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Journal of Computational and Applied Mathematics 223 (2009) 177–197

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Spectral approximation of variationally-posed eigenvalue problems by nonconforming methods

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Received 18 December 2006; received in revised form 21 December 2007

To the memory of our friend Nelly Ferretti

Abstract

This paper deals with the nonconforming spectral approximation of variationally posed eigenvalue problems. It is an extension to more general situations of known previous results about nonconforming methods. As an application of the present theory, convergence and optimal order error estimates are proved for the lowest order Crouzeix–Raviart approximation of the eigenpairs of two representative second-order elliptical operators.

Keywords: Nonconforming methods; Spectral approximation; Eigenvalue problems; Steklov eigenvalue problem

1. Introduction

The general results on spectral approximations for compact operators were first obtained in [5,16]. These results have been extended in [11,12] to consider the case of conforming discretizations of noncompact operators.

Nonconforming methods were also studied. The first approach was proposed in [15] and it is restricted to compact operators.

Later, in order to prove double order for the convergence of eigenfrequencies in fluid-structure vibration problems, Rodríguez and Solomin [17] extended classical results about finite element spectral approximation to nonconforming methods for noncompact operators. However, their theory does not cover many other practical situations since it assumes that the continuous and discrete bilinear forms appearing in the variational formulation of the considered problem coincide.

Very recently, discontinuous Galerkin approximations of the spectrum of the Laplace operator have been analysed in [2]. To do that, the authors adapted the theory presented in [11] to deal with nonconforming approximations of elliptical second order operators with compact inverse. Moreover, Buffa and Perugia [7] presented a theoretical framework for the analysis of discontinuous Galerkin approximations of the Maxwell eigenproblem.

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2008.01.008

The goal of this paper is to obtain some abstract results of spectral approximation that can be applied to a wide class of nonconforming methods for either compact or noncompact operators. These results are obtained by introducing suitable modifications in the theory developed in [11,12]. According to the fact that the approximations considered are nonconforming, consistency terms appear in our estimates which could be seen as a generalization of previous results obtained in [17].

The motivations for considering nonconforming finite element methods are several. For example, to avoid the necessity of smooth elements in fourth order problems or to deal with constrained minimization problems. Also, there is a closed relationship between mixed methods and nonconforming finite element methods for second order elliptical problems (see [1]). This relationship can be further exploited for deriving efficient solvers for the mixed formulations (see [9,3]).

We mention also that the present theory allows the analysis of a large class of discontinuous finite element methods when they are used for the approximation of spectral problems. This justifies the generality of our abstract approach.

The outline of the paper is as follows. In Section 2 we introduce the class of variationally posed eigenvalue problems we will consider and we define the approximation methods for these problems. The abstract results are presented and proved in Section 3. Finally, in Section 4 we illustrate the application of our theory by considering the nonconforming approximation of the eigenvalues and eigenfunctions of two representative second order elliptical problems. The analysis is carried out for the lowest order Crouzeix–Raviart finite element space. As in the conforming case, the order of convergence obtained for the eigenvalues doubles that for the eigenfunctions. To the best of the authors' knowledge, these estimates have not been proved before.

2. Statement of the eigenvalue problem

Let X be a complex Hilbert function space with norm $|\cdot|$. Let V be a subspace of X, with norm $\|\cdot\|$, such that the inclusion $V \hookrightarrow X$ is continuous.

Consider the eigenvalue problem:

Find $\mu \in \mathbb{C}$, $u \neq 0$, $u \in V$, *such that*

$$a(u, v) = \mu b(u, v), \quad \forall v \in V \tag{2.1}$$

where $a : V \times V \to \mathbb{C}$ is a continuous and coercive sesquilinear form and $b : X \times X \to \mathbb{C}$ is a continuous sesquilinear form.

Let $\mathbf{T}$ be the linear operator defined by

$$\mathbf{T} : X \to V \hookrightarrow X$$
$$x \mapsto u,$$

where $u \in V$ is the solution of

$$a(u, y) = b(x, y), \quad \forall y \in V. \tag{2.2}$$

Since a is elliptic, b is continuous and $V \hookrightarrow X$, Lax–Milgram's Lemma allows us to conclude that $\mathbf{T}$ is a bounded linear operator. It is simple to show that μ is an eigenvalue of (2.1) if and only if $\lambda = 1/\mu$ is an eigenvalue of the operator $\mathbf{T}$ and the corresponding associated eigenfunctions u coincide.

Now, let $\{V_h\}$ be a family of finite dimensional function subspaces of X not contained in V and consider the spaces $V + V_h$. We equip each space $V + V_h$ with the norms $\|\cdot\|_h$ and we assume that

$$\|v\| = \|v\|_h, \quad \forall v \in V \tag{2.3}$$

$$(V + V_h, \|\cdot\|_h) \hookrightarrow (X, |\cdot|), \quad \text{uniformly on } h. \tag{2.4}$$

Then, we consider the following discrete eigenvalue problem:

Find $\mu_h \in \mathbb{C}$, $u_h \neq 0$, $u_h \in V_h$, *such that*

$$a_h(u_h, v) = \mu_h b_h(u_h, v), \quad \forall v \in V_h. \tag{2.5}$$

Let us remark that since $V_h \not\subset V$, (2.5) represents a nonconforming approximation to (2.1).

From now and on, we shall consider that the domain of definition of the approximate sesquilinear forms a_h and b_h is $V + V_h$. We also assume that both discrete forms are continuous on $V + V_h$ uniformly on h and that a_h is coercive on $V + V_h$ uniformly on h. Finally, we assume that

$$a_h(v, w) = a(v, w), \quad \forall v, w \in V \tag{2.6}$$

$$b_h(v, w) = b(v, w), \quad \forall v, w \in V. \tag{2.7}$$

Then, we define the discrete analogue of the operator $\mathbf{T}$ as follows:

$$\mathbf{T}_h : X \to V_h$$
$$x \mapsto u_h,$$

where $u_h \in V_h$ is the solution of

$$a_h(u_h, y) = b_h(x, y), \quad \forall y \in V_h. \tag{2.8}$$

Once again, because of Lax–Milgram's Lemma, the operator $\mathbf{T}_h$ is bounded uniformly on h. As in the continuous case, it is simple to show that μ_h is an eigenvalue of problem (2.5) if and only if $\lambda_h = 1/\mu_h$ is an eigenvalue of the operator $\mathbf{T}_h$, and the corresponding associated eigenfunctions u_h coincide.

3. Spectral approximation

First, we introduce some notation that will be used in the sequel. For further information on eigenvalue problems we refer the reader to [4]. From now on, C denotes a generic constant not necessarily the same at each occurrence but always independent of h.

We denote by $\rho(\mathbf{T})$ the resolvent set of $\mathbf{T}$ and by $\sigma(\mathbf{T})$ the spectrum of $\mathbf{T}$. Now, for any $z \in \rho(\mathbf{T})$, $R_z(\mathbf{T}) = (z-\mathbf{T})^{-1}$ defines the resolvent operator.

Let us consider the restrictions $\mathbf{T}|_V$ and $\mathbf{T}|_{V+V_h}$. It can be proved that the knowledge of the spectrum of $\mathbf{T}|_{V+V_h}$ gives complete information about the spectrum of $\mathbf{T}|_V$. The proof closely follows the arguments used in the proof of Lemma 4.1 in [6].

Lemma 3.1. *The spectra of* $\mathbf{T}|_V$ *and* $\mathbf{T}|_{V+V_h}$ *satisfy*

$$\sigma(\mathbf{T}|_V) \cup \{0\} = \sigma(\mathbf{T}|_{V+V_h}).$$

Further, for any $z \in \rho(\mathbf{T}|_{V+V_h})$ *there is a constant* C*, independent of* h*, such that*

$$\|R_z(\mathbf{T}|_{V+V_h})\|_h \le C.$$

Proof. Let $z \notin \sigma(\mathbf{T}|_V), z \neq 0$. We are going to prove that $(z - \mathbf{T}|_{V+V_h}) : V + V_h \to V + V_h$ is one to one and onto. Suppose that $(z - \mathbf{T}|_{V+V_h})x = 0$. Since $\mathbf{T}|_{V+V_h}(V + V_h) \subset V, x = \frac{1}{z}\mathbf{T}|_{V+V_h}x \in V$ and then $(z - \mathbf{T}|_{V+V_h})x = (z - \mathbf{T}|_V)x = 0$. Since $z \notin \sigma(\mathbf{T}|_V)$, we can conclude that $x = 0$. Hence, $(z - \mathbf{T}|_{V+V_h})$ is one to one. Now, given $y \in V + V_h$ we can take $x = \frac{1}{z}(y + (z - \mathbf{T}|_V)^{-1}\mathbf{T}|_{V+V_h}y)$ and we have $(z - \mathbf{T}|_{V+V_h})x = y$. So, $(z - \mathbf{T}|_{V+V_h})$ is onto. Therefore, because of the open mapping theorem, $z \notin \sigma(\mathbf{T}|_{V+V_h})$.

Conversely, let $z \notin \sigma(\mathbf{T}|_{V+V_h})$. First, we have that $z \neq 0$ since $\mathbf{T}|_{V+V_h}(V + V_h) \subset V$ and so $\mathbf{T}|_{V+V_h}$ is not onto. Next, given $y \in V \subset V + V_h$, there exist a unique $x \in V + V_h$ such that $y = (z - \mathbf{T}|_{V+V_h})x$. Furthermore, $x = \frac{1}{z}(y + \mathbf{T}|_{V+V_h}x) \in V$. Hence, x is the unique element in V such that $(z - \mathbf{T}|_V)x = (z - \mathbf{T}|_{V+V_h})x = y$. Therefore, $(z - \mathbf{T}|_V) : V \to V$ is invertible and $z \notin \sigma(\mathbf{T}|_V)$.

