Completing Categorical Algebras
(Extended Abstract)

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Abstract. Let Σ be a ranked set. A categorical Σ-algebra, cΣa for short, is a small category C equipped with a functor $\sigma_C : C^n \longrightarrow C$, for each $\sigma \in \Sigma_n$, $n \geq 0$. A continuous categorical Σ-algebra is a cΣa which has an initial object and all colimits of ω-chains, i.e., functors $\mathbb{N} \longrightarrow C$; each functor σ_C preserves colimits of ω-chains. ($\mathbb{N}$ is the linearly ordered set of the nonnegative integers considered as a category as usual.)
We prove that for any cΣa C there is an ω-continuous cΣa C^ω, unique up to equivalence, which forms a "free continuous completion" of C. We generalize the notion of inequation (and equation) and show the inequations or equations that hold in C also hold in C^ω. We then find examples of this completion when

- C is a cΣa of finite Σ-trees
- C is an ordered Σ algebra
- C is a cΣa of finite A-sychronization trees
- C is a cΣa of finite words on A.

1 Introduction

Computer science is necessarily concerned with fixed point equations, and in finding settings in which fixed point equations may be solved. Such equations arise in well known ways, for example, in specifying both the syntax and semantics of programming languages. In many examples, the setting is some kind of ordered algebra A with the properties that A contains a least element $\bot$, and ω-chains, i.e., increasing sequences $a_0 \leq a_1 \leq \dots$ have least upper bounds. In this setting, the least solution of an equation

$$x = f(x),$$

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when $f : A \longrightarrow A$ preserves least upper bounds of ω-chains may be found as the least upper bound of

$$\perp \leq f(\perp) \leq f^2(\perp) \leq \dots$$

For one such example, if Σ is a *ranked alphabet*, i.e., a sequence Σ_n, $n \geq 0$, of pairwise disjoint sets, the collection of finite and infinite Σ-trees may be equipped with an ordering by adjoining a new label $\perp$ to Σ_0, and defining $s \leq t$ if t may be obtained from s by adjoining some trees to leaves of s labeled $\perp$ (see below, or [GTWW77, Gue81] and [BE93] for example).

Similarly, the category of all partial functions $X \longrightarrow X$ is naturally ordered by set-inclusion of graphs; then, the meaning of a looping construct, such as the `while-do`:

```
(while B? f)(x) =
    if (B? x) then (while B? f) (f x)
    else x
```

is the least upper bound of the sequence $f_0, f_1, \dots$, of partial functions, where $f_0 : X \longrightarrow X$ is the totally undefined function, and

$$f_{n+1}(x) = \ \texttt{if}\ (B?\ x)\ \texttt{then}\ (f_n(f(x))\ \texttt{else}\ x.$$

However, not all fixed point equations may be solved by means of least upper bounds. One example that plays an important role in the semantics of parallel computation is **synchronization trees**, see [Mil89, Win84][2] or [BE93]. For a fixed alphabet A, an A-synchronization tree is a finite or countable rooted tree, in which every edge is labeled by a letter in A; the collection of these trees forms a category $\mathcal{ST}_A$, in which a morphism $f : s \longrightarrow t$ is a function from the vertices of s to the vertices of t which preserves the root, the edge relation and the labeling. This category has an initial object $\perp$, the rooted tree with no edge, and is equipped with at least the operations of **prefixing** and **sum**. For each letter $a \in A$, and each synchronization tree t, $a : t$ is the tree obtained from t by adding a new root, r and an edge labeled a from r to the root of t. When s, t are synchronization trees, $s + t$ is the tree obtained from s, t by identifying their roots, and otherwise, keeping the vertices and edges of each. In this category, fixed point equations such as

$$x = (a : x) + x$$

have solutions, but there is no canonical ordering on the category in which least solutions exist. However, this category has all colimits of ω-diagrams; the right-side of fixed point equations determines a continuous endofunctor $F : \mathcal{ST}_A \longrightarrow \mathcal{ST}_A$. Further, the "initial fixed point" of the functor F is determined up to isomorphism as a colimit of the ω-diagram

[2] In [Win84], two complete partial orders are defined on synchronization trees. However, the definition depends on the concrete representation of trees and is thus not fully abstract.

$$\bot \xrightarrow{!} F(\bot) \xrightarrow{F(!)} F^2(\bot) \xrightarrow{F^2(!)} \ldots.$$

Thus, $\mathcal{ST}_A$ is an example of a continuous cΣa defined in the abstract (and immediately below). There are other examples which we will mention after stating our main results.

Although there are many kinds of completions in the category-theory literature, we were not able to find this particular completion, except for the case of linear orders. In volume 2 of [Ele02], Johnstone describes an "Ind-completion" of a category, which is certainly related to this one. However, Johnstone does not study algebraic structures on the category and thus does not consider (in)equations.

The notion of a cΣa probably occurs to all those familiar with both universal algebra and category theory, and the outline of an ω-completion result is probably obvious to many. Perhaps the "right" notion of the truth of an inequation in a cΣa is not obvious, and the details of the construction have turned out to be more delicate than expected. We think they merit exposition in this paper.

In this extended abstract, only a few proofs will be given. A version of this paper with full proofs may be found at
`www.cs.stevens.edu/~bloom/research/pubs2/ccafull.pdf`.

2 Some notation

$\mathbb{N}$ is the category whose objects are the nonnegative integers, in which there is a morphism $n \longrightarrow p$ exactly when $n \leq p$. If $f : X \longrightarrow Y$ is either a function or functor, we write

$$i\,f,\ f_i,\ f(i)$$

for the value of f on the argument i. The composite of $f : x \longrightarrow y$ and $g : y \longrightarrow z$ is written $fg : x \longrightarrow z$, where f, g are functions or functors.

