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# Analytical relation between two chaos indicators: FLI and MEGNO

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## ABSTRACT

We report an intrinsic relation between the mean exponential growth factor of nearby orbits (MEGNO) and the fast Lyapunov indicator (FLI), two variational methods that have been widely applied to stability issues in astronomy. For both continuous-time and discrete-time systems, we arrive at an analytic formula that expresses the MEGNO in terms of the FLI and its time-average. This connection, unknown for more than 10 years, allows us to understand the differences and similarities in the performance of both indicators.

**Key words:** chaos – instabilities – methods: analytical.

## 1 INTRODUCTION

Many astrophysical systems can be modelled by Hamiltonian differential equations. As examples we have solar systems (Quillen 2006; Makó et al. 2010), planets with moons (Rambaux, van Hoolst & Karatekin 2011), star clusters (Fujii et al. 2007), galaxies (Quillen 2003), clusters of galaxies and every other astronomical entity whose constitutive parts are related to each other by the gravitational force. Moreover, almost all of these deterministic conservative systems present divided phase space, in which there is coexistence of chaotic and regular orbits, such as the ones studied by Altmann et al. (2006).

As the stability of multidimensional deterministic systems with divided phase space is yet an unsolved problem and as its elucidation would help in understanding many astrophysical phenomena, in recent years it has become imperative to develop tools to comprehend the behaviour of non-integrable systems. For examples of stability studies in astronomy, see Kandrup et al. (2005); Nagy, Süli & Érdi (2006); Sándor, Kley & Klagyivik (2007); Goździewski, Breiter & Borczyk (2008); Libert & Tsiganis (2009) and Semerák & Suková (2010).

Among the existing dynamical methods, there is a group of chaos indicators that use the solution of the variational equations as input data. These methods have a strong theoretical background, are easy to implement numerically and perform the task quite fast. The fast Lyapunov indicator (FLI) and the mean exponential growth factor of nearby orbits (MEGNO) belong to this category and both have been largely used to estimate the degree of chaos in different astrophysical environments, according to the following recent publications: Pilat-Lohinger, Funk & Dvorak (2003); Astakhov & Farrelly (2004); Astakhov, Lee & Farrelly (2005); Breiter, Fouchard & Ratajczak (2008); Gayon, Marzari & Scholl (2008) and Goździewski & Migaszewski (2009).

This Letter is organized as follows. In Sections 2 and 3, we give the definitions of the FLI and MEGNO, respectively. In Section 4, we demonstrate the analytical relation between these two quantities for continuous-time systems and show how this discovery allows a comparison against each other. We also state the analogous link for the discrete-time case. Section 5 focuses on some numerical experiments with a symplectic map to test the validity of our results. In Section 6, we summarize this Letter. Finally, in Appendix A, we give a simple demonstration of the MEGNO–FLI relation for mappings.

## 2 DEFINITION OF THE FAST LYAPUNOV INDICATOR

Since its presentation in Froeschlé, Lega & Gonczi (1997b), the FLI has been slightly modified in some applications in order to answer different questions. However, the essence of the indicator has been basically kept the same.

It was originally conceived (Froeschlé, Gonczi & Lega 1997a) as the supremum of the norms of an evolving basis of deviation vectors of initial unitary length. Shortly afterwards, it was replaced in Froeschlé & Lega (2000) by a computationally cheaper definition that required the evolution of only one deviation vector ($\boldsymbol{v}$) of length $\delta(t) \equiv ||\boldsymbol{v}(t)||$:

$$\mathrm{FLI}(t) \equiv \ln ||\boldsymbol{v}(t)|| = \ln \delta(t). \tag{1}$$

This expression has been used to obtain analytical results by Froeschlé & Lega (2000) and Guzzo, Lega & Froeschlé (2002). Consequently, we will stick to it within this Letter, being the same for both Hamiltonian flows and symplectic maps.

