On the relationship between the *energy shaping* and the *Lyapunov constraint based* methods

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Abstract

In this paper, we make a review of the controlled Hamiltonians (CH) method and its related matching conditions, focusing on an improved version recently developed by D.E. Chang. Also, we review the general ideas around the Lyapunov constraint based (LCB) method, whose related partial differential equations (PDEs) were originally studied for underactuated systems with only one actuator, and then we study its PDEs for an arbitrary number of actuators. We analyze and compare these methods within the framework of Differential Geometry, and from a purely theoretical point of view. We show, in the context of underactuated systems defined by simple Hamiltonian functions, that the LCB method and the Chang's version of the CH method are equivalent stabilization methods (i.e. they give rise to the same set of control laws). In other words, we show that the Chang's improvement of the energy shaping method is precisely the LCB method. As a by-product, coordinate-free and connection-free expressions of Chang's matching conditions are obtained.

1 Introduction

Under the name of *energy shaping method*, several methods or procedures for achieving (asymptotic) stabilization of nonlinear underactuated Lagrangian and Hamiltonian systems are included: *potential shaping*, *kinetic shaping*, *total energy shaping*, *energy plus force shaping*, *IDA-PBC*, etc. See for instance [4, 6, 7, 8, 9, 31, 37, 42, 45], and [19, 40] for more recent works. They are based on the idea of *feedback equivalence* (see Ref. [20]), and their purpose is to construct, for a given underactuated mechanical system, a control law and a Lyapunov function for the resulting closed-loop system. To do that, a set of partial differential equations (PDEs), known as *matching conditions*, must be solved. Such PDEs have among their unknowns the aforementioned Lyapunov function.

All of these methods can be seen as particular versions of the so-called *controlled Lagrangians (CL) method* or the *controlled Hamiltonians (CH) method*, which in turn are equivalent stabilization methods, in a sense that has been carefully explained in Ref. [20].

The origin of the energy shaping method can be placed 35 years ago [2, 12, 41, 44], while the method in its more general form is around 15 years old [20]. More recently, 6 years ago, an alternative stabilization method for nonlinear underactuated mechanical systems has been presented: the *Lyapunov constraint based (LCB) method*. It appeared for the first time in [23], it was further developed in [25], and it was extended to systems with impulsive effects in Ref. [15]. The method is based on the idea of controlling actuated mechanical systems by imposing kinematic constraints (see [14, 22, 32, 33, 38, 39, 43]). It serves the same purpose as the energy shaping method (to construct a control law and a Lyapunov function for the resulting closed-loop system) and, in order to accomplish it, a set of PDEs must be solved too. It is worth mentioning that the LCB method has been originally developed for underactuated systems with only one actuator.

One of the aims of this paper is to extend the study of the LCB method to an arbitrary number of actuators, and to show that this method contains every version of the energy shaping method (and actually, every method that serves the same purpose), in the sense that the set of control laws that can be constructed with the energy shaping method is contained in the corresponding set of the LCB method (extending a result already presented in [25]).

Almost simultaneously with the appearance of [25], an improvement of the energy (plus force) shaping method, for underactuated systems defined by *simple* Lagrangian or Hamiltonian functions, was presented by Chang in [16, 17, 18]. It consists in an important simplification of the matching conditions. The main goal of the present paper is to show that such matching conditions are exactly the PDEs related to the LCB method, at least in the context of simple Hamiltonian functions. Moreover, we show in the same context that the Chang's version of the energy shaping method is equivalent to the LCB method, i.e. both methods give rise exactly, for a given underactuated system, to the same set of control laws. In other words, we show that the Chang's improvement of the energy shaping method is precisely the LCB method. Such a result is quite surprising for us, because the involved methods are based on very different ideas: "feedback equivalence" and "controlling by the imposition of kinematic constraints."

We can say that this article is similar in spirit to Ref. [20], where the equivalence between the CL and the CH methods was established. In particular, as in that paper, a substantial portion of the work is dedicated to describe, in a very precise way and by using the same language, the methods that we want to compare.

The paper is organized as follows. In Section 2, we present some basic facts about affine connections on general linear bundles, which will be used along all of the paper to write down coordinate-free expressions of the PDEs that we want to study. In Section 3, we make a review of the energy shaping method in a Hamiltonian language, i.e. the controlled Hamiltonians (CH) or IDA-PBC method. We begin with a rather general version of the method, then we progressively consider particular situations, and finally we present Chang's version of the method (see for instance [19]), with its related matching conditions. In Section 4, we recall the idea of controlling mechanical systems by the imposition of kinematic constraints. In particular, we review the idea of achieving (asymptotic) stability by means of the so-called *Lyapunov constraint*, which gives rise to the LCB method and its related set of PDEs. We show in the last section of the paper that such PDEs are exactly the matching conditions obtained by Chang [19], at least when underactuated systems defined by simple Hamiltonian functions are considered. Finally, we show the equivalence of the LCB method and the Chang's version of the CH method.

We assume that the reader is familiar with basic concepts of Differential Geometry [11, 30, 35], Hamiltonian systems in the context of Geometric Mechanics [1, 3, 34], Control Theory in a geometric language [10, 13], and Lyapunov theorems for (asymptotic) stability [29].

Basic notation. Along all of the paper, every manifold will be a smooth finite dimensional manifold, typically denoted by Q. By $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ we will denote the tangent and cotangent vector bundles, respectively. As it is customary, we indicate by $\langle \cdot, \cdot \rangle$ the natural pairing between T_q^*Q and T_qQ at every $q \in Q$, and by $\mathfrak{X}(Q)$ and $\Omega^1(Q)$ the sheaves of sections of τ_Q and π_Q, respectively. Unless a confusion may arise, we shall omit the subindex Q for τ_Q and π_Q. For a vector field $Y : Q \to TQ$, in order to indicate that its image is contained inside some subset W of TQ, we shall write, for simplicity, $Y \subset W$. Given a second manifold P and a smooth function $F : Q \to P$, we denote by F_* and F^* the push-forward map and its transpose, respectively.

Consider a local chart (U,φ) of Q, with $\varphi: U \to \mathbb{R}^n$. Given $q \in U$, we write $\varphi(q) = \left(q^1, ..., q^n\right) = \mathbf{q}$. For the induced local charts (TU, φ_*) and $\left(T^*U, (\varphi^*)^{-1}\right)$ on TQ and T^*Q, respectively, we write

$$\begin{aligned} \varphi_*(v) &= \left(q^1, ..., q^n, \dot{q}^1, ..., \dot{q}^n\right) = (\mathbf{q}, \dot{\mathbf{q}}), \\ (\varphi^*)^{-1}(\alpha) &= \left(q^1, ..., q^n, p_1, ..., p_n\right) = (\mathbf{q}, \mathbf{p}), \end{aligned} \tag{1}$$

or simply

$$\varphi_{*,q}(v) = \dot{\mathbf{q}} \quad \text{and} \quad \left(\varphi_q^*\right)^{-1}(\alpha) = \mathbf{p}, \tag{2}$$

for all $v \in TU$ and $\alpha \in T^*U$. On TT^*Q we shall consider the induced charts $\left(TT^*U, (\varphi^*)_*^{-1}\right)$, and write

$$(\varphi^*)_*^{-1}(V) = (\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}), \tag{3}$$

for all $V \in TT^*U$.

2 Some preliminary results

In this section we shall recall some results on vector bundles and affine connections that will enable us to write global expressions of the equations we want to study later. Most of these results were proved in Ref. [22]. Nevertheless, for the sake of completeness, we include some proofs here. Also, at the end of the section, we recall some basic facts about Lyapunov functions.

Let us consider a vector bundle $\Pi : \mathcal{U} \to Q$ and fix an affine connection $\nabla : \mathfrak{X}(Q) \times \Gamma(\mathcal{U}) \to \Gamma(\mathcal{U})$. Related to the latter we can define a diffeomorphism $\beta : T\mathcal{U} \to \mathcal{U} \oplus TQ \oplus \mathcal{U}$, given as follows (see Ref. [22]). For $V \in T\mathcal{U}$, consider a curve $u : (-\varepsilon, \varepsilon) \to \mathcal{U}$ passing through $\tau_{\mathcal{U}}(V)$ and with velocity V at $s = 0$, i.e. $u_*\left(d/ds|_0\right) = V$. Finally, define

$$\beta(V) := \tau_{\mathcal{U}}(V) \oplus \Pi_*(V) \oplus \frac{D}{Ds} u(0).$$

Fixing $q \in Q$ and a vector $X \in \mathcal{U}_q$ (i.e. $\Pi(X) = q$), we have the linear isomorphisms

$$\beta_X : T_X\mathcal{U} \to T_qQ \oplus \mathcal{U}_q \quad \text{and} \quad \beta_X^{-1} : T_qQ \oplus \mathcal{U}_q \to T_X\mathcal{U}, \tag{4}$$

given by

$$\beta_X(V) := \Pi_*(V) \oplus \frac{D}{Ds} u(0) \qquad \text{and} \qquad \beta_X^{-1}(Y \oplus Z) := u_*\left(d/ds|_0\right), \tag{5}$$

respectively, where we take u in the second equation to be a curve in $\mathcal{U}$ such that

$$u(0) = X, \qquad (\Pi \circ u)_*\left(d/ds|_0\right) = Y \qquad \text{and} \qquad \frac{D}{Ds} u(0) = Z.$$

We have in addition their corresponding transpose maps

$$\beta_X^* : T_q^*Q \oplus \mathcal{U}_q^* \to T_X^*\mathcal{U} \quad \text{and} \quad \beta_X^{*-1} : T_X^*\mathcal{U} \to T_q^*Q \oplus \mathcal{U}_q^*. \tag{6}$$

In terms of β, the horizontal and vertical subbundles related to ∇ at a point $X \in \mathcal{U}_q$ are, respectively,

$$\mathrm{Hor}_X := \beta_X^{-1}(T_qQ \oplus 0) \quad \text{and} \quad \mathrm{Ver}_X := \ker \Pi_{*,X} = \beta_X^{-1}(0 \oplus \mathcal{U}_q).$$

It can be shown that the *vertical lift isomorphism*

$$\mathrm{vlift}_X : \mathcal{U}_q \to \ker \Pi_{*,X} : Z \longmapsto \left.\frac{d}{ds}\right|_0 (X + s\, Z)$$

is related with β_X^{-1} by the formula

$$\mathrm{vlift}_X(Z) = \beta_X^{-1}(0 \oplus Z). \tag{7}$$

(This is true for any connection ∇.) Let us suppose that $\mathcal{U} = T^*Q$ and $\Pi = \pi_Q = \pi$. The related diffeomorphism

$$\beta : TT^*Q \to T^*Q \oplus TQ \oplus T^*Q \tag{8}$$

is given by

$$\beta(V) = \alpha \oplus \pi_*(V) \oplus \frac{D}{Ds}u(0), \quad \forall \alpha \in T^*Q, \quad V \in T_\alpha T^*Q,$$

where $u : (-\varepsilon, \varepsilon) \to T^*Q$ is a curve passing through α at $s = 0$ with velocity V. In a local chart (U, φ) of Q, it is easy to show that

$$\beta(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}, \dot{\mathbf{p}}) = (\mathbf{q}, \mathbf{p}) \oplus (\mathbf{q}, \dot{\mathbf{q}}) \oplus (\mathbf{q}, \dot{\mathbf{p}} + \Gamma(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}})) \tag{9}$$

(omitting in the last expression the map φ, just for simplicity), where $\Gamma(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}})$ is given by the Christoffel symbols $\Gamma_{il}^k(\mathbf{q})$ of ∇ (in the coordinate frame) as

$$\Gamma_i(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}) = \Gamma_{il}^k(\mathbf{q}) \; p_k \, \dot{q}^l.$$

Sum over repeated indices convention is assumed from now on. On the other hand, using the relationship between the vertical lift isomorphism $\mathrm{vlift}_\alpha : T^*_{\pi(\alpha)}Q \to \ker \pi_{*,\alpha}$ and the linear isomorphism $\beta_\alpha : T_\alpha T^*Q \to T_{\pi(\alpha)}Q \oplus T^*_{\pi(\alpha)}Q$ [see Eqs. (4), (5) and (7)], every vertical vector $Y_\alpha \in T_\alpha T^*Q$ may be identified with a unique covector $y_\alpha \in T^*_{\pi(\alpha)}Q$ in the following ways:

$$Y_\alpha = \mathrm{vlift}_\alpha(y_\alpha) = \beta_\alpha^{-1}(0 \oplus y_\alpha) = \beta^{-1}(\alpha \oplus 0 \oplus y_\alpha). \tag{10}$$

As a consequence, every vertical vector field $Y : T^*Q \to TT^*Q$ is defined by the unique fiber preserving map $y : T^*Q \to T^*Q$ such that

$$Y(\alpha) = \mathrm{vlift}_\alpha(y(\alpha)) = \beta^{-1}(\alpha \oplus 0 \oplus y(\alpha)). \tag{11}$$

Definition 1 *Given a function $F : \mathcal{U} \to \mathbb{R}$, the **fiber** and **base derivatives** of F are defined as the fiber-preserving maps $\mathbb{F}F : \mathcal{U} \to \mathcal{U}^*$ and $\mathbb{B}F : \mathcal{U} \to T^*Q$ given by*

$$\langle \mathbb{F}F(X), Z \rangle = \left. \frac{d}{ds} F(X + s\,Z) \right|_{s=0} \tag{12}$$

and

$$\langle \mathbb{B}F(X), Y \rangle = \left. \frac{d}{ds} F(u(s)) \right|_{s=0},$$

respectively, where $u : (-\varepsilon, \varepsilon) \to \mathcal{U}$ is a (horizontal) curve such that

$$u(0) = X, \quad (\Pi \circ u)_* \left(d/ds|_0 \right) = Y \quad \text{and} \quad \frac{D}{Ds}u(s) = 0. \tag{13}$$

Remark 2 *Note that $\mathbb{F}F$ is independent of ∇, but $\mathbb{B}F$ is not.*

Let us come back to the cotangent bundle of Q. Given a smooth function $F : T^*Q \to \mathbb{R}$, the fiber and base derivatives of F are bundle morphisms $\mathbb{F}F : T^*Q \to TQ$ and $\mathbb{B}F : T^*Q \to T^*Q$, respectively, which in local coordinates read

$$(\mathbb{F}F(\mathbf{q}, \mathbf{p}))^i = \frac{\partial F}{\partial p_i}(\mathbf{q}, \mathbf{p}) \tag{14}$$

and

$$(\mathbb{B}F(\mathbf{q}, \mathbf{p}))_i = \frac{\partial F}{\partial q^i}(\mathbf{q}, \mathbf{p}) + \Gamma_{il}^k(\mathbf{q}) \; \frac{\partial F}{\partial p_l}(\mathbf{q}, \mathbf{p}) \; p_k. \tag{15}$$

Regarding *basic functions* $F : \mathcal{U} \to \mathbb{R}$, i.e. functions for which there exists $f : Q \to \mathbb{R}$ such that $F = f \circ \Pi$, we have the next result.