3 The completion and characterization theorems

Let Σ be a ranked alphabet. A **categorical Σ-algebra** C consists of a small category C, and, for each letter $\sigma \in \Sigma_n$, a functor $\sigma_C : C^n \longrightarrow C$. A **morphism**

$$h : C \longrightarrow C'$$

of categorical Σ-algebras is a functor $h : C \longrightarrow C'$ such that for each $n \geq 0$ and each $\sigma \in \Sigma_n$, $C^n \xrightarrow{\sigma} C \xrightarrow{h} D$ and $C^n \xrightarrow{h^n} D^p \xrightarrow{\sigma} D$ are naturally isomorphic. A cΣa-morphism h is **strict** if the functors $\sigma \cdot h$ and $h^n \cdot \sigma$ are the same for all $\sigma \in \Sigma_n$.

Recall that a functor $h : D \longrightarrow D'$ is **ω-continuous**, or just "continuous", for short, if whenever a functor $f : \mathbb{N} \longrightarrow D$ has a colimit $(\nu_n : f_n \longrightarrow d)_n$ in D, then $(\nu_n h : f_n h \longrightarrow dh)_n$ is a colimit of $f \cdot h : \mathbb{N} \longrightarrow D'$.

A cΣa C is **(ω-)continuous** if

– C is ω-complete, i.e., C has an initial object $\perp$ and all functors $\mathbb{N} \longrightarrow C$ have colimits, and further,
– each functor $\sigma_C : C^n \longrightarrow C$ is continuous.

A (strict) **morphism of continuous cΣa's** is a continuous functor $F : C \longrightarrow D$ which preserves initial objects and is a (strict) cΣa morphism.

Remark 1. Categorical Σ-algebras are a generalization of ordered Σ-algebras and continuous cΣa's are a generalization of (order) continuous Σ-algebras, see [Blo76, GTWW77, Gue81] or below.

Let $\mathbf{Tm}_\Sigma(p)$ denote the collection of Σ-terms on p variables $x_1, \ldots, x_p$. Suppose that C is a cΣa. Any term $t \in \mathbf{Tm}_\Sigma(p)$ determines a functor $t_C : C^p \longrightarrow C$ as follows:

– $(x_i)_C : C^p \longrightarrow C$ is the i-th projection functor $(1 \leq i \leq p)$.
– If $\sigma \in \Sigma_k$, $0 \leq k$, $(\sigma(t_1, \ldots, t_k))_C$ is the composite

$$C^p \xrightarrow{\langle (t_1)_C, \ldots, (t_k)_C \rangle} C^k \xrightarrow{\sigma_C} C$$

A cΣa inequality is an expression

$$s \preceq t$$

where s, t are terms in $\mathbf{Tm}_\Sigma(p)$, for some $p \geq 0$. If C is a cΣa, we say C **is a model for** $s \preceq t$, in symbols,

$$C \models s \preceq t$$

if there is a natural transformation $s_C \longrightarrow t_C$ between the functors s_C and t_C. Similarly, we define a **cΣa equality** to be an expression $s \cong t$, where s, t are as before. We write

$$C \models s \cong t$$

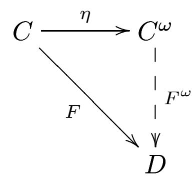
if there is a natural **isomorphism** $s_C \longrightarrow t_C$.

Our main results are about completions of cΣa's.

Theorem 1 (Completion theorem). *For any cΣa C having an initial object, there is a continuous cΣa C^ω, and a cΣa morphism*

$$\eta : C \longrightarrow C^\omega,$$

with the following properties. If D is a continuous cΣa, and if $F : C \longrightarrow D$ is any cΣa-morphism which preserves initial objects, then there is a morphism $F^\omega : C^\omega \longrightarrow D$ in the category of continuous cΣa's, unique up to a natural isomorphism, such that the functors $\eta \cdot F^\omega$ and F are naturally isomorphic.



It then follows that

- C^ω *is unique up to categorical equivalence.*
- η *is a full and faithful functor which is injective on objects, and which preserves initial objects.*
- *Any* $c\Sigma a$ *inequality or equality which holds in* C*, also holds in* C^ω*.*

Our characterization of C^ω involves the following notion.

Definition 1. *Suppose that* K *is a full subcategory of the category* D*.*

- K *is* ***compact in*** D *if for each object* c *in* K*, and each object* d *of* D*, if there is a colimiting cone*
$$(\tau_i^d : f_i \longrightarrow d)_i \tag{1}$$
where $f : \mathbb{N} \longrightarrow K$*, then any map* $c \longrightarrow d$ *factors through some* τ_i^d*.*
- D *is* ***compactly generated by*** K *if* K *is compact in* D *and for every object* d *of* D*, there is a functor* $f : \mathbb{N} \longrightarrow K$ *and a colimiting cone as in (1) in which each colimit morphism* $\tau_i^d : f_i \longrightarrow d$ *is monic.*

Using this notion, we describe those situations in which the induced functor F^ω in Theorem 1 is an equivalence.

Theorem 2 (Characterization theorem). *Suppose that* D *is a continuous* $c\Sigma a$ *and* $F : C \longrightarrow D$ *is a* $c\Sigma a$ *morphism which preserves initial objects. Then the induced functor* $F^\omega : C^\omega \longrightarrow D$ *is an equivalence iff* F *is full, faithful, and* D *is compactly generated by the image of* F*.*

We will outline the proofs after discussing some examples.

