Balanced many-to-one matching problems with preferences over colleagues

Juan C. Cesco[1 2]

[1] Instituto de Matemática Aplicada San Luis (IMASL), San Luis, Argentina,
jcesco@unsl.edu.ar

[2] Departamento de Matematica, Universdad Nacional de San Luis, San Luis, Argentina

Abstract. Matching problems is a well studied class of coalitions formation models. Several core-like type solutions have been proposed for these models. However, unlike what happens in game theory, no balancedness properties have been introduced to study existence problems so far. In this paper we state a balancedness condition on a many-to-one matching problem with preferences over colleagues which turns to be a necessary and sufficient condition to guarantee the non-emptiness of the set of core matchings. We use this result to improve a recent characterization about the existence of core matchings for the classical many-to-one matching problem without preferences over colleagues. Our approach has been carried out by using some techniques and results from the theory of hedonic games, which is another class of coalitions formation models.

1 Introduction

Matching problems is a well studied class of coalitions formation models since the seminal paper of Gale and Shapley [12]. The books of Gusfield and Irving [13] and Roth and Sotomayor [20] provide a very good introduction to several of the aspects that have been dealt with during the last years on this area. Among the models studied, many-to-one and many-to-many matching problems (see for instance Martinez *et al.* [16], Echenique and Oviedo [9], [10]) have received particular attention. A classical approach assumes that agents are endowed with preferences not depending on colleagues. However, Dutta and Massó [7], and more recently Echenique and Yenmez [11] and Dimitrov and Lazarova [6], incorporate dependence of the preferences over colleagues into the model and study existence problems of stable matchings. Dutta and Massó [7] consider some restrictions on the preference profiles and their main results are obtained under the assumption that individual preferences satisfy some lexicographical behavior. On the other hand, by using an algorithmic approach, Echenique and Yenmez [11] deal with the existence problem of stable matchings in a very general framework. The stability notions studied in these two papers are related to core concepts properly defined within the models considered. However, the conditions stated there to guarantee existence of the stable solutions are not related to any notion of balancedness of families of coalitions as the connection with core-type solutions could suggest. The main objective of this paper is to introduce a notion of balancedness for a matching problem, and then to prove that its is necessary and sufficient to guarantee the non-emptiness of the set of core matchings. In proving our results we are going to appeal to the notion of stable* matchings (Echenique and Oviedo [11]) in the framework of many-to-one matching problems with preferences over colleagues. Stability* is, in some sense, an intermediate notion of stability between that of stability and core-stability. It is worth noting that the set of stable* matchings coincides with the set of stable matchings in the classical many-to-one matching problem with substitutable preferences (Echenique and Oviedo [9]), while it also coincides with the set of core-partitions in the classical many-to-one matching problem (Echenique and Oviedo [9]). A stable* partition is a partition resistent to objections raised by a particular class of intermediate coalitions.

To get our objective we will associate, to each matching problem, a hedonic game. Then, we will show that stable* matchings are related to core-partitions of the associated game. Hedonic games, in the form that we are going to use here, were first dealt with in Banerjee et al. [1] and Bogomolnaia and Jackson [4], although hedonic aspect in economic models were previously studied, for instance, in Drèze and Greenberg [8]. The literature on hedonic or simple coalition formation games has grown considerably, in the last decade, dealing mainly with existence and uniqueness issues of core-partitions (see also Pápai [17], Iehlé [14]). We point out that Pycia [18] shares our objective ``... to develop our understanding when stable matchings exist.´´. Nevertheless, although the hedonic game associated to a matching problem we will use here to get our result is basically stated in his paper, his results have a very different flavor than ours. In a recent paper, Dimitrov and Lazarova [6], in the framework of many-to-many matching problems, also relate coalition formation models and matching problems in a way that resembles our approach. In this paper, total balancedness Bloch and Diamantoudi [3] of some related coalition formation game to a given many-to-many matching problem, shows up in connection to the existence problem of some kind of stable matchings in the latter problem. Finally, we would like to mention Kominers [15] who also relates a classical many-to-one matching problem to a many-to-one matching problem with preferences over colleagues to develop an algorithm to compute the set of all stable matchings for the classical matching problem. The matching problem with preferences over colleagues that he constructs, coincides with ours in Section 5 when dealing with the classical matching problem. In both cases, the preferences are lexicographical (see Section 5).