Proposition 3 *If $F : \mathcal{U} \to \mathbb{R}$ is basic, then*

$$\mathbb{F}F = 0 \quad \text{and} \quad \mathbb{B}F = df \circ \Pi. \tag{16}$$

Proof. Given $X, Z \in \mathcal{U}_q$ for some $q \in Q$, i.e. $\Pi(X) = \Pi(Z) = q$, we have that

$$\langle \mathbb{F}F(X), Z \rangle = \left. \frac{d}{ds} F(X + s\, Z) \right|_{s=0} = \left. \frac{d}{ds} f \circ \Pi(X + s\, Z) \right|_{s=0} = \left. \frac{d}{ds} f(q) \right|_{s=0} = 0.$$

On the other hand, given in addition $Y \in T_q Q$ and a curve u satisfying (13),

$$\langle \mathbb{B}F(X), Y \rangle = \left. \frac{d}{ds} F(u(s)) \right|_{s=0} = \left. \frac{d}{ds} f(\Pi(u(s))) \right|_{s=0} = \langle df(\Pi(X)), Y \rangle,$$

as we wanted to show. □

The isomorphisms β_X^{*-1} [see Eq. (6)] give rise to another diffeomorphism $\widetilde{\beta} : T^*\mathcal{U} \to \mathcal{U} \oplus T^*Q \oplus \mathcal{U}^*$, being $\widetilde{\beta}(\Sigma) = \beta_X^{*-1}(\Sigma)$ for all $X \in \mathcal{U}$ and $\Sigma \in T_X^*\mathcal{U}$. For the cotangent bundle, we have a diffeomorphism

$$\widetilde{\beta} : T^*T^*Q \to T^*Q \oplus T^*Q \oplus TQ. \tag{17}$$

Proposition 4 *Given $F : \mathcal{U} \to \mathbb{R}$ and $X \in \mathcal{U}$,*

$$\widetilde{\beta}(dF(X)) = X \oplus \mathbb{B}F(X) \oplus \mathbb{F}F(X). \tag{18}$$

Proof. We must show that, for all $q \in Q$, $Y \in T_q Q$ and $Z \in \mathcal{U}_q$,

$$\left\langle \beta_X^{*-1}(dF(X)), Y \oplus Z \right\rangle = \langle \mathbb{B}F(X), Y \rangle + \langle \mathbb{F}F(X), Z \rangle,$$

or equivalently, $\langle dF(X), V \rangle = \langle \mathbb{B}F(X), Y \rangle + \langle \mathbb{F}F(X), Z \rangle$, for $V = \beta_X^{-1}(Y \oplus Z)$. Let $u_1 : (-\varepsilon, \varepsilon) \to \mathcal{U}$ be a curve satisfying (13) and $u_2 : (-\varepsilon, \varepsilon) \to \mathcal{U}$ such that $u_2(s) := X + s\, Z$. Since

$$\beta_X\left((u_1)_*\left(d/ds|_0\right)\right) = Y \oplus 0 \quad \text{and} \quad \beta_X\left((u_2)_*\left(d/ds|_0\right)\right) = 0 \oplus Z,$$

then $(u_1)_*\left(d/ds|_0\right) + (u_2)_*\left(d/ds|_0\right) = V$. Consequently,

$$\langle dF(X), V \rangle = \left. \frac{d}{ds} F(u_1(s)) \right|_{s=0} + \left. \frac{d}{ds} F(X + s\, Z) \right|_{s=0},$$

what ends our proof. □

Remark 5 *Let us replace $\mathcal{U}$ by the Whitney sum of k copies of $\mathcal{U}$, which we shall denote $\mathcal{U} \times \cdots \times \mathcal{U}$, and consider on such a vector bundle the affine connection naturally induced by one fixed on $\mathcal{U}$. Then, given a function $F : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathbb{R}$, its fiber and base derivatives*

$$\mathbb{F}F : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{U}^* \times \cdots \times \mathcal{U}^* \quad \text{and} \quad \mathbb{B}F : \mathcal{U} \times \cdots \times \mathcal{U} \to T^*Q$$

will be defined by the formulae

$$\langle \mathbb{F}F(X_1, ..., X_k), (Z_1, ..., Z_k) \rangle = \left. \frac{d}{ds} F(X_1 + s\, Z_1, ..., X_k + s\, Z_k) \right|_{s=0}$$

and

$$\langle \mathbb{B}F(X_1, ..., X_k), Y \rangle = \left. \frac{d}{ds} F(u_1(s), ..., u_k(s)) \right|_{s=0},$$

respectively, where each $u_i : (-\varepsilon, \varepsilon) \to \mathcal{U}$ is a (horizontal) curve such that

$$u_i(0) = X_i, \quad (\Pi \circ u_i)_*\left(d/ds|_0\right) = Y \quad \text{and} \quad \frac{D}{Ds} u_i(s) = 0.$$

As usual, by a tensor on $\mathcal{U}$ we mean a function $T : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathbb{R}$, on the Whitney sum of copies of $\mathcal{U}$, which is multi-linear map when restricted to each fiber. When we write $T(X_1, ..., X_k)$, it is implicit that all X_i's are contained in the same fiber of $\mathcal{U}$.

Consider a tensor $\mathfrak{b} : T^*Q \times T^*Q \to \mathbb{R}$ and its related quadratic form $\mathfrak{q} : T^*Q \to \mathbb{R} : \alpha \longmapsto \mathfrak{b}(\alpha, \alpha)$. The following result is immediate.

Proposition 6 *For all $\alpha, \sigma \in T^*Q$,*

$$\langle \mathbb{F}\mathfrak{b}(\alpha, \alpha), (\sigma, \sigma) \rangle = \langle \mathbb{F}\mathfrak{q}(\alpha), \sigma \rangle \quad \text{and} \quad \mathbb{B}\mathfrak{b}(\alpha, \alpha) = \mathbb{B}\mathfrak{q}(\alpha). \tag{19}$$

Let ω be the canonical symplectic form on T^*Q.

Proposition 7 *Given $\alpha \in T^*Q$ and $V_1, V_2 \in T_\alpha T^*Q$, and writing $\beta_\alpha(V_1) = v_1 \oplus \sigma_1$ and $\beta_\alpha(V_2) = v_2 \oplus \sigma_2$, we have that*

$$\omega(V_1, V_2) = \langle \sigma_2, v_1 \rangle - \langle \sigma_1, v_2 \rangle + \langle \alpha, T(v_1, v_2) \rangle,$$

being T the torsion of ∇.

Proof. Fixing a local chart (U, φ) containing $q = \pi(\alpha)$ and writing

$$\varphi^{*-1}(\alpha) = (\mathbf{q}, \mathbf{p}) \quad \text{and} \quad \left(\varphi^{*-1}\right)_*(V_\gamma) = (\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_\gamma, \dot{\mathbf{p}}_\gamma), \quad \gamma = 1, 2,$$

it is well-known that $\omega(V_1, V_2) = \dot{p}_{2,i}\, \dot{q}_1^i - \dot{p}_{1,i}\, \dot{q}_2^i$. On the other hand, since

$$\beta(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_\gamma, \dot{\mathbf{p}}_\gamma) = (\mathbf{q}, \mathbf{p}) \oplus (\mathbf{q}, \dot{\mathbf{q}}_\gamma) \oplus (\mathbf{q}, \dot{\mathbf{p}}_\gamma + \Gamma(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_\gamma))$$

[see (9)], we have that $\varphi_*(v_\gamma) = (\mathbf{q}, \dot{\mathbf{q}}_\gamma)$ and $\varphi^{*-1}(\sigma_\gamma) = (\mathbf{q}, \dot{\mathbf{p}}_\gamma + \Gamma(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_\gamma))$, and consequently

$$\langle \sigma_2, v_1 \rangle - \langle \sigma_1, v_2 \rangle + \langle \alpha, T(v_1, v_2) \rangle = (\dot{p}_{2,i} + \Gamma_i(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_2))\, \dot{q}_1^i - (\dot{p}_{1,i} + \Gamma_i(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_1))\, \dot{q}_2^i + p_i\, T^i(\mathbf{q}, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2).$$

So, we must show that $\Gamma_i(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_2)\, \dot{q}_1^i - \Gamma_i(\mathbf{q}, \mathbf{p}, \dot{\mathbf{q}}_1)\, \dot{q}_2^i = -p_i\, T^i(\mathbf{q}, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2)$. But

$$T^i(\mathbf{q}, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) = \Gamma_{kl}^i(\mathbf{q})\; \dot{q}_1^k\, \dot{q}_2^l - \Gamma_{kl}^i(\mathbf{q})\; \dot{q}_2^k\, \dot{q}_1^l,$$

from which the wanted result easily follows. $\square$

Assume that ∇ is torsion-free, which we shall do from now on. In terms of the diffeomorphisms β and $\widetilde{\beta}$ we have the following result.

Proposition 8 *For all $v \in TQ$, $\alpha, \sigma \in T^*Q$, on the same base point,*

$$\beta \circ \omega^\sharp \circ \widetilde{\beta}^{-1}(\alpha \oplus \sigma \oplus v) = \alpha \oplus v \oplus (-\sigma). \tag{20}$$

Proof. Following the notation of the previous proposition, since $T = 0$ (the torsion-free condition), we have that

$$\omega\left(\beta_\alpha^{-1}(v_1 \oplus \sigma_1), \beta_\alpha^{-1}(v_2 \oplus \sigma_2)\right) = \langle \sigma_2, v_1 \rangle - \langle \sigma_1, v_2 \rangle,$$

or equivalently

$$\left\langle \left(\beta_\alpha^{*-1} \circ \omega^\flat \circ \beta_\alpha^{-1}\right)(v_1 \oplus \sigma_1), (v_2 \oplus \sigma_2) \right\rangle = \langle \sigma_2, v_1 \rangle - \langle \sigma_1, v_2 \rangle.$$

This implies that $\left(\beta_\alpha^{*-1} \circ \omega^\flat \circ \beta_\alpha^{-1}\right)(v \oplus \sigma) = -\sigma \oplus v$ for all v, σ on the base point of α. Finally, using the identity

$$\left(\beta_\alpha^{*-1} \circ \omega^\flat \circ \beta_\alpha^{-1}\right)^{-1} = \beta_\alpha \circ \omega^\sharp \circ \beta_\alpha^*,$$

the proof is done. $\square$

Since the canonical Poisson bracket on T^*Q is given by the formula

$$\{F,G\}(\alpha)=\left\langle dF(\alpha),\omega^{\sharp}\left(dG(\alpha)\right)\right\rangle ,\quad \forall F,G\in C^{\infty}\left(T^*Q\right),$$

using the last proposition and the Eq. (18) we easily arrive at the equation

$$\{F,G\}(\alpha)=\left\langle \mathbb{B}F(\alpha),\mathbb{F}G(\alpha)\right\rangle -\left\langle \mathbb{B}G(\alpha),\mathbb{F}F(\alpha)\right\rangle . \tag{21}$$

This identity will be central in the last section of the paper.

Finally, let us consider the next definition.

Definition 9 *Let P be a manifold and $X\in\mathfrak{X}(P)$ a vector field on P. Given a critical point $\alpha^{\bullet}\in P$ of X, i.e. $\alpha^{\bullet}$ such that $X\left(\alpha^{\bullet}\right)=0$, a* ***Lyapunov function*** *for X and $\alpha^{\bullet}$ is a smooth function $\hat{H}:P\rightarrow\mathbb{R}$ satisfying*

L1 *$\hat{H}$ is positive definite w.r.t. $\alpha^{\bullet}$ (i.e. non-negative and null only at $\alpha^{\bullet}$);*

L2 *$\left\langle d\hat{H}(\alpha),X(\alpha)\right\rangle \leq 0$ for all α.*

As it is well-known, if such a function exists, then $\alpha^{\bullet}$ is a stable point. Moreover, if the inequality in **L2** is strict for all $\alpha\neq\alpha^{\bullet}$, then $\alpha^{\bullet}$ is locally asymptotically stable, and if in addition $\hat{H}$ is a proper function and P is connected, then such a point is globally asymptotically stable. For a proof of these results, see Ref. [29].

3 Energy shaping method

We present in this section the Hamiltonian *side* of the energy shaping method: the *controlled Hamiltonians method* (as defined in [20]), also known as the *IDA-PBC method* [37]. We describe a quite general version of the method, with its related *matching conditions*, in terms that are more convenient for the present paper. For instance, we shall focus on Hamiltonian systems on a cotangent bundle only. We shall progressively consider particular situations to finally arrive at the case studied by Chang in Refs. [16]-[19], where particularly simple matching conditions can be derived.