3.1 Ordered Σ-algebras

When Σ is a ranked set, an **ordered Σ-algebra** consists of a partially ordered set $(A, \leq)$ equipped with a function
$$\sigma : A^n \longrightarrow A$$
which is order preserving. Such algebras are categorical Σ-algebras, in which the objects are the elements of A and in which there is a morphism $a \longrightarrow b$ exactly when $a \leq b$. Also, when s, t are in $\mathbf{Tm}_\Sigma(p)$, an inequation $s \leq t$ holds in A exactly when there is a natural transformation $s \longrightarrow t$.

In [Blo76], varieties of ordered algebras were considered, and it was shown that each variety V was closed under the free ω-completion of any algebra in V. Our main theorem is a significant generalization of this result.

3.2 Σ trees

As formalized in [BET93], a Σ-tree t is a partial function $t : \mathbb{N}_+^* \longrightarrow \Sigma$, with source the set $\mathbb{N}_+^*$ of finite sequences of positive integers, and target Σ, with the following properties.

- The domain of t is a nonempty, prefix-closed subset of $\mathbb{N}_+^*$.
- If $u \in \mathbb{N}_+^*$ is in the domain of t and if $t(u) \in \Sigma_n$, and i is a positive integer, then ui, the sequence obtained by putting i at the end of the sequence u, is in the domain of t iff $1 \leq i \leq n$. Thus, the leaves of t are those sequences u such that $t(u) \in \Sigma_0$.

We assume there is a distinguished letter $\perp \in \Sigma_0$. Then for trees s, t, we define $s \leq t$ if t may be obtained from s by attaching some trees to leaves of s labeled $\perp$. The collection Σ**tr** of Σ-trees is an ordered Σ-algebra, in which the letter $\sigma \in \Sigma_n$ denotes the "prefixing operation" which applied to the n-tuple of trees $(t_1, \ldots, t_n)$ produces the tree $\sigma(t_1, \ldots, t_n)$, with a new root labeled σ, whose immediate successors are the roots of $t_1, \ldots, t_n$, respectively. As a function, for $u, v \in \mathbb{N}_+^*$ and $i \in \mathbb{N}_+$,

$$\sigma(t_1, \ldots, t_n)(u) = \begin{cases} \sigma & \text{if } u \text{ is the empty sequence} \\ t_i(v) & \text{if } u = iv, \end{cases}$$

where iv is the sequence obtained by putting i on the front of the sequence v. Σ**tr** is an **ordered** cΣa, in that there is a morphism $s \longrightarrow t$, for any trees s, t iff $s \leq t$. It is well known that Σ**tr** is a continuous cΣa.

Let Σ**Ftr** denote the full subcategory of Σ**tr** determined by the finite trees (those whose domain is finite).

Proposition 1. Σ**tr** *is the completion of* Σ**Ftr**. □

Note that if D is any cΣa with an intial object $\perp_D$, there is a unique cΣa morphism Σ**Ftr** $\longrightarrow D$ taking $\perp$ to $\perp_D$. Thus,

Corollary 1. Σ**tr** *is the initial continuous cΣa in the category of all continuous cΣa's in which $\perp$ is the initial object: for any such continuous cΣa D there is a continuous cΣa-morphism* Σ**tr** $\longrightarrow D$*, unique up to an isomorphism.* □

3.3 Synchronization trees

We have shown in [BE93] that $\mathcal{ST}_A$ defined briefly in the introduction is an ω-continuous categorical Σ_A-algebra, where Σ is the signature having a constant symbol 0, denoting the initial object $\perp$, a unary function symbol a for each $a \in A$, denoting the prefixing operation, and a binary function symbol $+$, denoting the coproduct operation described above. See also [Mil89, Win84].

Let $\mathcal{FST}_A$ denote the full subcategory of $\mathcal{ST}_A$ determined by the finite synchronization trees. Note that $\mathcal{FST}_A$ is also a cΣa, a "categorical subalgebra" of $\mathcal{ST}_A$.

Proposition 2. *$\mathcal{ST}_A$ is the completion of $\mathcal{FST}_A$.* □

Let $\mathcal{V}$ be the collection of all cΣa's D in which 0 is an initial object which satisfy the following:

$$x + 0 \cong x$$
$$x + y \cong y + x$$
$$x + (y + z) \cong (x + y) + z$$

Then it is not hard to show that the subcategory $\mathcal{FST}_A$(mon) of $\mathcal{FST}_A$ with the same objects having only monics as morphisms is the initial cΣa in $\mathcal{V}$, in the following sense: for any cΣa in $\mathcal{V}$ there is a cΣa-morphism $F : \mathcal{FST}_A$(monics) $\longrightarrow D$, unique up to a natural isomorphism.

Corollary 2. *$\mathcal{FST}_A(mon)^\omega$ is initial in the category of all continuous cΣa's in $\mathcal{V}$.*

Proof. Let D be a continuous cΣa in $\mathcal{V}$. Then there is a cΣa morphism F : $\mathcal{FST}_A$(mon) $\longrightarrow D$, since $\mathcal{FST}_A$(mon) is initial in D. But then there is a continuous $F^\omega : \mathcal{FST}_A(\mathrm{mon})^\omega \longrightarrow D$, unique up to natural isomorphism, by the completion theorem. □

3.4 Words

We recall from [Cou78, BE05] that when A is a finite or countable set, a **word over** A (called an *arrangement* in [Cou78]) is a triple $u = (L_u, \leq_u, \lambda_u)$ consisting of a finite or countable linearly ordered set $(L_u, \leq_u)$ and a labeling function $\lambda : L_u \longrightarrow A$. A word u is finite if the set L_u is finite. A morphism between words $u = (L_u, \leq, \lambda_u)$ and $v = (L_v, \leq, \lambda_v)$ is an order and label preserving map $h : L_u \longrightarrow L_v$. It is clear that words over A and their morphisms form a category that we denote $\mathcal{W}_A$. The finite words over A determine a full subcategory of $\mathcal{W}_A$ denoted $\mathcal{FW}_A$.