Recently, Cesco [5], for classical many-to-one matching problems, states a necessary and sufficient condition for the existence of stable* matchings. The condition is given in terms of the pivotal balancedness (Iehlé [14]) of an associated hedonic game. The main result of this note (Theorem 5) improves that characterization by providing another one stated in terms of the p-balancedness (Section 3) of an associated many-to-one matching problem. This new problem has, however, preferences depending, in a lexicographical way, over colleagues. Besides the paper of Dutta and Massó [7], Dimitrov and Lazarova [6] also deal with different notions of lexicographical preferences but although they resemble closely the way we use to construct the lexicographical preferences in the just mentioned many-to-one matching problem with preferences over colleagues, none of the resulting domains of preferences coincide with ours, not even in the case that one of the set of agents has cardinality one. In Revilla [19], in the framework of matching problems with preferences over colleagues, two sufficient conditions guaranteeing the existence of stable matchings are provided but they are not however, related to our.

The paper is organized as follows. In the next section we introduce hedonic games with restrictions on the set of coalitions and the concept of core-partition as well in this setting. We also recall the notion of ordinal balancedness due to Bogomolnaia and Jackson [4] and that of pivotal balancedness due to Iehlé [14] and state conditions under which there exist core partitions in hedonic games with a restricted family of coalitions. Section 3 is devoted to many-to-one matching problems with preferences over colleagues and recall the notion of stable* matchings and introduce the definition of p-balanced matching problem. In Section 4 we construct, related to any many-to-one matching problem with preferences over colleagues, a hedonic game with restricted family of coalitions. We show a relationship between the set of stable* matchings of the matching problem with the weak core of the associated hedonic game. We use that relationship to prove our main theorem which states that the set of stable* matchings of a matching problem is non-empty if and only if the matching problem is p-balanced. In the next section we concentrate on the classical many-to-one matching problem. First we show how to construct its lexicographical extension, which is another many-to-one matching problem with preferences over colleagues. For this class of matching problems the results of Section 4 can be used to improve a characterization about the non-emptiness of the set of stable* matchings given in Cesco [5]. In a final Appendix we present the construction of a hedonic game associated to a classical many-to-one matching problem used in the characterization of Cesco [5].

2 Hedonic Games

A *hedonic game* with a family of admissible coalitions, also called a simple coalition formation model (Banerjee et al. [1], Papaí [17]) is defined by a non-empty finite set $N = \{1, ..., n\}$, the *players*, a collection $\mathcal{A}$ of non-empty subsets of N such that $\{i\} \in \mathcal{A}$ for each $i \in N$, the family of *admissible coalitions*, and a *preference profile* $\succeq = (\succeq^i)_{i \in N}$ where, for each $i \in N$, $\succeq^i$ is a complete[3] and transitive binary relation on $\mathcal{A}(i)$, the elements of $\mathcal{A}$ containing i. We will use $(N, \succeq; \mathcal{A})$ to denote a hedonic game with $\mathcal{A}$ as its family of admissible coalitions. For each $i \in N, \succ^i$ will stand for the strict preference relation derived from $\succeq_i$ ($x \succ^i y$ iff $x \succeq^i y$ but $y^i x$), and $\mathcal{P}^{\mathcal{A}}(N)$ will indicate the family of partitions of N such that

[3] In our formulation, completeness of a preference implies reflexivity.

all the elements in the partitions belong to $\mathcal{A}$. Given $\pi = \{\pi_1, ..., \pi_p\} \in \mathcal{P}^{\mathcal{A}}(N)$ and $i \in N, \pi(i)$ will denote the unique set in π containing player i. When $\mathcal{A} = \mathcal{N}$, the whole family of non-empty subsets of N, $(N, \succeq; \mathcal{A})$ is called simply a hedonic game and it is denoted by $(N, \succeq)$. The case $\mathcal{A} \neq \mathcal{N}$ was first studied in Papaí [17] and recently in Cesco [5].

Given $(N, \succeq; \mathcal{A})$ and a partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$, we say that $T \in \mathcal{A}$ *blocks* π if for each $i \in T, T \succ^i \pi(i)$.

Definition 1 The core $C(N, \succeq; \mathcal{A})$ of $(N, \succeq; \mathcal{A})$ is the set of partitions blocked by no coalition.

A partition π belonging to $C(N, \succeq; \mathcal{A})$ is a core-partition.