On the other hand, given $y \in V + V_h$, it is easy to show that $x = \frac{1}{z}(y + (z - \mathbf{T}|_V)^{-1}\mathbf{T}|_{V+V_h}y)$ is the unique element in $V + V_h$ such that $y = (z - T|_{V+V_h})x$. Now, since $T|_{V+V_h}y \in V$, $(z - \mathbf{T}|_V)^{-1}\mathbf{T}|_{V+V_h}y \in V$ and so, in view of our assumption (2.3), we can write $\|(z - \mathbf{T}|_V)^{-1}\mathbf{T}|_{V+V_h}y\|_h = \|(z - \mathbf{T}|_V)^{-1}\mathbf{T}|_{V+V_h}y\|$. Then, we obtain

$$\|x\|_h \le \frac{1}{|z|}\left(\|y\|_h + \|(z - \mathbf{T}|_V)^{-1}\mathbf{T}|_{V+V_h}y\|\right) \le \frac{1}{|z|}\left(\|y\|_h + \|(z - \mathbf{T}|_V)^{-1}\|\|\mathbf{T}|_{V+V_h}y\|\right).$$

Now, the continuous inclusion (2.4) implies that

$$\|\mathbf{T}|_{V+V_h} y\| \le C|y| \le C\|y\|_h.$$

Finally, combining the last two inequalities above, we can conclude the proof. □

Let λ be a nonzero isolated eigenvalue of $\mathbf{T}|_{V+V_h}$ with algebraic multiplicities m. Let Γ be a circle in the complex plane centred at λ which lies in $\rho(\mathbf{T}|_{V+V_h})$ and which encloses no other points of $\sigma(\mathbf{T}|_{V+V_h})$. The continuous spectral projector, $\mathbf{E} : V + V_h \to V + V_h$, relative to λ, is defined by

$$\mathbf{E} = \frac{1}{2\pi i}\int_\Gamma R_z(\mathbf{T}|_{V+V_h})\mathrm{d}z.$$

We assume the following properties to be satisfied:

P1:

$$\lim_{h\to 0} \|(\mathbf{T} - \mathbf{T}_h)|_{V_h}\|_h = 0.$$

P2. For each function x of $\mathbf{E}(V + V_h)$,

$$\lim_{h\to 0}\left(\inf_{x_h\in V_h} \|x - x_h\|_h\right) = 0.$$

P3:

$$\lim_{h\to 0} \|(\mathbf{T} - \mathbf{T}_h)|_{\mathbf{E}(V+V_h)}\|_h = 0.$$

We are going to give an extension of the theory developed in [11] to deal with nonconforming methods. Most of the proofs of the results stated below are slight modifications of those in [11], taking care of the fact that, here, $\|\cdot\|_h$ denotes the discrete norm associated with the nonconforming spaces.

Lemma 3.2. *Let $\mathcal{G}$ be a closed subset of $\rho(\mathbf{T}|_{V+V_h})$. Under assumption* **P1***, there exist positive constants C and h_0, independent of h, such that*

$$\|(z - \mathbf{T}_h|_{V_h})^{-1}\|_h \le C, \quad \forall z \in \mathcal{G},\ \forall h < h_0.$$

Proof. The proof is identical to that of Lemma 1 in [11]. □

Theorem 3.3. *Let $\Omega \subset \mathbb{C}$ be a compact set not intersecting $\sigma(\mathbf{T}|_{V+V_h})$. There exist $h_0 > 0$ such that, if $h < h_0$, then Ω does not intersect $\sigma(\mathbf{T}_h|_{V_h})$.*

Proof. The proof is a direct consequence of assumption **P1**, as it is shown in Theorem 1 in [11]. □

So, in virtue of the previous theorem, if h is small enough, $\Gamma \subset \rho(\mathbf{T}_h|_{V_h})$ and the discrete spectral projector, $\mathbf{E}_h : V_h \to V_h$, can be defined by

$$\mathbf{E}_h = \frac{1}{2\pi i}\int_\Gamma R_z(\mathbf{T}_h|_{V_h})\mathrm{d}z.$$

Let us recall the definition of the gap $\widehat{\delta}$ between two closed subspaces, Y and Z, of $V + V_h$. We define

$$\widehat{\delta}(Y, Z) := \max\{\delta(Y, Z), \delta(Z, Y)\},$$

where

$$\delta(Y, Z) := \sup_{\substack{y\in Y\\ \|y\|_h=1}}\left(\inf_{z\in Z}\|y - z\|_h\right).$$

The following theorem implies uniform convergence of $\mathbf{E}_h|_{V_h}$ to $\mathbf{E}|_{V_h}$ as h goes to 0:

Theorem 3.4. *Under assumption* **P1**,

$$\lim_{h\to 0} \|(\mathbf{E} - \mathbf{E}_h)|_{V_h}\|_h = 0.$$

Proof. It follows combining Lemma 3.2 with assumption **P1** and is essentially identical to that of Lemma 2 in [11]. □

Theorem 3.5. *Under the assumption* **P1**, *for all* $x \in \mathbf{E}_h(V_h)$ *there holds*

$$\lim_{h\to 0} \delta(x, \mathbf{E}(V + V_h)) = 0.$$

Proof. It is a direct consequence of Theorem 3.4. □

Theorem 3.6. *Under the assumptions* **P1** *and* **P2**, *for all* $x \in \mathbf{E}(V + V_h)$ *there holds*

$$\lim_{h\to 0} \delta(x, \mathbf{E}_h(V_h)) = 0.$$

Proof. The proof is identical to that of Theorem 3 in [11]. □

Theorem 3.7. *Under the assumptions* **P1** *and* **P2**,

$$\lim_{h\to 0} \widehat{\delta}(\mathbf{E}(V + V_h), \mathbf{E}_h(V_h)) = 0.$$

Proof. It is direct consequence of Theorems 3.5 and 3.6. □

As a consequence of the previous theorems, isolated parts of the spectrum of $\mathbf{T}$ are approximated by isolated parts of the spectrum of $\mathbf{T}_h$ (see [14,11]). More precisely, for any eigenvalue λ of $\mathbf{T}$ of finite multiplicity m, there exist exactly m eigenvalues $\lambda_{1h}, \cdots, \lambda_{mh}$ of $\mathbf{T}_h$, repeated according to their respective multiplicities, converging to λ as h goes to zero.

Now we are going to give estimates which show how the eigenvalues of $\mathbf{T}$ are approximated by those of $\mathbf{T}_h$. To attain this goal, the theory in [17] about nonconforming approximation for noncompact operators should be adapted to cover more general cases where the continuous and discrete sesquilinear forms do not coincide. By proceeding as in that reference, we extend the theory developed in [12], so that it can be applied to non conforming methods. By so doing, consistency terms arise in the error estimates. We shall give general expressions for these additional consistency terms.

We begin considering the bounded operator $\mathbf{T}_*$ defined by

$$\begin{aligned} \mathbf{T}_* &: X \to V \\ x &\mapsto u, \end{aligned}$$

where u is the solution of

$$a(y, u) = b(y, x), \quad \forall y \in V. \tag{3.1}$$

It is known that $\bar{\lambda}$ is an eigenvalue of $\mathbf{T}_*$ with the same multiplicity m as that of λ (see, for instance, [12]). We also consider the bounded operator $\mathbf{T}_{*h}$ defined by

$$\begin{aligned} \mathbf{T}_{*h} &: X \to V_h \\ x &\mapsto u_h, \end{aligned}$$

where u_h is the solution of

$$a_h(y, u_h) = b_h(y, x), \quad \forall y \in V_h. \tag{3.2}$$

Here, $\bar{\lambda}_{1h}, \cdots, \bar{\lambda}_{mh}$ are the eigenvalues of $\mathbf{T}_{*h}$ which converge to $\bar{\lambda}$ as h goes to zero.

Let $\mathbf{E}_*$ be the spectral projector of $\mathbf{T}_*|_{V+V_h}$ relative to $\bar{\lambda}$.

We also assume the following properties for $\mathbf{T}_*$ and $\mathbf{T}_{*h}$:

P4:

$$\lim_{h\to 0} \|(\mathbf{T}_* - \mathbf{T}_{*h})|_{V_h}\|_h = 0.$$

P5: For each function x of $\mathbf{E}_*(V+V_h)$,

$$\lim_{h\to 0}\left(\inf_{x_h\in V_h}\|x-x_h\|_h\right)=0.$$

P6:

$$\lim_{h\to 0}\|(\mathbf{T}_*-\mathbf{T}_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h=0.$$

We now need to introduce other operators.

Let $\Pi_h : V+V_h \to V+V_h$ be the projector with range V_h defined by

$$a_h(x-\Pi_h x, y)=0, \quad \forall y\in V_h. \tag{3.3}$$

Analogously, we define $\Pi_{*h} : V+V_h \to V+V_h$ by the relation

$$a_h(y, x-\Pi_{*h}x)=0, \quad \forall y\in V_h. \tag{3.4}$$

Moreover, since a_h is continuous and coercive on $V+V_h$, both uniformly on h, Π_h and Π_{*h} are bounded uniformly on h. Let us remark that for conforming methods $\mathbf{T}_h=\Pi_h\mathbf{T}$. This is assumed in the spectral approximation theory in [12] and used in the proofs therein. Obviously, for nonconforming methods, $\mathbf{T}_h$ and $\Pi_h\mathbf{T}$ do not coincide.

Let $\mathbf{B}_h := \mathbf{T}_h\Pi_h : V+V_h \to V+V_h$. Notice that $\sigma(\mathbf{T}_h)=\sigma(\mathbf{B}_h)$ and that, for any non null eigenvalue, the corresponding invariant subspaces coincide. Let $\mathbf{F}_h : V+V_h \to V+V_h$ be the spectral projector of $\mathbf{B}_h$ relative to its eigenvalues $\lambda_{1h},\cdots,\lambda_{mh}$. It can be proved that $\|R_z(\mathbf{B}_h)\|_h$ is bounded uniformly on h for $z\in\Gamma$ (see Lemma 1 in [12]). Consequently, the spectral projectors $\mathbf{F}_h$ are bounded uniformly on h.

Finally, let $\mathbf{B}_{*h} := \mathbf{T}_{*h}\Pi_{*h} : V+V_h \to V+V_h$ and let $\mathbf{F}_{*h}$ be the spectral projector of $\mathbf{B}_{*h}$ relative to $\bar{\lambda}_{1h},\cdots,\bar{\lambda}_{mh}$.

It is easy to show that $\mathbf{B}_{*h}$ is the actual adjoint of $\mathbf{B}_h$ with respect to a_h. In fact, for all x and $y\in V+V_h$, we have

$$a_h(\mathbf{B}_h x, y)=a_h(\mathbf{T}_h\Pi_h x, y)=a_h(\mathbf{T}_h\Pi_h x, \Pi_{*h}y)=b_h(\Pi_h x, \Pi_{*h}y).$$

Similarly, we get

$$a_h(x, \mathbf{B}_{*h}y)=b_h(\Pi_h x, \Pi_{*h}y).$$

Therefore, the spectral projector $\mathbf{F}_{*h}$ is also the adjoint of $\mathbf{F}_h$ with respect to a_h.

Let

$$\gamma_h := \delta(\mathbf{E}(V+V_h), V_h).$$

Property **P2** implies that $\gamma_h\to 0$ as $h\to 0$. Analogously, let

$$\gamma_{*h} := \delta(\mathbf{E}_*(V+V_h), V_h).$$

Here, because of **P4**, $\gamma_{*h}\to 0$ as $h\to 0$.