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For a future utilization, we keep in mind that the time-average of the FLI in the interval [0, $t$] for flows and maps is, respectively, given by

$$\overline{\mathrm{FLI}}(t) \equiv \frac{1}{t}\int_0^t \mathrm{FLI}(s)\mathrm{d}s = \frac{1}{t}\int_0^t \ln\delta(s)\mathrm{d}s \quad (2)$$

and

$$\overline{\mathrm{FLI}}(t) \equiv \frac{1}{t}\sum_{k=0}^{t} \mathrm{FLI}(k) = \frac{1}{t}\sum_{k=0}^{t} \ln\delta(k). \quad (3)$$

## 3 DEFINITION OF THE MEGNO

The MEGNO was first introduced by Cincotta & Simó (2000), for continuous-time systems, as

$$Y(t) \equiv \frac{2}{t}\int_0^t \frac{\dot{\delta}(s)}{\delta(s)} s\mathrm{d}s. \quad (4)$$

It is twice the time-weighted average of the relative divergence of orbits and it was created by taking advantage of the knowledge of the basic dynamics in Hamiltonian systems (Cincotta & Simó 2000). One of the objectives of its invention was to enhance the exponential sensitivity to initial conditions of chaotic orbits and, at the same time, to allow for a clear classification between chaotic, Kolmogorov–Arnold–Moser (KAM) regular and resonantly regular orbits.

Its discrete-time version appeared in Cincotta et al. (2003) according to the following expression:

$$Y(t) \equiv \frac{2}{t}\sum_{k=1}^{t} k \ln\left(\frac{\delta(k)}{\delta(k-1)}\right). \quad (5)$$

## 4 THE LINK

Now we will see how deeply related are the two chaos indicators defined above. Let us start analysing the case of Hamiltonian flow by rewriting equation (4) in the following way:

$$Y(t) = \frac{2}{t}\int_0^t s\frac{1}{\delta(s)}\frac{\mathrm{d}\delta(s)}{\mathrm{d}s}\mathrm{d}s = \frac{2}{t}\int_0^t s\frac{\mathrm{d}}{\mathrm{d}s}\left[\ln\delta(s)\right]\mathrm{d}s. \quad (6)$$

Integrating by parts, choosing $u(s) = s$ and $\mathrm{d}v(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left[\ln\delta(s)\right]\mathrm{d}s$, we have that $\mathrm{d}u(s) = \mathrm{d}s$ and $v(s) = \ln\delta(s)$, so that

$$Y(t) = \frac{2}{t}\left[u(s)v(s)\Big|_0^t - \int_0^t v(s)\mathrm{d}u(s)\right] \quad (7)$$

$$= 2\left[\ln\delta(t) - \frac{1}{t}\int_0^t \ln\delta(s)\mathrm{d}s\right], \quad (8)$$

where we have used $\delta(0) = 1$. Comparing this equation with equations (1) and (2), we arrive at the fact that the MEGNO is two times the difference between the FLI and its time-average in [0, $t$]:

$$Y(t) = 2\left[\mathrm{FLI}(t) - \overline{\mathrm{FLI}}(t)\right]. \quad (9)$$

This result allows us to understand two facts that have recently been mentioned in the literature.

One point is that the MEGNO criterion takes advantage of the dynamical information on the evolution of the tangent vector along the complete orbit, as pointed out by Valk et al. (2009) and Hinse et al. (2010). Equation (9) tells us exactly in which way it includes this information; at each time, the MEGNO subtracts from the FLI the average value of the latter.

The other point to discuss, explicitly mentioned in Breiter et al. (2008) and Barrio, Borczyk & Breiter (2009), is the reason why the MEGNO gives the degree of the chaoticity of an orbit in an absolute scale, while the FLI just gives relative values; that is, in the case of regular orbits, the MEGNO tends asymptotically towards a constant value (2), while the FLI behaves logarithmically, not allowing a time-independent criterion to establish the threshold that separates chaotic from regular orbits. That is why if we want to decide whether an orbit is chaotic or not, at a fixed time $t$, applying the FLI criterion, we must first conduct tests of its behaviour on orbits with already known regular character.