The basic operation on words is **concatenation** $u, v \mapsto u; v$ defined as follows. Without loss of generality we may assume that L_u and L_v are disjoint. Then the *concatenation* $u \cdot v$ is the word whose underlying linear order is $(L_u \cup L_v, \leq)$ where $x \leq y$ for all $x \in L_u$ and $y \in L_v$ and such that the restriction of $\leq$ to L_u agrees with $\leq_u$ and the restriction of $\leq$ to L_v agrees with $\leq_v$. The labeling function λ is given by

$$\lambda(x) = \begin{cases} \lambda_u(x) \text{ if } x \in L_u \\ \lambda_v(x) \text{ if } x \in L_u. \end{cases}$$

We extend concatenation to a functor. Given $f : u \longrightarrow u'$ and $g : v \longrightarrow v'$, we define the morphism $f \cdot g : u \cdot v \longrightarrow u' \cdot v'$ so that it agrees with f on the elements of L_u and with g on the elements of L_v.

Let Σ be the signature with a constant symbol a, for each $a \in A$, denoting the constant functor $\mathcal{W}_A^0 \longrightarrow \mathcal{W}_A$ whose value is the singleton word labeled

a, a symbol 0 in Σ_0 denoting the constant functor whose value is the empty word, and a binary function symbol ; denoting the concatenation functor. The following fact was essentially shown in [Cou78].

Proposition 3. *$\mathcal{W}_A$ is a continuous cΣa.*

In $\mathcal{W}_A$, one can solve such equations as $x = a; x$ and $x = x; a; x$. The initial solution to the second, is the word $[\![a]\!]^\eta$ whose underlying order is isomorphic to the rationals, with every point labeled a. (There doesn't seem to be an ordering of $\mathcal{W}_A$ such that $[\![a]\!]^\eta$ is the least upper bound of a sequence of finite approximations.)

Let $\mathcal{FW}_A$ be the full subcategory of $\mathcal{W}_A$ determined by the finite words.

Proposition 4. *$\mathcal{W}_A$ is the completion of $\mathcal{FW}_A$.*

Let $\mathcal{FW}_A$(mon) be the subcategory of $\mathcal{FW}_A$ with the same objects, having only the monics as morphisms. Define the category $\mathcal{M}$ having as objects all cΣa's with an initial object 0 which satisfy the monoid equations

$$0; x \cong x$$
$$x; 0 \cong x$$
$$x; (y; z) \cong (x; y); z$$

It is not hard to show that $\mathcal{FW}_A$(mon) is freely generated by A in $\mathcal{M}$ in the sense that for any cΣa D in $\mathcal{M}$, and any function $f : A \longrightarrow \mathrm{obj}(D)$, mapping 'letters' in A to objects in D, there is a functor $F : \mathcal{FW}_A(\mathrm{mon}) \longrightarrow D$, unique up to a natural isomorphism, such that $F(0)$ is initial and $F(a) = f(a)$, for each $a \in A$. Thus,

Corollary 3. *$\mathcal{FW}_A(mon)^\omega$ is freely generated by A in the category of all continuous cΣa's in $\mathcal{M}$.* □

4 Weak maps and compact generation

An endofunctor $m : \mathbb{N} \longrightarrow \mathbb{N}$ is just a nondecreasing function. We say an endofunctor is **unbounded** if for each $i \in \mathbb{N}$, $i \leq jm$, for some $j \in \mathbb{N}$.

When $m : \mathbb{N} \longrightarrow \mathbb{N}$ is an endofunctor and $f : \mathbb{N} \longrightarrow C$ is a chain, we write mf for the composite

$$\mathbb{N} \xrightarrow{m} \mathbb{N} \xrightarrow{f} C.$$

Thus, on the object $i \in \mathbb{N}$, $(mf)_i = f_{im}$.

When f, g are chains, a **weak map** $\alpha : f \longrightarrow g$ is a natural transformation

$$\alpha : f \longrightarrow mg$$

for some unbounded endofunctor m on $\mathbb{N}$. We define the composite of weak maps $\alpha : f \longrightarrow m_\alpha g$ and $\beta : g \longrightarrow m_\beta h$ as

$$\alpha \circ \beta := f \xrightarrow{\alpha} m_\alpha g \xrightarrow{m_\alpha \beta} (m_\alpha m_\beta)h.$$

Definition 2. *For weak maps* $\alpha : f \longrightarrow m_\alpha g$ *and* $\beta : f \longrightarrow m_\beta g$*, define* $\alpha \simeq \beta$ *by: for all* $i \geq 0$ *there is some* $j \geq im_\alpha, im_\beta$ *such that*

$$\alpha_i \cdot g(im_\alpha, j) = \beta_i \cdot g(im_\beta, j). \tag{2}$$

It is clear that $\simeq$ is an equivalence relation on the weak maps with the same source and target. Let $[\alpha] : f \longrightarrow g$ denote the $\simeq$-equivalence class of the weak map $\alpha : f \longrightarrow g$. This equivalence relation is compatible with composition.

Proposition 5. *If* $\alpha \simeq \alpha' : f \longrightarrow g$ *and* $\beta \simeq \beta' : g \longrightarrow h$*, then* $\alpha \circ \beta \simeq \alpha' \circ \beta'$*.* □

We will need the following fact about $\alpha \simeq \beta$.