Lemma 3.8.

$$\|(\mathbf{I}-\Pi_h)|_{\mathbf{E}(V+V_h)}\|_h \le C\gamma_h,$$
$$\|(\mathbf{I}-\Pi_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h \le C\gamma_{*h}.$$

Proof. Let $x\in\mathbf{E}(V+V_h)$. Since a_h is coercive on $V+V_h$ uniformly on h, we have

$$\|(\mathbf{I}-\Pi_h)x\|_h^2 \le Ca_h((\mathbf{I}-\Pi_h)x, (\mathbf{I}-\Pi_h)x)=Ca_h((\mathbf{I}-\Pi_h)x, x-y_h), \quad \forall y_h\in V_h,$$

where the last equality yields from the definition of Π_h. Now, taking into account that a_h is continuous on $V+V_h$ uniformly on h, we obtain

$$\|(\mathbf{I}-\Pi_h)x\|_h \le C\inf_{y_h\in V_h}\|x-y_h\|_h,$$

which allows us to conclude the proof of the first estimation. Analogous proof is valid for the second one. □

Lemma 3.9.

$$\|(\mathbf{E}-\mathbf{F}_h)|_{\mathbf{E}(V+V_h)}\|_h \le C\|(\mathbf{T}-\mathbf{B}_h)|_{\mathbf{E}(V+V_h)}\|_h,$$
$$\|(\mathbf{E}_*-\mathbf{F}_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h \le C\|(\mathbf{T}_*-\mathbf{B}_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h.$$

Proof. The proof is identical to that of Lemma 3 in [12]. □

Now, let

$$\delta_h := \gamma_h + \|(\mathbf{T}-\mathbf{T}_h)|_{\mathbf{E}(V+V_h)}\|_h.$$

From properties **P2** and **P3**, it is easily seen that $\delta_h \to 0$ as $h \to 0$. Analogously, let

$$\delta_{*h} := \gamma_{*h} + \|(\mathbf{T}_*-\mathbf{T}_{*h})|_{\mathbf{E}(V+V_h)}\|_h.$$

Because **P5** and **P6**, $\delta_{*h} \to 0$ as $h \to 0$.

Lemma 3.10.

$$\|(\mathbf{T}-\mathbf{B}_h)|_{\mathbf{E}(V+V_h)}\|_h \le C\delta_h,$$
$$\|(\mathbf{T}_*-\mathbf{B}_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h \le C\delta_{*h}.$$

Proof. Let $x \in \mathbf{E}(V+V_h)$ with $\|x\| = 1$. We have

$$\begin{aligned}\|(\mathbf{T}-\mathbf{B}_h)x\|_h &\le \|(\mathbf{T}-\mathbf{T}_h)x\|_h + \|\mathbf{T}_h(\mathbf{I}-\Pi_h)x\|_h \\ &\le \|(\mathbf{T}-\mathbf{T}_h)|_{\mathbf{E}(V+V_h)}\|_h + \|\mathbf{T}_h\|_h\|(\mathbf{I}-\Pi_h)|_{\mathbf{E}(V+V_h)}\|_h \\ &\le C(\|(\mathbf{T}-\mathbf{T}_h)|_{\mathbf{E}(V+V_h)}\|_h + \gamma_h),\end{aligned}$$

where the last inequality follows from Lemma 3.8 and the fact that $\|\mathbf{T}_h\|_h$ is uniformly bounded with respect to h.
Analogous proof is valid for the second estimate of the lemma. □

Let

$$\Lambda_h := \mathbf{F}_h|_{\mathbf{E}(V+V_h)} : \mathbf{E}(V+V_h) \to \mathbf{F}_h(V+V_h).$$

Lemma 3.11. *For h small enough, Λ_h is a bijection and $\|\Lambda_h^{-1}\|_h$ is bounded uniformly on h.*

Proof. See the proof of Theorem 1 in [12]. □

Theorem 3.12.

$$\widehat{\delta}(\mathbf{F}_h(V+V_h), \mathbf{E}(V+V_h)) \le C\delta_h.$$

Proof. The proof is identical to that of Theorem 1 in [12]. □

Let us now define the operator $\hat{\mathbf{T}} := \mathbf{T}|_{\mathbf{E}(V+V_h)} : \mathbf{E}(V+V_h) \to \mathbf{E}(V+V_h)$ and $\hat{\mathbf{B}}_h := \Lambda_h^{-1}\mathbf{B}_h\Lambda_h : \mathbf{E}(V+V_h) \to \mathbf{E}(V+V_h)$. From these definitions, it follows that $\hat{\mathbf{T}}$ has a unique eigenvalue λ of algebraic multiplicity m and that $\hat{\mathbf{B}}_h$ has the eigenvalues $\lambda_{1h}, \cdots, \lambda_{mh}$.

Let us consider the following consistency terms:

$$M_h = \sup_{\substack{x\in\mathbf{E}(V+V_h)\\ \|x\|_h=1}} \sup_{\substack{y\in\mathbf{E}_*(V+V_h)\\ \|y\|_h=1}} |a_h(\mathbf{T}x, \Pi_{*h}y) - b_h(x, \Pi_{*h}y)|,$$

$$M_{*h} = \sup_{\substack{x\in\mathbf{E}(V+V_h)\\ \|x\|_h=1}} \sup_{\substack{y\in\mathbf{E}_*(V+V_h)\\ \|y\|_h=1}} |a_h(\Pi_h x, \mathbf{T}_*y) - b_h(\Pi_h x, y)|.$$

Theorem 3.13.

$$\|\hat{\mathbf{T}} - \hat{\mathbf{B}}_h\| \leq C\left(\delta_h \delta_{*h} + M_h + M_{*h}\right).$$

Proof. We have

$$\begin{aligned}
\|\hat{\mathbf{T}} - \hat{\mathbf{B}}_h\| &= \sup_{\substack{x\in \mathbf{E}(V+V_h)\\ \|x\|_h=1}} \|(\hat{\mathbf{T}} - \hat{\mathbf{B}}_h)x\|_h \\
&\leq C \sup_{\substack{x\in \mathbf{E}(V+V_h)\\ \|x\|_h=1}} \sup_{\substack{y\in V\\ \|y\|_h=1}} |a((\hat{\mathbf{T}} - \hat{\mathbf{B}}_h)x, y)| \\
&= C \sup_{\substack{x\in \mathbf{E}(V+V_h)\\ \|x\|_h=1}} \sup_{\substack{y\in V\\ \|y\|_h=1}} |a(\mathbf{E}(\hat{\mathbf{T}} - \hat{\mathbf{B}}_h)x, y)| \\
&= C \sup_{\substack{x\in \mathbf{E}(V+V_h)\\ \|x\|_h=1}} \sup_{\substack{y\in V\\ \|y\|_h=1}} |a((\hat{\mathbf{T}} - \hat{\mathbf{B}}_h)x, \mathbf{E}_* y)| \\
&\leq C \sup_{\substack{x\in \mathbf{E}(V+V_h)\\ \|x\|_h=1}} \sup_{\substack{y\in \mathbf{E}_*(V+V_h)\\ \|y\|_h=1}} |a((\hat{\mathbf{T}} - \hat{\mathbf{B}}_h)x, y)| \\
&= C \sup_{\substack{x\in \mathbf{E}(V+V_h)\\ \|x\|_h=1}} \sup_{\substack{y\in \mathbf{E}_*(V+V_h)\\ \|y\|_h=1}} |a_h((\hat{\mathbf{T}} - \hat{\mathbf{B}}_h)x, y)|.
\end{aligned} \tag{3.5}$$

Now, using that $(\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})\mathbf{T}|_{\mathbf{E}(V+V_h)} = 0$ and that $\mathbf{B}_h$ commutes with its spectral projector $\mathbf{F}_h$, it follows that

$$\hat{\mathbf{T}} - \hat{\mathbf{B}}_h = (\mathbf{T} - \mathbf{B}_h)|_{\mathbf{E}(V+V_h)} + (\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{T} - \mathbf{B}_h)|_{\mathbf{E}(V+V_h)}. \tag{3.6}$$

Let $x \in \mathbf{E}(V + V_h)$ and $y \in \mathbf{E}_*(V + V_h)$, with $\|x\|_h = \|y\|_h = 1$. Since $\mathbf{F}_h(\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I}) = 0$ and $\mathbf{F}_{*h}$ is the adjoint of $\mathbf{F}_h$ with respect to a_h, we have

$$\begin{aligned}
&|a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{T} - \mathbf{B}_h)x, y)| \\
&\quad = |a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{T} - \mathbf{B}_h)x, y) - a_h(\mathbf{F}_h(\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{T} - \mathbf{B}_h)x, y)| \\
&\quad = |a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{T} - \mathbf{B}_h)x, y) - a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{T} - \mathbf{B}_h)x, \mathbf{F}_{*h}y)| \\
&\quad = |a_h((\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I})(\mathbf{T} - \mathbf{B}_h)x, (\mathbf{I} - \mathbf{F}_{*h})y)| \\
&\quad \leq C\|\Lambda_h^{-1}\mathbf{F}_h - \mathbf{I}\|_h \|(\mathbf{T} - \mathbf{B}_h)|_{\mathbf{E}(V+V_h)}\|_h \|(\mathbf{I} - \mathbf{F}_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h \\
&\quad \leq C\delta_h\delta_{*h}.
\end{aligned} \tag{3.7}$$

The last inequality in (3.7) follows from Lemmas 3.9–3.11 and the fact that a_h is continuous on V_h independently of h and that $\mathbf{F}_h$ is bounded uniformly on h.

On the other hand,

$$a_h((\mathbf{T} - \mathbf{B}_h)x, y) = a_h((\mathbf{T} - \mathbf{B}_h)x, \Pi_{*h}y) + a_h((\mathbf{T} - \mathbf{B}_h)x, (\mathbf{I} - \Pi_{*h})y). \tag{3.8}$$

To bound the second term in the right-hand side of (3.8), we use Lemmas 3.8 and 3.10. We thus obtain

$$\begin{aligned}
|a_h((\mathbf{T} - \mathbf{B}_h)x, (\mathbf{I} - \Pi_{*h})y)| &\leq C\|(\mathbf{T} - \mathbf{B}_h)|_{\mathbf{E}(V+V_h)}\|_h \|(\mathbf{I} - \Pi_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h \\
&\leq C\delta_h\gamma_{*h}.
\end{aligned} \tag{3.9}$$

For the first term, we write

$$a_h((\mathbf{T} - \mathbf{B}_h)x, \Pi_{*h}y) = a_h((\mathbf{T} - \mathbf{T}_h)x, \Pi_{*h}y) + a_h((\mathbf{T}_h - \mathbf{B}_h)x, \Pi_{*h}y). \tag{3.10}$$

Now,

$$|a_h((\mathbf{T} - \mathbf{T}_h)x, \Pi_{*h}y)| = |a_h(\mathbf{T}x, \Pi_{*h}y) - b_h(x, \Pi_{*h}y)| \leq M_h, \tag{3.11}$$

and

$$a_h((\mathbf{T}_h - \mathbf{B}_h)x, \Pi_{*h}y) = a_h(\mathbf{T}_h(\mathbf{I} - \Pi_h)x, \Pi_{*h}y) = b_h((\mathbf{I} - \Pi_h)x, \Pi_{*h}y)$$
$$= b_h((\mathbf{I} - \Pi_h)x, y) - b_h((\mathbf{I} - \Pi_h)x, (\mathbf{I} - \Pi_{*h})y). \quad (3.12)$$

The first term in the right-hand side of (3.12) can be written as

$$b_h((\mathbf{I} - \Pi_h)x, y) = b(x, y) - b_h(\Pi_h x, y) = a(x, T_* y) - b_h(\Pi_h x, y)$$
$$= [a_h(\Pi_h x, \mathbf{T}_* y) - b_h(\Pi_h x, y)] - a_h((\Pi_h - \mathbf{I})x, \mathbf{T}_* y)$$
$$= [a_h(\Pi_h x, \mathbf{T}_* y) - b_h(\Pi_h x, y)] - a_h((\Pi_h - \mathbf{I})x, (\mathbf{I} - \Pi_{*h})\mathbf{T}_* y), \quad (3.13)$$

where in the last equality we have used that $a_h((\Pi_h - \mathbf{I})x, \Pi_{*h}\mathbf{T}_* y) = 0$, which follows easily from (3.3) and the fact that $\Pi_{*h}\mathbf{T}_* y \in V_h$.