In order to explain this situation with an example, let us analyse the case of an ideal KAM regular orbit. Due to the well-known differential rotation that happens in non-linear integrable systems, we do the approximation that for ordered motion in KAM orbits the norms of the deviation vectors satisfy $\delta(t) = 1 + \lambda t$ ($\lambda \neq 0$, a constant). In this case we have that

$$\mathrm{FLI}(t) = \ln(1 + \lambda t) \quad (10)$$

and

$$\overline{\mathrm{FLI}}(t) = \ln(1 + \lambda t) + \frac{\ln(1 + \lambda t)}{\lambda t} - 1. \quad (11)$$

Therefore, considering equation (9),

$$Y(t) = 2\left[1 - \frac{\ln(1 + \lambda t)}{\lambda t}\right], \quad (12)$$

so we rediscover the already mentioned asymptotic limit of the MEGNO for regular orbits.

On the other hand, in the case of an ideal chaotic orbit, with $\delta(t) = \mathrm{e}^{\sigma t}$ ($\sigma$ being the maximum characteristic Lyapunov exponent), the MEGNO–FLI relation allows us to prove that both indicators perform similarly. In fact, both of them behave linearly with time with a slope equal to $\sigma$.

In the case of symplectic maps, the relation between the MEGNO and the FLI is slightly different. In Appendix A, we demonstrate that the MEGNO for maps satisfies equation (9) with an error of the order of $\sigma$, that is,

$$Y(t) = 2\left[\mathrm{FLI}(t) - \overline{\mathrm{FLI}}(t)\right] + \mathcal{O}(\sigma). \quad (13)$$

It is important to remark that in both regular and chaotic situations, the value of $\sigma$ is negligible with respect to the other terms, as we show with numerical experiments in the next section.

## 5 EXPERIMENTS WITH THE STANDARD MAP

In these experiments, we test the validity of equation (13) for both ordered and disordered motion, working in the mod $2\pi$ standard map (SM), where the time-evolution of the two variables $I$ and $\theta$ is, respectively, determined by

$$I_{t+1} = I_t + K\sin\theta_t \quad \text{and} \quad \theta_{t+1} = \theta_t + I_{t+1}. \quad (14)$$

We have worked with a parameter value $K = 0.90$, which is a little smaller than the critical value (Greene 1979) for which the last invariant KAM torus has been destroyed.

In Fig. 1, we show, for $t \in [0, 2000]$, a chaotic and a regular orbit with the red and blue points, respectively. The chaotic trajectory belongs to the separatrix, with the initial condition

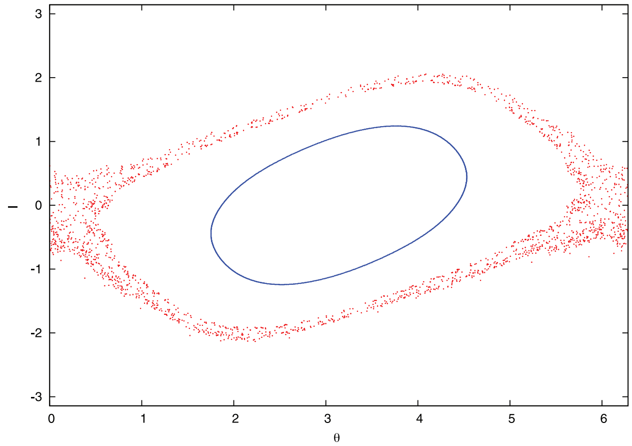


**Figure 1.** A chaotic (red) and a regular (blue) orbit of the SM ($K = 0.90$).

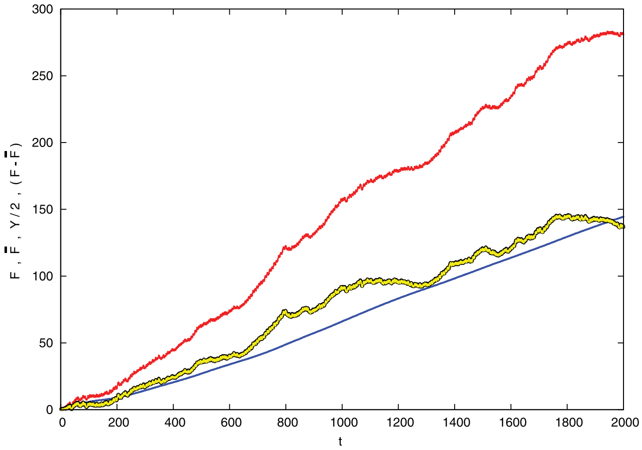


**Figure 2.** Values of FLI($t$) (red), $\overline{\mathrm{FLI}}(t)$ (blue), $Y(t)/2$ (black) and FLI($t$) − $\overline{\mathrm{FLI}}(t)$ (yellow) for a chaotic orbit of the SM ($K = 0.90$). The abbreviation F ≡ FLI has been applied in the label of the vertical axis.