An individual preference $\succeq^i$ is strict if $A \succeq^i B$ and $A \neq B$ implies that $B^i A$. When all the preferences in the preference profile $\succeq$ of a hedonic game $(N, \succeq; \mathcal{A})$ are strict, then $\pi \in C(N, \succeq; \mathcal{A})$ if and only if there is no $T \in \mathcal{A}$ such that $T \succeq^i \pi(i)$ for all $i \in T$, and $T \succ^i \pi(i)$ for at least one $i \in T$ (weak blocking). We denote with $slC_W(N, \succeq; \mathcal{A})$ the set of partition weakly blocked by no coalition. Clearly, always $slC_W(N, \succeq; \mathcal{A}) \subseteq slC(N, \succeq; \mathcal{A})$.

Let $b = (b^A)_{\phi \neq A \in \mathcal{A}}$ be a family of non-negative vectors in R^n satisfying: $\mathrm{b}_i^A = 0$ *for all* $i \notin A$, and $\mathrm{b}_i^N > 0$ *for all* $i \in N$. A non-empty subfamily $\mathcal{B} \subseteq \mathcal{A}$ is *b-balanced* (Billera [2]) if there exist positive weights $\lambda = (\lambda_A)_{A \in \mathcal{B}}$ such that $\sum\limits_{A \in \mathcal{B}} \lambda_A . b^A = b^N$. Given a coalition $A \in \mathcal{A}$, χ^A will stand for the n-dimensional *indicator vector* of A, namely, $\chi_i^A = 1$ if $i \in A$ and if $i \notin A$. In the case that for each coalition $A \in \mathcal{A}, b^A = \chi^A$, and $b^N = \chi^N$ (provided $N \notin \mathcal{A}$), b-balancedness coincides with the classical notion of *balancedness* (see Shapley [21]).

A family $\mathcal{I} = (\mathcal{I}(A))_{A \in \mathcal{A}}$ is called an $\mathcal{A}$-*distribution*, or simply a distribution (Iehlé (2007)) if, for each coalition $A \in \mathcal{A}$, $\mathcal{I}(A) \in \mathcal{A}$ and $\mathcal{I}(A) \subseteq A$. Given a distribution $\mathcal{I}$, a family $\mathcal{B} \subseteq \mathcal{A}$ is $\mathcal{I}$-balanced if it is b-balanced with respect to $b = (\chi^{\mathcal{I}(A)})_{A \in \mathcal{A}}$.

Definition 2 $(N, \succeq; \mathcal{A})$ is ordinally balanced if for any balanced family $\mathcal{B} \subseteq \mathcal{A}$ there exists $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ such that, for each $i \in N, \pi(i) \succeq^i B$ for some $B \in \mathcal{B}(i)$, being $\mathcal{B}(i)$ the set of coalitions in $\mathcal{B}$ containing player i. Ordinal balancedness is due to Bogomolnaia and Jackson [4]) for the case $\mathcal{A} = \mathcal{N} \backslash \{\phi\}$.

Definition 3 $(N, \succeq; \mathcal{A})$ is pivotally balanced with respect to an $\mathcal{A}$-distribution $\mathcal{I}$ if for each $\mathcal{I}$-balanced family $\mathcal{B}$, there exists a partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ such that, for each $i \in N, \pi(i) \succeq^i B$ for some $B \in \mathcal{B}(i)$. The game is pivotally balanced if it is pivotally balanced with respect to some distribution $\mathcal{I}$.

This general concept of balancedness is due to Iehlé [14] for the case $\mathcal{A} = \mathcal{N} \backslash \{\phi\}$).

The first part of the following theorem is a sufficient condition for the existence of core-partitions for hedonic games with coalitional restrictions which parallels the first part of Theorem 1 in Bogomolnaia and Jackson [4], while the second part parallels the characterization given in Theorem 3 of Iehlé [14], and whose proofs are carried out in a similar way.

Theorem 1 Let $(N, \succeq; \mathcal{A})$ be a hedonic game.

a) If the game is ordinally balanced, and has strict individual preferences, then $C(N, \succeq; \mathcal{A})$ is non-empty.

b) $C(N, \succeq; \mathcal{A})$ is non-empty if and only if the game is pivotally balanced.