3.1 The controlled Hamiltonians

Fix a manifold Q, a function $H:T^*Q\rightarrow\mathbb{R}$ and a vertical subbundle $\mathcal{W}\subset\ker\pi_*\subset TT^*Q$ of the tangent bundle on T^*Q. Denote by $X_H:T^*Q\rightarrow TT^*Q$ the Hamiltonian vector field of H w.r.t. the canonical symplectic structure ω on T^*Q, i.e. $X_H:=\omega^{\sharp}\circ dH\in\mathfrak{X}\left(T^*Q\right)$. Fix also a critical point $\alpha^{\bullet}\in T^*Q$ of X_H. Note that the pair $(H,\mathcal{W})$ defines an underactuated Hamiltonian system on Q (with Hamiltonian function H and space of actuators $\mathcal{W}$). It is clear that the rank of $\mathcal{W}$ represents the number of actuators. Suppose that we want to solve the following problem.

P. Find a control signal $Y\subset\mathcal{W}$, i.e. a vertical vector field $Y\in\mathfrak{X}\left(T^*Q\right)$ with image inside $\mathcal{W}$, such that the closed loop system defined by X_H+Y is stable at $\alpha^{\bullet}$.

We shall call *stabilization method* to any "systematic procedure" that enables us to solve the problem **P**. To be more precise, let us consider the definitions below.

Definition 10 *Fix a manifold Q and let $\mathfrak{U}$ be a subset of triples $(H, \mathcal{W}, \alpha^\bullet)$, where $(H, \mathcal{W})$ is an underactuated system on Q and $\alpha^\bullet \in T^*Q$ is a critical point of X_H. Given a triple $(H, \mathcal{W}, \alpha^\bullet) \in \mathfrak{U}$, denote by $\mathcal{S}_{H,\mathcal{W},\alpha^\bullet} \subset \mathfrak{X}(T^*Q)$ the subset of all the vector fields $Y \in \mathfrak{X}(T^*Q)$ solving **P**. We shall call **stabilization method** on $\mathfrak{U}$ to any function[1] $\digamma$ from $\mathfrak{U}$ to the power set of $\mathfrak{X}(T^*Q)$, such that $\digamma(H, \mathcal{W}, \alpha^\bullet) \subset \mathcal{S}_{H,\mathcal{W},\alpha^\bullet}$. In addition, we shall say that $\digamma$ is **Lyapunov based** if for each element $Y \in \digamma(H, \mathcal{W}, \alpha^\bullet)$ a Lyapunov function for $X_H + Y$ and $\alpha^\bullet$ can be exhibited (or at least exists).*[2]

Definition 11 *Given two stabilization methods $\digamma$ and $\digamma'$ on the subsets $\mathfrak{U}$ and $\mathfrak{U}'$, respectively, we shall say that $\digamma$ is **included** in $\digamma'$ if*

$$\digamma(H, \mathcal{W}, \alpha^\bullet) \subset \digamma'(H, \mathcal{W}, \alpha^\bullet), \quad \forall (H, \mathcal{W}, \alpha^\bullet) \in \mathfrak{U} \cap \mathfrak{U}'.$$

*If both inclusions hold, we shall say that $\digamma$ and $\digamma'$ are **equivalent** on $\mathfrak{U} \cap \mathfrak{U}'$.*

Remark 12 *For methods on some common subset $\mathfrak{U}$, the inclusion relation we just presented defines a partial order. For such a partially ordered set, a maximal element represents the most general way to systematically stabilize underactuated systems in $\mathfrak{U}$.*

The same can be said for the subset of Lyapunov based stabilization methods.

The intention behind above definitions is to give a precise framework to compare different methods and to establish what we mean by an equivalence between them. Nonetheless, we will not construct the functions $\digamma$ when we describe the methods involved in this paper, but give a synthetic explanation of the procedure they give rise to instead.

In the following, we shall define a Lyapunov based stabilization method on the whole set of triples $(H, \mathcal{W}, \alpha^\bullet)$, known as the *controlled Hamiltonians method*.

Assume that we are given a function $\hat{H} \in C^\infty(T^*Q)$, an anti-symmetric tensor $B : T^*T^*Q \times T^*T^*Q \to \mathbb{R}$ (i.e. an almost-Poisson structure) on T^*Q and two vertical vector fields $Z_g, Z_d \in \mathfrak{X}(T^*Q)$ such that:

1. $\hat{H}$ is positive-definite w.r.t. $\alpha^\bullet$,
2. $\left\langle d\hat{H}(\alpha), Z_g(\alpha)\right\rangle = 0$ for all $\alpha \in T^*Q$, i.e. Z_g is a *gyroscopic* force,
3. $B^\sharp \circ d\hat{H} + Z_g - X_H \subset \mathcal{W}$,
4. $\left\langle d\hat{H}(\alpha), Z_d(\alpha)\right\rangle \leq 0$ for all $\alpha \in T^*Q$, i.e. Z_d is a *dissipative* force,
5. $Z_d \subset \mathcal{W}$,
6. and $Z_d(\alpha^\bullet) = -Z_g(\alpha^\bullet)$.

Note that 1 implies $d\hat{H}(\alpha^\bullet) = 0$. Then, because of 6, defining $\hat{X}_{\hat{H}} := B^\sharp \circ d\hat{H}$ and $\hat{X} := \hat{X}_{\hat{H}} + Z_g + Z_d$, the point $\alpha^\bullet$ is a critical point of $\hat{X}$. Also, for all $\alpha \in T^*Q$,

$$\left\langle d\hat{H}(\alpha), \hat{X}(\alpha)\right\rangle = \left\langle d\hat{H}(\alpha), \hat{X}_{\hat{H}}(\alpha) + Z_g(\alpha) + Z_d(\alpha)\right\rangle = \left\langle d\hat{H}(\alpha), Z_d(\alpha)\right\rangle \leq 0, \tag{22}$$

[1] Note that a stabilization method on $\mathfrak{U}$ can also be seen as relation in $\mathfrak{U} \times \mathfrak{X}(T^*Q)$.

[2] The Massera's theorem [36] (and its various generalizations -see [28] for a review-) ensures that, if a smooth closed-loop system is asymptotically stable, then a smooth Lyapunov function exists for such a system. But the same can not be ensured if the system is just stable (see [5]).

because of 2, 4 and the fact that $\left\langle d\hat{H}(\alpha), \hat{X}_{\hat{H}}(\alpha)\right\rangle = B\left(d\hat{H}(\alpha), d\hat{H}(\alpha)\right) = 0$. As a consequence, since 1 coincides with **L1** and Eq. (22) coincides with **L2** (see Definition 9), $\hat{H}$ is a Lyapunov function for the dynamical system defined by $\hat{X}$ and the critical point $\alpha^{\bullet}$. This says that such a system is stable at $\alpha^{\bullet}$ (see Ref. [29]). So, defining

$$Y := \hat{X} - X_H, \tag{23}$$

which belongs to $\mathcal{W}$ because of the points 3 and 5, the problem **P** is solved. In particular, we have that

$$\left\langle d\hat{H}(\alpha), X_H(\alpha) + Y(\alpha)\right\rangle \leq 0. \tag{24}$$

All that gives rise to the following procedure.

Definition 13 *Given an underactuated system $(H, \mathcal{W})$ on Q and a critical point $\alpha^{\bullet} \in T^*Q$ of X_H, the **controlled Hamiltonians (CH) method** consists in finding $\hat{H}$, B and Z_g satisfying* 1 *and* 2 *and solving the equation [see point* 3*]*

$$B^{\sharp} \circ d\hat{H} + Z_g - \omega^{\sharp} \circ dH \subset \mathcal{W}; \tag{25}$$

finding Z_d satisfying 4*,* 5 *and* 6*; and defining [see* (23)*]*

$$Y := B^{\sharp} \circ d\hat{H} + Z_g + Z_d - \omega^{\sharp} \circ dH. \tag{26}$$

It is clear that above procedure defines, according to the Definition 10, a Lyapunov based stabilization method: its function $\digamma$ assigns to every triple $(H, \mathcal{W}, \alpha^{\bullet})$ a set of vector fields $\digamma(H, \mathcal{W}, \alpha^{\bullet}) \subset \mathfrak{X}(T^*Q)$ given by Eq. (26) (and consequently solving the problem **P**), where $\hat{H}$, B, Z_g and Z_d must fulfill the properties summarized in the last definition.

Remark 14 *The usual way of presenting the CH method is through the idea of* feedback equivalence *[20]. We shall not explore this point of view here.*

3.2 A particular version

The core of the CH method is Eq. (25), which is a system of PDEs for $\hat{H}$, with unknown "parameters" B and Z_g. These PDEs are usually called *matching conditions*.[3] Different assumptions on the original underactuated system $(H, \mathcal{W})$, and particular ansatzs for the unknowns $\hat{H}$, B and Z_g, give rise to particular forms of (25) and, consequently, to particular versions of the method. (In terms of Definition 11, we have in this way different included methods.) For instance, let us assume that $H, \hat{H} : T^*Q \to \mathbb{R}$ are hyper-regular, i.e. $\mathbb{F}H, \mathbb{F}\hat{H} : T^*Q \to TQ$ are linear bundle isomorphisms [see (12)], and also

$$\mathbb{F}H = \mathbb{F}H^* \quad \text{and} \quad \mathbb{F}\hat{H} = \mathbb{F}\hat{H}^*, \tag{27}$$

i.e. their fiber derivatives are symmetric. In addition, fix a torsion-free connection on T^*Q and assume that [recall Eqs. (8) and (17)]

$$\beta \circ B^{\sharp} \circ \widetilde{\beta}^{-1}(\alpha \oplus \sigma \oplus v) = \alpha \oplus \Psi(v) \oplus \Psi^*(-\sigma), \tag{28}$$

for some fiber bundle morphism $\Psi : TQ \to TQ$ [compare to Eq. (20)].

Remark 15 *Note that, according to* (10)*, each $\mathcal{W}_\alpha \subset T_\alpha T^*Q$ is defined by the unique subspace $W_\alpha \subset T^*_{\pi(\alpha)}Q$ such that*

$$\mathcal{W}_\alpha = \text{vlift}_\alpha(W_\alpha) = \beta^{-1}(\alpha \oplus 0 \oplus W_\alpha). \tag{29}$$

[3] The rest of the equations, the conditions 4, 5 and 6, define algebraic conditions for Z_d that will be studied later.

Also [see Eq. (11)*], the vertical vector field Z_g can be written*

$$Z_g(\alpha) = \beta^{-1}(\alpha \oplus 0 \oplus z_g(\alpha)), \tag{30}$$

for a unique fiber preserving map $z_g : T^*Q \to T^*Q$. (Idem Z_d.)

Proposition 16 *Under above assumptions and notation, the matching conditions* (25) *reduce to [see Eq.* (30)*]*

$$\left\langle \mathbb{B}\hat{H}(\alpha), \mathbb{F}H(\sigma) \right\rangle - \left\langle \mathbb{B}H(\alpha), \mathbb{F}\hat{H}(\sigma) \right\rangle - \left\langle z_g(\alpha), \mathbb{F}\hat{H}(\sigma) \right\rangle = 0, \quad \forall \sigma \in \hat{W}_\alpha, \tag{31}$$

where [see Eq. (29)*]*

$$\hat{W}_\alpha := \left(\mathbb{F}\hat{H}(W_\alpha)\right)^0 = \mathbb{F}\hat{H}^{-1}\left(W_\alpha^0\right), \quad \forall \alpha \in T^*Q. \tag{32}$$

In particular, the unknowns are $\hat{H}$ and z_g ***only****.*

Proof. From Eqs. (18) and (28), we have that

$$\begin{aligned}\beta\left(\hat{X}_{\hat{H}}(\alpha)\right) &= \beta \circ B^\sharp\left(d\hat{H}(\alpha)\right) = \beta \circ B^\sharp \circ \widetilde{\beta}^{-1}\left(\widetilde{\beta}\left(d\hat{H}(\alpha)\right)\right) = \beta \circ B^\sharp \circ \widetilde{\beta}^{-1}\left(\alpha \oplus \mathbb{B}\hat{H}(\alpha) \oplus \mathbb{F}\hat{H}(\alpha)\right) \\ &= \alpha \oplus \Psi\left(\mathbb{F}\hat{H}(\alpha)\right) \oplus \Psi^*\left(-\mathbb{B}\hat{H}(\alpha)\right).\end{aligned} \tag{33}$$

Similarly, from Eqs. (18) and (20),

$$\beta(X_H(\alpha)) = \alpha \oplus \mathbb{F}H(\alpha) \oplus (-\mathbb{B}H(\alpha)). \tag{34}$$

As a consequence, using Equations (29), (30), (33) and (34), it easily follows that (25) reduces to

$$\Psi\left(\mathbb{F}\hat{H}(\alpha)\right) = \mathbb{F}H(\alpha) \quad \text{and} \quad -\Psi^*\left(\mathbb{B}\hat{H}(\alpha)\right) + z_g(\alpha) + \mathbb{B}H(\alpha) \in W_\alpha,$$

for all $\alpha \in T^*Q$. This implies that $\Psi = \mathbb{F}H \circ \mathbb{F}\hat{H}^{-1}$ and, taking Eq. (27) into account,

$$-\mathbb{F}\hat{H}^{-1} \circ \mathbb{F}H \circ \mathbb{B}\hat{H}(\alpha) + z_g(\alpha) + \mathbb{B}H(\alpha) \in W_\alpha.$$

It only rests to use the Eq. (32) in order to end the proof. □

Remark 17 *It is clear from* (34) *that $\alpha^\bullet$ is a critical point of X_H if and only if $\mathbb{F}H(\alpha^\bullet) = 0$ and $\mathbb{B}H(\alpha^\bullet) = 0$.*

Let us mention that, according to Eqs. (11) and (26), each control law Y of the method is now given by the vertical lift of the fiber preserving map

$$y(\alpha) := z_d(\alpha) + z_g(\alpha) - \mathbb{F}\hat{H}^{-1} \circ \left(\mathbb{F}H \circ \mathbb{B}\hat{H}(\alpha) - \mathbb{F}\hat{H} \circ \mathbb{B}H(\alpha)\right). \tag{35}$$

Also,

$$\left\langle z_g(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle = 0 \tag{36}$$

from point 2 above (the gyroscopic condition),

$$\left\langle z_d(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle \leq 0 \quad \text{and} \quad z_d(\alpha) \in W_\alpha \tag{37}$$

according to points 4 and 5, and

$$z_d(\alpha^\bullet) = -z_g(\alpha^\bullet) \tag{38}$$

from point 6.