The last term of the right-hand side above can be easily bounded by

$$|a_h((\Pi_h - \mathbf{I})x, (\mathbf{I} - \Pi_{*h})\mathbf{T}_* y)| \le C\|\mathbf{T}_*\| \|(\mathbf{I} - \Pi_{*h})|_{\mathbf{E}_*(V+V_h)}\|_h \|(\Pi_h - \mathbf{I})|_{\mathbf{E}(V+V_h)}\|_h. \quad (3.14)$$

Then, Lemma 3.8, (3.13) and (3.14) immediately yield

$$|b_h((\mathbf{I} - \Pi_h)x, y)| \le C(M_{*h} + \gamma_h \gamma_{*h}). \quad (3.15)$$

Finally, the theorem is a consequence of formulae (3.5)–(3.15). □

By using the previous theorem, we deduce the following result about the approximation of the eigenvalue λ:

Theorem 3.14. (i) $\left|\lambda - \frac{1}{m}\sum_{i=1}^{m} \lambda_{ih}\right| \le C\,(\delta_h \delta_{*h} + M_h + M_{*h})$

(ii) $\max_{i=1,\cdots,m} |\lambda - \lambda_{ih}| \le C\,(\delta_h \delta_{*h} + M_h + M_{*h})^{1/\alpha}$ *where α is the ascent of the eigenvalue λ of $\hat{\mathbf{T}}$.*

Proof. Taking into account that $\sigma(\hat{\mathbf{T}}) = \lambda$ and that $\lambda_{1h}, \cdots, \lambda_{mh}$ are the eigenvalues of $\hat{\mathbf{B}}_h$, we have $\mathrm{tr}(\hat{\mathbf{T}}) = m\lambda$ and $\mathrm{tr}(\hat{\mathbf{B}}_h) = \sum_{i=1}^{m} \lambda_{ih}$. Then, from the continuity of the traces

$$\left|\lambda - \frac{1}{m}\sum_{i=1}^{m} \lambda_{ih}\right| = \frac{1}{m}|\mathrm{tr}(\hat{\mathbf{T}}) - \mathrm{tr}(\hat{\mathbf{B}}_h)| \le C\|\hat{\mathbf{T}} - \hat{\mathbf{B}}_h\|.$$

On the other hand, it is known that

$$|\lambda - \lambda_{ih}|^{\alpha} \le C\|\hat{\mathbf{T}} - \hat{\mathbf{B}}_h\|,$$

for any $1 \le i \le m$. Therefore, we can conclude (i) and (ii) directly from Theorem 3.13. □

Remark 3.15. In many applications, the operator $\hat{\mathbf{T}}$ is selfadjoint. In this case, if μ is a nonzero eigenvalue of $\hat{\mathbf{T}}$, the ascent α of $(\mu - \hat{\mathbf{T}})$ is one. So, the space of generalized eigenvectors $\mathbf{E}(V + V_h)$ coincides with the space of the actual eigenvectors corresponding to μ (see [4]).

4. Examples

In this section we apply the abstract theory results obtained above to two representative problems.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected and bounded domain with polygonal boundary $\partial\Omega = \Gamma$.

Let $(\cdot, \cdot)$ be the scalar product in $L^2(\Omega)$ and let $\|\cdot\|_0$ denote the corresponding L^2 norm. Further, let $H^s(\Omega)$ denote the standard Sobolev spaces with the usual norms $\|\cdot\|_s$ and seminorms $|\cdot|_s$. We also denote $H^1_\Gamma(\Omega)$ the subspace of functions in $H^1(\Omega)$ with a vanishing trace on Γ. We use a circumflex above a function space to denote the subspace of elements with mean value zero.

Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω such that any two triangles share at most a vertex or an edge. We also assume that the family $\{\mathcal{T}_h\}$ is regular in the sense of the minimum angle condition (see [8], for instance). Finally, let $\mathcal{E}_h$ denote the set of all the edges of triangles $T \in \mathcal{T}_h$.

With the triangulation $\mathcal{T}_h$, we consider the lowest-order Crouzeix–Raviart finite element spaces:

$$CR_h := \{v_h \in L^2(\Omega) : v_h|_T \in \mathcal{P}_1(T), \forall T \in \mathcal{T}_h, v_h \text{continuous at midpoints of all } \ell \in \mathcal{E}_h\}.$$

4.1. A Steklov eigenvalue problem

Eigenvalue problems of the Steklov type, in which the eigenvalue parameter appears in the boundary conditions, arise in a number of applications. Let us mention, for instance, the problem of determining the vibrations modes of liquids in moving containers, the so-called sloshing problem.

We consider the following spectral problem:

Find $\lambda \in \mathbb{R}$ and $u \neq 0$ *such that*

$$\begin{cases} -\mathrm{div}(\alpha \nabla u) + \beta u = 0 & \text{in } \Omega, \\ \alpha \dfrac{\partial u}{\partial \mathbf{n}} = \lambda u & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where the coefficients $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are bounded by above and below by positive constants. We assume that $\alpha \in C^1(\bar{\Omega})$.

Let $W := H^1(\Omega)$. Let a^* and b^* be the symmetrical bilinear forms defined by

$$a^*(u, v) := \int_\Omega \alpha \nabla u \cdot \nabla v + \int_\Omega \beta u v, \quad \forall u, v \in W,$$

$$b^*(u, v) := \int_\Gamma u v, \quad \forall u, v \in W.$$

Since α and β are bounded in $\bar{\Omega}$, we have that a^* is continuous and coercive on W.

Then, the variational formulation of the spectral problem (4.1) is given by:

Find $\lambda \in \mathbb{R}$ *and* $u \in W$, $u \neq 0$, *such that*

$$a^*(u, v) = \lambda b^*(u, v), \quad \forall v \in W. \tag{4.2}$$

From the classical theory of abstract elliptical eigenvalue problems, we can infer that problem (4.2) attains a sequence of finite multiplicity eigenvalues $\lambda_n > 0$, $n \in \mathbb{N}$, diverging to $+\infty$, with corresponding $L^2(\Gamma)$-orthonormal eigenfunctions u_n belonging to W.

We introduce the following spaces:

$$X := L^2(\Omega) \times L^2(\Gamma)$$

$$V := \{(u, \xi) \in H^1(\Omega) \times H^{1/2}(\Gamma) : \xi = u|_\Gamma\},$$

endowed with the norms defined by

$$|(u, \xi)| := (\|u\|_0^2 + \|\xi\|_{0,\Gamma}^2)^{1/2},$$

$$\|(u, \xi)\| := (\|u\|_1^2 + \|\xi\|_{0,\Gamma}^2)^{1/2}.$$

Let a be the bilinear and continuous form defined on $V \times V$ by

$$a((u, \xi), (v, \eta)) := \int_\Omega \alpha \nabla u \cdot \nabla v + \int_\Omega \beta u v + \int_\Gamma \xi \eta.$$

Note that a is V-elliptical. Let b be the bilinear and continuous form defined on $X \times X$ by

$$b((u, \xi), (v, \eta)) := \int_\Gamma \xi \eta.$$

Now, we consider the following spectral problem:

Find $\lambda \in \mathbb{R}$ *and* $(u, \xi) \in V$, $u \neq 0$, *such that*

$$a((u, \xi), (v, \eta)) = (\lambda + 1) b((u, \xi), (v, \eta)), \quad \forall (v, \eta) \in V. \tag{4.3}$$

For $\lambda \neq 0$, variational problems (4.2) and (4.3) are equivalent to problem (4.1). In fact, the solution of (4.1) satisfy Eqs. (4.2) and (4.3). Conversely, by testing these two equations with adequate smooth functions, it is easy to show that any solution of each of them, corresponding to a nonzero eigenvalue, also satisfy (4.1).

As in Section 2, we consider the bounded linear operator $\mathbf{T} : X \to X$ defined by $\mathbf{T}(f, \tau) = (u, \xi) \in V$ and

$$a((u, \xi), (y, \zeta)) = b((f, \tau), (y, \zeta)), \quad \forall (y, \zeta) \in V. \tag{4.4}$$

By virtue of Lax–Milgram Lemma, we have

$$\|(u, \xi)\| \le C|(f, \tau)|.$$

Since a and b are symmetrical, $\mathbf{T}$ is self-adjoint with respect to a. Clearly, $(\lambda, (u, \xi))$ is a solution of problem (4.3) if and only if $(\frac{1}{1+\lambda}, (u, \xi))$ is an eigenpair of $\mathbf{T}$.

The following proposition states a priori estimates for the solution of problem (4.4) depending on the regularity of the data.

Proposition 4.1. *Let (u, ξ) be the solution of problem (4.4). There exist constants $r \in (1/2, 1]$ and $C > 0$ such that*

- *if $\tau \in L^2(\Gamma)$, $u \in H^{1+r/2}(\Omega)$ and*
$$\|u\|_{1+r/2} \le C\|\tau\|_{0,\Gamma}, \tag{4.5}$$
- *if $\tau \in H^{\epsilon}(\Gamma)$, with $\epsilon \in (0, r - 1/2)$, $u \in H^{3/2+\epsilon}(\Omega)$ and*
$$\|u\|_{3/2+\epsilon} \le C\|\tau\|_{\epsilon,\Gamma}, \tag{4.6}$$
- *if $\tau \in H^{1/2}(\Gamma)$, $u \in H^{1+r}(\Omega)$ and*
$$\|u\|_{1+r} \le C\|\tau\|_{1/2,\Gamma}. \tag{4.7}$$

Proof. It follows directly from classical regularity results (see [10]). □

In the previous proposition, $r = 1$ if Ω is a convex region and $r < \frac{\pi}{\theta}$, with θ being the largest interior angle of Ω, otherwise (see [13]). Notice that, as a consequence, the eigenfunctions (u_n, ξ_n) of $\mathbf{T}$ belong to $H^{1+r}(\Omega) \times H^{1/2+r}(\Gamma)$ and satisfy

$$\|u_n\|_{1+r} \le C\|(u_n, \xi_n)\|. \tag{4.8}$$

Now we introduce the nonconforming finite element spaces

$$\mathcal{L}_h := \{\xi_h \in L^2(\Gamma) : \xi_h|_\ell \in \mathcal{P}_1(\ell), \forall \ell \subset \Gamma\}.$$

and

$$V_h := \{(v_h, \eta_h) \in CR_h \times \mathcal{L}_h : \eta_h|_\ell = v_h|_\ell, \forall \ell \subset \Gamma\}.$$