3 Many-to-one matching problems

A many-to-one matching problem with preferences over colleagues consists of two disjoint finite sets of agents, the sets C of *colleges* and the set S of *students*. For any family $\mathcal{B}$ of subsets of $S \cup C$, and for any $f \in S \cup C, \mathcal{B}(f)$ will stand for the subfamily of $\mathcal{B}$ of those sets containing b. Let $\mathcal{S}$ denote the family of subsets of S. We will use $\mathcal{CS}$ to denote the collection of all sets of the form $\{c\} \cup A$ for some $c \in C, A \in \mathcal{S}$, along with the individual sets $\{s\}, s \in S$. In the model it is assumed that each college $c \in C$ is endowed with a preference over $\mathcal{CS}(c)$, and that each student $s \in S$ has a preference over $\mathcal{CS}(s)$. Individual preferences are assumed to be strict, complete and transitive on their corresponding domains.

We denote a *many-to-one matching problem* by $(C, S; \gg_C, \gg_S)$, where $\gg_C = (\gg_c)_{c \in C}$ and $\gg_S = (\gg_s)_{s \in S}$ are the preference profiles of the colleges and students respectively. A *matching* in $(C, S; \gg_C, \gg_S)$ is a function $\mu : C \cup S \to \mathcal{CS}$ satisfying:

a) For each $c \in C, \mu(c) = \{c\} \cup A$ with $A \in \mathcal{S}$.

b) For each $s \in S$, if $\mu(s) = \{c\} \cup A$ for some $c \in C$, then $A \in \mathcal{S}(s)$.

c) For each $c \in C, s \in S$, if $s \in \mu(c)$, then $\mu(s) = \mu(c)$, and if $c \in \mu(s)$, then $\mu(s) = \mu(c)$.

$\mu(s) = \{s\}$ means that student s wants to stand alone rather to join any college c. Similarly, $\mu(c) = \{c\}$ means that college c wants no student to be joined to him.

It is easy to show that this definition is equivalent to the definition of matching in many-to-one matching problems with preferences over colleagues given by Etchenique and Yenmez [11]. They also introduce two solution concepts, the core and the set of stable* matchings, whose definitions we recall below adapted to the notation we are using in this paper.

A matching μ is blocked by an individual agent $s \in S$ if $\{s\} >_s \mu(s)$, and by an agent $c \in C$ if $\{c\} >_c \mu(c)$. A matching μ is *individually stable* if it is blocked by no individual agent.

Given a many-to-one matching problem, $(C, S; \gg_C, \gg_S)$, the *core* $C(C, S; \gg_C, \gg_S)$ is the set of matchings for which there is no $\hat{C} \subseteq C, \hat{S} \subseteq S, \hat{C} \cup \hat{S} \neq \phi$, and a matching $\hat{\mu}$ such that, for all $c \in \hat{C}, s \in \hat{S}$ it holds that:

a) $\hat{\mu}(c) = \{c\} \cup A$ for some $A \subseteq \hat{S}, \hat{\mu}(s) = \{c'\} \cup A$ for some $c' \in \hat{C}$, $A \subseteq \hat{S}, s \in A$.

b) $\hat{\mu}(c) \gg_c \mu(c)$.

c) $\hat{\mu}(s) \gg_s \mu(s)$.

d) $\hat{\mu}(f) >_f \mu(f)$ for at least one $f \in \hat{C} \cup \hat{S}$.

A matching μ is *blocked*^* by $c \in C, B \subseteq S$ if $B \cap \mu(c) = \phi$ and there exists $A \subseteq \mu(c)$ so that for every $s \in A \cup B, \{c\} \cup A \cup B >_s \mu(s)$ and $\{c\} \cup A \cup B >_c \mu(c)$. A matching μ is *stable*^* if it is individually stable and there is no pair $c \in C, \phi \neq$

$B \subseteq S$ blocking* μ. $S^*(C, S; \gg_C, \gg_S)$ will stand for the set of stable* matchings of $(C, S; \gg_C, \gg_S)$. The following result, which states that the core and the set of stable* matchings coincide, is also due to Echenique and Yenmez [11], Lemma 3.2).

Theorem 2 Let $(C, S; \gg_C, \gg_S)$ be a many-to-one matching problem. Then, $S^*(C, S; \gg_C, \gg_S) = C(C, S; \gg_C, \gg_S)$.