3.3 The *kinetic* and *potential* matching conditions

Let us further restrict the original underactuated system $(H, \mathcal{W})$ and the unknowns $\hat{H}$ and z_g of (31). Assume first that $H : T^*Q \to \mathbb{R}$ is a *simple* Hamiltonian function, i.e.

$$H = \mathfrak{H} + h \circ \pi \quad \text{with} \quad \mathfrak{H}(\alpha) := \frac{1}{2} \left\langle \alpha, \rho^\sharp(\alpha) \right\rangle, \quad \forall \alpha \in T^*Q,$$

where $h \in C^\infty(Q)$ and ρ is a Riemannian metric on Q. The first and second terms in H are known as the *kinetic* and *potential* terms. Note that $\mathfrak{H}$ is the quadratic form of the tensor

$$\mathfrak{h} : T^*Q \times T^*Q \to \mathbb{R} : (\alpha, \sigma) \longmapsto \frac{1}{2} \left\langle \alpha, \rho^\sharp(\sigma) \right\rangle.$$

Also note that $\mathbb{F}H = \mathbb{F}\mathfrak{H} = \rho^\sharp$ [for the first equality, recall the Eq. (16) of Proposition 3]. This implies that $\mathbb{F}H$ is a symmetric linear bundle isomorphism. Choosing a coordinate chart (U, φ) on Q and its induced one on T^*Q [see (1) and (2)], i.e. choosing canonical coordinates, and denoting by $\mathbb{H}(\mathbf{q})$ the coordinate matrix representation at q of the Riemannian metric ρ, we can write

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} p_i \, \mathbb{H}^{ij}(\mathbf{q}) \, p_j + h(\mathbf{q}). \tag{39}$$

Of course, the symmetric condition $\mathbb{F}H = \mathbb{F}H^*$ translates to $\mathbb{H}$ as $\mathbb{H}^{ij} = \mathbb{H}^{ji}$.

Remark 18 *As it is well-known, for Hamiltonian systems defined by a simple function, the critical points are of the form* $\alpha^\bullet = (q^\bullet, 0)$, *with* $dh(q^\bullet) = 0$ *(use Proposition 3 and Remark 17).*

Regarding the unknowns of (31), assume that $\hat{H}$ is simple too. We shall use for $\hat{H}$ an analogue notation to that we used for H. For instance, we shall write $\hat{H} = \hat{\mathfrak{H}} + \hat{h} \circ \pi$. Thus, $\mathbb{F}\hat{H} = \mathbb{F}\hat{\mathfrak{H}}$. Also, assume that $z_g : T^*Q \to T^*Q$ is given by

$$\langle z_g(\alpha), v \rangle = \mathfrak{Z}_g \left(\alpha, \alpha, \mathbb{F}\hat{\mathfrak{H}}^{-1}(v) \right), \quad \forall q \in Q, \ \alpha \in T_q^*Q, \ v \in T_qQ, \tag{40}$$

for some tensor field $\mathfrak{Z}_g : T^*Q \times T^*Q \times T^*Q \to \mathbb{R}$. This particular choice for the map z_g implies that it is quadratic in α. Note that the tensor field $\mathfrak{Z}_g$ can be assumed symmetric in its first two arguments, i.e.

$$\mathfrak{Z}_g(\alpha_1, \alpha_2, \alpha) = \mathfrak{Z}_g(\alpha_2, \alpha_1, \alpha). \tag{41}$$

In addition, using the gyroscopic condition [see Eq. (36)], it is clear that

$$\mathfrak{Z}_g(\alpha, \alpha, \alpha) = 0, \tag{42}$$

for all $\alpha \in T_q^*Q$. Reciprocally, any tensor satisfying (42) gives rise, through (30) and (40), to a gyroscopic force.

Coming back to the matching conditions, since the kinetic terms of H and $\hat{H}$ are quadratic functions and their potential terms are basic functions, the next result can be easily proved.

Proposition 19 *Under above assumptions and notation, the matching conditions* (31) *decompose into two equations:*

$$\left\langle \mathbb{B}\hat{\mathfrak{H}}(\alpha), \mathbb{F}\mathfrak{H}(\sigma) \right\rangle - \left\langle \mathbb{B}\mathfrak{H}(\alpha), \mathbb{F}\hat{\mathfrak{H}}(\sigma) \right\rangle - \mathfrak{Z}_g(\alpha, \alpha, \sigma) = 0, \tag{43}$$

the kinetic matching conditions*, and*

$$\left\langle d\hat{h}(\pi(\sigma)), \mathbb{F}\mathfrak{H}(\sigma) \right\rangle - \left\langle dh(\pi(\sigma)), \mathbb{F}\hat{\mathfrak{H}}(\sigma) \right\rangle = 0, \tag{44}$$

the potential matching conditions*. They must be satisfied for all*

$$\alpha \in T^*Q \quad \text{and} \quad \sigma \in \hat{W}_\alpha. \tag{45}$$

Remark 20 *Note that [see Eq. (32)] $\hat{W}_\alpha = \mathbb{F}\hat{\mathfrak{H}}^{-1}\left(W_\alpha^0\right)$ is the orthogonal complement of W_α w.r.t. the bilinear $\hat{\mathfrak{b}}$.*

In local coordinates (U,φ), combining (14), (15) and (39),

$$\begin{aligned}\left\langle \mathbb{B}\hat{\mathfrak{H}}\left(\alpha\right), \mathbb{F}\mathfrak{H}\left(\sigma\right)\right\rangle - \left\langle \mathbb{B}\mathfrak{H}\left(\alpha\right), \mathbb{F}\hat{\mathfrak{H}}\left(\sigma\right)\right\rangle = \frac{1}{2}\left(\frac{\partial \hat{\mathbb{H}}^{ij}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial \mathbb{H}^{ij}(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_i p_j \widetilde{p}_l \\ + \Gamma^j_{ks}(\mathbf{q})\,\left(\hat{\mathbb{H}}^{is}(\mathbf{q})\,\mathbb{H}^{kl}(\mathbf{q}) - \mathbb{H}^{is}(\mathbf{q})\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_i p_j \widetilde{p}_l\end{aligned} \tag{46}$$

and

$$\left\langle d\hat{h}\left(\pi\left(\sigma\right)\right), \mathbb{F}\mathfrak{H}\left(\sigma\right)\right\rangle - \left\langle dh\left(\pi\left(\sigma\right)\right), \mathbb{F}\hat{\mathfrak{H}}\left(\sigma\right)\right\rangle = \left(\frac{\partial \hat{h}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial h(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right)\widetilde{p}_l, \tag{47}$$

for $\alpha = \varphi^*\left(\mathbf{q},\mathbf{p}\right)$ and $\sigma = \varphi^*\left(\mathbf{q},\widetilde{\mathbf{p}}\right)$. Thus, Eqs. (43) and (44) translate to

$$\begin{aligned}\left(\frac{\partial \hat{\mathbb{H}}^{ij}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial \mathbb{H}^{ij}(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q}) - 2\,\mathfrak{z}_g^{ijl}(\mathbf{q})\right) p_i p_j \widetilde{p}_l \\ +2\,\Gamma^j_{ks}(\mathbf{q})\,\left(\hat{\mathbb{H}}^{is}(\mathbf{q})\,\mathbb{H}^{kl}(\mathbf{q}) - \mathbb{H}^{is}(\mathbf{q})\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_i p_j \widetilde{p}_l = 0\end{aligned} \tag{48}$$

and

$$\left(\frac{\partial \hat{h}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial h(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right)\widetilde{p}_l = 0,$$

for all $q \in U$, $\mathbf{p} \in \left(\varphi_q^*\right)^{-1}\left(T_q^*Q\right)$ and $\widetilde{\mathbf{p}} \in \left(\varphi_q^*\right)^{-1}\left(\hat{W}_{\varphi_q^*(\mathbf{p})}\right)$. (Of course, the numbers $\mathfrak{z}_g^{ijl}(\mathbf{q})$ are the coefficients of the coordinate matrix representation of $\mathfrak{z}_g$ at q).

Equations (43), (44) and (45) define a generalized version of the traditional IDA-PBC method [37], also known as the *energy plus force shaping method*. In the following subsection, we shall make two more assumptions that will drive us to another set of matching conditions, originally[4] studied by Chang in [18, 19] (see also [16, 17] for the Lagrangian counterpart).

To end this section, note that under all above assumptions [see Eq. (35)]

$$\begin{aligned}y\left(\alpha\right) &= z_d\left(\alpha\right) + \mathfrak{z}_g\left(\alpha,\alpha,\mathbb{F}\hat{H}^{-1}\left(\cdot\right)\right) - \mathbb{F}\hat{H}^{-1}\circ\left(\mathbb{F}H\circ\mathbb{B}\hat{H}\left(\alpha\right) - \mathbb{F}\hat{H}\circ\mathbb{B}H\left(\alpha\right)\right) \\ &= z_d\left(\alpha\right) + \mathfrak{z}_g\left(\alpha,\alpha,\mathbb{F}\hat{\mathfrak{H}}^{-1}\left(\cdot\right)\right) - \mathbb{F}\hat{\mathfrak{H}}^{-1}\circ\left(\mathbb{F}\mathfrak{H}\circ\mathbb{B}\hat{\mathfrak{H}}\left(\alpha\right) - \mathbb{F}\hat{\mathfrak{H}}\circ\mathbb{B}\mathfrak{H}\left(\alpha\right)\right) \\ &\quad - \mathbb{F}\hat{\mathfrak{H}}^{-1}\circ\left(\mathbb{F}\mathfrak{H}\circ d\hat{h}\left(\pi\left(\alpha\right)\right) - \mathbb{F}\hat{\mathfrak{H}}\circ dh\left(\pi\left(\alpha\right)\right)\right).\end{aligned} \tag{49}$$

3.4 *Simple* matching conditions

First, assume that there exists a subbundle $W \subset T^*Q$ such that [recall Eq. (29)]

$$\mathcal{W}_\alpha = \mathrm{vlift}_\alpha\left(W_{\pi(\alpha)}\right), \quad \forall \alpha \in T^*Q. \tag{50}$$

Following (32), let us define another subbundle of T^*Q,

$$\hat{W} := \left(\mathbb{F}\hat{\mathfrak{H}}\left(W\right)\right)^0 = \mathbb{F}\hat{\mathfrak{H}}^{-1}\left(W^0\right). \tag{51}$$

Remark 21 *Note that, according to Remark 20, $\hat{W}$ is the orthogonal complement of W w.r.t. to the tensor $\hat{\mathfrak{b}}$. In particular, we have that $T^*Q = W \oplus \hat{W}$.*

[4] Actually, such matching conditions first appeared in [27], but without a derivation.

Under these assumptions, Chang showed in [16] that there exists a solution $\left(\hat{\mathfrak{H}}, \mathfrak{Z}_g\right)$ of (43) if and only if $\hat{\mathfrak{H}}$ satisfies

$$\left\langle \mathbb{B}\hat{\mathfrak{H}}(\sigma), \mathbb{F}\mathfrak{H}(\sigma)\right\rangle - \left\langle \mathbb{B}\mathfrak{H}(\sigma), \mathbb{F}\hat{\mathfrak{H}}(\sigma)\right\rangle = 0, \quad \forall \sigma \in \hat{W}.$$

Moreover, it can be shown by using elementary tensor algebra that:

Proposition 22 $\left(\hat{\mathfrak{H}}, \mathfrak{Z}_g\right)$ *is a solution of* (43) *if and only if* $\hat{\mathfrak{H}}$ *satisfies the above equation and* $\mathfrak{Z}_g$ *is given as follows:*

1. *define* $\Upsilon : T^*Q \times T^*Q \times T^*Q \to \mathbb{R}$ *as*

$$\Upsilon(\alpha_1, \alpha_2, \alpha_3) := \left\langle \mathbb{B}\hat{\mathfrak{b}}(\alpha_1, \alpha_2), \mathbb{F}\mathfrak{H}(\alpha_3)\right\rangle - \left\langle \mathbb{B}\mathfrak{b}(\alpha_1, \alpha_2), \mathbb{F}\hat{\mathfrak{H}}(\alpha_3)\right\rangle; \tag{52}$$

2. *fix a tensor* $A : W \times W \times \hat{W} \to \mathbb{R}$ *satisfying*

$$A(\alpha_1, \alpha_2, \alpha) = -A(\alpha_2, \alpha_1, \alpha), \quad \forall \alpha_1, \alpha_2 \in W,\ \alpha \in \hat{W},$$

and a tensor $B : W \times W \times W \to \mathbb{R}$ *satisfying* (41) *and* (42) *along* W*;*