Let a_h and b_h be the symmetrical bilinear forms defined by

$$a_h((u, \xi), (v, \eta)) := \sum_{T \in \mathcal{T}_h} \int_T \alpha \nabla u \cdot \nabla v + \int_\Omega \beta u v + \int_\Gamma \xi \eta, \quad \forall (u, \xi), (v, \eta) \in V + V_h,$$

$$b_h((u, \xi), (v, \eta)) := b((u, \xi), (v, \eta)), \quad \forall (u, \xi), (v, \eta) \in X.$$

Then, the discretization of the spectral problem (4.3) is given by
Find $\lambda_h \in \mathbb{R}$ and $(u_h, \xi_h) \in V_h$, $(u_h, \xi_h) \neq (0, 0)$, such that

$$a_h((u_h, \xi_h), (v_h, \eta_h)) = (\lambda_h + 1) b((u_h, \xi_h), (v_h, \eta_h)), \quad \forall (v_h, \eta_h) \in V_h. \tag{4.9}$$

By choosing

$$\|(v_h, \eta_h)\|_h = \left(\sum_{T \in \mathcal{T}_h} |v_h|_{1,T}^2 + \|v_h\|_0^2 + \|\eta_h\|_{0,\Gamma}^2 \right)^{1/2}$$

as a norm over the space $V + V_h$, the continuity of the imbedding (2.4) follows immediately and the condition (2.3) is obviously satisfied. Also, it follows from the definition of the discrete norms $\|\cdot\|_h$ that the approximate bilinear forms a_h are coercive uniformly on $V + V_h$.

Now, we consider the bounded linear operator $\mathbf{T}_h : X \to V + V_h$ defined by $\mathbf{T}_h(f, \tau) \in V_h$ and

$$a_h(\mathbf{T}_h(f, \tau), (v_h, \eta_h)) = b((f, \tau), (v_h, \eta_h)), \quad \forall (v_h, \eta_h) \in V_h. \tag{4.10}$$

The spectral convergence results rely on properties **P1**, **P2** and **P3**. The proofs of these properties for this nonconforming finite element approximation are standard but we include them for the sake of completeness.

Theorem 4.2. **(P2)** *For each eigenfunction* (u, ξ) *of* $\mathbf{T}$ *associated to* λ, *there exists a strictly positive constant* C *such that*

$$\inf_{(v_h,\eta_h)\in V_h} \|(u, \xi) - (v_h, \eta_h)\|_h \le Ch^r \|u\|_{1+r}.$$

Proof. Since $u \in H^{1+r}(\Omega)$, $u \in C^0(\Omega)$. So, the piecewise linear Lagrange interpolant of u, u^I, is well defined (see [8], for instance). Moreover, $u^I \in V_h \cap H^1(\Omega)$. Then, we can choose $v_h = u^I$. By using a suitable trace theorem and standard interpolation results, we obtain

$$\begin{aligned}\|\xi - \eta_h\|_{0,\ell} = \|u|_\ell - v_h|_\ell\|_{0,\ell} &\le C\left[h^{-1/2}\|u - v_h\|_{0,T} + h^{1/2}|u - v_h|_{1,T}\right]\\ &\le Ch^{r+1/2}\|u\|_{1+r,T}, \quad \forall \ell \subset \Gamma\end{aligned} \tag{4.11}$$

and

$$\|u - v_h\|_{1,T} \le Ch^r \|u\|_{1+r,T}. \tag{4.12}$$

Now, combining (4.11) and (4.12) with the definition of $\|\cdot\|_h$, we can write

$$\|(u, \xi) - (v_h, \eta_h)\|_h \le Ch^r \|u\|_{1+r}.$$

So, taking the infimum with respect to $(v_h, \eta_h) \in V_h$, we can conclude the proof. □

Theorem 4.3. **(P1)** *There exists a positive constant* C *such that*

$$\|(\mathbf{T} - \mathbf{T}_h)(f, \tau)\|_h \le Ch^{r/2}\|(f, \tau)\|_h, \quad \forall (f, \tau) \in V_h.$$

Proof. Let $(u, \eta) := \mathbf{T}(f, \tau)$ and $(u_h, \eta_h) := \mathbf{T}_h(f, \tau)$, for any $(f, \tau) \in V_h$. From the second Strang's Lemma (see [8]) we have

$$\begin{aligned}\|(u, \xi) - (u_h, \xi_h)\|_h \le C\Bigg(&\inf_{(v_h,\eta_h)\in V_h} \|(u, \xi) - (v_h, \eta_h)\|_h\\ &+ \sup_{(w_h,\chi_h)\in V_h} \frac{a_h((u, \xi), (w_h, \chi_h)) - b((f, \tau), (w_h, \chi_h))}{\|(w_h, \chi_h)\|_h}\Bigg).\end{aligned} \tag{4.13}$$

To bound the first term in the right-hand side of inequality (4.13), we repeat the arguments used in the proof of Theorem 4.2. By using estimate (4.5), we obtain

$$\inf_{(v_h,\eta_h)\in V_h} \|(u, \xi) - (v_h, \eta_h)\|_h \le Ch^{r/2}\|(f, \tau)\|_h.$$

Now we are going to bound the second term. By testing (4.4) with adequate smooth function, it is simple to show that u satisfy the following strong problem:

$$\begin{cases} -\operatorname{div}(\alpha\nabla u) + \beta u = 0 & \text{in } \Omega,\\ \alpha\dfrac{\partial u}{\partial \mathbf{n}} + \xi = \tau & \text{on } \Gamma, \end{cases} \tag{4.14}$$

where $\xi = u|_\Gamma$.

Multiplying Eq. (4.14) by $w_h \in CR_h$ and integrating by parts, it is straightforward to see that

$$a_h((u, \xi), (w_h, \chi_h)) - b((f, \tau), (w_h, \chi_h)) = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \alpha\frac{\partial u}{\partial \mathbf{n}} w_h,$$

where $\chi_h = w_h|_\Gamma$. Notice that, since $f \in CR_h$, $\tau \in H^\epsilon(\Gamma)$. So, from estimate (4.6), we have $u \in H^{3/2+\epsilon}(\Omega)$ which gives a meaning to the integrals $\int_\ell \alpha \frac{\partial u}{\partial \mathbf{n}} w_h$ on each edge ℓ.

Let us denote by $[\![\cdot]\!]$ the jump across an inner edge $\ell \in \mathcal{E}_h$. Then, we can write

$$a_h((u,\xi),(w_h,\chi_h)) - b((f,\tau),(w_h,\chi_h)) = \sum_{\substack{\ell\in\mathcal{E}_h\\ \ell\not\subset\Gamma}} \int_\ell \alpha \frac{\partial u}{\partial \mathbf{n}} [\![w_h]\!]. \tag{4.15}$$

Let P_ℓ denote the $L^2(\ell)$-projection of $H^\epsilon(\ell)$ onto the constants. For $\ell \in \mathcal{E}_h$, let $T_1, T_2 \in \mathcal{T}_h$ be such that $T_1 \cap T_2 = \ell$. Since $[\![w_h]\!]$ is a linear function vanishing at the midpoint of ℓ, we have

$$\begin{aligned}\left|\int_\ell \alpha \frac{\partial u}{\partial \mathbf{n}} [\![w_h]\!]\right| &= \left|\int_\ell \left(\alpha \frac{\partial u}{\partial \mathbf{n}} - P_\ell\left(\alpha \frac{\partial u}{\partial \mathbf{n}}\right)\right)[\![w_h]\!]\right| \\ &= \left|\int_\ell \left(\alpha \frac{\partial u}{\partial \mathbf{n}} - P_\ell\left(\alpha \frac{\partial u}{\partial \mathbf{n}}\right)\right)(w_h|_{T_1}) - \int_\ell \left(\alpha \frac{\partial u}{\partial \mathbf{n}} - P_\ell\left(\alpha \frac{\partial u}{\partial \mathbf{n}}\right)\right)(w_h|_{T_2})\right| \\ &\le \sum_{i=1,2} \left|\int_\ell \left(\alpha \frac{\partial u}{\partial \mathbf{n}} - P_\ell\left(\alpha \frac{\partial u}{\partial \mathbf{n}}\right)\right)[(w_h|_{T_i}) - P_\ell(w_h|_{T_i})]\right|.\end{aligned}$$

Now, if P_T denotes the $L^2(T)$-projection of $H^{\epsilon+1/2}(T)$ onto the constants, by using a trace theorem and standard error estimates for the L^2-projection, we obtain

$$\begin{aligned}\left|\int_\ell \alpha \frac{\partial u}{\partial \mathbf{n}} [\![w_h]\!]\right| &\le \sum_{i=1,2} \|\alpha\nabla u\cdot\mathbf{n} - P_T(\alpha\nabla u\cdot\mathbf{n})\|_{0,\ell}\|(w_h|_{T_i}) - P_T(w_h|_{T_i})\|_{0,\ell} \\ &\le C\sum_{i=1,2}\left(\ell^{r/2-1/2}\|\nabla u\|_{r/2,T_i}\right)\left(\ell^{1/2}\|w_h\|_{1,T_i}\right).\end{aligned} \tag{4.16}$$

Summing up on all the edges $\ell \in \mathcal{E}_h$ and using estimate (4.5), we can write

$$|a_h((u,\xi),(w_h,\chi_h)) - b((f,\tau),(w_h,\chi_h))| \le Ch^{r/2}\|(f,\tau)\|_h\|(w_h,\chi_h)\|_h,$$

which allows us to conclude the proof. □

By virtue of the previous theorems, the spectrum of $\mathbf{T}_h$ furnishes the approximations of the spectrum of $\mathbf{T}$ as we stated in Section 3.

Theorem 4.4. **(P3)** *There exists a positive constant C such that*

$$\|(\mathbf{T} - \mathbf{T}_h)|_{\mathbf{E}(V+V_h)}\|_h \le Ch^r.$$

Proof. For $(x,\zeta) \in \mathbf{E}(V + V_h)$, $(u,\xi) = \mathbf{T}(x,\zeta) \in H^{1+r}(\Omega) \times H^{1/2+r}(\Gamma)$. Then, we proceed as in the proof of Theorem 4.3 with (f,τ) substituted by (x,ζ). □

Observe that, since $\mathbf{T}$ and $\mathbf{T}_h$ are self-adjoint, properties **P4**, **P5** and **P6** are equally valid.

Now, we are going to estimate the consistency terms appearing in Theorem 3.14. Notice that M_h and M_{*h} also coincide because of the symmetry of a_h and b.