4 Balanced matching problems

In some cases, the core of a many-to-many matching problem can be the empty set, so a natural question is if there is a characterization of the class of problems having non-empty core. Moreover, since there is a close relationship between matching models and hedonic games, it seems to be natural too to ask if such a condition involves some kind of balancedness characteristic. The aim of this section is to introduce a balancedness condition which characterizes the class of many-to-many matching problems, with preferences over colleagues, having non-empty core. The condition that we introduce below resembles that of pivotal balancedness, which characterizes hedonic games with non-empty set of core-partitions (Ihelé [14]).

Definition 4 A many-to-one matching problem with preferences over colleagues $(C, S; \gg_C, \gg_S)$ is p-balanced if there exists a distribution $\mathcal{I} \subseteq \mathcal{CS}$ such that, for each $\mathcal{I}$-balanced family of coalition $\mathcal{B} \subseteq \mathcal{CS}$ there exists a matching μ such that, for each $f \in C \cup S, \mu(f) \gg_f B$ for at least one $B \in \mathcal{B}(f)$. $(C, S; \gg_C, \gg_S)$ is balanced if $\mathcal{I} = \mathcal{CS}$.

We want now to show that p-balancedness characterizes matchings problems with non-empty core. To this end, to any given many-to-one matching problem $(C, S; \gg_C, \gg_S)$, with preferences over colleages, we associate a hedonic game whose construction is described below.

Let $N = C \cup F$ and $\mathcal{A} = \mathcal{CS}$. We now define the following preference profile related to the members of N. For each $i \in N \cap C$, namely, if $i = c$ for some $c \in C$, and A and B are two coalitions in $\mathcal{A}(i)$,we say that

$$A \succeq^i B \text{ if and only if } A \gg_c B. \quad (1)$$

On the other hand, if $i \in N \cap S$, namely, if $i = s$ for some $s \in S$, and A and B are two coalitions in $\mathcal{A}(i)$, we say that

$$A \succeq^i B \text{ if and only if } A \gg_s B. \quad (2)$$

Thus, if $\succeq = (\succeq^i)_{i \in N}$, $(N, \succeq; \mathcal{A})$ is a game related to the many-to-one matching problem having strict individual preferences.

Given a partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$, there is a related matching μ^π for the many-to-one matching problem, namely,

$$\mu^\pi(c) = \pi(c), \quad (3)$$

and

$$\mu^{\pi}(s) = \pi(s). \tag{4}$$

Conversely, related to any matching μ of a many-to-one matching problem there is a partition $\pi^{\mu} \in \mathcal{P}^{\mathcal{A}}(N)$ defined as follows. For each $c \in C$, let

$$\pi^{\mu}_{c} = \mu(c), \tag{5}$$

and for each $s \in S$ such that $\mu(s) = \{s\}$, let $\pi^{\mu}_{s} = \{s\}$. Then $\pi^{\mu} = \{\pi^{\mu}_{f} : f \in F\} \cup \{\pi^{\mu}_{s} : s \in S \text{ and } \mu(s) = \{s\}\}$. It is clear that, for each $s \in S$,

$$\pi^{\mu}(s) = \mu(s). \tag{6}$$

Lemma 3 Let $(C, S; \gg_C, \gg_S)$ be a many-to-one matching problem, μ a matching and π^{μ} its related $\mathcal{A}$-partition in $(N, \succeq; \mathcal{A})$. If μ is individually stable in $(C, S; \gg_C, \gg_S)$ then π^{μ} is blocked by no individual coalition in $(N, \succeq; \mathcal{A})$ (individual rationality). Conversely, if π is individually rational in $(N, \succeq; \mathcal{A})$ then μ^{π} is individually stable in $(C, S; \gg_C, \gg_S)$.

Proof Let μ be individually stable. Then, for each $s \in S, \mu(s) \gg_s \{s\}$. But since $\pi^{\mu}(s) = \mu(s)$ it follows that π^{μ} cannot be blocked by any individual coalition $\{s\}, s \in S$. On the other hand, assume that there is $\{c\} \succ^{c} \pi^{\mu}(c)$ in $(N, \succeq; \mathcal{A})$ for some $c \in C$. This would imply that $\{c\} >_c \mu(c)$. But this contradicts the individual stability of μ. Thus, π^{μ} is individually rational.

Conversely, let π be individually rational in $(N, \succeq; \mathcal{A})$. This means that for all $i \in N, \pi(i) \succeq^{i} \{i\}$. But this means that $\mu^{\pi}(i) \gg_i \{i\}$, either $i \in C$ or $i \in S$. But this implies that μ^{π} is individually stable.

