European Journal of Combinatorics 46 (2015) 161–175

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Toll convexity

Liliana Alcón [a], Boštjan Brešar [b,c], Tanja Gologranc [c], Marisa Gutierrez [d], Tadeja Kraner Šumenjak [e,c], Iztok Peterin [f,c], Aleksandra Tepeh [f,g]

[a] Department of Mathematics, National University of La Plata, Argentina
[b] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
[c] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[d] Department of Mathematics, National University of La Plata, CONICET, Argentina
[e] FKBV, University of Maribor, Slovenia
[f] Faculty of Electrical Engineering and Computer Science, University of Maribor, Slovenia
[g] Faculty of Information Studies, Novo mesto, Slovenia

ARTICLE INFO

Article history:
Received 14 May 2014
Accepted 6 January 2015
Available online 27 January 2015

ABSTRACT

A walk W between two non-adjacent vertices in a graph G is called tolled if the first vertex of W is among vertices from W adjacent only to the second vertex of W, and the last vertex of W is among vertices from W adjacent only to the second-last vertex of W. In the resulting interval convexity, a set $S \subset V(G)$ is toll convex if for any two non-adjacent vertices $x, y \in S$ any vertex in a tolled walk between x and y is also in S. The main result of the paper is that a graph is a convex geometry (i.e. satisfies the Minkowski–Krein–Milman property stating that any convex subset is the convex hull of its extreme vertices) with respect to toll convexity if and only if it is an interval graph. Furthermore, some well-known types of invariants are studied with respect to toll convexity, and toll convex sets in three standard graph products are completely described.

E-mail addresses: liliana@mate.unlp.edu.ar (L. Alcón), bostjan.bresar@um.si (B. Brešar), tanja.gologranc@gmail.com (T. Gologranc), marisa@mate.unlp.edu.ar (M. Gutierrez), tadeja.kraner@um.si (T.K. Šumenjak), iztok.peterin@um.si (I. Peterin), aleksandra.tepeh@um.si (A. Tepeh).

http://dx.doi.org/10.1016/j.ejc.2015.01.002
0195-6698/

1. Introduction

Theory of convex structures has considerably developed in the last few decades; cf. the seminal monograph due to van de Vel [28]. Several classical theorems of combinatorial flavor on convex sets in Euclidean spaces have been studied in a more general context. In abstract convexity theory three axioms determine the pair $(X, \mathcal{C})$, where $\mathcal{C}$ is a collection of subsets (called convex sets) of X, as a convex space: (i) $\emptyset$ and X are convex, (ii) intersection of any two convex sets is convex, and (iii) nested unions of convex sets are convex. All these axioms hold in particular for the so-called *interval-convexities*, where an interval $I : X \times X \to 2^X$ has the property that $x, y \in I(x, y)$, and convex sets are defined as the sets S such that $I(x, y) \in S$ for any $x, y \in S$, see Calder [7]. Intervals in graphs usually arise from some types of paths, like the shortest (also called geodesic) or the induced (also called monophonic) paths; see the recent monograph [26] on geodesic convexity in graphs. Some early influential papers in the area were published by Farber and Jamison [16,17], and Duchet [14]. They considered various types of graph invariants and other properties (like the Carathéodory, Tietze, Helly and Radon type theorems), which arise from the prototypical Euclidean convexity, in relation with geodesic and monophonic paths in graphs. Later several other types of paths and intervals were used, yielding other types of convexities, like the all-paths convexity [9], the triangle path convexity [10], the longest path convexity [11], and many others (see Mulder [25]).

In [16] the authors study the problem from the abstract convexity theory, which is some times referred to as Minkowski–Krein–Milman property or convex geometry property. Recall that a vertex s from a convex set S is an *extreme vertex* of S, if $S - \{s\}$ is also convex. A graph G is called a *convex geometry* with respect to a given convexity, if any convex set of G is the convex hull of its extreme vertices. An alternative definition of convex geometries, using the so-called anti-exchange axiom is also often used (cf. [1], where convex geometries are studied in the context of lattices). In the case of monophonic convexity exactly chordal graphs are convex geometries, while in the geodesic convexity these are precisely Ptolemaic graphs (i.e. distance-hereditary chordal graphs). In a similar way, totally balanced hypergraphs and strongly chordal graphs have been characterized as convex geometries of some particular (hyper)graph convexities [16]. More recently, the so-called Steiner convexity was introduced, and it was shown that precisely 3-fan-free chordal graphs are convex geometries with respect to this convexity [6].

In this paper we focus on the well-known class of interval graphs, i.e. the intersection graphs of the real-line intervals. Following the concept from [2], where tolled paths were used in a characterization of interval graphs, we introduce the so-called tolled walks, which are walks, having a special restriction on their two end-vertices. In the resulting toll convexity the interval graphs are precisely the graphs which are convex geometry. Then we focus on two standard invariants, in relation with this newly introduced type of graph convexity (for a study of these invariants in relation with geodesic convexity see the survey [5]). Finally, we describe the structure of toll convex sets in three graph products, which has also been done for some other types of convexities [3,27].

In the following section we present main definitions, needed in the sequel, in particular the definition of tolled walks and toll convexity. Then, in Section 3, we prove the main result that a graph is a convex geometry with respect to toll convexity if and only if it is an interval graph. In Section 4 we study some invariants that arise from toll convexity, notably the so-called toll number and t-hull number. In particular, we determine these two numbers in arbitrary trees. Unlike as for the monophonic convexity (see [16]) we prove that the Carathéodory number for toll convexity can be bigger than 2 even in chordal graphs. Finally, in Section 5, we give a full description of the structure of the proper convex subsets of three standard graph products, the lexicographic, the strong and the Cartesian product.

2. Basic and main concepts

Let G be a graph (by which we mean an undirected graph without loops or multiple edges). The *distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path (alias geodesic) between u and v in G. The *geodesic interval* $I_G(u, v)$ between vertices u and v is the set of all vertices that lie on some shortest path between u and v in G, i.e. $I_G(u, v) = \{x \in V(G) : d_G(u, x) + d_G(x, v) = d_G(u, v)\}$. A subset S of $V(G)$ is *geodesically convex* (or *g-convex*) if $I_G(u, v) \subseteq S$ for all $u, v \in S$. Similarly one defines

$J_G(u, v) = \{x \in V(G) : x$ lies on an induced path between u and $v\}$ to be the *monophonic interval* between u and v in G. In the resulting *monophonic convexity* a subset S of $V(G)$ is *monophonically convex* (or *m-convex*) if $J_G(u, v) \subseteq S$ for all $u, v \in S$. (Indices above may be omitted, whenever the graph G is understood from the context.)

Next, we introduce the concept of *toll convexity*. Let u and v be two different non adjacent vertices in G. A *tolled walk* T between u and v in G is a sequence of vertices of the form

$$T : u, w_1, \dots, w_k, v,$$

where $k \geq 1$, which enjoys the following three conditions:

- $w_i w_{i+1} \in E(G)$ for all i,
- $uw_i \in E(G)$ if and only if $i = 1$,
- $vw_i \in E(G)$ if and only if $i = k$.

In other words, a tolled walk is any walk between u and v such that u is adjacent only to the second vertex of the walk, and v is adjacent only to the second-to-last vertex of the walk. The name tolled arises from understanding that the edges uw_1 and w_kv may be passed only by "paying the toll" that no other vertex of the walk, except for w_1 (resp. w_k), will be adjacent to u (resp. to v). For $uv \in E(G)$ we let $T : u, v$ be a tolled walk as well and the only tolled walk that starts and ends in the same vertex v is v itself. We define $T_G(u, v) = \{x \in V(G) : x$ lies on a tolled walk between u and $v\}$ to be the *toll interval* between u and v in G. Finally, a subset S of $V(G)$ is *toll convex* (or *t-convex*) if $T_G(u, v) \subseteq S$ for all $u, v \in S$.

Note that any vertex and any adjacent pair of vertices form a convex subset in any of the above convexities, as does the whole vertex set of a graph. Also, any toll convex subset is also monophonically convex subset, and any monophonically convex subset is also geodesically convex. On the other hand, for instance, a set S of vertices inducing a P_3 in C_5 is geodesically convex, but not monophonically convex. To see that monophonic convexity is not the same as toll convexity, consider the graph obtained from $P_5 : x_0x_1x_2x_3x_4$ by attaching additional leaf a to the central vertex x_2 of P_5. The set of vertices S which induces P_5 is monophonically convex, but is not toll convex, as $x_0, x_1, x_2, a, x_2, x_3, x_4$ is a tolled walk between x_0 and x_4 that contains $a \notin S$.