Lemma 1 (Inflation Lemma). *Suppose that* $\alpha : f \longrightarrow mg$ *and that* $m' : \mathbb{N} \longrightarrow \mathbb{N}$ *is any functor satisfying*

$$k\,m \leq km',$$

for all $k \geq 0$*. Define the natural transformation*

$$\alpha' : f \longrightarrow m'g$$

by

$$\alpha'_i := f_i \xrightarrow{\alpha_i} g_{im} \xrightarrow{g(im,\ im')} g_{im'}.$$

Then

$$\alpha \simeq \alpha'.$$

4.1 Compact generation

Recall Definition 1. Note the similarity of this notion to that of the definition in [CCL80] of a continuous lattice.

The following lemma indicates where compact generation arises.

Lemma 2. *Let* C *be a full subcategory of* D*. Suppose that* $f, f' : \mathbb{N} \longrightarrow C$ *and that* $(\tau_i^d : f_i \longrightarrow d)_i$ *and* $(\tau_i^{d'} : f'_i \longrightarrow d')_i$ *are colimiting cones in* D*. Then*

1. *A weak map* $\gamma : f \longrightarrow mf'$ *determines the map*

   $$\kappa(\gamma) : d \longrightarrow d'$$

   as the unique morphism $d \longrightarrow d'$ *such that*

   $$\tau_i^d \cdot \kappa(\gamma) = \gamma_i \cdot \tau_{i\,m}^{d'}$$

   for all $i \geq 0$*.*

2. *If* $\gamma : f \longrightarrow mf'$ *and* $\overline{\gamma} : f \longrightarrow \overline{m}f'$ *are weak maps such that* $\gamma \simeq \gamma'$*, then*

$$\kappa(\gamma) = \kappa(\gamma').$$

3. *Suppose that* D *is compactly generated by* C *and that for* $i \geq 0$*, the morphisms* τ_i^d *and* $\tau_i^{d'}$ *are monics. Then, for any map*

$$h : d \longrightarrow d'$$

in D *there is a weak map* $\gamma : f \longrightarrow mf'$ *such that*

$$\kappa(\gamma) = h.$$

4. *Suppose that* D *is compactly generated by* C *and for* $i \geq 0$*, the morphisms* τ_i^d *and* $\tau_i^{d'}$ *are monics. If* $\gamma : f \longrightarrow mf'$ *and* $\overline{\gamma} : f \longrightarrow \overline{m}f'$ *are weak maps, and* $\kappa(\gamma) = \kappa(\overline{\gamma})$*, then*

$$\gamma \simeq \overline{\gamma}.$$

Now, we give a condition sufficient to obtain a colimit of a functor $G : \mathbb{N} \longrightarrow D$.

Lemma 3. *We assume the following hypotheses.*

– *For* $i \geq 0$, $f^i : \mathbb{N} \longrightarrow D$ *is a functor with colimiting cone* $(\tau_j^i : f_j^i \longrightarrow \kappa(f^i))_j$.
– *For each* $i \leq j$, $\beta^{i,j} : f^i \longrightarrow f^j$ *is a natural transformation such that* $\beta^{i,i} = \mathbf{1}_{f^i}$ *and, when* $i \leq j \leq k$,

$$\beta^{i,j} \cdot \beta^{j,k} = \beta^{i,k}.$$

Thus, $G : \mathbb{N} \longrightarrow D$ *is a functor, where* $G_i = \kappa(f^i)$*, and* $G(i,j) = \kappa(\beta^{i,j})$*, for all* $0 \leq i \leq j$.
– $g : \mathbb{N} \longrightarrow D$ *is the diagonal functor, defined by* $g_i = f_i^i$ *and, for* $i \leq j$,

$$\begin{aligned} g(i,j) &:= f^i(i,j) \cdot \beta_j^{i,j} \\ &= \beta_i^{i,j} \cdot f^j(i,j). \end{aligned}$$

– *Let* $\mu_i(j) = \max(i,j)$*, and let* $\delta^i : f^i \longrightarrow \mu_i g$ *be the weak map*

$$\delta_j^i := \begin{cases} f^i(j,i) & j \leq i \\ \beta_j^{i,j} & i < j. \end{cases}$$

– *Suppose that* $(\tau_i^g : g_i \longrightarrow \kappa(g))_i$ *is a colimiting cone.*

Then, it follows that $(\kappa(\delta^i) : \kappa(f^i) \longrightarrow \kappa(g))_i$ *is a colimiting cone over* G*, where* $\kappa(\delta^i)$ *is the unique map satisfying the conditions that*

$$\tau_j^i \cdot \kappa(\delta^i) = \delta_j^i \cdot \tau_{\mu_i(j)}^g, \tag{3}$$

for all j.

The following Proposition is quite useful.

Proposition 6. *Suppose that D is compactly generated by the full subcategory C. Then:*

1. *C has initial object iff D has.*
2. *D has colimits of all ω-diagrams iff each functor $\mathbb{N} \longrightarrow C$ has a colimit in D.*
3. *A functor $F : D \longrightarrow D'$ is continuous iff it preserves colimits of all functors $\mathbb{N} \longrightarrow C$.*

Proof. We prove only the second two statements.

Proof of 2. Suppose that each functor $\mathbb{N} \longrightarrow C$ has a colimit in D. We show that if $G : \mathbb{N} \longrightarrow D$ is a functor, G has a colimit in D.

For each $n \geq 0$, let $f^n : \mathbb{N} \longrightarrow C$ be a functor such that $(\tau_i^n : f_i^n \longrightarrow G_n)_i$ is a colimiting cone in D.