3. *and finally define* $\mathfrak{Z}_g : T^*Q \times T^*Q \times T^*Q \to \mathbb{R}$ *as*[5]

$$\begin{cases} \mathfrak{Z}_g(\alpha_1, \alpha_2, \sigma) := \Upsilon(\alpha_1, \alpha_2, \sigma), \quad \mathfrak{Z}_g(\sigma_1, \sigma_2, \gamma) := -\Upsilon(\gamma, \sigma_2, \sigma_1) - \Upsilon(\gamma, \sigma_1, \sigma_2), \\ \mathfrak{Z}_g(\gamma_1, \sigma, \gamma_2) := \mathfrak{Z}_g(\sigma, \gamma_1, \gamma_2) := -\frac{1}{2}\left[\Upsilon(\gamma_1, \gamma_2, \sigma) + A(\gamma_1, \gamma_2, \sigma)\right], \\ \mathfrak{Z}_g(\gamma_1, \gamma_2, \gamma_3) := B(\gamma_1, \gamma_2, \gamma_3), \end{cases} \tag{53}$$

with $\alpha_i \in T^*Q$*,* $\sigma, \sigma_i \in \hat{W}$ *and* $\gamma, \gamma_i \in W$*.*

Then, a solution $\left(\hat{\mathfrak{H}}, \hat{h}, \mathfrak{Z}_g\right)$ of (43), (44) and (45) can be found if and only if we solve the equations

$$\left\langle \mathbb{B}\hat{\mathfrak{H}}(\sigma), \mathbb{F}\mathfrak{H}(\sigma)\right\rangle - \left\langle \mathbb{B}\mathfrak{H}(\sigma), \mathbb{F}\hat{\mathfrak{H}}(\sigma)\right\rangle = 0, \tag{54}$$

$$\left\langle d\hat{h}(\pi(\sigma)), \mathbb{F}\mathfrak{H}(\sigma)\right\rangle - \left\langle dh(\pi(\sigma)), \mathbb{F}\hat{\mathfrak{H}}(\sigma)\right\rangle = 0, \quad \forall \sigma \in \hat{W}, \tag{55}$$

for $\hat{\mathfrak{H}}$ and $\hat{h}$ **only**. These are the new matching conditions that we mentioned above, which we shall call the *Chang's* or *simple matching conditions.* The local counterpart reads

$$\left(\frac{\partial \hat{\mathbb{H}}^{ij}(\mathbf{q})}{\partial q^k}\, \mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial \mathbb{H}^{ij}(\mathbf{q})}{\partial q^k}\, \hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_i p_j p_l = 0 \tag{56}$$

[see Eq. (46) for $\alpha = \sigma$ and use the torsion-free condition] and

$$\left(\frac{\partial \hat{h}(\mathbf{q})}{\partial q^k}\, \mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial h(\mathbf{q})}{\partial q^k}\, \hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_l = 0 \tag{57}$$

[see Eq. (47)], for all $q \in U$ and $\mathbf{p} \in \left(\varphi_q^*\right)^{-1}\left(\hat{W}_q\right)$, or equivalently [see (51)] $\varphi_{*,q}^{-1}\left(\hat{\mathbb{H}}(\mathbf{q}) \cdot \mathbf{p}\right) \in W_q^0$.

Remark 23 *Above equations are (up to a sign) Eqs.* (2.21) *and* (2.29) *of [19] for*

$$M^{ij} = \mathbb{H}^{ij}, \quad \widehat{M}^{ij} = \hat{\mathbb{H}}^{ij}, \quad V = h, \quad \widehat{V} = \hat{h}, \quad \text{and} \quad G^{\perp} = W^0.$$

Now, let us study the Eqs. (37) and (38) in the present situation.

[5] Chang made a similar construction to show the existence of solutions of (43) (see Eqs. 2.8 to 2.11 in Ref. [19], and replace Υ and $\mathfrak{Z}_g$ by S and C, respectively), but taking $A = 0$ and $B = 0$.

Remark 24 *If* $\alpha^{\bullet} = (q^{\bullet}, 0)$ *is a critical point for* X_H*, since* $\mathfrak{Z}_g\left(0, 0, \mathbb{F}\hat{\mathfrak{H}}^{-1}(\cdot)\right) = 0$*, condition* (38) *for* z_d *reduces to* $z_d(\alpha^{\bullet}) = 0$.

According to the last remark and using that

$$\left\langle z_d(\alpha), \mathbb{F}\hat{H}(\alpha)\right\rangle = \left\langle z_d(\alpha), \mathbb{F}\hat{\mathfrak{H}}(\alpha)\right\rangle = \hat{\mathfrak{b}}(z_d(\alpha), \alpha),$$

such equations can be written

$$\hat{\mathfrak{b}}(z_d(\alpha), \alpha) \leq 0, \quad z_d(\alpha) \in W \quad \text{and} \quad z_d(\alpha^{\bullet}) = 0. \tag{58}$$

Suppose that we have a solution z_d of (58), and consider the orthogonal projection P with image W (see Remark 21). Then,

$$\hat{\mathfrak{b}}(z_d(\alpha), \alpha) = \hat{\mathfrak{b}}(z_d(\alpha), P(\alpha)) = -\mu(\alpha)$$

for some non-negative function $\mu : T^*Q \to \mathbb{R}$ such that $\mu(\sigma) = 0$ for all $\sigma \in \hat{W}$. Fixing $\alpha_0 \notin \hat{W}$, i.e. $P(\alpha_0) \neq 0$, and using elementary linear algebra, it follows that

$$z_d(\alpha_0) = x_0 - \frac{\hat{\mathfrak{b}}(x_0, P(\alpha_0)) + \mu(\alpha_0)}{\hat{\mathfrak{b}}(P(\alpha_0), P(\alpha_0))} P(\alpha_0), \quad \text{for some} \quad x_0 \in W.$$

In addition, since the complementary subset of $\hat{W}$ in T^*Q is an open dense submanifold, this means that z_d must be given by the expression

$$x(\alpha) - \frac{\hat{\mathfrak{b}}(x(\alpha), P(\alpha)) + \mu(\alpha)}{\hat{\mathfrak{b}}(P(\alpha), P(\alpha))} P(\alpha), \quad \forall \alpha \notin \hat{W}, \tag{59}$$

for some fiber preserving map $x : T^*Q \to T^*Q$ with image inside W. And since $z_d(\alpha^{\bullet}) = 0$,

$$\operatorname*{Lim}_{\alpha \to \alpha^{\bullet}} \left(x(\alpha) - \frac{\hat{\mathfrak{b}}(x(\alpha), P(\alpha)) + \mu(\alpha)}{\hat{\mathfrak{b}}(P(\alpha), P(\alpha))} P(\alpha) \right) = 0. \tag{60}$$

Reciprocally, it is easy to see that, if z_d is a smooth map given by the formula (59) and satisfies (60), then it satisfies (58). Concluding,

Proposition 25 *If we:*

i. *fix a non-negative function* $\mu : T^*Q \to \mathbb{R}$ *such that* $\mu(\sigma) = 0$ *for all* $\sigma \in \hat{W}$*;*

ii. *fix a fiber preserving map* $x : T^*Q \to T^*Q$ *with image inside* W *and such that the formula* (59)*:*

- **(a)** *defines a smooth application on all of* T^*Q*,*
- **(b)** *satisfies* (60)*;*

iii. *define* $z_d : T^*Q \to T^*Q$ *by the formula* (59)*;*

then we have a solution of (58)*. Moreover, every solution of* (58) *can be constructed in this way.*

For instance, we can take

$$\mu(\alpha) := \hat{\mathfrak{b}}(P(\alpha), P(\alpha)) \quad \text{and} \quad x(\alpha) := -\mu(\alpha)\, \xi(\pi(\alpha)),$$

being $\xi : Q \to T^*Q$ any 1-form on Q with image inside W (this is possible, since W is a linear subbundle).

Summing up, we have a new method for solving the problem **P**. Assume that a connection was already chosen on T^*Q.

Definition 26 *Given an underactuated system* $(H,\mathcal{W})$*, with* $H=\mathfrak{H}+h\circ\pi$ *simple and* $\mathcal{W}$ *defined by a subbundle* W *[see Eq.* (50)*], and given a critical point* $\alpha^{\bullet}\in T^{*}Q$ *of* X_{H}*, the* ***simple CH method*** *consists in:*

- *finding a solution* $\hat{H}=\hat{\mathfrak{H}}+\hat{h}\circ\pi$ *of Eqs.* (54) *and* (55)*, with* $\hat{H}$ *positive definite w.r.t.* $\alpha^{\bullet}$*;*
- *fixing a tensor* $\mathfrak{Z}_{g}:T^{*}Q\times T^{*}Q\times T^{*}Q\rightarrow\mathbb{R}$ *through the steps* 1 *to* 3 *above;*
- *fixing a fiber preserving map* $z_{d}:T^{*}Q\rightarrow T^{*}Q$ *through the steps i to iii above;*
- *and defining a fiber preserving map* $y:T^{*}Q\rightarrow T^{*}Q$ *as in* (49) *and* $Y\in\mathfrak{X}(T^{*}Q)$ *as the vertical lift of* y*.*

Of course, the simple CH method is Lyapunov based. In particular, $\hat{H}$ and Y satisfy (24). And it is easy to show that the simple CH method is included (in the sense of Definition 11) in the (general) CH method (see Definition 13).

Coming back to the new matching conditions (54) and (55) [with local versions (56) and (57)], the improvement or simplification accomplished by Chang [w.r.t. to the matching conditions (43) and (44)] is two-fold. On the one hand, the three unknown $\hat{\mathfrak{H}}$, $\hat{h}$ and $\mathfrak{Z}_{g}$ in (43) and (44) have been *decoupled*.[6] On the other hand, the Eq. (48) [the local version of (43)] has been replaced by the Eq. (56), a too much simple set of equations. It is simpler not only because of the form, but also because of the number of equations that contains. In fact, it can be shown that the number of equations in (48) and (56) are, respectively,

$$\frac{n\ (n+1)\ (n-m)}{2} \tag{61}$$

and

$$\frac{(n-m+2)\ (n-m+1)\ (n-m)}{6}, \tag{62}$$

being $n:=\dim Q$ and m the rank of W.

Remark 27 *Regarding the last improvement, it was shown in Reference [21] that, even for the traditional IDA-PBC method (where the unknowns* $\mathfrak{Z}_{g}^{ijk}$ *adopt a particular form), the number of equations in* (48) *can also be reduced from* (61) *to* (62) *by using the freedom one has in choosing each function* $\mathfrak{Z}_{g}^{ijk}$*. Thus, the main contribution of Chang in that respect, perhaps, was not to reduce the number of equations, but to give a precise, simple and useful prescription to do that.*

Equations (56) and (57) [and consequently (54) and (55)] were independently obtained in [25] [see Equations (67) and (68) of [25] for $\mathbb{V}^{ij}=\hat{\mathbb{H}}^{ij}$ and $v=\hat{h}$], almost simultaneously with the paper of Chang [16], in the context of the so-called *Lyapunov constraint based (LCB) method* for underactuated systems with only one actuator. We will see in the last section of this paper that the same equations are obtained (in the mentioned context) for an arbitrary number of actuators.

4 The LCB method

In this section we extend the Lyapunov constraint based method for the stabilization of underactuated systems, originally presented in [23] (and then further developed in [25]) for systems with one degree of actuation, to systems with an arbitrary number of actuators. To do that, we firstly recall, within the Hamiltonian framework, the idea of

[6] More precisely, we can firstly find $\hat{\mathfrak{H}}$ by solving (54), then, for such a solution $\hat{\mathfrak{H}}$, we can find $\hat{h}$ by solving (55), and finally, using $\hat{\mathfrak{H}}$ and $\hat{h}$, we can construct $\mathfrak{Z}_{g}$ by following the steps 1 to 3 listed above.

controlling underactuated mechanical systems by imposing kinematic constraints [32, 33, 43] (see also [22, 38, 39] for further examples), and the deep relationship between constrained and closed-loop mechanical systems. It is worth mentioning that we shall focus on a Hamiltonian formulation of the method, although a Lagrangian one is equally possible.

4.1 Second order constraints and closed-loop systems

Following [23], a *second order constrained system* (SOCS) on Q is a triple $(H, \mathcal{P}, \mathcal{W})$ where

1. $H : T^*Q \to \mathbb{R}$ is a smooth function defining an (unconstrained) Hamiltonian system,
2. $\mathcal{P} \subset TT^*Q$ is a submanifold defining the *second order kinetic constraints*[7] imposed on the system, and
3. $\mathcal{W}$ is a vertical subbundle of the tangent bundle TT^*Q defining the *subspace of constraint forces.*

In this paper, by a trajectory[8] of $(H, \mathcal{P}, \mathcal{W})$ we mean an integral curve $\Gamma : I \subset \mathbb{R} \to T^*Q$ of a vector field $X \in \mathfrak{X}(T^*Q)$ that satisfies

$$X \subset \mathcal{P} \quad \text{and} \quad X - X_H \subset \mathcal{W}. \tag{63}$$

Of course, any trajectory must satisfy $\Gamma'(t) \in \mathcal{P}$, for all $t \in I$. As in the previous section, $X_H \in \mathfrak{X}(T^*Q)$ is the Hamiltonian vector field of H w.r.t. the canonical symplectic form of T^*Q. The vector field $Y := X - X_H$ is called the *constraint force* related to X.

Remark 28 *Note that X is a solution of* (63) *if and only if $Y = X - X_H$ is a solution of*

$$X_H + Y \subset \mathcal{P} \quad \textit{and} \quad Y \subset \mathcal{W}. \tag{64}$$

On the other hand, a *closed-loop mechanical system* (CLMS) is defined in [24] as a dynamical system on Q given by

1. a smooth function $H : T^*Q \to \mathbb{R}$, describing a (non actuated) Hamiltonian system,
2. a vertical subbundle $\mathcal{W} \subset TT^*Q$, representing the *actuation subspace* and defining, together with H, the *underactuated system* $(H, \mathcal{W})$, and
3. a vector field Y on T^*Q such that $Y \subset \mathcal{W}$: the *control law.*

We will denote such a system by $(H, Y)_{\mathcal{W}}$. By a trajectory of $(H, Y)_{\mathcal{W}}$ we mean an integral curve of the vector field $X_H + Y$.