A graph G is *chordal* if it contains no induced cycles of length greater than 3. A vertex v of a graph G is called *simplicial* if $N(v)$ is a clique (i.e. a complete subgraph) in G. A *perfect elimination ordering* of G is an ordering $v_1, \dots, v_n$ of $V(G)$ in which for every $i \in \{1, \dots, n\}$, v_i is simplicial in the subgraph of G induced by $v_i, \dots, v_n$.

Theorem 2.1 ([13,18]). *A graph is chordal if and only if it has a perfect elimination ordering.*

An *interval representation* of a graph is a family of intervals of the real line assigned to vertices so that vertices are adjacent if and only if the corresponding intervals intersect. A graph is an *interval graph* if it has an interval representation. See the monographs [4,24] for more on interval graphs, chordal graphs and related classes of graphs.

Three vertices of a graph form an *asteroidal triple* if between any pair of them there exists a path that avoids the neighborhood of the third vertex.

Theorem 2.2 ([23]). *A graph is an interval graph if and only if it is a chordal graph with no asteroidal triple.*

We start with two basic observations on t-convex sets that will be used several times in the paper. Recall that the set $S \subset V(G)$ *separates* a vertex $a \in V(G)$ from a vertex $b \in V(G)$ if every path from a to b passes through a vertex from S.

Lemma 2.3. *A vertex v is in some tolled walk between two non-adjacent vertices x and y if and only if $N[x] - \{v\}$ does not separate v from y and $N[y] - \{v\}$ does not separate v from x.*

Proof. It suffices to notice that there exists a tolled walk between x and y containing v if and only if x and y are nonadjacent and there exists a path between x and v in $G - (N[y] - \{v\})$ and a path between v and y in $G - (N[x] - \{v\})$. $\square$

The following general result on t-convex sets in graphs follows from Lemma 2.3.

Proposition 2.4. *Let G be a graph. A subset C of $V(G)$ is t-convex if and only if for every $x, y \in C$ and every $v \in V(G) - C$ the set $N[x] - \{v\}$ separates v from y or the set $N[y] - \{v\}$ separates v from x.*

Proof. Let C be a t-convex set in a graph G, and let $x, y \in C$ and $v \in V(G) - C$ be arbitrary vertices. Since C is t-convex there is no tolled walk between x and y that contains v. By Lemma 2.3 we derive that $N[x] - \{v\}$ separates v from y or $N[y] - \{v\}$ separates v from x, as desired.

Conversely, suppose that C is not t-convex. Thus there exists a tolled walk W from x to y for some $x, y \in C$ that contains some $v \in V(G) - C$. Since W is tolled, the subwalk of W that lies between x and v does not pass through $N[y] - \{v\}$. Hence $N[y] - \{v\}$ does not separate v from x. By the same reason $N[x] - \{v\}$ does not separate v from y. □

The *t-convex hull* of a set $S \subseteq V(G)$ is the smallest set of vertices in G that contains S and is t-convex (alternatively, it is the intersection of all t-convex sets that contain S). As mentioned above, a vertex s from a t-convex set S of a graph G is an *extreme vertex* of S if $S - \{s\}$ is also a t-convex set in G. The set of extreme vertices of $V(G)$ will be denoted by $Ext(G)$.

3. Interval graphs as convex geometry

In order to prove that a graph G is a convex geometry with respect to some convexity, it is important to know what are extreme vertices of convex sets in that convexity. This is easy in the case of g-convexity and m-convexity, as the extreme vertices of a convex set are exactly the simplicial vertices with respect to this set. However, this is not always the case with t-convexity. For instance, let G be the graph obtained from the triangle by adding two leaves, say, a and b, adjacent to two distinct vertices of the triangle. Note that the simplicial vertices in G are both leaves a and b, and also the third vertex x of the triangle, i.e. the unique vertex of degree 2 in G. Clearly, $V(G)$ is a t-convex set in G, but x is not its extreme vertex, since there is a tolled walk from a to b, which passes through x, implying $V(G) - \{x\}$ is not t-convex. Nevertheless, we can prove the following observation about extreme vertices.

Lemma 3.1. *Let C be a toll convex set of a graph G. If x is an extreme vertex in C, then x is a simplicial vertex in C.*

Proof. For the purpose of getting a contradiction, suppose that x is an extreme vertex of C that is not simplicial in C. Then there exist two neighbors of x in C, say u and v, which are not mutually adjacent. But then u, x, v is a tolled walk in G. Hence $C - \{x\}$ is not convex, which is the desired contradiction. □

For the proof of the main result we need several lemmas.

Lemma 3.2. *Let vertices a, b and c form an asteroidal triple in a graph G. If C is the t-convex hull of the set $\{a, b, c\}$, then C has no extreme vertices.*

Proof. Since vertices a, b and c form an asteroidal triple in G, there exists a path P_{ca} from c to a such that none of its vertices belong to the closed neighborhood of b; also there exists a path P_{ab} from a to b, so that none of its vertices belong to the closed neighborhood of c. We may assume that P_{ca} and P_{ab} are induced paths. Consider the walk W from c to b, obtained by concatenating P_{ca} with P_{ab}. Since both paths are induced, c is adjacent only to one vertex of P_{ca} and to no vertices of P_{ab}, and b is adjacent only to one vertex of P_{ab} and to no vertices of P_{ca}. We deduce that W is a tolled walk from c to b that passes a. Clearly, W lies in the t-convex hull C of the set $\{a, b, c\}$, and so a is not an extreme vertex of C. In an analogous way we prove that b and c are not extreme vertices of C. Let x be any vertex of C, different from a, b and c. If x is an extreme vertex in C, then $C - \{x\}$ is also a convex set, properly included in C, and containing a, b and c. This is in a contradiction with C being the t-convex hull of $\{a, b, c\}$, thus x is not an extreme vertex in C. As x was arbitrarily chosen, we derive that C has no extreme vertices. □

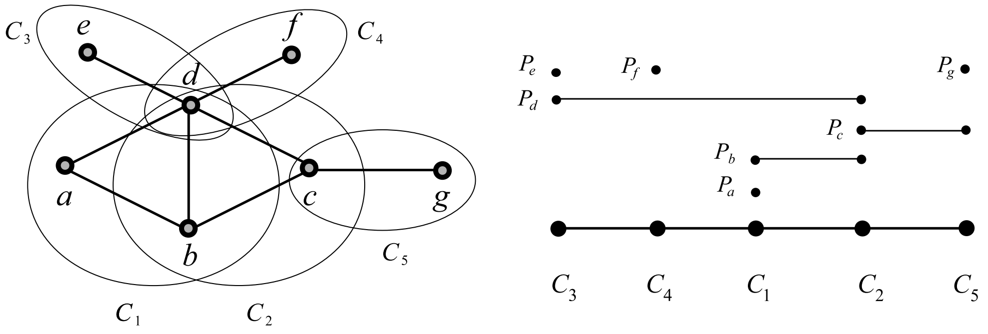

Fig. 1. An interval graph G and its canonical representation C_3, C_4, C_1, C_2, C_5 (note that C_4, C_3, C_1, C_2, C_5 is another canonical representation of G).

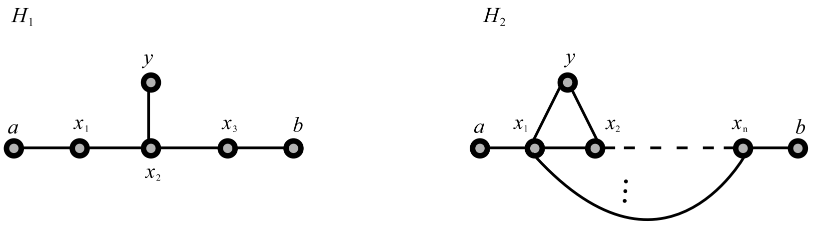

Fig. 2. Forbidden induced subgraphs for end-simplicial vertex y. (Note that in H_2, x_1 is adjacent to all other x_i; it is allowed that $n = 2$, and so the smallest H_2 is a triangle with two pendant vertices added.)

Let $\mathcal{C}(G)$ be the set of all maximal cliques in G. A *canonical representation* of an interval graph G is a linear order $C_1, \dots, C_k$ of the set $\mathcal{C}(G)$ in which, for each vertex v of G, the set $\{C \in \mathcal{C}(G) : v \in C\}$ occurs consecutively in the linear order. Note that every canonical representation provides an interval representation of G by considering for each vertex $v \in V(G)$, the interval $P_v = \{C \in \mathcal{C}(G) : v \in C\}$ of the total order $C_1, \dots, C_k$. Cliques C_1 and C_k are called *end-cliques* of the representation. Clearly every end-clique contains a simplicial vertex. Every interval graph admits a canonical representation [18,22], usually more than one. A simplicial vertex v of G is called *end-simplicial* whenever it belongs to an *end-clique* of some canonical representation of G. (Fig. 1 presents an example of an interval graph, together with one of its canonical representations. Note that the representation implies that vertices e and g are end-simplicial vertices of G. An alternative canonical representation C_4, C_3, C_1, C_2, C_5 yields that f is also end-simplicial in G.)