$(I_0, \theta_0) = (\pi/100, 0.628)$, while the regular trajectory is in the oscillatory regime, with the initial condition $(I_0, \theta_0) = (\pi/100, 1.884)$. In Fig. 2, we display, for the former orbit, the values of FLI($t$), $\overline{\mathrm{FLI}}(t)$, $Y(t)/2$ and FLI($t$) − $\overline{\mathrm{FLI}}(t)$, in the colours red, blue, black and yellow, respectively. We note that the agreement with equation (13) is perfect, even from the first time-steps. In Fig. 3, we can see the evolution of the same quantities, in the same colours, for the other orbit. In this case, the initial difference is more evident between $Y(t)/2$ and FLI($t$) − $\overline{\mathrm{FLI}}(t)$, for times shorter than one hundred units.

## 6 CONCLUSIONS

In this Letter, we have shown the strong analytical link between the MEGNO and the FLI, both rather well-known indicators of chaos and order in astronomical systems. In addition, we have studied the implications of this connection and we have presented numerical results that support our theoretical estimations.

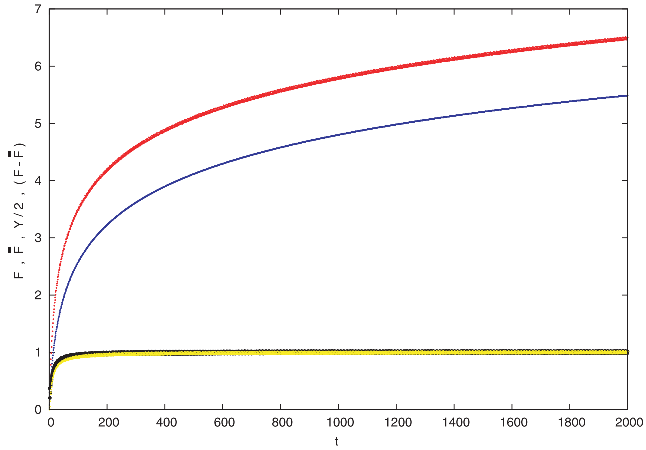


**Figure 3.** Values of FLI($t$) (red), $\overline{\mathrm{FLI}}(t)$ (blue), $Y(t)/2$ (black) and FLI($t$) − $\overline{\mathrm{FLI}}(t)$ (yellow) for a regular orbit of the SM ($K = 0.90$). The abbreviation F ≡ FLI has been applied in the label of the vertical axis.

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## APPENDIX A: DEMONSTRATION OF THE MEGNO–FLI RELATION FOR MAPPINGS

Here, we proceed to show the validity of equation (13). Starting from equation (5)

$$\begin{aligned} Y(t) &= \frac{2}{t}\left[\sum_{k=1}^{t} k \ln \delta(k) - \sum_{k=1}^{t} k \ln \delta(k-1)\right] \\ &= \frac{2}{t}\left[\sum_{k=1}^{t} k \ln \delta(k) - \sum_{k=0}^{t-1} (k+1) \ln \delta(k)\right] \\ &= 2\left[\ln \delta(t) - \frac{1}{t}\sum_{k=0}^{t} \ln \delta(k) + \frac{1}{t}\ln \delta(t)\right], \end{aligned} \tag{A1}$$

where in the last step we have again used $\delta(0) = 1$. Then, comparing with equations (1) and (3), we arrive at

$$Y(t) = 2\left[\mathrm{FLI}(t) - \overline{\mathrm{FLI}}(t)\right] + (2/t) \ln \delta(t), \tag{A2}$$

and as the second term on the right-hand side tends, asymptotically, to $2\sigma$, we recover equation (13).

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