By Lemma 2, each $0 \leq i \leq j$, each morphism $G(i,j) : G_i \longrightarrow G_j$ is determined by a weak map

$$\beta^{i,j} : f^i \longrightarrow m_{i,j} f^j.$$

For ease of notation, let's assume that all functors $m_{i,j}$ are the identity, so that for each $0 \leq i \leq j$, $\beta^{i,j} : f^i \longrightarrow f^j$ is a natural transformation.

Define $g : \mathbb{N} \longrightarrow C$ by:

$$\begin{aligned} g_i &:= f_i^i \\ g(i,j) &:= f^i(i,j) \cdot \beta_j^{i,j} \\ &= \beta_i^{i,j} \cdot f^j(i,j). \end{aligned}$$

Since every functor $\mathbb{N} \longrightarrow C$ has a colimit in D, let $(\tau_i^g : g_i \longrightarrow d)_i$ be a colimit in D.

For each $i \geq 0$, there is a weak map $\delta^i : f^i \longrightarrow \mu_i g$ defined by

$$\delta_j^i := \begin{cases} f^i(j,i) & j \leq i \\ \beta_j^{i,j} & i < j. \end{cases}$$

(As above, $\mu_i(j) = \max(i,j)$.) Thus, there is a unique map $\kappa(\delta^i) : G_i \longrightarrow d$ such that for all $j \geq 0$, (3) holds. In particular, letting $j = i$,

$$\begin{aligned} \tau_i^g &= \delta_i^i \cdot \tau_i^g \\ &= \tau_i^i \cdot \kappa(\delta^i). \end{aligned} \tag{4}$$

Claim. $(\kappa(\delta^i) : G_i \longrightarrow d)_i$ is a colimiting cone. Indeed, any cone $(\nu_i : G_i \longrightarrow e)_i$ over G determines the cone

$$(\tau_i^i \cdot \nu_i : g_i \longrightarrow e)_i$$

over g, and hence, there is a unique map

$$\nu^{\#} : d \longrightarrow e$$

such that for all i,

$$\tau_i^g \cdot \nu^{\#} = \tau_i^i \cdot \nu_i.$$

We show that for all $i \geq 0$,

$$\nu_i = \kappa(\delta^i) \cdot \nu^{\#}. \tag{5}$$

Indeed, for fixed i, the maps

$$\alpha_j := \tau_j^i \cdot \kappa(\delta^i) \cdot \nu^{\#}$$

form a cone over $f^i : \mathbb{N} \longrightarrow C$, so that there is unique map $\alpha^{\#} : G_i \longrightarrow e$ such that for all j,

$$\tau_j^i \cdot \alpha^{\#} = \tau_j^i \cdot \kappa(\delta^i) \cdot \nu^{\#}.$$

But ν_i is one such map. Hence $\alpha^{\#} = \nu_i$.

We now show $\nu^{\#} : g \longrightarrow e$ is the unique map such that for all i, (5) holds. Indeed, suppose

$$\nu_i = \kappa(\delta^i) \cdot \alpha,$$

for all $i \geq 0$. Then, for each i, j,

$$\begin{aligned} \tau_j^i \cdot \nu_i &= \tau_j^i \cdot \kappa(\delta^i) \cdot \alpha \\ &= \delta_j^i \cdot \tau_{\mu_i(j)}^g \cdot \alpha. \end{aligned}$$

But if $i = j$,

$$\tau_i^i \cdot \nu_i = \tau_i^g \cdot \alpha,$$

and $\nu^{\#}$ is the unique such map. □

Proof of 3. Suppose that $F : D \longrightarrow D'$ preserves the colimits of all functors $\mathbb{N} \longrightarrow C$. We show that F preserves the colimits of all functors $\mathbb{N} \longrightarrow D$. We use Lemma 3. Indeed, suppose that $G : \mathbb{N} \longrightarrow D$ is a functor. Using the notation of the previous part, we have shown that

$$(\kappa(\delta^i) : G_i \longrightarrow g)_i$$

is a colimit of G, where, for each $i \geq 0$, $f^i : \mathbb{N} \longrightarrow C$ is a functor and

$$(\tau_j^i : f_j^i \longrightarrow G_i)_j$$

is a colimit in D, and where g is the diagonal functor, with colimiting cone

$$(\tau_i^g : g_i \longrightarrow g)_i.$$

But now, applying F, the assumptions imply that

$$(\tau^i_j F : f^i_j F \longrightarrow G_i F)_j$$

is a colimiting cone, as is

$$(\tau^g_i F : g_i F \longrightarrow gF)_i.$$

It then follows from Lemma 3 that

$$([\kappa(\delta^i)F] : G_i F \longrightarrow gF)_i$$

is a colimiting cone in D'. □

5 Construction of C^ω

We now describe the cΣa C^ω as a quotient of the functor category $C^{\mathbb{N}}$.

5.1 Step 1.

We assume C has an initial object (if necessary, we adjoin one freely.)

Let $C^{\mathbb{N}}$ be the category whose objects are all functors $f : \mathbb{N} \longrightarrow C$; a morphism $\alpha : f \longrightarrow g$ is a natural transformation. We usually denote the components of a natural transformation $\alpha : f \longrightarrow g$ by

$$\alpha_n : f_n \longrightarrow g_n,$$

for $n \geq 0$.

We impose the structure of a cΣa on $C^{\mathbb{N}}$ by 'lifting' the functors $\sigma : C^p \longrightarrow C$ to $\mathbb{N}$.