Now, let us see how a SOCS gives rise to a CLMS. Let us suppose that a triple $(H, \mathcal{P}, \mathcal{W})$ defines a SOCS that admits a solution X of (63) (which is unique, for instance, in the case of *normal* SOCSs -see [24]-). Because of Remark 28, this is the same as saying that it admits a solution Y of (64). Then, from the SOCS $(H, \mathcal{P}, \mathcal{W})$ we can define the CLMS $(H, Y)_{\mathcal{W}}$, with $Y := X - X_H$. Note that

- both systems have the same trajectories: the integral curves of the vector field $X = X_H + Y$;
- the role of Y is two fold: a constraint force for $(H, \mathcal{P}, \mathcal{W})$ and a control law for $(H, Y)_{\mathcal{W}}$.

[7]Note that, in canonical coordinates (q, p), the submanifold $\mathcal{P}$ is defined, among other things, by restrictions on $\dot{q}$ and $\dot{p}$. In the Lagrangian formalism, applying the Legendre transformation, this gives rise to restrictions on $\dot{q}$ and $\ddot{q}$. This is why we talk about *second* order constraints.

[8]These curves were called *type III trajectories* in [23], where another types of trajectories were also considered.

This construction tells us that, in order to design a control strategy for controlling a given underactuated system $(H, \mathcal{W})$, we can "think of constraints," i.e. we can think of the possible constraints $\mathcal{P}$ that give rise to the desirable behavior, and then obtain the control law as the related constraint force $Y \subset \mathcal{W}$.

It was shown in [24] that every CLMS can be constructed from a SOCS as we did above, i.e. every control law may be seen as the constraint force of a given set of second order constraints. This result reveals a deep connection between closed-loop and constrained mechanical systems and, from the point of view of the applications to automatic control, the result says that, in order to synthesize a state feedback for a given underactuated system, we **always** (i.e. without loss of generality) can "think of constraints."

4.2 (Asymptotic) stability and related constraints

Let us consider a dynamical system on a manifold P defined by a vector field $X \in \mathfrak{X}(P)$. Given a critical point $\alpha^{\bullet} \in P$ of X, let $\hat{H} : P \to \mathbb{R}$ be a Lyapunov function for X and $\alpha^{\bullet}$ (recall Definition 9). Note that, given a trajectory $\Gamma : I \subset \mathbb{R} \to P$ of X, condition **L2** of Definition 9 implies that

$$\left\langle d\hat{H}(\Gamma(t)), \Gamma'(t) \right\rangle = -\mu(\Gamma(t)), \quad \forall t \in I, \tag{65}$$

where $\mu : P \to \mathbb{R}$ is the non-negative function given by

$$\mu\left(\alpha\right) := -\left\langle d\hat{H}(\alpha), X(\alpha) \right\rangle, \quad \forall \alpha \in P.$$

Remark 29 *Observe that* $\mu^{-1}(0)$ *is the* La'Salle surface *related to* $\hat{H}$ *(see [29]), and* $\alpha^{\bullet} \in \mu^{-1}(0)$.

That is, condition **L2** may be interpreted as a kinematic constraint on the system. Hence, roughly speaking, if we want to stabilize a dynamical system, we can think of imposing a constraint of the form (65), for appropriate non-negative functions $\hat{H}$ and μ. We shall call it **Lyapunov constraint**. Of course, depending on the conditions we impose on $\hat{H}$ and μ, we shall have different stability properties. For instance if, besides condition **L1** for $\hat{H}$, we ask μ to be such that the singleton $\{\alpha^{\bullet}\}$ is the bigger invariant subset of $\mu^{-1}(0)$, the *La'Salle invariance principle* would ensure (local) asymptotic stability for $\alpha^{\bullet}$. This is true, for example, if we assume that property **L1** also holds for μ, what would imply that

$$\left\langle d\hat{H}(\alpha), X(\alpha) \right\rangle < 0 \quad \text{for all} \quad \alpha \neq \alpha^{\bullet}.$$

If in addition we ask $\hat{H}$ to be a proper function (and P to be connected), then global asymptotic stability for $\alpha^{\bullet}$ would be ensured. (For a proof of these results, see [29] again.)

Now, let us focus our attention on Hamiltonian systems. Take $P = T^*Q$ for some Q, fix a smooth function $H : T^*Q \to \mathbb{R}$ and consider the Hamiltonian system on Q defined by H. Given a point $\alpha^{\bullet} \in T^*Q$ and non-negative functions $\hat{H}, \mu : T^*Q \to \mathbb{R}$, let us impose the constraint (65) on this system. In other words, let us define the submanifold

$$\mathcal{P} := \left\{ V \in TT^*Q : \left\langle d\hat{H}(\tau_{T^*Q}(V)), V \right\rangle = -\mu(\tau_{T^*Q}(V)) \right\},$$

and impose the constraint $\Gamma'(t) \in \mathcal{P}$ on the trajectories.

Remark 30 *Notice that, if* (U, φ) *is a coordinate chart of* Q*, in terms of the induced chart on* TT^*Q *(see Eq. (3)) the submanifold* $\mathcal{P}$ *is locally given by the equation*

$$\frac{\partial \hat{H}}{\partial q^i}(\mathbf{q}, \mathbf{p})\, \dot{q}^i + \frac{\partial \hat{H}}{\partial p_i}(\mathbf{q}, \mathbf{p})\, \dot{p}_i = -\mu(\mathbf{q}, \mathbf{p}).$$

Suppose that we want to implement this constraint by exerting forces lying inside a vertical subbundle $\mathcal{W} \subset TT^*Q$. All that defines the SOCS $(H, \mathcal{P}, \mathcal{W})$. Assume that this SOCS admits a solution X of (63), or equivalently, admits a solution Y of (64). (In other words, assume that the Lyapunov constraint $\mathcal{P}$ can be implemented by a constraint force $Y \subset \mathcal{W}$.) This is the same as saying that there exists $Y \in \mathfrak{X}(T^*Q)$ such that

$$\left\langle d\hat{H}(\alpha), X_H(\alpha) + Y(\alpha) \right\rangle = -\mu(\alpha) \quad \text{and} \quad Y(\alpha) \in \mathcal{W}, \tag{66}$$

or equivalently

$$\mathrm{i}_{X_H+Y}\, d\hat{H} = -\mu \quad \text{and} \quad Y \subset \mathcal{W}. \tag{67}$$

Since μ is non-negative, then $\left\langle d\hat{H}(\alpha), X_H(\alpha) + Y(\alpha) \right\rangle \leq 0$, i.e. $\hat{H}$ satisfies **L2**. In addition, if $X_H(\alpha^\bullet) + Y(\alpha^\bullet) = 0$ and $\hat{H}$ satisfies **L1** for $\alpha^\bullet$, then $\hat{H}$ is a Lyapunov function for $X_H + Y$ and $\alpha^\bullet$, and consequently the underactuated system $(H, \mathcal{W})$ can be stabilized at $\alpha^\bullet$ by the control law Y. Of course, if stronger conditions are imposed on $\hat{H}$ and μ (as discussed at the beginning of this section), stronger stability properties can be ensured for the system defined by $X_H + Y$.

Remark 31 *In this way, as we have seen in the previous section, we are constructing the CLMS $(H, Y)_{\mathcal{W}}$ from the SOCS $(H, \mathcal{P}, \mathcal{W})$.*

In conclusion, if a solution Y exists for Equation (66), for some functions $\hat{H}$ and μ, different assertions about the stabilizability around $\alpha^\bullet$ of the underactuated system $(H, \mathcal{W})$ can be made, depending on the properties of $\hat{H}$ and μ.

Remark 32 *Also, if a solution Y of* (66) *exists along an open subset $\pi^{-1}(U) = T^*U \subset T^*Q$ containing $\alpha^\bullet$, namely a* local solution *of* (66)*, the same assertions can be made, just replacing Q by U.*

4.3 A *maximal* stabilization method

The discussion in the previous section drives us to another method for (asymptotic) stabilization of non-linear underactuated mechanical systems.

Definition 33 *Given an underactuated system $(H, \mathcal{W})$ on Q and a critical point $\alpha^\bullet \in T^*Q$ of X_H, the* ***Lyapunov constraint based (LCB) method*** *consists in finding two functions $\hat{H}, \mu : T^*Q \to \mathbb{R}$ and a vector field $Y \in \mathfrak{X}(T^*Q)$ such that $\hat{H}$ is positive definite w.r.t. $\alpha^\bullet$, μ is non-negative, $Y(\alpha^\bullet) = 0$ and Eq.* (66) *is solved.*

Note that the method is a Lyapunov based stabilization method, and it is essentially defined by Eq. (66). So, we can identify the method with this equation. Let us write it in other terms. Since $\mathcal{W}$ is a vertical subbundle and Y is a vertical vector field, we can write $\mathcal{W}_\alpha = \mathrm{vlift}_\alpha(W_\alpha)$ and $Y(\alpha) = \mathrm{vlift}_\alpha(y(\alpha))$, for a unique subspace $W_\alpha \subset T^*_{\pi(\alpha)}Q$ and a unique fiber preserving map $y : T^*Q \to T^*Q$. Under this notation,

$$\left\langle d\hat{H}(\alpha), Y(\alpha) \right\rangle = \left\langle y(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle$$

and conditions $Y(\alpha^\bullet) = 0$ and $Y(\alpha) \in \mathcal{W}_\alpha$ translate to $y(\alpha^\bullet) = 0$ and $y(\alpha) \in W_\alpha$, respectively. On the other hand,

$$\left\langle d\hat{H}(\alpha), X_H(\alpha) \right\rangle = \{\hat{H}, H\}(\alpha),$$

being $\{\hat{H}, H\}$ the canonical Poisson bracket between $\hat{H}$ and H. Combining all that, we can write (66) as

$$\left\langle y(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle = -\mu(\alpha) - \left\{\hat{H}, H\right\}(\alpha) \quad \text{and} \quad y(\alpha) \in W. \tag{68}$$

To conclude the section, let us mention the important fact that any stabilization method for $(H, \mathcal{W})$ which gives rise to a control law $Y \subset \mathcal{W}$ and a Lyapunov function $\hat{H}$ for the related closed-loop system $X_H + Y$ (and some critical point of X_H), as every version of the energy shaping method does, can be reduced to the LCB method, i.e. to solve Eq. (66) [or equivalently, to solve Eq. (68)]. More precisely,

Theorem 34 *Let $(H, \mathcal{W})$ be an underactuated system and $\alpha^\bullet \in T^*Q$ a critical point of X_H. If we are given a vector field $Y \subset \mathcal{W}$ and a Lyapunov function $\hat{H}$ for $X_H + Y$ and $\alpha^\bullet$, then $\hat{H}$ is positive definite w.r.t. $\alpha^\bullet$, $\mu := -\mathfrak{i}_{X_H+Y}\, d\hat{H}$ is non-negative and $Y(\alpha^\bullet) = 0$. In particular, Y is given by the LCB method.*

Proof. Given a vector field $Y \subset \mathcal{W}$, if $\hat{H}$ is a Lyapunov function for $X := X_H + Y$ and $\alpha^\bullet$, the theorem easily follows from the fact that $\alpha^\bullet$ must be critical for X [from which $Y(\alpha^\bullet) = 0$], the item **L1** and the combination of **L2** and the Eq. (67). □

In terms of Definitions 10 and 11, this theorem says that the LCB method includes all the Lyapunov based stabilization methods, i.e. it is ***maximal*** among such methods. In other words, the LCB method is the most general method among the Lyapunov based stabilization methods (see Remark 12). In particular, any version of the CH method (see Definition 13) is included in the LCB method.

5 The LCB method for simple functions

In Ref. [25], a deep study of Eq. (68) has been done for underactuated systems with only one actuator. In this section we shall extend such a study to an arbitrary number of actuators. More precisely, given an underactuated system $(H, \mathcal{W})$ and non-negative functions $\hat{H}$ and μ, we shall study under which conditions there exists a fiber preserving map y solving (68) (thinking of $\hat{H}$ and μ as data of (68), instead of unknowns). We shall focus in the case in which H and $\hat{H}$ are simple functions. In this case, we show that the mentioned existence problem is governed by a set of PDEs for $\hat{H}$, which define what we have called the ***simple*** *LCB method* in Ref. [25]. We shall see that these equations are exactly the matching conditions (54) and (55) obtained by Chang in [18, 19] (related to the simple CH method of Definition 26). Finally, we show that the simple LCB and the simple CH method are equivalent stabilization methods.

5.1 The *kinetic* and *potential* equations

Following the same notation as in Section 3.3, consider two simple Hamiltonian functions $H = \mathfrak{H} + h \circ \pi$ and $\hat{H} = \hat{\mathfrak{H}} + \hat{h} \circ \pi$ on the cotangent bundle T^*Q. Then, given a torsion-free connection on T^*Q, the canonical Poisson bracket between H and $\hat{H}$ can be written, for all $\alpha \in T^*Q$,

$$\left\{\hat{H}, H\right\}(\alpha) = \left\langle \mathbb{B}\hat{\mathfrak{H}}(\alpha), \mathbb{F}\mathfrak{H}(\alpha)\right\rangle - \left\langle \mathbb{B}\mathfrak{H}(\alpha), \mathbb{F}\hat{\mathfrak{H}}(\alpha)\right\rangle + \left\langle d\hat{h}(q), \mathbb{F}\mathfrak{H}(\alpha)\right\rangle - \left\langle dh(q), \mathbb{F}\hat{\mathfrak{H}}(\alpha)\right\rangle,$$

with $q = \pi(\alpha)$. This is a direct consequence of Eqs. (16) and (21). In a local chart (U, φ), using (46) and (47) (and the fact that our connection is torsion-free),

$$\left\{\hat{H}, H\right\}(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(\frac{\partial \hat{\mathbb{H}}^{ij}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial \mathbb{H}^{ij}(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_i p_j p_l + \left(\frac{\partial \hat{h}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial h(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_l. \quad (69)$$

Looking at above expression, the next result easily follows.

