            N(i0,i1)=objf;
            f=zeros(1,xn);


        end


        if objf > objbest
        objbest=objf;
        m0best=m0;
        m1best=m1;
        fbest=f;
        ubest=u;
        evolutionG=Gplot;
        end


    end


end


Plot simulated pdf
Final=sort(evolutionG);
pvalues=(-.5+(1:SimN+1))/(SimN+2);
data=[Final;pvalues];
i=2;
while i<=length(data)
    if data(1,i)==data(1,i-1);
         le=length(data);
        datal=data(:,1:i-2);
        datar=data(:,i:le);
        data=[datal, datar];
    else
        i=i+1;
    end
end


xN2=200;
xmin2=min(Final);
```

```
xmax2=max(Final);
xstep2=(xmax2-xmin2)/xN2;
xvalues=xmin2+xstep2*((1:xN2));


yvalues=zeros(1,xN2);
for i=1:xN2
    yvalues(i)=interp1(data(1,:),data(2,:),xvalues(i));
end


opt=max(N);
NPV=max(opt);
mBA=m0best;
mAB=m1best;


figure(1);
contourf(xrange, xrange, max(0,N)',50)
xlabel('$m_{BA}'$');
ylabel('$m_{AB}'$');


figure(2);
plot(xvalues(1:xN2-1),diff(yvalues)/xstep2);
xlabel('Pricedifference value');
ylabel('Simulated Probabilitydensity function');
hold on
plot(xrange,fbest,'r');
ylabel('Analytycal Probability density function');
```

The following two functions show the changes of the objective function, and thresholds with respect to changes in costs and capacity. Note that both use CAMfunction and that are thought to generate figures like 3.5 and 3.6. The two parameters are: mod (when=1 means we are interested in J^*; when=2 we are interested in the thresholds) and N (limit value of cost or capacity to be evaluated).

COST PLOTS

```
function RES=cmod(mod,N)


        RES1=[CAMfunction(mode1,mode2,0,q,SimN,xn,mv0,mv1,qv0,qv1)(mod)];
        RES2=[CAMfunction(mode1,mode2,0,q',SimN,xn,mv0,mv1,qv0,qv1)(mod)];
        for i=1:N
        Res1=CAMfunction(mode1,mode2,i,q,SimN,xn,mv0,mv1,qv0,qv1)(mod);
```

```
        Res2=CAMfunction(mode1,mode2,i,q',SimN,xn,mv0,mv1,qv0,qv1)(mod);
        RES1=[RES1 Res1];
        RES2=[RES2 Res2];
    end


    if(mod==1)
        figure(1);
        plot(RES1,'r');
        hold on
        plot(RES2,'b');
        xlabel('Cost c');
        ylabel('J*');
    end


    if(mod==2)
        figure(1);
        plot(RES1,'r');
        hold on
        plot(RES2,'b');
        xlabel('Cost c');
        ylabel('$m^{AB}=-m^{BA}$');
    end
```

CAPACITY PLOTS

```
function RES=qmod(mod,N)


    RES1=[CAMfunction(mode1,mode2,c,0.01,SimN,xn,mv0,mv1,qv0,qv1)(mod)];
    RES2=[CAMfunction(mode1,mode2,c',0.01,SimN,xn,mv0,mv1,qv0,qv1)(mod)];
    for i=1:N
        Res1=CAMfunction(mode1,mode2,c,i,SimN,xn,mv0,mv1,qv0,qv1)(mod);
        Res2=CAMfunction(mode1,mode2,c',i,SimN,xn,mv0,mv1,qv0,qv1(mod));
        RES1=[RES1 Res1];
        RES2=[RES2 Res2];
    end


    if(mod==1)
        figure(1);
        plot(RES1,'r');
        hold on
        plot(RES2,'b');
        xlabel('Capacity q');
```

```
        ylabel('J*');
    end


if(mod==2)
    figure(2);
    plot(RES1,'r');
    hold on
    plot(RES2,'b');
    xlabel('Capacity q');
    ylabel('$m^{AB}=-m^{BA}$');
end
```

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