Theorem 3.3 ([19]). *A simplicial vertex y of an interval graph G is end-simplicial if and only if G contains as an induced subgraph none of the graphs in Fig. 2 with y as the designated vertex.*

It follows from Theorem 3.3 that in an interval graph G every extreme vertex of the t-convex set $V(G)$ is end-simplicial. We now prove that the converse of this statement is also true.

Lemma 3.4. *Every end-simplicial vertex of an interval graph G is an extreme vertex of the t-convex set $V(G)$.*

Proof. Let $C_1, \dots, C_k$ be a canonical representation of G and let v_1 be a simplicial vertex such that $C_1 = N[v_1]$. Assume, in order to obtain a contradiction, that $V(G) - v_1$ is not t-convex. Then there exists a tolled walk T between two non-adjacent vertices x and y of $G - v_1$ containing v_1. Since x and y are non-adjacent, we can assume $1 \le \min\{i : x \in C_i\} \le \max\{i : x \in C_i\} < \min\{i : y \in C_i\} \le k$; thus any path between v_1 and y contains some neighbor of x. This implies that $N[x] - \{v_1\}$ separates v_1 from y, contradicting Lemma 2.3. □

From the above two results we derive the following.

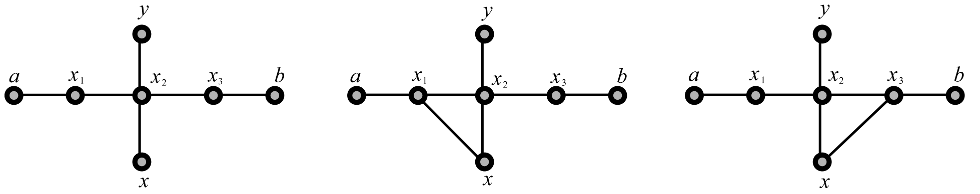

Fig. 3. The case in the proof of Lemma 3.6 when H is isomorphic to H_1.

Corollary 3.5. *A vertex v of an interval graph G is an extreme vertex of the t-convex set $V(G)$ if and only if v is end-simplicial.*

Lemma 3.6. *Let $C_1, \ldots, C_k$ be a canonical representation of an interval graph G. If for some $s < t < k$ there exists $x \in C_s - C_t$ such that $C_t \cap C_{t+1} \subseteq N(x)$, then C_t contains a simplicial vertex y. Even more, if x is end-simplicial then y is end-simplicial.*

Proof. Let $C_1, \ldots, C_k$ be a canonical representation of an interval graph G, and let C_s, C_t and $x \in C_s - C_t$ be as in the statement of the lemma. Since C_t is a maximal clique, there exists $y \in C_t - C_{t-1}$. We claim that y is simplicial, and in order to obtain a contradiction, suppose that $y \in C_{t+1}$. Since $y \in C_t \cap C_{t+1} \subseteq N(x)$ and $x \notin C_t$ it follows that $y \in C_{t-1}$ which contradicts the choice of y. Hence $N[y] = C_t$ and y is simplicial.

Assume that x is an end-simplicial vertex, while y is not. It follows from Theorem 3.3 that G has an induced subgraph H isomorphic either to H_1 or H_2 depicted in Fig. 2, where y is the designated vertex. Note that y is an end-simplicial vertex of the graph G', obtained from G by removing the vertices from $(\cup_{i\geq t+1} C_i) - N(x)$. Thus H is not an induced subgraph of G', which means that some vertex u of H belongs to $(\cup_{i\geq t+1} C_i) - N(x)$. It is clear that u is not adjacent to y. On the other hand, since $C_t \cap C_{t+1} \subseteq N(x)$, every path in G (and thus in H) between any vertex of $(\cup_{i\geq t+1} C_i) - N(x)$ and y (in particular, between u and y) contains some vertex v adjacent to both y and x.

We will consider two cases: H is isomorphic to H_1 or H is isomorphic to H_2.

In the first case, it is clear that $v = x_2$ and thus x_2 is adjacent to x. Since x is simplicial, one of the three graphs depicted in Fig. 3 appear as an induced subgraph in G. Observe that the graph on the left has H_1 as an induced subgraph with x as the designated vertex; the other two graphs contain H_2 as an induced subgraph with x as the designated vertex. As x is an end-simplicial vertex of G we get a contradiction with Theorem 3.3.

When H is isomorphic to H_2, we get in an analogous way that x_1 or x_2 is adjacent to x. Note that here it is also possible that both x_1 and x_2 are adjacent to x.

First let x be adjacent to x_1 and to no other vertex x_i, $2 \leq i \leq n$, of H_2. Then vertices x_i, $i \in \{2, \ldots, n\}$ and b are contained in G', since there exists a path between any such vertex and y that has no vertex adjacent to both x and y. This implies that the graph depicted in Fig. 4(i) is an induced subgraph of G'. Since y is an end-simplicial vertex of G' we get a contradiction with Theorem 3.3.

Now let x be adjacent to x_1 and to another vertex x_i, $2 \leq i \leq n$, of H_2. If x is adjacent to more than one such vertex, then let x_i be the vertex adjacent to x with $i > 1$ as big as possible. Hence the graph depicted in Fig. 4(ii), is an induced subgraph of G and is isomorphic to H_2. This contradicts Theorem 3.3, since x is an end-simplicial vertex of G.

Finally, if x is not adjacent to x_1, then there is a path between any vertex of $H_2 - \{v\}$ and y with no vertex adjacent to both x and y. This means that every vertex of H_2 (also v, as v is adjacent to both x and y) belongs to G', which again contradicts the fact y is an end-simplicial vertex of G'. □

Two vertices u and v of a graph G are called *twins* if $N[u] = N[v]$.

Theorem 3.7. *A graph is a convex geometry with respect to toll convexity if and only if it is an interval graph.*

Proof. Let us assume that G is a graph which is a convex geometry with respect to toll convexity. We start the proof by showing that G is a chordal graph, for which we use induction on the order of G. The

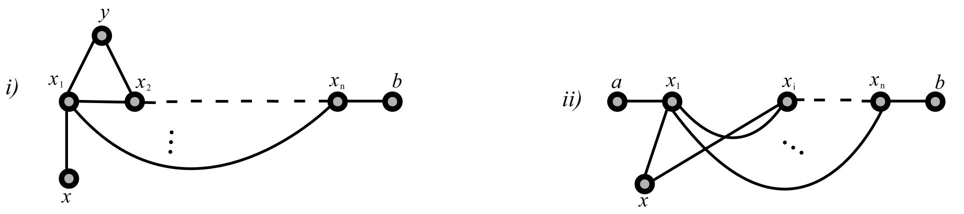

Fig. 4. The case in the proof of Lemma 3.6 when H is isomorphic to H_2: (i) induced subgraph of G', (ii) induced subgraph of G.

claim is trivially true if G has only one or two vertices. Let us assume that G has n vertices and that the claim is true for all graphs with fewer that n vertices. Note that $V(G)$ itself (as all its t-convex subsets) is the convex hull of its extreme vertices. In particular, this implies that $V(G)$ has extreme vertices, and let x be any extreme vertex of $V(G)$. As x is extreme, the subset $V(G) - \{x\}$ is also t-convex in G, and so the graph $G - x$ is a convex geometry (using that any subset of $V(G) - \{x\}$ is t-convex in $G - x$ if and only if it is t-convex in G). Thus, by induction hypothesis $G - x$ is chordal. By Lemma 3.1, x is a simplicial vertex in G, so G is also chordal by Theorem 2.1. Now, we claim that G has no asteroidal triples. Indeed, if there is an asteroidal triple in G, then by Lemma 3.2, the convex hull C of the triple has no extreme vertices. This readily implies that C is not the convex hull of its extreme vertices, which is a contradiction with G being a convex geometry. Thus G is a chordal graph with no asteroidal triple, and so it is an interval graph, by Theorem 2.2.

Every convex subset of an interval graph G induces an interval graph. Thus it suffices to show that $V(G)$ is the convex hull of its extreme vertices. We will prove that every vertex of $V(G) - Ext(G)$ belongs to a tolled walk between two vertices of $Ext(G)$. Let x and y be twins. If x is an extreme vertex, then also $y \in Ext(G)$. If $x \in V(G) - Ext(G)$ and $x \in T(a, b)$ for $a, b \in Ext(G)$, then also $y \in T(a, b)$. Hence we may assume without loss of generality that G contains no twins.