Proposition 4 Let $(C, S; \gg_C, \gg_S)$ be a many-to-one matching problem and $(N, \succeq; \mathcal{A})$ its related hedonic game. If $\mu \in slS^{*}(C, S; \gg_C, \gg_S)$ then $\pi^{\mu} \in slC_W(N, \succeq; \mathcal{A})$. Conversely, if $\pi \in C(N, \succeq; \mathcal{A})$, $\mu^{\pi} \in slS^{*}(C, S; \gg_C, \gg_S)$

Proof From Lemma 3 we get that π^{μ} is individually rational. So, let us assume that there is a blocking coalition in $\mathcal{A}$ of the form $\{c\} \cup \hat{A}$. Then $\{c\} \cup \hat{A} \succeq^{b} \pi^{\mu}(b)$ for all $b \in \{c\} \cup \hat{A}$, and $\{c\} \cup \hat{A} \succ^{b} \pi^{\mu}(b)$ for at least one $b \in \{c\} \cup \hat{A}$. But, since the individual preferences are strict, $\{c\} \cup \hat{A} \succ^{b} \pi^{\mu}(b)$ for all $b \in \{c\} \cup \hat{A}$ indeed. Otherwise, $\{c\} \cup \hat{A} = \pi^{\mu}(b)$ for all $b \in \{c\} \cup \hat{A}$ and therefore, it could not be a blocking coalition to π^{μ}. Now, $B = (\{c\} \cup \hat{A}) \backslash \mu(c)$. Certainly, $B \subseteq S$ and $B \cap \mu(c) = \phi$. Let us define $A = ((\{c\} \cup \hat{A}) \cap \mu(c)) \backslash \{c\}$. Then $A \subseteq \mu(c)$ and $A \cup B = \hat{A}$. Moreover, since $\{c\} \cup \hat{A} \succ^{b} \pi^{\mu}(b)$ for all $b \in \{c\} \cup \hat{A}$ we conclude that the pair c and B blocks* μ, a contradiction which proves that $\pi^{\mu} \in C_W(N, \succeq; \mathcal{A})$.

Conversely, let $\pi \in C(N, \succeq; \mathcal{A})$. Once more, because of Lemma 3, μ^{π} is individually stable. Now, let $c \in C$ and $B \in S$ be a blocking pair to μ^{π}. Then $B \cap \mu(c) = \phi$ and there exists $A \subseteq \mu(c)$ such that for each $s \in A \cup B$, $\{c\} \cup A \cup B >_s \mu^{\pi}(s)$, $\{c\} \cup A \cup B >_c \mu^{\pi}(c)$, But, since $\{c\} \cup A \cup B$ belongs to $\mathcal{A}$, the latter relationships imply that $\{c\} \cup A \cup B$ blocks π, which is a contradiction. This proves that μ^{π} is stable*.

Theorem 5 Let $(C, S; \gg_C, \gg_S)$ be a many-to-one matching with preferences over colleagues, and $(N, \succeq; \mathcal{A})$ its related hedonic game. Then the following conditions are equivalent.

$a) slS^*(C, S; \gg_C, \gg_S) \neq \phi$.

$b) C(C, S; \gg_C, \gg_S) \neq \phi$.

$c)(N, \succeq; \mathcal{A})$ is pivotally balanced.

$d)(C, S; \gg_C, \gg_S)$ is p-balanced.

Proof The equivalence between $a)$ and $b)$ is given in Theorem 2. The equivalence between $a)$ and $c)$ follows from Proposition 4 and part $b)$ of Theorem 1. So, we are going to show now the equivalence between $c)$ and $d)$. Let us assume first that $(N, \succeq; \mathcal{A})$ is pivotally balanced with respect to an $\mathcal{A}$- distribution $\mathcal{I}$. We claim that $(C, S; \gg_C, \gg_S)$ is p-balanced with respect to the same distribution $\mathcal{I}$. To see this, let $\mathcal{B}$ be an $\mathcal{I}$-balanced family of coalition in $\mathcal{CS}$ (we recall that $\mathcal{A} = \mathcal{CS}$). Since $(N, \succeq; \mathcal{A})$ is pivotally balanced, there is a partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ such that, for each $i \in N, \pi(i) \succeq_i B$ for at least one $B \in \mathcal{B}(i)$. But, taking into account the definition of μ^π (see (3) and (4)) it follows that $\mu^\pi(i) = \pi(i) \gg^i B$, showing the pivotal balancedness of $(C, S; \gg_C, \gg_S)$. The converse is proved similarly by taking into account the association between any matching μ in the matching problem and the partition π^μ (see (5) and (6)) in the associated hedonic game.