For example, if $\sigma \in \Sigma_2$, and $f, g : \mathbb{N} \longrightarrow C$, $\sigma_{C^{\mathbb{N}}}(f, g) : \mathbb{N} \longrightarrow C$ is the functor whose value on n is

$$\sigma_C(f_n, g_n).$$

The value on the arrow $n \leq p$ in $\mathbb{N}$ is:

$$\sigma_C(f(n,p), g(n,p)) : \sigma_C(f_n, g_n) \longrightarrow \sigma_C(f_p, g_p).$$

So, now, for every term s in $\mathbf{Tm}_\Sigma(p)$, $s_{C^{\mathbb{N}}}$ is defined. (We usually will drop subscripts.) For example, if $p = 2$, and $\alpha : f \longrightarrow f'$ and $\beta : g \longrightarrow g'$ are arrows in $C^{\mathbb{N}}$ (i.e., natural transformations),

$$s(\alpha, \beta) : s(f, g) \longrightarrow s(f', g')$$

is the natural transformation with components

$$(s(\alpha, \beta))_n = s(\alpha_n, \beta_n) : s(f_n, g_n) \longrightarrow s(f'_n, g'_n).$$

Definition 3 (η_0 defined). *Let*

$$\eta_0 : C \longrightarrow C^{\mathbb{N}}$$

be the functor taking the object x in C to the functor $\eta_0(x)$ with $\eta_0(x)_n = x$, and $\eta_0(x)(n,p) = \mathbf{1}_x$, the identity morphism $x \longrightarrow x$, for all $0 \leq n \leq p$. On the morphism $g : x \longrightarrow y$ in C, the value of $\eta_0(g)$ is the natural transformation $\eta_0(x) \longrightarrow \eta_0(y)$, each of whose components is g.

Proposition 7. *The functor $\eta_0 : C \longrightarrow C^{\mathbb{N}}$ is a strict cΣa-morphism, which is full and faithful, and injective on objects. If $\perp$ is an initial object in C, $\eta_0(\perp)$ is initial in $C^{\mathbb{N}}$.* □

Now for the next step.

5.2 Step 2.

Definition 4. *Let C^ω be the category whose objects are those of $C^{\mathbb{N}}$ in which a morphism $[\alpha] : f \longrightarrow g$ is an $\simeq$-equivalence class of a weak map $\alpha : f \longrightarrow mg$.*

We define the canonical embedding of C into C^ω.

Definition 5 (η defined). *Let $\eta : C \longrightarrow C^\omega$ be the functor taking $f : x \longrightarrow y$ in C to $[\eta_0(f)] : \eta_0(x) \longrightarrow \eta_0(y)$ in C^ω.*

We would like to impose the structure of a cΣa on C^ω. The first problem is that if $\sigma \in \mathbf{Tm}_\Sigma(2)$, say, and if $\alpha : f \longrightarrow m_\alpha g$ and $\beta : f' \longrightarrow m_\beta g'$, when $m_\alpha \neq m_\beta$, how should we define $\sigma([\alpha],[\beta]) : \sigma(f,f') \longrightarrow \sigma(g,g')$, since $\sigma(\alpha,\beta)$ may not be weak map! Indeed, for $i \in \mathbb{N}$, if $im_\alpha \neq im_\beta$, we have

$$\sigma(\alpha,\beta)_i = \sigma(\alpha_i,\beta_i) : \sigma(f_i,f'_i) \longrightarrow \sigma(g_{im_\alpha}, g'_{im_\beta}),$$

which is not a weak map. However, if $m_\alpha = m_\beta$, this equation does define a weak map $\sigma(\alpha,\beta) : \sigma(f,f') \longrightarrow \sigma(g,g')$.

We have a simple alternative, using the Inflation Lemma 1, above.

Lemma 4. *Suppose that m, m' are unbounded endofunctors on $\mathbb{N}$ with $jm \leq jm'$, for all $j \geq 0$. Suppose also that $\alpha_i : f_i \longrightarrow mg_i$, $\beta_i : f_i \longrightarrow m'g_i$ are natural transformations such that $\alpha_i \simeq \beta_i$, for each $i = 1, \ldots, n$. Then if $\sigma \in \Sigma_n$, we have the natural transformations*

$$\sigma(\alpha_1,\ldots,\alpha_n) := \langle \alpha_1,\ldots,\alpha_n\rangle \cdot \sigma : \sigma(f_1,\ldots,f_n) \longrightarrow m\sigma(g_1,\ldots,g_n)$$
$$\sigma(\beta_1,\ldots,\beta_n) := \langle \beta_1,\ldots,\beta_n\rangle \cdot \sigma : \sigma(f_1,\ldots,f_n) \longrightarrow m'\sigma(g_1,\ldots,g_n).$$

With these assumptions,

$$\sigma(\alpha_1,\ldots,\alpha_n) \simeq \sigma(\beta_1,\ldots,\beta_n). \quad \square$$

Definition 6 (C^ω as cΣa). *Suppose $\sigma \in \Sigma_n$ and $n \geq 0$. For any n-tuple $[\alpha_1], \ldots, [\alpha_n]$, where $[\alpha_i]$ is an equivalence class of a weak map $\alpha_i : f_i \longrightarrow g_i$, $i = 1, \ldots, n$, choose some $m : \mathbb{N} \longrightarrow \mathbb{N}$ and some $\beta_i : f_i \longrightarrow mg_i$, for $i = 1, \ldots, n$ such that*

- *$\alpha_i \simeq \beta_i$, for each i;*
- *$\beta_i : f_i \longrightarrow mg_i$, for each i.*

*The existence of such β_i and m follows by the Inflation Lemma. Now **define***

$$\sigma_{C^\omega}([\alpha_1], \ldots, [\alpha_n]) : \sigma(f_1, \ldots, f_n) \longrightarrow \sigma(g_1, \ldots, g_n)$$

as

$$[\sigma(\beta_1, \ldots, \beta_n)],$$

the equivalence class of the weak map $\sigma(\beta_1, \ldots, \beta_n)$. (We write just σ for $\sigma_{C^{\mathbb{N}}}$.)