Let $C_1, \ldots, C_k$ be a given canonical representation of G. Denote by $I' = \{i :$ there exists $x \in Ext(G)$ such that $C_i = N[x]\}$. It is clear that $\{1, k\} \subseteq I'$. Since there are no twins and since for every $x \in Ext(G)$ there exists a unique $i \in I'$ such that $N[x] = C_i$, we can write $Ext(G) = \{x_i : i \in I'\}$. Let u be any vertex of $V(G) - Ext(G)$. Let $i_m = \min\{i : u \in C_i\}$ and $i_M = \max\{i : u \in C_i\}$. Observe that u is simplicial if and only if $i_m = i_M$.

Let $s = \max\{i \in I' : i \le i_m\}$ and $r = \min\{i \in I' : i_M \le i\}$. We can assume $s \ne r$, and note that $1 \le s \le i_m \le i_M \le r \le k$.

Assume that u is in no tolled walk between x_s and x_r. Then, by Lemma 2.3, we can assume without loss of generality that $N[x_s] - u$ separates u from x_r. In particular, u is not adjacent to x_r, which implies that $i_M < r$. It follows that there exists t with $i_M \le t < r$ such that $C_t \cap C_{t+1} \subseteq N(x_s)$. Thus, by Lemma 3.6, there exists an end-simplicial vertex y such that $N[y] = C_t$. Then $t \in I'$ and $i_M \le t < r$, this contradicts the fact that $r = \min\{i \in I' : i_M \le i\}$. It follows that u is in a tolled walk between x_s and x_r, and the proof is complete. □

4. Some invariants arising from toll convexity

In this section we consider some standard invariants with respect to toll convexity that have been extensively studied for other (graph) convexities. We first consider the so-called t-hull number and toll number of a graph, and at the end we give some remarks on the Carathéodory number with respect to toll convexity.

Recall that a set S of vertices of a graph G is a *geodetic* (resp. *monophonic*) *set* if every vertex of G lies in a geodetic (resp. monophonic) interval between two vertices from S. (See the survey on geodetic sets in graphs [5].) The *geodetic* (*monophonic*) *number* $g(G)$ ($mn(G)$) of a graph G is the minimum cardinality of a geodetic (monophonic) set in G.

Toll interval is defined on pairs of vertices. The definition can be generalized to an arbitrary subset S of $V(G)$, so that $T_G(S) = \cup_{u,v \in S} T_G(u, v)$. If $T_G(S) = V(G)$, we call S a *toll set* of a graph G. The order of a minimum toll set in G is called the *toll number* of G and is denoted by $tn(G)$.

Since every geodetic set is monophonic, and every monophonic set is a toll set, we have $tn(G) \leq mn(G) \leq g(G)$. For any non-trivial connected graph G we obviously have $2 \leq tn(G) \leq n$. Moreover, $tn(G) = n$ if and only if G is a complete graph K_n.

As mentioned earlier the *t-convex hull* of a set $S \subseteq V(G)$ is defined as the intersection of all t-convex sets that contain S, and we will denote this set by $[S]_t$. A set S is a *t-hull set* of G if its t-convex hull $[S]_t$ coincides with $V(G)$. The *t-hull number* of G, denoted by $th(G)$, is the minimum among all cardinalities of t-hull sets. (Compare these definitions with those of the hull set and the monophonic hull set and corresponding hull numbers, which were defined analogously for the geodetic, respectively monophonic convexity in graphs [21].) Given a set $S \subset V(G)$ we define $T^k(S)$ as follows: $T^0(S) = S$ and $T^{k+1}(S) = T(T^k(S))$ for any $k \geq 1$. Note that $[S]_t = \bigcup_{k \in \mathbb{N}} T^k(S)$. From definitions we immediately infer that every toll set is a t-hull set, and hence $th(G) \leq tn(G)$.

Let S be a toll set of G and let x be an extreme vertex of $V(G)$ (i.e. $V(G) - \{x\}$ is convex). If $x \notin S$, then, as S is a toll set, there exist $u, v \in S$ such that $x \in T_G(u, v)$. But then $V(G) - \{x\}$ is not t-convex, which is a contradiction with x being an extreme vertex. Hence all extreme vertices of a graph G are contained in every toll set of G. (This holds for extreme vertices in any convexity, see e.g. [12].) In a similar way one can show that each extreme vertex of G belongs to every t-hull set of G (see [15] for the case of geodetic and monophonic convexities).

From the proof of Theorem 3.7 follows that the toll number (as well as the t-hull number) of an interval graph coincides with the number of its extreme vertices. Indeed, it is proved that every vertex, which is not extreme (i.e. end-simplicial), lies on the toll interval between two extreme vertices. We infer the following:

Proposition 4.1. *Let G be an interval graph. Then $tn(G) = th(G) = |Ext(G)|$.*

A *caterpillar* is a tree for which the set of vertices obtained by deleting all leaves induces a path, called the *spine* of the caterpillar. It is well known that among trees only caterpillars are interval graphs. By the above proposition the toll number of a caterpillar equals the cardinality of its extreme vertices which are exactly the leaves adjacent to the end-vertices of the caterpillar's spine.

To consider the toll number of trees that are not interval graphs we need the notion of a *support vertex* in a tree, which is defined as a vertex, adjacent to at least one leaf of the tree.

Theorem 4.2. *Let G be a tree not isomorphic to a caterpillar. We have,*

(i) *if G has at least two support vertices of degree 2 then $tn(G) = 2$, otherwise $tn(G) = 3$;*
(ii) *$th(G) = 2$.*

Proof. (i) Let G be a tree not isomorphic to a caterpillar. Suppose that there exist two support vertices u and v in G of degree 2, and let a and b be the two leaves such that $av, bu \in E(T)$. Clearly every vertex of $V(G) - \{a, b\}$ lies on some tolled walk between a and b, hence $tn(G) \leq 2$, and since G is a nontrivial graph, $tn(G) = 2$.

Now consider the case when G has at most one support vertex of degree 2. First, we claim that $tn(G) > 2$. Suppose to the contrary that $S = \{\ell_1, \ell_2\}$ is a toll set of G, and assume that one of ℓ_1 and ℓ_2, say ℓ_1, is not a leaf. Then for the neighbor v of ℓ_1, for which the v, ℓ_2-path contains ℓ_1, we have that $N[\ell_1] - \{v\}$ separates v from ℓ_2. By Lemma 2.3 this implies that v does not lie on a tolled walk between ℓ_1 and ℓ_2, a contradiction. Hence we derive that both ℓ_1 and ℓ_2 are leaves. It is also clear that ℓ_1 and ℓ_2 cannot be adjacent to the same support vertex (this would only be possible if T was isomorphic to a path on three vertices, which is not the case). We may assume without loss of generality that the degree of the support vertex x of ℓ_1 is at least 3. Let y be a neighbor of x that does not lie on the shortest path between ℓ_1 and ℓ_2. Since $N[\ell_1]$ separates y from ℓ_2, Lemma 2.3 implies that y does not lie on a tolled walk between ℓ_1 and ℓ_2, again a contradiction, which proves the claim.

To conclude the proof we will show that $tn(G) \leq 3$. Note that G is a chordal graph that is not an interval graph, hence it has an asteroidal triple by Theorem 2.2. We will show that the set $S = \{a, b, c\}$, consisting of vertices that form an asteroidal triple in G, is a toll set of G. Let $v \in V(G) - S$. If $N[a] - \{v\}$ separates v from b and c, then $v \in T(b, c)$. Similarly, if $N[b] - \{v\}$ separates v from a and c or $N[c] - \{v\}$ separates v from a and b, then $v \in T(a, c)$ or $v \in T(a, b)$, respectively. In the still remaining case v lies

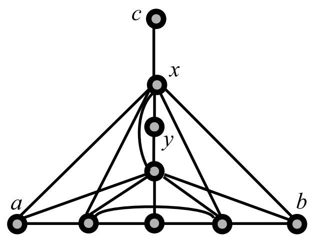

Fig. 5. A chordal graph with Carathéodory number equal to 3.

on a tolled walk between any two vertices in S. Combining $tn(T) > 2$ and S is a toll set with cardinality 3, we get $tn(T) = 3$.

(ii) Let G be a tree not isomorphic to a caterpillar. Then, as above, we find that G has an asteroidal triple a, b, c. We claim that $S = \{a, b\}$ is a t-hull set of G. In the proof of Lemma 3.2, we showed that $c \in T(a, b)$. Thus c lies in the t-convex hull of $\{a, b\}$. Moreover, from the proof of Theorem 4.2(i) it follows that the t-convex hull of $\{a, b, c\}$ is $V(G)$, and therefore $th(T) = 2$. □

We propose a further study of these two invariants in general graphs. In particular, we pose the question, in which graphs G, $|Ext(G)| = tn(G)$, and when is $|Ext(G)| = th(G)$. Note that $|Ext(G)|$ is a lower bound for both invariants as is number 2.