Lemma 35 *If H and $\hat{H}$ are simple functions on T^*Q, then $\left\{\hat{H}, H\right\}$ is an odd function (when restricted to each fiber of T^*Q), i.e.*

$$\left\{\hat{H}, H\right\}(-\alpha) = -\left\{\hat{H}, H\right\}(\alpha) \qquad \textit{for all} \quad \alpha \in T^*Q. \quad (70)$$

Now, fix an underactuated system $(H, \mathcal{W})$ with H simple and $\mathcal{W}$ given by a subbundle $W \subset T^*Q$ [see (50)].

Proposition 36 *Consider a simple function $\hat{H} : T^*Q \to \mathbb{R}$ and a non-negative function $\mu : T^*Q \to \mathbb{R}$. If there exists a solution $y : T^*Q \to T^*Q$ of Eq.* (68) *for $\hat{H}$ and μ, then*

$$\left\{\hat{H}, H\right\}(\sigma) = 0, \qquad \forall \sigma \in \hat{W}, \tag{71}$$

where $\hat{W} := \left(\mathbb{F}\hat{\mathfrak{H}}(W)\right)^0 = \mathbb{F}\hat{\mathfrak{H}}^{-1}\left(W^0\right)$.

Proof. By hypothesis [see Eq. (68)]

$$\left\langle \alpha, \mathbb{F}\hat{\mathfrak{H}}(y(\alpha))\right\rangle = -\mu(\alpha) - \{\hat{H}, H\}(\alpha) \quad \text{and} \quad y(\alpha) \in W.$$

It is clear that $\left\langle \sigma, \mathbb{F}\hat{\mathfrak{H}}(y(\sigma))\right\rangle = 0$ for all $\sigma \in \hat{W}$, and consequently

$$\mu(\sigma) + \{\hat{H}, H\}(\sigma) = 0, \qquad \forall \sigma \in \hat{W}.$$

Suppose that for some $\sigma_0 \in \hat{W}$ we have that $\{\hat{H}, H\}(\sigma_0) \neq 0$. If $\{\hat{H}, H\}(\sigma_0) > 0$, then $\mu(\sigma_0) < 0$. But this is not possible, so $\{\hat{H}, H\}(\sigma_0) < 0$. According to Eq. (70),

$$\{\hat{H}, H\}(-\sigma_0) = -\{\hat{H}, H\}(\sigma_0) > 0.$$

Since $-\sigma_0 \in \hat{W}$, it follows that $\mu(-\sigma_0) + \{\hat{H}, H\}(-\sigma_0) = 0$, what implies $\mu(-\sigma_0) < 0$. As a consequence, $\{\hat{H}, H\}(\sigma) = 0$ for all $\sigma \in \hat{W}$. □

We can write (71) in an equivalent way.

Lemma 37 *If H and $\hat{H}$ are simple functions on T^*Q and $V \subset T^*Q$ is a linear subbundle, the following conditions are equivalent.*

1. $\left\{\hat{H}, H\right\}(\sigma) = 0$, $\forall \sigma \in V$.
2. *Given a connection on T^*Q, for all $q \in Q$ and $\sigma \in V_q$, the Eqs.* (54) *and* (55) *hold, with $\hat{W}$ replaced by V.*
3. *Given a local chart (U, φ), for all $q \in U$ and $\mathbf{p} \in \left(\varphi_q^*\right)^{-1}(V_q)$, the Eqs.* (56) *and* (57) *hold.*

Proof. The equivalence between 2 and 3 is given by the Eqs. (46) and (47). We only need to prove that 1 implies 3 (the converse is immediate). If $\{\hat{H}, H\}(\sigma) = 0$, $\forall \sigma \in V$, it follows from Equation (69) that, in a local chart (U, φ),

$$\frac{1}{2}\left(\frac{\partial \hat{\mathbb{H}}^{ij}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial \mathbb{H}^{ij}(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_i p_j p_l + \left(\frac{\partial \hat{h}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial h(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) p_l = 0, \tag{72}$$

for all $q \in U$ and $\mathbf{p} \in \left(\varphi_q^*\right)^{-1}(V_q)$. We must show that each term vanishes. Fixing $\widehat{\mathbf{p}} \in \left(\varphi_q^*\right)^{-1}(V_q)$, define

$$A := \left(\frac{\partial \hat{\mathbb{H}}^{ij}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial \mathbb{H}^{ij}(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) \widehat{p}_i \widehat{p}_j \widehat{p}_l \quad \text{and} \quad B := \left(\frac{\partial \hat{h}(\mathbf{q})}{\partial q^k}\,\mathbb{H}^{kl}(\mathbf{q}) - \frac{\partial h(\mathbf{q})}{\partial q^k}\,\hat{\mathbb{H}}^{kl}(\mathbf{q})\right) \widehat{p}_l.$$

Then, if we replace $\mathbf{p}$ by $\lambda\,\widehat{\mathbf{p}}$ in Eq. (72), with $\lambda \in \mathbb{R}$, we have that $A\,\lambda^3/2 + B\,\lambda = 0$ for all $\lambda \in \mathbb{R}$. But this is possible only if $A = B = 0$. □

Combining last lemma and Proposition 36, we have the following result.

Proposition 38 *Consider a simple function $\hat{H} : T^*Q \to \mathbb{R}$ and a non-negative function $\mu : T^*Q \to \mathbb{R}$. If there exists a solution $y : T^*Q \to T^*Q$ of Eq.* (68) *for $\hat{H}$ and μ, then, for every connection on T^*Q,*

$$\left\langle \mathbb{B}\hat{\mathfrak{H}}(\sigma), \mathbb{F}\mathfrak{H}(\sigma) \right\rangle - \left\langle \mathbb{B}\mathfrak{H}(\sigma), \mathbb{F}\hat{\mathfrak{H}}(\sigma) \right\rangle = 0, \tag{73}$$

$$\left\langle d\hat{h}(\pi(\sigma)), \mathbb{F}\mathfrak{H}(\sigma) \right\rangle - \left\langle dh(\pi(\sigma)), \mathbb{F}\hat{\mathfrak{H}}(\sigma) \right\rangle = 0, \quad \forall \sigma \in \hat{W}. \tag{74}$$

Notice that Equations (73) and (74) are exactly the *simple* matching conditions (54) and (55) derived by Chang. It is quite surprising for us that these two methods, the LCB and the simple CH methods, give rise to the same set of equations, in spite of they arise from very different ideas: "feedback equivalence" and "controlling by imposing kinematic constraints."

Remark 39 *This means, using again Lemma 37, that the simple matching conditions can be written as in* (71)*, or equivalently*

$$\left\{\hat{H}, H\right\} \circ \mathbb{F}\hat{H}^{-1}(v) = 0, \qquad \forall v \in W^0,$$

which is a coordinate-free and connection-free equation.

Propositions 36 and 38 say that Equation (71), or equivalently Equations (73) and (74), which have a function $\hat{H} = \hat{\mathfrak{H}} + \hat{h} \circ \pi$ as unknown, define necessary conditions for the existence of a solution of (66) for some non-negative function μ. Let us see that they also define sufficient conditions. We saw in Section 3.4 that if a simple function $\hat{H}$ satisfies (73) and (74), then there exists a vector field $Y \subset \mathcal{W}$ satisfying (24). (In fact, Y can be constructed by following the procedure described in Definition 26). Thus, Y satisfies Eq. (66) for $\hat{H}$ and some non-negative function μ. Concluding,

Theorem 40 *Given a simple function $\hat{H} : T^*Q \to \mathbb{R}$, there exists a solution y of Eq.* (68)*, for $\hat{H}$ and some non-negative function μ, if and only if* (71) *is fulfilled for $\hat{H}$.*

Remark 41 *In contrast to the situation considered in Ref. [25], since here the subbundle W is not 1-dimensional in general, we can not ensure the uniqueness of solutions y of Eq.* (68) *(for $\hat{H}$ and μ fixed). It is shown in [25] that, if a global generator $\xi : Q \to T^*Q$ of the 1-dimensional subbundle W is given, the solution is necessarily*

$$y(\alpha) := -\frac{\mu(\alpha) + \left\{\hat{H}, H\right\}(\alpha)}{\left\langle \xi(\pi(\alpha)), \mathbb{F}\hat{H}(\alpha) \right\rangle} \, \xi(\pi(\alpha)).$$

Taking into account the maximality of the LCB method (see Theorem 34), the theorem above says that (71), and consequently the simple matching conditions, are a ***key ingredient*** that must be present (explicitly or not) in any stabilization method that involves the construction of a simple Lyapunov function. More precisely, if by means of some stabilization method we construct a control law Y and a simple Lyapunov function $\hat{H}$ for $X_H + Y$, then $\hat{H}$ must satisfy Eq. (71).

5.2 The simple LCB method

Let us see how the procedure described in Definition 33 can be reformulated when simple Hamiltonian functions are involved. Fix an underactuated system $(H, \mathcal{W})$ with H simple and $\mathcal{W}$ given by a subbundle $W \subset T^*Q$. Fix also a critical point $\alpha^\bullet = (q^\bullet, 0)$ of X_H (see Remark 18). Suppose that we have a function $\hat{H}$, simple and positive-definite w.r.t. $\alpha^\bullet$, and a fiber preserving map $y : T^*Q \to T^*Q$ such that $y(\alpha^\bullet) = 0$. It is clear that y can be written

$$y(\alpha) := z(\alpha) - \mathbb{F}\hat{H}^{-1} \circ \left(\mathbb{F}H \circ \mathbb{B}\hat{H}(\alpha) - \mathbb{F}\hat{H} \circ \mathbb{B}H(\alpha)\right), \tag{75}$$

being $z : T^*Q \to T^*Q$ some fiber preserving map. Consider the orthogonal decomposition $T^*Q = W \oplus \hat{W}$ (recall Remark 21), given by the tensor $\hat{\mathfrak{b}}$, and write $z(\alpha) = z_{\shortparallel}(\alpha) - z_{\perp}(\alpha)$, with $z_{\shortparallel}(\alpha) \in W$ and $z_{\perp}(\alpha) \in \hat{W}$.

Proposition 42 *Under above assumptions and notation, the map y given by* (75) *satisfies* (68)*, for some non-negative function μ, if and only if $\hat{H}$ satisfies* (71) *and z satisfies*

$$\hat{\mathfrak{b}}(z(\alpha), \alpha) = -\mu(\alpha), \quad \hat{\mathfrak{b}}(z(\alpha), \sigma) = \Upsilon(\alpha, \alpha, \sigma) \qquad \text{and} \quad z(\alpha^\bullet) = 0, \tag{76}$$

[where $\Upsilon : T^*Q \times T^*Q \times T^*Q \to \mathbb{R}$ is the tensor field defined in (52)*] for all $\alpha \in T^*Q$ and $\sigma \in \hat{W}$; or equivalently, $z_\perp$ is given by*

$$\hat{\mathfrak{b}}(z_\perp(\alpha), \sigma) = \begin{cases} 0, & \sigma \in W, \\ -\Upsilon(\alpha, \alpha, \sigma), & \sigma \in \hat{W}, \end{cases} \tag{77}$$

and $z_{\shortparallel}$ fulfills

$$\hat{\mathfrak{b}}(z_{\shortparallel}(\alpha), \alpha) = \hat{\mathfrak{b}}(z_\perp(\alpha), \alpha) - \mu(\alpha), \quad z_{\shortparallel}(\alpha) \in W \quad \text{and} \quad z_{\shortparallel}(\alpha^\bullet) = 0. \tag{78}$$

Proof. Let us first recall that, from the results of the previous section (see Theorem 40), there exists a solution y of (68) for $\hat{H}$ (and for some non-negative function μ) if and only if $\hat{H}$ satisfies (71), or equivalently (73) and (74). If such a solution is given by (75), let us see what that means in terms of z. Using (21), we have that

$$\left\langle \mathbb{F}\hat{H}^{-1} \circ \left(\mathbb{F}H \circ \mathbb{B}\hat{H}(\alpha) - \mathbb{F}\hat{H} \circ \mathbb{B}H(\alpha) \right), \mathbb{F}\hat{H}(\alpha) \right\rangle = \left\langle \alpha, \mathbb{F}H \circ \mathbb{B}\hat{H}(\alpha) - \mathbb{F}\hat{H} \circ \mathbb{B}H(\alpha) \right\rangle =$$

$$= \left\langle \mathbb{B}\hat{H}(\alpha), \mathbb{F}H(\alpha) \right\rangle - \left\langle \mathbb{B}H(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle = \left\{ \hat{H}, H \right\}(\alpha).$$

Then $\left\langle y(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle = \left\langle z(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle - \left\{ \hat{H}, H \right\}(\alpha)$ and, consequently, the first part of (68) implies that

$$\left\langle z(\alpha), \mathbb{F}\hat{H}(\alpha) \right\rangle = -\mu(\alpha). \tag{79}$$