Remark 1 The characterization of $C(C, S; \gg_C, \gg_S)$ in terms of the set of fixed point of the T algorithm given by Echenique and Yenmez [11] indicates that any of the statements in Theorem 5 is equivalent to the following one: $e)$ The set of fixed point of T is non-empty.

Remark 2 Since balancedness for a matching problem $(C, S; \gg_C, \gg_S)$ clearly implies its p-balancedness, it turns to be that balancedness is a sufficient condition for the non-emptiness of $C(C, S; \gg_C, \gg_S)$.

5 The classical many-to-one matching problem

A classical many-to-one matching problem is a matching problem $(C, S; \succeq_C, \succeq_S)$ where the individual preference for each student is a strict preference over the set of colleges. In this case, the preferences do not depend on colleagues.

We recall that a matching in $(C, S; \succeq_C, \succeq_S)$ is a function $\mu : C \cup S \to C \cup \mathcal{W}$ satisfying:

$a)$ For each $c \in C$,if $\mu(c) \neq \phi$, then $\mu(c) \in \mathcal{W}$.

$b)$ For each $s \in W$, if $|\mu(s)| = 1$, then $\mu(s) \in C$ and $\mu(s) = c$ if and only if $s \in \mu(c)$.

A matching μ is blocked* by $c \in C, \phi \neq B \subseteq S$ if $c \succ_s \mu(s)$ for all $s \in B$, and there exists $A \subseteq \mu(c)$ such that $A \cup B \succ_c \mu(c)$. A matching μ is stable* (Echenique and Oviedo (2004)) if it is individually rational (see Roth and Sotomayor (1990)) and there is no pair $c \in C, \phi \neq B \subseteq S$ blocking* μ. $slS^*(C, S; \succeq_C, \succeq_S)$ will stand for the set of stable* matchings of $(C, S; \succeq_C, \succeq_S)$.

With each classical many-to-one matching problem we associate its lexicographical extension $(C, S; \gg_C, \gg_S)$, which is another many-to-one matching problem but having preferences depending on colleagues where, for each $c \in C$, and $A, B \in \mathcal{CS}(c)$, A$\gg_c B$ *if and only if* $(A \cap S) \succeq_c (B \cap S)$ and, for each $s \in S$ and $A, B \in \mathcal{CS}(s), A \gg_s B$ *if and only if*

$$(A \cap C) \succeq_s (B \cap C) if (A \cap C) \neq (B \cap C), or$$
$$(A \cap S) \succeq_c (B \cap S) if (A \cap C) = (B \cap C) = \{c\}.$$

Remark 2 Although Dutta and Massó [7] also study a many-to-one matching model with lexicographical preferences, we point out that their $\mathcal{W}$-lexicographical preferences for the members of $\mathcal{W}$ ($\mathcal{S}$ in our model) are very different of ours. Indeed, they assume that each $w \in \mathcal{W}$ has an ordering on the subsets of $\mathcal{W}$ which prevails on any ordering that w could have on the members of $\mathcal{F}$, while in our model each $s \in \mathcal{S}$ ($\mathcal{W}$ in Dutta-Massó's model) first care about the members of $\mathcal{C}$ ($\mathcal{F}$ in their model) and later, on the subsets of $\mathcal{S}$ if two coalitions sharing the same $c \in \mathcal{C}$ are being compared. But even in this latter case, the comparing ordering is borrowed from the common agent c, and thus, there is not a unique ordering on the subsets of $\mathcal{S}$ like in the Dutta and Massó model.

Theorem 6 Let $(C, S; \succeq_C, \succeq_S)$ be a many-to-one matching classical problem and let $(C, S; \gg_C, \gg_S)$ be its lexicographical extension. Then the following conditions are equivalent.

$a)slS^*(C, S; \succeq_C, \succeq_S) \neq \phi$.

$b)(C, S; \gg_C, \gg_S)$ is p-balanced.