Next we focus on the invariant that arises from the classical theorem in geometry, due to Carathéodory [8]. The *Carathéodory number* c is the smallest integer (if it exists) such that for any subset S of $V(G)$ and any vertex $p \in [S]_t$, there exists $F \subseteq S$ with $|F| \le c$ and $p \in [F]_t$. For the monophonic convexity it was first proved that this number is 2 in chordal graphs [16], and then this result was extended to all connected graphs, except complete graphs [14]. For the case of toll convexity we show that this is not the case and present an example of a chordal graph with Carathéodory number bigger than 2.

First let us recall a general property of Carathéodory number, which holds in any interval-convexity space. (Note that the toll convexity is an instance of an interval-convexity.) It uses a so-called *redundant set* $A \subseteq V(G)$, which is defined as a nonempty set with the following property (we use the t-hull here, although it is applicable for the convex hull in any convexity):

$$[A]_t = \bigcup_{a \in A} [A - a]_t.$$

The following connection between Carathéodory number and redundant sets was proved in [14].

Proposition 4.3. *In any convexity space, the Carathéodory number is the smallest integer c such that every $(c + 1)$-element set is redundant.*

Consider the graph G from Fig. 5, and its vertices a, b, c. Note that $[a, c]_t = \{a, c, x\}$, $[b, c]_t = \{b, c, x\}$ and $[a, b]_t = V(G) - \{y, c\}$. On the other hand $[a, b, c]_t$ contains also y, hence the set $\{a, b, c\}$ is not redundant. This implies that the Carathéodory number of this graph is at least 3 (it is easy to see that is in fact equal to 3).

Another two graph convexity invariants, which arise from classical convexity theorems are the Radon number and the Helly number. For instance in the monophonic convexity context they were considered by Duchet [14]. We leave the study of these and other known invariants, set in the context of toll convexity, for future work.

5. Toll convex sets in product graphs

In this section we study the structure of t-convex sets in some standard graph products. In fact, among the four standard graph products we consider all, except the direct product.

Recall that for all of the standard graph products, the vertex set of the product of graphs G and H is equal to $V(G) \times V(H)$, In the *lexicographic product* $G \circ H$ (also denoted by $G[H]$), vertices (g_1, h_1) and

(g_2, h_2) are adjacent if either $g_1g_2 \in E(G)$ or ($g_1 = g_2$ and $h_1h_2 \in E(H)$). In the *strong product* $G \boxtimes H$ of graphs G and H vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever ($g_1g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1h_2 \in E(H)$) or ($g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$). Finally, in the *Cartesian product* $G\Box H$ of graphs G and H two vertices (g_1, h_1) and (g_2, h_2) are adjacent when ($g_1g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1h_2 \in E(H)$). Hence in general we have $E(G\Box H) \subseteq E(G \boxtimes H) \subseteq E(G \circ H)$.

Let G and H be graphs and $*$ be one of the three graph products under consideration. For a vertex $h \in V(H)$, we call the set $G^h = \{(g, h) \in V(G * H) : g \in V(G)\}$ a *G-layer* of $G * H$. By abuse of notation we will also consider G^h as the corresponding induced subgraph. Clearly G^h is isomorphic to G. For $g \in V(G)$, the *H-layer* ${}^gH$ is defined as ${}^gH = \{(g, h) \in V(G * H) : h \in V(H)\}$. We will again also consider ${}^gH$ as an induced subgraph and note that it is isomorphic to H. A map $p_G : V(G*H) \to V(G)$, $p_G(g, h) = g$ is the *projection* onto G and $p_H : V(G * H) \to V(H)$, $p_H(g, h) = h$ the *projection* onto H. We say that $G * H$ is *nontrivial* if both factors are graphs on at least two vertices.

The lexicographic product $G \circ H$ is associative but not commutative. The one-vertex graph is the unit for the operation and nontrivial graph $G \circ H$ is connected if and only if G is connected. For more fundamental properties of the lexicographic product see [20]. In [3] all nontrivial g-convex and m-convex subsets in lexicographic product of graphs have been characterized. We continue this line of research by studying t-convexity in lexicographic products.

We need yet another definition of a property of subsets of $V(G \circ H)$, obtained in a similar, yet modified fashion, as in [3]. A set Y, where $Y \subset V(G\circ H)$, is said to be *non-extreme complete* if ${}^gH \cap Y = {}^gH$ holds for all non-extreme vertices g of $p_G(Y)$.

Theorem 5.1. *Let $G \circ H$ be a nontrivial, connected lexicographic product. A proper subset Y of $V(G \circ H)$, which does not induce a complete graph, is t-convex if and only if the following conditions hold:*

(a) *$p_G(Y)$ is t-convex in G,*
(b) *Y is non-extreme complete, and*
(c) *H is complete.*

Proof. Suppose first that (a), (b) and (c) hold for a proper subset Y of $V(G \circ H)$. Since H is complete, any two vertices (g, h_1) and (g, h_2) from Y are adjacent and thus form a t-convex subset of Y. Consider non adjacent vertices $(g_1, h_1), (g_2, h_2) \in Y$. Let W be a tolled walk between (g_1, h_1) and (g_2, h_2) in $G \circ H$. Then $p_G(W)$ is a tolled walk between g_1 and g_2 in G. Let (g, h) be an arbitrary inner vertex of W. We would like to see that $(g, h) \in Y$. Since $p_G(Y)$ is t-convex, ${}^gH \cap Y \neq \emptyset$. Moreover, ${}^gH \cap Y = {}^gH$, since (g, h) is an inner vertex of a tolled walk W and Y is non-extreme complete. Therefore $(g, h) \in Y$ and hence $W \subseteq Y$. We conclude that Y is t-convex.

Conversely, let Y be a t-convex subset of $G \circ H$. Since Y does not induce a complete graph (by theorem's assumption), Y contains three vertices that induce a path, say $P = (g_1, h_1)(g_2, h_2)(g_3, h_3)$. First we will show that $g_1 \neq g_3$.

Assume $g_1 = g_3$. Since $d_Y((g_1, h_1), (g_3, h_3)) = 2$, H is not complete in this case. Let g be an arbitrary neighbor of g_1 in G. Then ${}^gH$ is a subset of $T_{G\circ H}((g_1, h_1),\ (g_3, h_3))$ and hence, since Y is t-convex, ${}^gH \subseteq Y$. Also, for any neighbor x of $g \in N_G(g_1)$, the layer ${}^xH$ is a subset of Y. Since $G\circ H$ is connected and thus G is connected, we can prove by using induction on the distance from g_1 in G that $Y = V(G \circ H)$, which is a contradiction with theorem's assumption.

We may thus assume that $g_1 \neq g_3$. Since P is induced and g_2 is a common neighbor of g_1 and g_3 in G, we have $d_G(g_1, g_3) = 2$. In particular, G is not complete. Clearly ${}^{g_2}H$ is included in $T_{G\circ H}((g_1, h_1), (g_3, h_3))$ and thus ${}^{g_2}H \subseteq Y$. If H is not complete there exists an induced path of length 2 in ${}^{g_2}H$ and we can continue as above by concluding that $Y = V(G \circ H)$, a contradiction. Thus H is complete.

To prove that Y is non-extreme complete, consider a non-extreme vertex g of $p_G(Y)$. As g is non-extreme in $p_G(Y)$, there exists a tolled walk W in $p_G(Y)$ between g' and g'' that contains g, and there exist vertices $(g', h'), (g, h), (g'', h'')$ that belong to Y for some $h, h', h'' \in V(H)$. Clearly, a walk W' from (g', h') to (g'', h'') such that the first coordinates of vertices from W' coincide with W in G is a tolled walk in $G \circ H$. In particular, for any $y \in V(H)$, (g, y) can lie in such W'. Hence $(g, y) \in Y$ for all $y \in V(H)$ and so Y is non-extreme complete.

Finally, if $p_G(Y)$ would not be t-convex in G, Y would clearly not be t-convex. □

Next we focus on the strong product of graphs G and H. The commutativity of $G \boxtimes H$ follows from symmetry in the definition of adjacency. All g-convex sets among strong product of graphs have been recently characterized in [27], however as one can see from the next theorem, the result on the t-convexity does not generalize the result on g-convexity as in the case of lexicographic product. Since $G \circ K_n$ is isomorphic to $G \boxtimes K_n$, Theorem 5.1 already gives a hint for the strong product.

Recall that a vertex of a graph G is *universal* if it is adjacent to all other vertices of G.