Lemma 4.5. *Let*

$$M_h = \sup_{\substack{(x,\zeta)\in\mathbf{E}(V+V_h)\\ \|(x,\zeta)\|_h=1}} \sup_{\substack{(y,\varphi)\in\mathbf{E}(V+V_h)\\ \|(y,\varphi)\|_h=1}} |a_h(\mathbf{T}(x,\zeta),\Pi_h(y,\varphi)) - b((x,\zeta),\Pi_h(y,\varphi))|,$$

with Π_h being the projection onto V_h with respect to a_h, defined by Eq. (3.3). There exists a positive constant C such that

$$M_h \le Ch^{2r}.$$

Proof. Let $(x, \zeta) \in \mathbf{E}(V + V_h)$ with $\|(x, \zeta)\|_h = 1$ and let $(w, \chi) = \mathbf{T}(x, \zeta)$. From (4.7), we have $(w, \chi) \in H^{1+r}(\Omega) \times H^{1/2+r}(\Gamma)$ and

$$\|w\|_{1+r} \leq C\|(x, \zeta)\|_h \leq C.$$

Proceeding as in Theorem 4.3, it can be shown that (w, χ) satisfy the following strong problem:

$$\begin{cases} -\operatorname{div}(\alpha \nabla w) + \beta w = 0 & \text{in } \Omega, \\ \alpha \dfrac{\partial w}{\partial \mathbf{n}} + \chi = \zeta & \text{on } \Gamma, \end{cases} \tag{4.17}$$

where $\chi = w|_\Gamma$.

Now, consider any function $(y, \varphi) \in \mathbf{E}(V + V_h)$ with $\|(y, \varphi)\|_h = 1$. We denote by $(y_h, \varphi_h) = \Pi_h(y, \varphi)$. Multiplying Eq. (4.17) by y_h and integrating by parts, it is straightforward to see that

$$a_h((w, \chi), \Pi_h(y, \varphi)) - b_h((x, \zeta), \Pi_h(y, \varphi)) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} \alpha \nabla w \cdot \mathbf{n} y_h.$$

Since $y_h \in CR_h$, we proceed in this case as in the proof of the previous theorem, with $[\![w_h]\!]$ substituted by $[\![y_h]\!]$, and we obtain

$$\begin{aligned} &|a_h((w, \chi), \Pi_h(y, \varphi)) - b_h((x, \zeta), \Pi_h(y, \varphi))| \\ &\leq C \left(\sum_{\substack{\ell \in \mathcal{E}_h \\ \ell \not\subset \Gamma}} \sum_{i=1,2} \|\alpha \nabla w \cdot \mathbf{n} - P_T(\alpha \nabla w \cdot \mathbf{n})\|_{0,\ell} \|y_h|_{T_i} - P_T(y_h|_{T_i})\|_{0,\ell} \right). \end{aligned} \tag{4.18}$$

We can write

$$\|y_h - P_T(y_h)\|_{0,\ell} \leq \|(y_h - y) - P_T(y_h - y)\|_{0,\ell} + \|y - P_T y\|_{0,\ell}. \tag{4.19}$$

Since $(y, \varphi) \in \mathbf{E}(V + V_h)$, $y \in H^{1+r}(\Omega)$. Then, the terms in the right-hand side of inequality (4.19) can be bounded directly. In fact, using standard error estimates for the P_T-projection, we have

$$\|(y_h - y) - P_T(y_h - y)\|_{0,\ell} \leq Ch^{1/2}\|y_h - y\|_{1,T} \tag{4.20}$$

and

$$\|y - P_T y\|_{0,\ell} \leq Ch^{1/2+r}\|y\|_{1+r,T}. \tag{4.21}$$

On the other hand, $w \in H^{1+r}(\Omega)$ for $r > 1/2$. So, once again, by using standard error estimates for the P_T-projection, we have

$$\|\alpha \nabla w \cdot \mathbf{n} - P_T(\alpha \nabla w \cdot \mathbf{n})\|_{0,\ell} \leq Ch^{r-1/2}\|w\|_{1+r,T}. \tag{4.22}$$

Thus, combining inequalities (4.18)–(4.22), summing up on all the edges $\ell \in \mathcal{E}_h$, using Theorem 4.2 and estimate (4.8) we conclude the proof. □

Theorem 4.6. *There exists a positive constant C such that*

$$\max_{i=1,\cdots,m} |\lambda - \lambda_{ih}| \leq Ch^{2r}.$$

Proof. It is an immediate consequence of properties **P2** and **P3** and the previous lemma. □

4.2. Eigenvalue problem for a system of partial differential equations

Now we consider the following spectral problem:

Find $\lambda \in \mathbb{R}$ *and* $(u_1, \mathbf{u}_2) \neq (0, \mathbf{0})$ *such that*

$$\begin{cases} -\Delta u_1 - \operatorname{div} \mathbf{u}_2 = \lambda u_1 & \text{in } \Omega, \\ \nabla u_1 + \mathbf{u}_2 = \lambda \mathbf{u}_2 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma. \end{cases} \tag{4.23}$$

The same problem is considered in [11,12] where a conforming finite element method was proposed and analysed and optimal order error estimates were proven. Here, we introduce a low order nonconforming space for dealing with problem (4.23). By applying the abstract theory developed in Section 3, we prove that this method yields the same order of accuracy.

We begin by giving a thorough spectral characterization of this problem.

The second equation in (4.23) implies

$$-\Delta u_1 - \operatorname{div} \mathbf{u}_2 = -\lambda \operatorname{div} \mathbf{u}_2.$$

Hence, if $\lambda \neq 0$, u_1 is a solution of the following problem:

$$\begin{cases} -\Delta u_1 = (\lambda - 1)u_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma. \end{cases} \tag{4.24}$$

Let (α_n, ϕ_n) denote the eigenpairs of the Laplace equation with Dirichlet boundary condition. Then, $\lambda_n = 1 + \alpha_n$ is an eigenvalue of problem (4.23) with $(\phi_n, \lambda_n^{-1}\nabla\phi_n)$ being the associated eigenfunction.

Now, if $\lambda = 1$, the unique solution of problem (4.24) is $u_1 = 0$ and, by using the first equation in (4.23), it follows that our problem has eigenfunctions of the form $(0, \mathbf{curl}\,\psi)$ for any $\psi \in H^1(\Omega)$.

Finally, from the second equation in (4.23), it is immediate to see that the eigenfunctions associated to the eigenvalue $\lambda = 0$ have the form $(\xi, -\nabla\xi)$, with $\xi \in H^1_\Gamma(\Omega)$.

Let $\mathbf{X} := (L^2(\Omega))^3$ and $|(u_1, \mathbf{u}_2)|$ be the standard $\mathbf{L}^2$-norm. Let $\mathbf{V}$ be the subspace of $\mathbf{X}$ defined by $\mathbf{V} := H^1_\Gamma(\Omega) \times (L^2(\Omega))^2$, endowed with the usual product norm $\|(v_1, \mathbf{v}_2)\| := (\|v_1\|_1^2 + \|\mathbf{v}_2\|_0^2)^{1/2}$.

Let a be the symmetrical bilinear form defined on $\mathbf{V} \times \mathbf{V}$ by

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla u_1 \cdot \nabla v_1 + \int_\Omega \nabla u_1 \cdot \mathbf{v}_2 + \int_\Omega \mathbf{u}_2 \cdot \nabla v_1 + \int_\Omega u_1 v_1 + 2\int_\Omega \mathbf{u}_2 \cdot \mathbf{v}_2.$$

Is simple to show that the form a is continuous on $\mathbf{V}$. Moreover, using the inequality

$$2\int_\Omega \nabla u_1 \cdot \mathbf{v}_2 \geq -\left(\varepsilon^2\|\nabla u_1\|^2 + \frac{1}{\varepsilon^2}\|\mathbf{u}_2\|^2\right),$$

with $\varepsilon > 0$, the coercivity of a on this space follows directly. Let b be the bilinear and continuous form defined on $\mathbf{X} \times \mathbf{X}$ by

$$b(\mathbf{u}, \mathbf{v}) := \int_\Omega u_1 v_1 + \int_\Omega \mathbf{u}_2 \cdot \mathbf{v}_2.$$

The variational formulation of the spectral problem (4.23) is given by:

Find $\lambda \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbf{V}$, $\mathbf{u} \neq \mathbf{0}$, *such that*

$$a(\mathbf{u}, \mathbf{v}) = (\lambda + 1)b(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}. \tag{4.25}$$

In order to analyse the spectral problem (4.25), let us introduce the following bounded linear operator:

$$\mathbf{T} : \mathbf{X} \to \mathbf{X}$$
$$(f, \mathbf{g}) \mapsto (u_1, \mathbf{u}_2),$$

with $(u_1, \mathbf{u}_2)$ being the solution of the elliptical problem

$$a((u_1, \mathbf{u}_2), (v_1, \mathbf{v}_2)) = b((f, \mathbf{g}), (v_1, \mathbf{v}_2)), \quad \forall (v_1, \mathbf{v}_2) \in \mathbf{V}. \tag{4.26}$$

Then, $\mathbf{T}$ is self-adjoint and positive definite with respect to a and b. It is clear that $(\lambda, \mathbf{u})$ is a solution of problem (4.25) if and only if $(\frac{1}{\lambda+1}, \mathbf{u})$ is an eigenpair of $\mathbf{T}$. Since the eigenvalues of problem (4.25) are positive, then those of $\mathbf{T}$ satisfy $0 < \frac{1}{\lambda+1} \leq 1$.

The operator $\mathbf{T}$ is not compact. In fact, the eigenspace corresponding to eigenvalue $\lambda = 1$ is $\mathbf{K}_1 = \{(0, \mathbf{curl}\,\xi), \xi \in H^1(\Omega)\}$ and the corresponding one to $\lambda = 0$ is $\mathbf{K}_2 = \{(\varphi, -\nabla\varphi), \varphi \in H^1_\Gamma(\Omega)\}$, both having infinite algebraic multiplicity, proving the noncompactness of $\mathbf{T}$.