Proof According to the equivalence between $c)$ and $d)$ stated in Theorem 5, $(C, S; \gg_C, \gg_S)$ is p-balanced if and only if the associated hedonic game $(N, \succeq; \mathcal{A})$ to $(C, S; \gg_C, \gg_S)$ is pivotally balanced. On the other hand, according to Theorem 9 in Cesco [5][5], $slS^*(C, S; \succeq_C, \succeq_S) \neq \phi$ if and only if $(N, \hat{\succeq}; \mathcal{A})$ is pivotally balanced where the latter is also a hedonic game associated to the classical many-to-one matching problem, and whose definition is given in the Appendix. Thus, the equivalence between $a)$ and $b)$ is completed by noting that $(N, \hat{\succeq}; \mathcal{A}) = (N, \succeq; \mathcal{A})$.

Remark 3 Because of Theorem 5 and Theorem 2 of Echenique and Oviedo [9], it turns to be that a necessary and sufficient condition for the core of a classical many-to-one matching problem $(C, S; \succeq_C, \succeq_S)$ be non-empty is that its lexicographical extension $(C, S; \gg_C, \gg_S)$ be p-balanced. Moreover, for this class of matching problems, the statements in Theorem 6 are also equivalent to the following ones:

$c)slS^*(C, S; \succeq_C, \succeq_S) \neq \phi$.

$d)$The set of fixed point of the T algorithm is non-empty.

If the matching problem has substitutable preferences, then all of them are also equivalent to

$e)slS(C, S; \succeq_C, \succeq_S) \neq \phi$, where $slS(C, S; \succeq_C, \succeq_S)$ indicates the set of stable matchings.

X

We refer the reader to Echenique and Oviedo [9] where these equivalences have been proven.

6 Appendix

Cesco [5] associates a hedonic game $(N, \hat{\succeq}; \mathcal{A})$ to a given classical many-to-one matching problem $(C, S; \succeq_C, \succeq_S)$ and show that $slS^*(C, S; \succeq_C, \succeq_S)$ is non-empty if and only if the game has has non-empty core. Here we indicate how to construct the game.

Let $(C, S; \succeq_C, \succeq_S)$ be a classical many-to-one matching problem.

We recall that a *matching* in $(C, S; \succeq_C, \succeq_S)$ is a function $\mu : C \cup S \to C \cup \mathcal{W}$ satisfying:

a) For each $c \in C$,if $\mu(c) \neq \phi$, then $\mu(c) \in \mathcal{W}$.

b) For each $s \in W$, if $|\mu(s)| = 1$, then $\mu(s) \in C$ and $\mu(s) = c$ if and only if $s \in \mu(c)$.

A matching μ is *blocked** by $c \in C, \phi \neq B \subseteq S$ if $c \succ_s \mu(s)$ for all $s \in B$, and there exists $A \subseteq \mu(c)$ such that $A \cup B \succ_c \mu(c)$. A matching μ is stable* if it is individually rational (see Roth and Sotomayor (1990)) and there is no pair $c \in C, \phi \neq B \subseteq S$ blocking* μ.

We now define the associated hedonic game $(N, \hat{\succeq}; \mathcal{A})$ to $(C, S; \succeq_C, \succeq_S)$. Let $N = C \cup S$ and $\mathcal{A} = \mathcal{CS}$ (see Section 3). The preference profile $\hat{\succeq} = (\hat{\succeq}_i)_{i \in N}$ is defined as follows. For each $i \in N$ such that $i = c$ for some $c \in C$, for each pair A, B of coalitions in $\mathcal{A}(i)$, we say that A$\hat{\succeq}_i B$ *if and only if* $(A \cap S) \succeq_c (B \cap S)$. On the other hand, for each $i \in N$ such that $i = s$ for some $s \in S$, for each pair A, B of coalitions in $\mathcal{A}(i)$, $A \hat{\succeq}_i B$ if and only if

$$
\begin{array}{l}
(A \cap C) \succeq_s (B \cap C) if (A \cap C) \neq (B \cap C) and \\
(A \cap S) \succeq_c (B \cap S) if (A \cap C) = (B \cap C) = \{c\},
\end{array}
$$

and if we agree in putting $\{s\} \succeq_s \{s\}$, it turns out that $\hat{\succeq}_i$ is a strict, reflexive, complete and transitive preference on $\mathcal{A}(i)$.

Clearly the game $(N, \hat{\succeq}; \mathcal{A})$ coincides with the game $(N, \hat{\succeq}; \mathcal{A})$ defined in Section 4.

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