Theorem 5.2. *Let $G \boxtimes H$ be a nontrivial, connected strong product of graphs G and H, and let U_G and U_H be the sets of all universal vertices of G and H, respectively. A proper subset Y of $V(G \boxtimes H)$, which does not induce a complete graph, is t-convex if and only if*

(i) *$p_G(Y)$ is t-convex in G, Y is non-extreme complete and H is complete; or*
(ii) *both U_G and U_H are nonempty proper subsets of $V(G)$ and $V(H)$, respectively, $Y = (U_G \times V(H)) \cup (V(G) \times U_H)$ and every component of the graphs induced by $V(G) - U_G$ and by $V(H) - U_H$ is isomorphic to a complete graph.*

Proof. If (i) is true for Y, then $G \boxtimes H$ is isomorphic to $G \circ H$ and Y is t-convex by Theorem 5.1. So assume that (ii) is fulfilled. Let (g, h) and (g', h') be arbitrary nonadjacent vertices of Y. Note that both must be either in $U_G \times (V(H) - U_H)$ or in $(V(G) - U_G) \times U_H$. Since both possibilities are symmetric we can assume that $(g, h), (g', h') \in U_G \times (V(H) - U_H)$. In this case we have that $h \neq h'$ and $hh' \notin E(H)$. If there exists a vertex (g'', h'') from $V(G \boxtimes H) - Y$ which is adjacent to both (g, h) and (g', h'), then $hh'', h'h'' \in E(H)$, and h, h', h'' are distinct vertices that belong to the same component of the graph induced by $V(H) - U_H$, which is not possible by assumption since $hh' \notin E(H)$. Hence every vertex (g'', h'') from $V(G \boxtimes H) - Y$ is adjacent to at most one of (g, h) and (g', h'). If (g'', h'') is adjacent to (g, h), then $N[(g, h)] - \{(g'', h'')\}$ separates (g'', h'') from (g', h') and if (g'', h'') is adjacent to (g', h'), then $N[(g', h')] - \{(g'', h'')\}$ separates (g'', h'') from (g, h). If (g'', h'') is adjacent to none of (g, h) and (g', h'), then h, h', h'' lie in different components C, C', C'', respectively, of the graph induced by $V(H) - U_H$. Note that

$$N[(g, h)] - \{(g'', h'')\} = N[(g, h)] = V(G) \times U_H \cup V(G) \times V(C).$$

Therefore $G \boxtimes H - N[(g, h)]$ is not connected and it consists of components induced by $V(G) \times V(C_i)$, where C_i is a component of the graph induced by $V(H) - U_H$. Thus $N[(g, h)] - \{(g'', h'')\}$ separates (g'', h'') from (g', h') (and by symmetric argumentation also $N[(g', h')] - \{(g'', h'')\}$ separates (g'', h'') from (g, h)). Hence Y is t-convex by Proposition 2.4.

Conversely let Y be a proper t-convex subset of $V(G \boxtimes H)$ which does not induce a complete graph. Strong product of two complete graphs is complete, hence G and H are not both complete. If one factor, say H, is complete, then (i) follows by Theorem 5.1. So let us assume that both factors are not complete. There exist at least three vertices in Y, since Y does not induce a complete graph. Let (g, h), (g', h') and (g'', h'') be these vertices and in addition, we may assume that (g, h) and (g'', h'') are not adjacent, but have (g', h') as their common neighbor. We split the remaining part of the proof into three cases, in which we omit two symmetric cases due to commutativity.

Case 1: $g = g''$ and $d_H(h, h'') = 2$.

If there exists $x \in V(G)$ which is at distance 2 from g with y as a common neighbor of g and x, then $(g, h), (y, h), (x, h), (x, h'), (x, h''), (y, h''), (g, h'')$ as well as $(g, h), (y, h'), (g, h'')$ and $(g, h), (g, h'), (g, h'')$ are tolled walks, see Fig. 6. Hence $\{g, y, x\} \times \{h, h', h''\} \subseteq Y$ and we call this the $\sqsupset$-argument for (g, h) and (g, h''). The product $G \boxtimes H$ is connected, which yields that both G and H are connected. If we apply $\sqsupset$-argument for any (w, h) and (w, h'') already in Y, we get $V(G) \times \{h, h', h''\} \subseteq Y$. Hence $G^h, G^{h'}, G^{h''} \subseteq Y$. Now we can repeat the process for H, since $(g, h), (y, h), (x, h) \in Y$ and $d_G(g, x) = 2$. As G is connected and is not complete every vertex of $V(G)$ lies on an induced path of length two. Therefore, by repeating this process for every such path, we derive that $Y = V(G \boxtimes H)$, a contradiction with the assumption.

Thus there is no vertex in G at the distance 2 to g, which yields that $g \in U_G$. Moreover there are at least two nonadjacent vertices a and b in $V(G) - U_G$, since G is not complete. Clearly (g, h), $(w, h'), (g, h'')$ is a tolled walk for every $w \in V(G)$ and we have that $G^{h'} \subset Y$. But then $(a, h'), (b, h')$

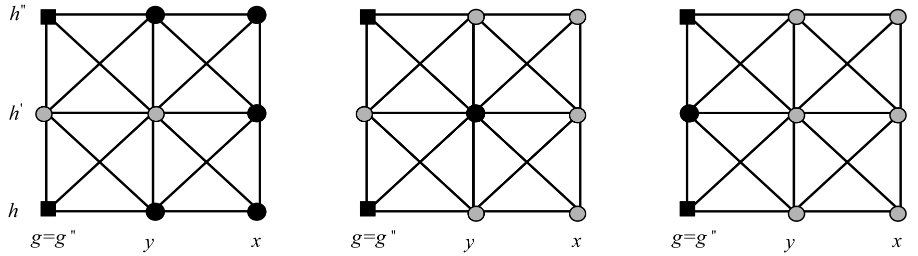

Fig. 6. Tolled walks for ⊐-argument. (Square vertices are in t-convex set and black vertices form a tolled walk between them.)

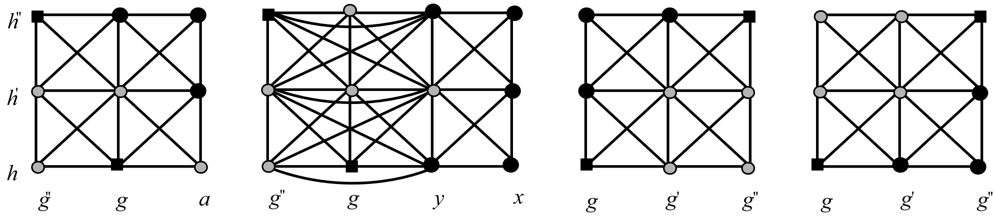

Fig. 7. Tolled walks from Case 2 and Case 3. (Square vertices are in t-convex set and black vertices form a tolled walk between them.)

$\in Y$ and $d_G(a, b) = 2$. By repeating ⊐-argument for (a, h') and (b, h') we get the same contradiction as above if there exists any vertex in H at distance 2 to h'. Therefore $h' \in U_H$ and ${}^gH \subset Y$. Also any common neighbor z of h and h'' in H must be a universal vertex and $G^z \subset Y$ as well as any common neighbor t of a and b in G must be a universal vertex and ${}^tH \subset Y$. With this we have

$$(U_G \times V(H)) \cup (V(G) \times U_H) \subseteq Y.$$

Suppose, in order to get a contradiction, that there exists a component C of the graph induced by $V(H) - U_H$ which is not complete. Let $h_1, h_2, h_3 \in V(C)$ where $h_1h_2, h_2h_3 \in E(C)$ and $h_1h_3 \notin E(C)$. This yields a tolled walk $(g, h_1), (w, h_2), (g, h_3)$ for every $w \in V(G)$ and so the whole layer G^{h_2} is contained in Y. Since $h_2 \notin U_H$ there exists a vertex $h_4 \in V(H)$ with $d_H(h_2, h_4) = 2$ and we can use ⊐-argument for any pair (g_1, h_2) and (g_2, h_2), where g_1 and g_2 are any nonadjacent vertices from $V(G) - U_G$ (note that such vertices exist since G is not complete). After using ⊐-argument for all pairs of such vertices in $V(G) - U_G$ we have that G^{h_4} is contained in Y. Repeating this process for all vertices in H at distance 2 from h_2 and at distance 2 from $h_4 \in V(H)$, we obtain a contradiction with the assumption that $Y \neq V(G \boxtimes H)$. We get the same contradiction by symmetry of all arguments if there exists such a component in the graph induced by $V(G) - U_G$. Therefore all components in graphs induced by $V(G) - U_G$ and $V(H) - U_H$ must be isomorphic to complete graphs.