On the other hand, by virtue of Lax–Milgram Lemma, we have

$$\|(u_1, \mathbf{u}_2)\| \leq C|(f, \mathbf{g})|. \tag{4.27}$$

As a consequence of the classical a priori estimates for the Laplace problem, the eigenvectors of problem (4.25), not corresponding to $\lambda = 0$ or $\lambda = 1$, satisfy some further regularity. In fact, $(u_1, \mathbf{u}_2) \in H^{1+r}(\Omega) \times (H^r(\Omega))^2$ for some $r > 1/2$, depending on the geometry of Ω, and there holds

$$\|u_1\|_{1+r} + \|\mathbf{u}_2\|_r \leq C|(u_1, \mathbf{u}_2)|. \tag{4.28}$$

Denoting by $\mathcal{P}_1(T)$ the set of functions on T which are the restrictions of linear polynomials, we introduce the following finite element spaces:

$$S_h := \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1(T), \quad \forall T \in \mathcal{T}_h\},$$
$$R_h := \{v_h \in S_h : v_h = 0 \text{ on } \Gamma\},$$
$$\mathbf{U}_{1h} := \{\nabla v_h : v_h \in R_h\},$$
$$\mathbf{U}_{2h} := \{\mathbf{curl}\, v_h : v_h \in S_h\}.$$

Next, we consider the discontinuous finite element space

$$W_{1h} := \{v_h \in CR_h : v_h = 0 \text{ at the midpoints of all } \ell \subset \Gamma\}$$

and we define the spaces

$$\mathbf{W}_{2h} := \mathbf{U}_{1h} \oplus \mathbf{U}_{2h}$$
$$\mathbf{V}_h := W_{1h} \times \mathbf{W}_{2h}.$$

Let a_h and b_h be the symmetrical bilinear forms defined by

$$a_h(\mathbf{u}, \mathbf{v}) := \sum_{T \in \mathcal{T}_h} \left(\int_T \nabla u_1 \cdot \nabla v_1 + \int_T \nabla u_1 \cdot \mathbf{v}_2 + \int_T \mathbf{u}_2 \cdot \nabla v_1 \right) + \int_\Omega u_1 v_1 + 2 \int_\Omega \mathbf{u}_2 \cdot \mathbf{v}_2, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V} + \mathbf{V}_h,$$
$$b_h(\mathbf{u}, \mathbf{v}) := b(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}.$$

Now, we are in order to define a discrete analogue of problem (4.25).
Find $\lambda_h \in \mathbb{R}$ and $\mathbf{u}_h \in \mathbf{V}_h$, $\mathbf{u}_h \neq \mathbf{0}$, such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\lambda_h + 1) b(\mathbf{u}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{4.29}$$

By choosing

$$\|\mathbf{v}_h\|_h = \left(\sum_{T \in \mathcal{T}_h} |v_{1h}|_{1,T}^2 + \|v_{1h}\|_0^2 + \|\mathbf{v}_{2h}\|_0^2 \right)^{1/2}$$

as a norm over the space $\mathbf{V} + \mathbf{V}_h$, the continuity of the imbedding (2.4) follows immediately and the condition (2.3) is obviously satisfied. Also, it follows from the definition of the discrete norms $\|\cdot\|_h$ that the approximate bilinear forms a_h are coercive uniformly on $\mathbf{V} + \mathbf{V}_h$.

Let $\mathbf{T}_h$ be the linear bounded operator given by

$$\mathbf{T}_h : \mathbf{X} \to \mathbf{V}_h$$
$$(f, \mathbf{g}) \mapsto (u_{1h}, \mathbf{u}_{2h}),$$

with $(u_{1h}, \mathbf{u}_{2h})$ being the solution of the discretized source problem

$$a_h((u_{1h}, \mathbf{u}_{2h}), (v_{1h}, \mathbf{v}_{2h})) = b((f, \mathbf{g}), (v_{1h}, \mathbf{v}_{2h})), \quad \forall (v_{1h}, \mathbf{v}_{2h}) \in \mathbf{V}_h. \tag{4.30}$$

As is shown below, the spectra of these discrete operators provide good approximations of the spectrum of $\mathbf{T}$. Moreover, the operators $\mathbf{T}_h$ have eigenspaces providing good approximations of the infinite dimensional eigenspaces $\mathbf{K}_1$ and $\mathbf{K}_2$ of $\mathbf{T}$ with exactly the same eigenvalues.

Theorem 4.7. *For $\lambda_h = 1$, $\mu_h = \frac{1}{1+\lambda_h} = \frac{1}{2}$ is an eigenvalue of $\mathbf{T}_h$. Furthermore, if $\mathbf{K}_{1h}$ denotes the corresponding eigenspace, then*

$$\mathbf{K}_1 \cap \mathbf{V}_h = \{(0, \mathbf{curl}\,\xi_h), \xi_h \in S_h\} \subset \mathbf{K}_{1h},$$

and

$$\mathbf{K}_1 \cap \mathbf{V}_h = \mathbf{K}_{1h} \cap (R_h \times \mathbf{W}_{2h}).$$

Proof. We note that every $(u_{1h}, \mathbf{u}_{2h}) \in \mathbf{K}_{1h}$ is clearly an eigenvector of $\mathbf{T}_h$ associated to $\lambda_h = 1$. In fact, from (4.29)

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{u}_{2h}\cdot\nabla v_{1h} = \sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\,\xi_h\cdot\nabla v_{1h} = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{curl}\,\xi_h\cdot\mathbf{n}v_{1h}$$
$$= \sum_{\substack{\ell\in\mathcal{E}_h\\ \ell\not\subset\Gamma}}\int_\ell \mathbf{curl}\,\xi_h\cdot\mathbf{n}[\![v_{1h}]\!] + \sum_{\ell\subset\Gamma}\int_\ell \mathbf{curl}\,\xi_h\cdot v_{1h}.$$

Now, since $\xi_h|_T \in \mathcal{P}_1(T)$, $\mathbf{curl}\,\xi_h\cdot\mathbf{n}$ takes constant values on each edge of the triangulation. On the other hand, for any $v_{1h}\in W_{1h}$, $[\![v_{1h}]\!]|_\ell$ is a linear function vanishing at the midpoint of ℓ. So, we get

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\,\xi_h\cdot\nabla v_{1h} = 0. \tag{4.31}$$

Now, let $(u_{1h}, \mathbf{u}_2) \in \mathbf{V}_h$ such that $\mathbf{T}_h(u_{1h}, \mathbf{u}_{2h}) = \frac{1}{2}(u_{1h}, \mathbf{u}_2)$. From Eq. (4.29), it satisfies

$$\sum_{T\in\mathcal{T}_h}\int_T \nabla u_{1h}\cdot\mathbf{v}_{2h} = 0, \quad \forall(0, \mathbf{v}_{2h})\in\mathbf{V}_h. \tag{4.32}$$

Since we are assuming that $u_{1h}\in R_h$, $\nabla u_{1h}\in\mathbf{W}_{2h}$. Then, from (4.32) it follows that u_{1h} is piecewise constant. But, $u_{1h}\in W_{1h}$, so $u_{1h}=0$ in Ω. Now, testing (4.29) with $(v_{1h}, \mathbf{0})$ we have

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{u}_{2h}\cdot\nabla v_{1h} = 0, \quad \forall(v_{1h}, \mathbf{0})\in\mathbf{V}_h.$$

Since $\mathbf{u}_{2h}\in\mathbf{W}_{2h}$, we have $\mathbf{u}_{2h} = \nabla\varphi_h + \mathbf{curl}\,\xi_h$, with $\varphi_h\in R_h$ and $\xi_h\in S_h$. Taking into account the orthogonality (4.31), we conclude that

$$\sum_{T\in\mathcal{T}_h}\int_T \nabla\varphi_h\cdot\nabla v_{1h} = 0, \quad \forall(v_{1h}, \mathbf{0})\in\mathbf{V}_h.$$

Then, $\nabla\varphi_h = \mathbf{0}$ and hence $\mathbf{u}_h = (0, \mathbf{curl}\,\xi_h)$. □

Theorem 4.8. *For $\lambda_h = 0$, $\mu_h = \frac{1}{1+\lambda_h} = 1$ is an eigenvalue of $\mathbf{T}_h$ with corresponding eigenspace*

$$\mathbf{K}_{2h} = \mathbf{K}_2\cap\mathbf{V}_h = \{(u_{1h}, \mathbf{u}_{2h}), u_{1h}\in R_h, \mathbf{u}_{2h} = -\nabla u_{1h}\}.$$

Proof. We note that if $\lambda_h = 0$ and $(u_{1h}, \mathbf{u}_{2h})\in\mathbf{V}_h$ satisfies $\mathbf{T}_h(u_{1h}, \mathbf{u}_{2h}) = (u_{1h}, \mathbf{u}_{2h})$, from (4.29) we get

$$\sum_{T\in\mathcal{T}_h}\int_T (\nabla u_{1h} + \mathbf{u}_{2h})\cdot\mathbf{v}_{2h} = 0, \quad \forall(0, \mathbf{v}_{2h})\in\mathbf{V}_h. \tag{4.33}$$

Since $\mathbf{u}_{2h}\in\mathbf{W}_{2h}$, it can be written as $\mathbf{u}_{2h} = \nabla\varphi_h + \mathbf{curl}\,\xi_h$, with $\varphi_h\in R_h$ and $\xi_h\in S_h$. Then, by taking $\mathbf{v}_{2h} = \mathbf{curl}\,\psi_h$ and using the orthogonality (4.31), we can obtain

$$\sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\,\xi_h\cdot\mathbf{curl}\,\psi_h = 0, \quad \forall\psi_h\in S_h. \tag{4.34}$$

So, from (4.34) it follows that $\mathbf{curl}\,\xi_h = \mathbf{0}$.

Now, by considering $\mathbf{v}_{2h} = \nabla\phi_h$, we further obtain

$$\sum_{T\in\mathcal{T}_h}\int_T (\nabla u_{1h} + \nabla\varphi_h)\cdot\nabla\phi_h = 0, \quad \forall\phi_h \in R_h. \tag{4.35}$$

On the other hand, by testing (4.29) with $(v_{1h}, \mathbf{0})$, we have

$$\sum_{T\in\mathcal{T}_h}\int_T (\nabla u_{1h} + \nabla\varphi_h)\cdot\nabla v_{1h} = 0, \quad \forall v_{1h} \in W_{1h}. \tag{4.36}$$

By using $\phi_h = \varphi_h$ in (4.35) and $v_{1h} = u_{1h}$ in (4.36), we may write

$$\sum_{T\in\mathcal{T}_h}\int_T (\nabla u_{1h} + \nabla\varphi_h)\cdot(\nabla u_{1h} + \nabla\varphi_h) = 0. \tag{4.37}$$

So, $(u_{1h} + \varphi_h)$ is piecewise constant and since $(u_{1h} + \varphi_h) \in W_{1h}$, we deduce directly that $(u_{1h} + \varphi_h)|_T = 0, \forall T \in \mathcal{T}_h$. □

Now, we are going to prove that the eigenvalues of $\mathbf{T}$ in $(0, 1/2)$ and their eigenfunctions are well approximated by the nonconforming discretization considered here. To do that, we need to prove properties **P1**, **P2** and **P3**.