Let us study the second part of (68), i.e. the condition $y \subset W$. First note that, for all $\alpha, \sigma \in T^*Q$ on the same fiber,

$$\left\langle y(\alpha), \mathbb{F}\hat{H}(\sigma) \right\rangle = \left\langle z(\alpha), \mathbb{F}\hat{H}(\sigma) \right\rangle - \left\langle \mathbb{B}\hat{H}(\alpha), \mathbb{F}H(\sigma) \right\rangle + \left\langle \mathbb{B}H(\alpha), \mathbb{F}\hat{H}(\sigma) \right\rangle. \tag{80}$$

To find a more convenient expression of the above equation, let us write H and $\hat{H}$ in terms of their kinetic and potential terms, and consider the tensor Υ defined in (52) and the tensor $\psi : T^*Q \to \mathbb{R}$ given by

$$\psi(\alpha) := \left\langle d\hat{h}(\pi(\alpha)), \mathbb{F}\mathfrak{H}(\cdot) \right\rangle - \left\langle dh(\pi(\alpha)), \mathbb{F}\hat{\mathfrak{H}}(\cdot) \right\rangle.$$

According to the second part of (19), Eq. (52) says that

$$\Upsilon(\alpha, \alpha, \sigma) = \left\langle \mathbb{B}\hat{\mathfrak{b}}(\alpha, \alpha), \mathbb{F}\mathfrak{H}(\sigma) \right\rangle - \left\langle \mathbb{B}\mathfrak{b}(\alpha, \alpha), \mathbb{F}\hat{\mathfrak{H}}(\sigma) \right\rangle = \left\langle \mathbb{B}\hat{\mathfrak{H}}(\alpha), \mathbb{F}\mathfrak{H}(\sigma) \right\rangle - \left\langle \mathbb{B}\mathfrak{H}(\alpha), \mathbb{F}\hat{\mathfrak{H}}(\sigma) \right\rangle,$$

for all $\alpha, \sigma \in T^*Q$ on the same fiber. Then, Eqs. (73) and (74) read

$$\Upsilon(\sigma, \sigma, \sigma) = \psi(\sigma) = 0, \quad \forall \sigma \in \hat{W}.$$

On the other hand, since

$$\left\langle \mathbb{B}\hat{H}(\alpha), \mathbb{F}H(\sigma) \right\rangle - \left\langle \mathbb{B}H(\alpha), \mathbb{F}\hat{H}(\sigma) \right\rangle = \left\langle \mathbb{B}\hat{\mathfrak{H}}(\alpha), \mathbb{F}\mathfrak{H}(\sigma) \right\rangle - \left\langle \mathbb{B}\mathfrak{H}(\alpha), \mathbb{F}\hat{\mathfrak{H}}(\sigma) \right\rangle +$$

$$+ \left(\left\langle d\hat{h}(q), \mathbb{F}\mathfrak{H}(\cdot) \right\rangle - \left\langle dh(q), \mathbb{F}\hat{\mathfrak{H}}(\cdot) \right\rangle \right) = \Upsilon(\alpha, \alpha, \sigma) + \psi(\sigma),$$

we have from (80) the equality

$$\left\langle y\left(\alpha\right),\mathbb{F}\hat{H}\left(\sigma\right)\right\rangle = \left\langle z\left(\alpha\right),\mathbb{F}\hat{H}\left(\sigma\right)\right\rangle - \Upsilon\left(\alpha,\alpha,\sigma\right) - \psi\left(\sigma\right). \tag{81}$$

Coming back to (68), the condition $y \subset W$ implies that, for all $\sigma \in \hat{W} = \mathbb{F}\hat{H}^{-1}\left(W^{0}\right)$ [see Eq. (81)],

$$0 = \left\langle y\left(\alpha\right),\mathbb{F}\hat{H}\left(\sigma\right)\right\rangle = \left\langle z\left(\alpha\right),\mathbb{F}\hat{H}\left(\sigma\right)\right\rangle - \Upsilon\left(\alpha,\alpha,\sigma\right) - \psi\left(\sigma\right).$$

But, since $\psi\left(\sigma\right) = 0$ for all $\sigma \in \hat{W}$,

$$\left\langle z\left(\alpha\right),\mathbb{F}\hat{H}\left(\sigma\right)\right\rangle = \Upsilon\left(\alpha,\alpha,\sigma\right), \quad \forall \alpha \in T^{*}Q,\ \sigma \in \hat{W}. \tag{82}$$

Thus, we have for z the equations [see (79) and (82)]

$$\left\langle z\left(\alpha\right),\mathbb{F}\hat{H}\left(\alpha\right)\right\rangle = -\mu\left(\alpha\right) \quad \text{and} \quad \left\langle z\left(\alpha\right),\mathbb{F}\hat{H}\left(\sigma\right)\right\rangle = \Upsilon\left(\alpha,\alpha,\sigma\right),$$

or equivalently, in terms of the (non-degenerate) tensor $\hat{\mathfrak{b}}$, we have the first two equations in (76). Since $\hat{H}$ is positive-definite w.r.t. $\alpha^{\bullet}$, and consequently $d\hat{H}\left(\alpha^{\bullet}\right) = 0$, it is easy to show from (75) that the condition $y\left(\alpha^{\bullet}\right) = 0$ implies $z\left(\alpha^{\bullet}\right) = 0$. This gives us the last part of (76).

Now, writing $z\left(\alpha\right) = z_{\shortparallel}\left(\alpha\right) - z_{\perp}\left(\alpha\right)$ as explained above, Eq. (76) translates to

$$\hat{\mathfrak{b}}\left(z_{\shortparallel}\left(\alpha\right),\alpha\right) = \hat{\mathfrak{b}}\left(z_{\perp}\left(\alpha\right),\alpha\right) - \mu\left(\alpha\right) \quad \text{and} \quad \hat{\mathfrak{b}}\left(z_{\perp}\left(\alpha\right),\sigma\right) = -\Upsilon\left(\alpha,\alpha,\sigma\right), \tag{83}$$

for all $\alpha \in T^{*}Q$ and $\sigma \in \hat{W}$. The second part of (83) says precisely that $z_{\perp}$ is given by (77). In particular, $z_{\perp}\left(0\right) = 0$ on any fiber of $T^{*}Q$, so $z_{\perp}\left(\alpha^{\bullet}\right) = 0$, and consequently $z_{\shortparallel}\left(\alpha^{\bullet}\right) = 0$ also. We conclude then that $z_{\shortparallel}$ must satisfies all the conditions appearing in (78).

Reciprocally, it is easy to see that, if z (resp. $z_{\perp}$ and $z_{\shortparallel}$) satisfies (76) [resp. the Equations (77) and (78)], and $\hat{H}$ is a solution of (71), reversing the steps above we have that y is a solution of (68). □

From all that, we have another stabilization method (included in the LCB method).

Definition 43 *Given an underactuated system $(H, \mathcal{W})$, with H simple and $\mathcal{W}$ defined by a subbundle $W \subset T^{*}Q$, and a critical point $\alpha^{\bullet} \in T^{*}Q$ of X_{H}, the **simple LCB method** consists in:*

- *finding a simple function $\hat{H}$ that solves (71), with $\hat{H}$ positive definite w.r.t. $\alpha^{\bullet}$;*
- *defining $z_{\perp}$ by the Eq. (77);*
- *fixing a fiber preserving map $z_{\shortparallel} : T^{*}Q \to T^{*}Q$ by following the steps ii and iii of Proposition 25, replacing $\mu\left(\alpha\right)$ by $\mu\left(\alpha\right) - \hat{\mathfrak{b}}\left(z_{\perp}\left(\alpha\right),\alpha\right)$ in formula (59);*
- *defining $z := z_{\shortparallel} - z_{\perp}$, y as in Eq. (75) and finally $Y \in \mathfrak{X}\left(T^{*}Q\right)$ as the vertical lift of y.*

5.3 Maximality and equivalence

The following theorem is a direct consequence of Theorem 34 and the calculations of the two previous subsections.

Theorem 44 *Consider an underactuated system $(H, \mathcal{W})$, with H simple, $\mathcal{W}$ given by a subbundle $W \subset T^{*}Q$, and a critical point $\alpha^{\bullet} \in T^{*}Q$ of X_{H}. If we are given a vector field $Y \subset \mathcal{W}$ and a simple Lyapunov function for $X_{H} + Y$ and $\alpha^{\bullet}$, then Y is given by the simple LCB method (see Definition 43).*

This result says that the simple LCB method is maximal among the Lyapunov based stabilization methods for which the related Lyapunov functions can be chosen simple. In particular, it says that the simple CH method (see Definition 26) is included in the simple LCB method (recall Definition 11). We show below that the other inclusion, and consequently the equivalence, also holds. This is a very remarkable fact, since the form of the control laws given by the (simple) CH method seems to be not too general, mainly because of the rather special form of the gyroscopic forces. But, as we show below, this is a wrong impression.

Theorem 45 *The simple LCB and CH methods are equivalent, in the sense of Definition 11.*

Proof. Consider the set $\mathfrak{U}$ formed out by the triples $(H, \mathcal{W}, \alpha^\bullet)$, where H is simple, $\mathcal{W}$ is given by a subbundle $W \subset T^*Q$, and $\alpha^\bullet \in T^*Q$ is a critical point of X_H. Note that the simple LCB and CH methods are stabilization methods on $\mathfrak{U}$ (see Definition 10). Denote by $\digamma_{LCB}$ and $\digamma_{CH}$ their corresponding functions. Theorem 44 tells us that

$$\digamma_{CH}(H, \mathcal{W}, \alpha^\bullet) \subset \digamma_{LCB}(H, \mathcal{W}, \alpha^\bullet), \quad \forall (H, \mathcal{W}, \alpha^\bullet) \in \mathfrak{U}.$$

Let us show the other inclusions also hold. Let $Y \in \digamma_{LCB}(H, \mathcal{W}, \alpha^\bullet)$. Then, Y is the vertical lift of a fiber preserving map y given by (75), with $\hat{H}$ simple, positive definite w.r.t. $\alpha^\bullet$, solving (71), and z satisfying (76). We want to see that $Y \in \digamma_{CH}(H, \mathcal{W}, \alpha^\bullet)$. This would mean, comparing (49) and (75), that

$$z(\alpha) = z_d(\alpha) + \mathfrak{Z}_g\left(\alpha, \alpha, \mathbb{F}\hat{H}^{-1}(\cdot)\right)$$

for some fiber preserving map z_d satisfying (58) and some tensor $\mathfrak{Z}_g$ given by (53). So, it is enough to take

$$\begin{cases} \mathfrak{Z}_g(\alpha_1, \alpha_2, \sigma) := \Upsilon(\alpha_1, \alpha_2, \sigma), \quad \mathfrak{Z}_g(\sigma_1, \sigma_2, \gamma) := -\Upsilon(\gamma, \sigma_2, \sigma_1) - \Upsilon(\gamma, \sigma_1, \sigma_2), \\ \mathfrak{Z}_g(\gamma_1, \sigma, \gamma_2) := \mathfrak{Z}_g(\sigma, \gamma_1, \gamma_2) := -\frac{1}{2}\Upsilon(\gamma_1, \gamma_2, \sigma), \quad \mathfrak{Z}_g(\gamma_1, \gamma_2, \gamma_3) := 0, \end{cases}$$

with $\alpha_i \in T^*Q$, $\sigma, \sigma_i \in \hat{W}$ and $\gamma, \gamma_i \in W$, and

$$z_d(\alpha) \coloneqq z(\alpha) - \mathfrak{Z}_g\left(\alpha, \alpha, \mathbb{F}\hat{H}^{-1}(\cdot)\right).$$

In fact, using (76) and the definition of $\mathfrak{Z}_g$,

$$\hat{\mathfrak{b}}(z_d(\alpha), \alpha) \coloneqq \hat{\mathfrak{b}}(z(\alpha), \alpha) - \mathfrak{Z}_g(\alpha, \alpha, \alpha) = -\mu(\alpha) + 0 = -\mu(\alpha),$$

for all $\alpha \in T^*Q$,

$$\hat{\mathfrak{b}}(z_d(\alpha), \sigma) \coloneqq \hat{\mathfrak{b}}(z(\alpha), \sigma) - \mathfrak{Z}_g(\alpha, \alpha, \sigma) = \Upsilon(\alpha, \alpha, \sigma) - \Upsilon(\alpha, \alpha, \sigma) = 0,$$

for all $\sigma \in \hat{W}$, and

$$z_d(\alpha^\bullet) \coloneqq z(\alpha^\bullet) - \mathfrak{Z}_g\left(\alpha^\bullet, \alpha^\bullet, \mathbb{F}\hat{H}^{-1}(\cdot)\right) = 0 - 0 = 0,$$

what implies that z_d satisfies (58). $\square$

Let us mention a straightforward, but important, implication of the Theorems 44 and 45 about the energy shaping method: given a triple $(H, \mathcal{W}, \alpha^\bullet) \in \mathfrak{U}$, any control law $Y \subset \mathcal{W}$ such that $X_H + Y$ has a simple Lyapunov function, related to the point $\alpha^\bullet$, can be constructed with the simple CH method, and consequently (by inclusion), with the (general) CH method. Roughly speaking, in the realm of simple Hamiltonian systems with actuators, if we want to stabilize one of them by means of finding a simple Lyapunov function, the energy shaping method is the most general way to do it. This claim, to the best of our knowledge, is not previously mentioned in the literature.

As a last comment, we want to say that, in spite of the equivalence, it is not convenient to discard one of these methods. Although they give rise to the same sets of control laws, the latter are constructed in rather different ways. Also, the ideas behind both methods are completely different, so, one of them can be more appropriate than the other in certain situations. For instance, in Ref. [26] we were able to improve a result of [17] about stabilizability of underactuated Hamiltonian systems with two degrees of freedom. More precisely, in Reference [17], by using the energy shaping method, a set of conditions that ensures the stabilizability of such systems has been established. In Reference [26], by using the LCB method, we have shown that such conditions not only ensure the stabilizability, but also the asymptotic stabilizability.

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