To finish the proof of Case 1 suppose, in order to get a contradiction, that there exists $(p, q) \in Y$ where p and q are not universal vertices of G and H, respectively. Suppose that $p' \in V(G)$ is not adjacent to p in G and $q' \in V(H)$ is not adjacent to q in H. Since $(p, q), (g, q), (p', q), (p', h'), (g, q')$ is a tolled walk, also $(p', q) \in Y$ and we can apply ⊐-argument for (p, q) and (p', q) since $q' \in V(H)$ is not adjacent to q. This again results in $Y = V(G \boxtimes H)$ which is not possible by assumption. Hence $Y = (U_G \times V(H)) \cup (V(G) \times U_H)$ and (ii) follows.

Case 2: $gg'' \in E(G)$ and $d_H(h, h'') = 2$.

Clearly $(g, h'), (g'', h') \in Y$. If there exists a neighbor a of g which is nonadjacent to g'', then $(g'', h''), (g, h''), (a, h''), (a, h'), (g, h)$ is a tolled walk, see the left graph in Fig. 7. We can start to use ⊐-argument on vertices (a, h'') and (g'', h'') as in Case 1, which finally results in $Y = V(G \boxtimes H)$ which is not possible. If there exists a neighbor b of g'' which is nonadjacent to g, then $(g'', h''), (b, h'), (b, h), (g'', h), (g, h)$ is a tolled walk. We can again start to use ⊐-argument on vertices (b, h) and (g, h) which finally results in the same contradiction. Hence $N_G[g] = N_G[g'']$. If g is not a universal vertex, then there exists $x \in V(G)$ with $d_G(g, x) = 2$ and y is a common neighbor of g and

x. Note that also g'' is not adjacent to x since $N_G[g] = N_G[g'']$. A walk $(g, h), (y, h), (x, h), (x, h'), (x, h''), (y, h''), (g'', h'')$ is tolled, see the second to the left graph of Fig. 7, and $(x, h), (x, h'), (x, h'') \in Y$. Thus we can start with ⊐-argument on (x, h) and (x, h''), which yields the same contradiction again. Thus $g, g'' \in U_G$ and consequently $G^{h'} \subset Y$. Since G is not complete there exists an induced path $(u, h')(v, h')(w, h')$ and we have symmetric version of Case 1 for these three vertices.

Case 3: $d_G(g, g'') = 2 = d_H(h, h'')$.

Walks $(g, h), (g', h), (g'', h), (g'', h'), (g'', h'')$ and $(g, h), (g, h'), (g, h''), (g', h''), (g'', h'')$ are tolled, see the two graphs on right of Fig. 7. Thus we see that $\{g, g', g''\} \times \{h, h', h''\} \subseteq Y$. We can apply ⊐-argument for (g, h) and (g, h'') since $d_G(g, g'') = 2$. By repeating ⊐-argument for any other pair that occurs as in Case 1 we end with $Y = V(G \boxtimes H)$ which gives a final contradiction and completes the proof. □

As the strong product, the Cartesian product operation is also commutative and associative, see [20].

Theorem 5.3. *Let $G \Box H$ be a nontrivial, connected Cartesian product. A proper subset Y of $V(G \Box H)$ which does not induce a complete graph is t-convex if and only if $Y = V(G_1) \times V(H_1)$ where one factor, say H_1, equals H, which is a complete graph, and G_1 is isomorphic to P_k, $k \geq 2$, where every inner vertex of P_k has degree 2 in G.*

Proof. Let Y be a proper t-convex subset of $V(G \Box H)$, which does not induce a complete graph. If $Y \neq V(G_1 \Box H_1)$ for some subgraphs G_1 and H_1 of G and H, respectively, then there exists $(g, h) \notin Y$ such that $(g, h'), (g', h) \in Y$ for some $h' \in V(H) - \{h\}$ and $g' \in V(G) - \{g\}$. Let P be a shortest $(g, h), (g', h)$-path in G^h and Q be a shortest $(g, h), (g, h')$-path in ${}^gH$. Concatenation of P and Q gives a shortest $(g, h'), (g', h)$-path in $G \Box H$, which is also a tolled walk, a contradiction with t-convexity of Y. Hence $Y = V(G_1 \Box H_1)$.

Suppose next that $|V(G_1)| = 1$. Since Y induces a non-complete graph, we have $|V(H_1)| \geq 3$ and at least two vertices of $V(H_1)$ are nonadjacent. Suppose that $h, h', h'' \in V(H_1)$, h and h'' are nonadjacent and h' is their common neighbor. For a neighbor g' of $g \in V(G)$ is $(g, h), (g', h), (g', h'), (g', h''), (g, h'')$ a tolled walk in $G \Box H$ which starts and ends in Y, but is not contained in Y, see the left graph of Fig. 8. This is a contradiction, which implies that $|V(G_1)| > 1$ and by commutativity of the Cartesian product also $|V(H_1)| > 1$.

Suppose now that $V(G_1) \neq V(G)$ and $V(H_1) \neq V(H)$. We may choose the notation in such a way, that g has a neighbor g'' outside of $V(G_1)$ and h a neighbor h'' outside of $V(H_1)$. Let $gg' \in E(G_1)$ and $hh' \in E(H_1)$. If $h'h'' \in E(H)$, then $(g', h), (g', h''), (g, h''), (g, h')$ is a tolled walk violating t-convexity of Y, see the middle graph of Fig. 8. Similarly, if $h'h'' \notin E(H)$, then

$$(g', h), (g', h''), (g, h''), (g'', h''), (g'', h), (g'', h'), (g, h')$$

is a tolled walk contradicting t-convexity of Y, see the right graph of Fig. 8. Hence the set of vertices of one factor, say $V(H_1)$, equals $V(H)$, and consequently $V(G_1)$ is a proper subset of $V(G)$ by the assumption. Also if there exist $h, h'' \in V(H)$ which are not adjacent (and have h' as a common neighbor) and $gg' \in E(G)$ where $g \in V(G_1)$ and $g' \in V(G - G_1)$, then the tolled walk

$$(g, h), (g', h), (g', h'), (g', h''), (g, h'')$$

violates t-convexity of Y, see the left graph of Fig. 8. Therefore, H must be a complete graph.

If G_1 is not a path, then it is a cycle or there exists a vertex of G_1 with at least three neighbors in G_1. First, suppose G_1 is isomorphic to C_k for $k \geq 3$, and let $v \in V(G_1)$ be a vertex with a neighbor u in $G - G_1$. Let x and y be neighbors of v on G_1. If x and y are nonadjacent with u, then the walk $W : (x, h), (v, h), (u, h), (u, h'), (v, h'), (y, h')$ is a tolled walk which violates t-convexity of Y for any $hh' \in E(H)$, see the upper left graph of Fig. 9. If x or y are adjacent to u, then we shorten W for (v, h) or (v, h'), respectively, and still get the contradiction with t-convexity of Y.

If there exists $v \in V(G_1)$ with $\deg_{G_1}(v) \geq 3$, then let u, x, y be the neighbors of v in G_1. We can assume without loss of generality that either v has a neighbor $z \notin V(G_1)$ or v is the closest to a vertex w outside G_1 among all vertices in G_1 with more than three neighbors from G_1. In addition we may

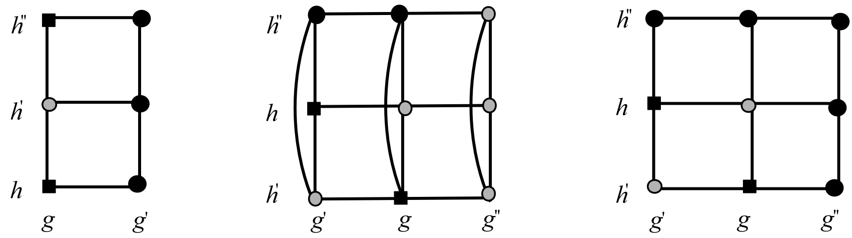

Fig. 8. Square vertices are in t-convex set and black vertices form a tolled walk between them.