Theorem 4.9. **(P2)** *For each eigenfunction* $\mathbf{u}$ *of* $\mathbf{T}$ *associated to* $\lambda \in (0, 1/2)$, *there exists a strictly positive constant* C *such that*

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_h \le Ch^r|\mathbf{u}|.$$

Proof. Since $u_1 \in H^{1+r}(\Omega)$, $u_1 \in C^0(\Omega)$. So, u_1^I, the piecewise linear Lagrange interpolant of u_1, is well defined (see [8], for instance). Moreover, $u_1^I \in W_{1h} \cap H^1_\Gamma(\Omega)$. Then, we can choose $v_{1h} = u_1^I$. By using standard error estimates, we get

$$\|u_1 - v_{1h}\|_{1,T} \le Ch^r\|u_1\|_{1+r,T}. \tag{4.38}$$

Since $\mathbf{u}_2 = \lambda^{-1}\nabla u_1$, we can take $\mathbf{v}_{2h} = \lambda^{-1}\nabla v_{1h}$ and we get

$$\|\mathbf{u}_2 - \mathbf{v}_{2h}\|_{0,\Omega} \le Ch^r\|\mathbf{u}_2\|_{r,\Omega}. \tag{4.39}$$

Now, combining (4.38) and (4.39) with the definition of $\|\cdot\|_h$ and estimation (4.28), we can write

$$\|\mathbf{u} - \mathbf{v}_h\|_h \le Ch^r|\mathbf{u}|.$$

So, taking the infimum with respect to $\mathbf{v}_h \in \mathbf{V}_h$, we can conclude the proof. □

Theorem 4.10. *(P1) There exists a positive constant* C *such that*

$$\|(\mathbf{T} - \mathbf{T}_h)(f, \mathbf{g})\|_h \le Ch^r\|(f, \mathbf{g})\|_h, \quad \forall(f, \mathbf{g}) \in \mathbf{V}_h.$$

Proof. Let $\mathbf{u} := \mathbf{T}(f, \mathbf{g})$ and $\mathbf{u}_h := \mathbf{T}_h(f, \mathbf{g})$, for any $(f, \mathbf{g}) \in \mathbf{V}_h$. Since $\mathbf{V}_h \not\subset \mathbf{V}$, we apply the second Strang's Lemma (see [8]), which in this case reads:

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le C\left(\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{w}_h\in\mathbf{V}_h}\frac{a_h(\mathbf{u}, \mathbf{w}_h) - b((f, \mathbf{g}), \mathbf{w}_h)}{\|\mathbf{w}_h\|_h}\right). \tag{4.40}$$

We begin the proof by bounding the first term in the right-hand side of inequality (4.40). To do that, we may test Eq. (4.26) separately with $(0, \mathbf{v}_2)$ and $(v_1, \mathbf{0})$, for any $\mathbf{v}_2 \in \mathbf{L}^2(\Omega)$ and $v_1 \in H^1_\Gamma(\Omega)$. Then, we obtain that $(u_1, \mathbf{u}_2)$ satisfies

$$\nabla u_1 + 2\mathbf{u}_2 = \mathbf{g} \tag{4.41}$$

and that

$$\int_\Omega \nabla u_1 \cdot \nabla v_1 + \int_\Omega \mathbf{u}_2 \cdot \nabla v_1 + \int_\Omega u_1 v_1 = \int_\Omega f v_1, \quad \forall v_1 \in H^1_\Gamma(\Omega). \tag{4.42}$$

Now, since $\mathbf{g} \in \mathbf{W}_{2h}$, we can write the orthogonal decomposition

$$\mathbf{g} = \nabla\varphi_h + \mathbf{curl}\,\xi_h, \tag{4.43}$$

with $\varphi_h \in R_h$ and $\xi_h \in S_h$. So, by using Eqs. (4.41)–(4.43), we have

$$\int_\Omega \nabla u_1 \cdot \nabla v_1 + \frac{1}{2}\int_\Omega (\nabla\varphi_h - \nabla u_1) \cdot \nabla v_1 + \int_\Omega u_1 v_1$$
$$= \int_\Omega f v_1, \quad \forall v_1 \in H^1_\Gamma(\Omega). \tag{4.44}$$

Let $w \in H^1_\Gamma(\Omega)$ be defined by $w = u_1 + \varphi_h$. Then, w satisfies

$$\frac{1}{2}\int_\Omega \nabla w \cdot \nabla v_1 + \int_\Omega w v_1 = \int_\Omega (f + \varphi_h) v_1, \quad \forall v_1 \in H^1_\Gamma(\Omega), \tag{4.45}$$

and is shown to be the solution of the following problem

$$\begin{cases} -\frac{1}{2}\Delta w + w = f + \varphi_h & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma. \end{cases} \tag{4.46}$$

From the a priori estimates for this problem, it turns out that $w \in H^{1+r}(\Omega)$ with

$$\|w\|_{1+r} \le C\|f + \varphi_h\|_0 \le C(\|f\|_0 + \|\nabla\varphi_h\|_0) \le C|(f, \mathbf{g})|. \tag{4.47}$$

Arguing again as in Theorem 4.9, w^I, the piecewise linear Lagrange interpolant of w, is proved to be well defined. Choosing,

$$u_1^I = w^I - \varphi_h \quad \text{and} \quad \mathbf{u}_2^I = \frac{1}{2}(\mathbf{g} - \nabla u_1^I),$$

we can obtain

$$\left(\sum_{T\in\mathcal{T}_h} \|u_1 - u_1^I\|^2_{1,T}\right)^{1/2} \le Ch^r|(f, \mathbf{g})|,$$

$$\|\mathbf{u}_2 - \mathbf{u}_2^I\|_0 \le \frac{1}{2}\|\nabla u_1 - \nabla u_1^I\|_0 \le Ch^r|(f, \mathbf{g})|$$

and, finally,

$$\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_h \le Ch^r|(f, \mathbf{g})|.$$

It remains to bound the second term in Eq. (4.40). We multiply Eq. (4.46) by $v_{1h} \in W_{1h}$ and integrate to obtain

$$\frac{1}{2}\sum_{T\in\mathcal{T}_h}\left(\int_T \nabla w \cdot \nabla v_{1h} - \int_{\partial T} \nabla w \cdot \mathbf{n} v_{1h}\right) + \int_\Omega w v_{1h} = \int_\Omega (f + \varphi_h) v_{1h}. \tag{4.48}$$

Now, taking into account that

$$w = u_1 + \varphi_h \quad \text{and} \quad \nabla u_1 = \mathbf{g} - 2\mathbf{u}_2$$

and replacing $\mathbf{g}$ by the orthogonal decomposition (4.43), we deduce

$$\sum_{T\in\mathcal{T}_h}\left(\int_T \nabla u_1 \cdot \nabla v_{1h} + \int_T \mathbf{u}_2 \cdot \nabla v_{1h}\right) + \int_\Omega u_1 v_{1h} - \int_\Omega f v_{1h}$$

$$= \sum_{T\in\mathcal{T}_h} \left(\int_{\partial T} \nabla w \cdot \mathbf{n} v_{1h} + \frac{1}{2} \int_T \mathbf{curl}\, \xi_h \cdot \nabla v_{1h} \right). \tag{4.49}$$

Now, as shown in the proof of Theorem 4.7, for every $\xi_h \in S_h$,

$$\sum_{T\in\mathcal{T}_h} \int_T \mathbf{curl}\, \xi_h \cdot \nabla v_{1h} = 0.$$

Multiplying Eq. (4.41) by $\mathbf{v}_{2h} \in \mathbf{W}_{2h}$ and adding to Eq. (4.49), we have

$$a_h(\mathbf{u}, \mathbf{v}_h) - b((f, \mathbf{g}), \mathbf{v}_h) = \sum_{T\in\mathcal{T}_h} \int_{\partial T} \nabla w \cdot \mathbf{n} v_{1h}.$$

Now, because $w \in H^{1+r}(\Omega)$, we can proceed as in the proof of Theorem 4.3 to obtain

$$|a_h(\mathbf{u}, \mathbf{v}_h) - b((f, \mathbf{g}), \mathbf{v}_h)| \le C h^r |(f, \mathbf{g})| \|\mathbf{v}_h\|_h, \tag{4.50}$$

which allows us to conclude the theorem. □

Theorem 4.11. (P3) *Let $\mathbf{E}(V + V_h)$ be the direct sum of the eigenspaces of $\mathbf{T}$ associated to its eigenvalues $\lambda \in (0, 1/2)$. There exists a positive constant C such that*

$$\|(\mathbf{T} - \mathbf{T}_h)|_{\mathbf{E}(V+V_h)}\|_h \le C h^r.$$

Proof. For $(f, \mathbf{g}) \in \mathbf{E}(\mathbf{V} + \mathbf{V}_h)$, $\mathbf{u} = \mathbf{T}(f, \mathbf{g}) \in H^{1+r}(\Omega) \times (H^r(\Omega))^2$. From second Strang's Lemma (see [8]) we have

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le C \left(\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{w}_h \in \mathbf{V}_h} \frac{a_h(\mathbf{u}, \mathbf{w}_h) - b((f, \mathbf{g}), \mathbf{w}_h)}{\|\mathbf{w}_h\|_h} \right). \tag{4.51}$$

Because of Theorem 4.9, it only remains to bound the second term in the right-hand side above.

By testing (4.26) with adequate smooth function $\mathbf{v}$, it is simple to show that $\mathbf{u}$ satisfy the following strong problem:

$$\begin{cases} -\Delta u_1 - \operatorname{div} \mathbf{u}_2 + u_1 = f & \text{in}\,\Omega, \\ \nabla u_1 + 2\mathbf{u}_2 = \mathbf{g} & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma. \end{cases} \tag{4.52}$$

Multiplying the first equation by $v_{1h} \in W_{1h}$, the second one by $\mathbf{v}_{2h} \in \mathbf{W}_{2h}$ and integrating by parts we obtain

$$\begin{aligned} |a_h(\mathbf{u}, \mathbf{v}_h) - b((f, \mathbf{g}), \mathbf{v}_h)| &= \sum_T \left(\int_T \frac{\partial u_1}{\partial \mathbf{n}} v_{1h} + \int_T \mathbf{u}_2 \cdot \mathbf{n} v_{1h} \right) \\ &= \sum_{\substack{\ell\in\mathcal{E}_h \\ \ell\not\subset\Gamma}} \left(\int_\ell \frac{\partial u_1}{\partial \mathbf{n}} [\![v_{1h}]\!] + \int_\ell \mathbf{u}_2 \cdot \mathbf{n} [\![v_{1h}]\!] \right) + \sum_{\ell\subset\Gamma} \left(\int_\ell \frac{\partial u_1}{\partial \mathbf{n}} v_{1h} + \int_\ell \mathbf{u}_2 \cdot \mathbf{n} v_{1h} \right). \end{aligned}$$

So, proceeding identically as in the proof of Theorem 4.3, we are able to prove an inequality similar to (4.50). Then, we conclude the proof. □

Lemma 4.12. *There exists a positive constant C such that*

$$M_h = \sup_{\substack{\mathbf{x}\in\mathbf{E}(\mathbf{V}+\mathbf{V}_h) \\ \|\mathbf{x}\|_h=1}} \sup_{\substack{\mathbf{y}\in\mathbf{E}(\mathbf{V}+\mathbf{V}_h) \\ \|\mathbf{y}\|_h=1}} |a_h(\mathbf{T}\mathbf{x}, \Pi_h \mathbf{y}) - b(\mathbf{x}, \Pi_h \mathbf{y})| \le C h^{2r},$$

with Π_h being the projection onto $\mathbf{V}_h$ with respect to a_h, defined by Eq. (3.3).

Proof. The proof runs almost identically to that of Theorem 4.11. □

Theorem 4.13. *There exists a positive constant C such that*

$$\max_{i=1,\cdots,m} |\lambda - \lambda_{ih}| \le C h^{2r}.$$

Proof. It is an immediate consequence of properties **P2** and **P3** and the previous lemma. □

Acknowledgment

The authors wish to thank Rodolfo Rodríguez for many valuable comments on this paper.

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