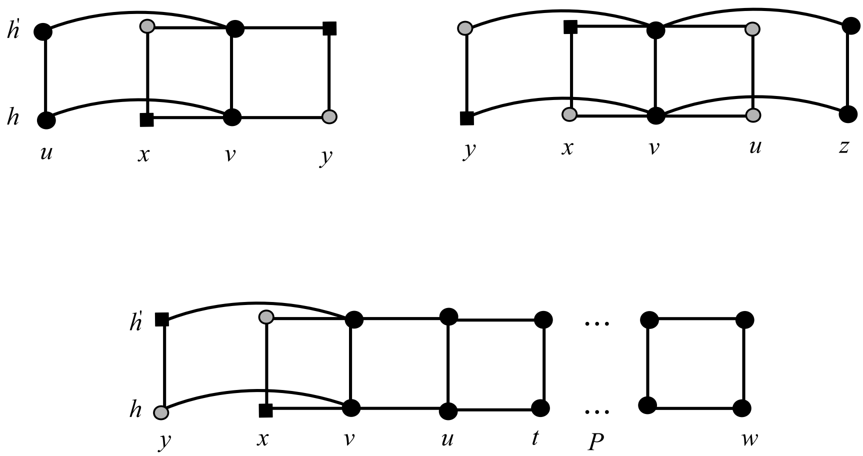

Fig. 9. Square vertices are in t-convex set and black vertices form a tolled walk between them.

also choose the notation so that the shortest v, w-path P contains u. First, let $vz \in E(G)$ for $z \notin V(G_1)$. The walk $W : (y, h), (v, h), (z, h), (z, h'), (v, h'), (x, h')$ is a tolled walk contradicting the t-convexity of Y whenever y and x are not adjacent to z, see the upper right graph of Fig. 9. If x or y are adjacent to z, then we shorten W for (v, h') or (v, h), respectively, and get the same contradiction.

In the second case we let the walk W start with $(x, h), (v, h), (u, h)$, and then continue along the vertices of P in the layer G^h to reach (w, h), then move to (w, h'), and follow P back in the layer $G^{h'}$ to (u, h'), and end with $(v, h'), (y, h')$, see the lower graph of Fig. 9. Note that, since P is a shortest path, x and y can be adjacent (in addition to v) only to u and to the neighbor $t \neq v$ of u on P. If none of these edges appears, then W is a tolled walk (by the choice of v) that contradicts t-convexity of Y whenever $hh' \in E(G)$. If some of the pairs xu, xt, yu, yt form edges in G, then we shorten W in a similar way as in the above cases and again obtain a contradiction. We derive that G_1 is isomorphic to P_k.

If there exists an inner vertex $v \in V(P_k)$ with $\deg_G(v) \geq 3$, then let x, y, u be neighbors of v, where $x, y \in P_k$ and $u \notin P_k$. A walk

$$W : (x, h), (v, h), (u, h), (u, h'), (v, h'), (y, h'),$$

$hh' \in E(H)$, is again a tolled walk whenever y and x are not adjacent to u, see the upper left graph of Fig. 8. Again we can shorten this walk if ux or uy are edges of G and obtain a tolled walk, which provides the final contradiction with t-convexity of Y for the proof of this implication.

For the converse suppose that $Y = V(G_1 \Box H)$ where $H \cong K_n$, G_1 is isomorphic to P_k and every inner vertex of G_1 has degree 2 in G. Let $P = v_1, \ldots, v_k$. Clearly $T_{G \Box H}((v_i, h), (v_i, h')) \subseteq Y$ for any $h, h' \in V(H)$ and every $i \in \{1, \ldots, k\}$ since H is complete. So let $i < j$ and let (u, v) be an arbitrary vertex of $V(G \Box H) - Y$. If u is closer to v_i than to v_j in G, then $N[(v_i, h)] - \{(u, v)\}$ separates (u, v) from (v_j, h'). Otherwise, if u is closer to v_j than to v_i in G, then $N[(v_j, h')] - \{(u, v)\}$ separates (u, v) from (v_i, h). Hence Y is t-convex by Proposition 2.4. $\Box$

Note that a proper non-complete t-convex set of vertices of $G\Box H$ does not contain any extreme vertices with respect to t-convexity.

6. Concluding remarks

In this paper we introduce a new type of interval-convexity in graphs that we call toll convexity. The main result of the paper is that a graph is a convex geometry with respect to toll convexity if and only if it is an interval graph. Thus a new convexity type characterization of the well-known class of interval graphs is presented. In addition, we prove that the toll number of every tree different than caterpillar equals 2 or 3, and that the toll hull number of every tree different than caterpillar equals 2. We also present an example of a non-complete graph showing that the Carathéodory number can be greater than 2, which differs from the situation in geodesic and monophonic convexity. It would be interesting to investigate other convexity type invariants with respect to the new convexity, such as the convexity number, the Radon number, and the Helly number. In addition, the concepts of Carathéodory number, the toll number, and the toll hull number should be further explored.

At the end of the paper we characterize toll convex sets of several products of graphs, i.e. the lexicographic, the strong, and the Cartesian product. We conclude the paper with the open problem of characterizing proper toll convex subsets in the direct product of two graphs.

Acknowledgments

We thank both anonymous referees for their useful remarks. This work was realized in the frame of MINCYT (Argentina)–ARRS (Slovenia) (BI-AR/12-14-013) bilateral projects (Graph operators and products), and the support of both agencies is appreciated. Boštjan Brešar, Tanja Gologranc, Tadeja Kraner Šumenjak and Iztok Peterin were also supported by the Slovenian Ministry of Education, Science and Sport under grant P1-0297.

References

[1] K. Adaricheva, Representing finite convex geometries by relatively convex sets, European J. Combin. 37 (2014) 68–78.
[2] L. Alcón, A note on dominating paths (submitted for publication).
[3] B.S. Anand, M. Changat, S. Klavžar, I. Peterin, Convex sets in lexicographic products of graphs, Graphs Combin. 28 (2012) 77–84.
[4] A. Brandstädt, V.B. Le, J.P. Spinrad, Graphs Classes: A Survey, SIAM, Philadelphia, 1999.
[5] B. Brešar, M. Kovše, A. Tepeh, Geodetic sets in graphs, in: M. Dehmer (Ed.), Structural Analysis of Complex Networks, Birkhäuser, Boston, 2010, pp. 197–218.
[6] J. Cáceres, A. Marquez, M.L. Puertas, Steiner distance and convexity in graphs, European J. Combin. 29 (2008) 726–736.
[7] J.R. Calder, Some elementary properties of interval convexities, J. Lond. Math. Soc. (2) 3 (1971) 422–428.
[8] C. Carathéodory, Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen, Rend. Circ. Mat. Palermo 32 (1911) 193–217.
[9] M. Changat, S. Klavžar, H.M. Mulder, The all-paths transit function of a graph, Czechoslovak Math. J. 51 (2001) 439–448.
[10] M. Changat, J. Mathew, On triangle path convexity in graphs, Discrete Math. 206 (1999) 91–95.
[11] M. Changat, G.N. Prasanth, I.M. Pelayo, The longest path transit function of a graph and betweenness, Util. Math. 82 (2010) 111–127.
[12] G. Chartrand, F. Harary, P. Zhang, On the geodetic number of a graph, Networks 39 (2002) 1–6.
[13] G. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg 25 (1961) 71–76.
[14] P. Duchet, Convex sets in graphs II. Minimal path convexity, J. Combin. Theory Ser. B 44 (1986) 307–316.
[15] M.G. Everett, S.B. Seidman, The hull number of a graph, Discrete Math. 57 (1985) 217–223.
[16] M. Farber, R.E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Algebr. Discrete Math. 7 (1986) 433–444.
[17] M. Farber, R.E. Jamison, On local convexity in graphs, Discrete Math. 66 (1987) 231–247.
[18] D.R. Fulkerson, O.A. Gross, Incidence matrices and interval graphs, Pacific J. Math. 15 (1965) 835–855.
[19] J. Gimbel, End vertices in interval graphs, Discrete Appl. Math. 21 (1988) 257–259.
[20] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs, second ed., CRC Press, Boca Raton, FL, 2011.
[21] C. Hernando, T. Jiang, M. Mora, I. Pelayo, C. Seara, On the Steiner, geodetic and hull numbers of graphs, Discrete Math. 293 (2005) 139–154.
[22] J. Köbler, S. Kuhnert, B. Laubner, O. Verbitsky, Interval graphs: canonical representation in logspace, in: Lecture Notes in Comput. Sci., vol. 6198, 2010, pp. 384–395.
[23] C.B. Lekkerkerker, J.C. Boland, Representation of finite graphs by a set of intervals on the real line, Fund. Math. 51 (1962) 45–64.
[24] T.A. McKee, F.R. McMorris, Topics in Intersection Graph Theory, SIAM, Philadelphia, 1999.
[25] H.M. Mulder, Transit functions on graphs (and posets), in: M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar (Eds.), Convexity in Discrete Structures, in: Lecture Notes Ser., vol. 5, Ramanujan Math. Soc., 2008, pp. 117–130.
[26] I.M. Pelayo, Geodesic Convexity in Graphs, Springer, New York, 2013.
[27] I. Peterin, Intervals and convex sets in strong product of graphs, Graphs Combin. 29 (2013) 705–714.
[28] M.L.J. van de Vel, Theory of Convex Structures, North Holland, Amsterdam, 1993.