The fact that $\sigma([\alpha_1], \ldots, [\alpha_n])$ is independent of the choice of m follows Lemma 4.

It should be checked that with this definition, σ is indeed a functor $C^\omega \times \ldots \times C^\omega \longrightarrow C^\omega$. But this is easy. We have thus constructed the cΣa C^ω.

We omit the proof of the following fact.

Proposition 8. *The functor η is a strict cΣa morphism which preserves the initial object, and is full, faithful and injective on objects.* □

In the next section we will prove that C^ω is an ω-continuous cΣa.

6 C^ω has the required properties

In the previous section we defined the categorical Σ-algebra C^ω and the embedding $\eta : C \longrightarrow C^\omega$. In this section, we prove that the construction satisfies all properties required in Theorem 1.

We will show that C^ω is compactly generated by $\eta(C)$, and then apply Proposition 6.

Lemma 5. *If $f : \mathbb{N} \longrightarrow C$ is any functor, then f is the colimit object in C^ω of the diagram*

$$f \cdot \eta = \eta(f_0) \xrightarrow{\eta(f(0,1))} \eta(f_1) \xrightarrow{\eta(f(1,2))} \ldots$$

via the colimit morphisms

$$[\tau_n^f] : \eta(f_n) \longrightarrow f$$

where, for each n, τ_n^f has the components

$$\tau_n^f(i) := f(n, \max\{i, n\}). \quad (6)$$

Further, each morphism $[\tau_n^f]$ is monic.

Proof. First, we show each morphism τ_n^f is monic. Suppose that $f, g : \mathbb{N} \longrightarrow C$ are objects in C^ω, and $\alpha, \beta : g \longrightarrow f_n\eta$ are weak maps such that

$$[\alpha] \cdot [\tau_n^f] = [\beta] \cdot [\tau_n^f].$$

By the Inflation Lemma, we may assume that $\alpha, \beta : g \longrightarrow mf_n$ for some unbounded endofunctor $m : \mathbb{N} \longrightarrow \mathbb{N}$. Thus,

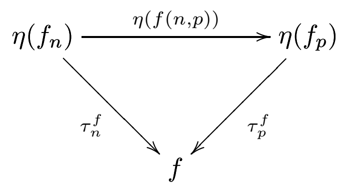
$$\alpha \circ \tau_n^f \simeq \beta \circ \tau_n^f$$

so that for each i there is some $j \geq n + im$ such that

$$\alpha_i \cdot f(n, j) = \beta_i \cdot f(n, j),$$

But this implies $\alpha \simeq \beta$, and hence $[\alpha] = [\beta]$.

It is clear that for $n \leq p$, the diagram



commutes.

Now suppose that g is any object in C^ω, $([\nu_i] : \eta(f_i) \longrightarrow g)_i$ is a cocone over the diagram $f\eta$. But defining

$$\nu^\# : f \longrightarrow g$$

as the weak map with components

$$(\nu^\#)_i := \nu_i,$$

we have

$$\nu_i = \tau_i^f \cdot \nu^\#,$$

for each i. □

We now consider the factorization property.

Lemma 6 (C^ω has the factorization property). *Suppose that c is an object in C, $f : \mathbb{N} \longrightarrow C$ is an object in C^ω, and $[\alpha] : c\eta \longrightarrow f$ is a morphism in C^ω. Then $[\alpha]$ factors as*

$$[\alpha] = [g\eta] \cdot [\tau_n^f],$$

for some $n \geq 0$, and some morphism $g : c \longrightarrow f_n$ in C.

Proof. If $\alpha : c\eta \longrightarrow mf$ is any weak map, then, for any i, since $(c\eta)(0,i) = \mathbf{1}_c$,

$$\alpha_i = c \xrightarrow{\alpha_0} f_{0m} \xrightarrow{f(0m,\, im)} f_{im}.$$

If $g = \alpha_0 : f_0 \longrightarrow f_{0m}$ in C, we have

$$[\alpha] = [g] \cdot [\tau^f_{0m}]. \quad \square$$

Proposition 9. *C^ω is compactly generated by $\eta(C)$.*

Proof. By Lemmas 5 and 6.

Corollary 4. *C^ω is ω-complete.*

Proof. By Proposition 6 and Proposition 9. $\square$

We now show C^ω is a continuous cΣa.

Proposition 10. *For each $\sigma \in \Sigma$, the functor σ_{C^ω} is continuous.*

Proof. For ease of notation, assume that $\sigma \in \Sigma_1$. We have to show that if $([\tau^i] : f^i \longrightarrow g)_i$ is a colimit of the ω-diagram Δ, then $([\sigma(\tau^i)] : \sigma(f^i) \longrightarrow \sigma(g))_i$ is a colimit of $\sigma(\Delta)$, i.e., the diagram

$$\sigma(f^0) \xrightarrow{[\sigma(\beta^0)]} \sigma(f^1) \xrightarrow{[\sigma(\beta^1)]} \sigma(f^2) \longrightarrow \ldots$$

But this fact follows just as above, since the colimit of this diagram is the diagonal, which is $\sigma(g)$.

There is an alternative argument using the fact that for each $n \geq 1$, $(C^\omega)^n$ is compactly generated by C^n. Then, by Proposition 6, we need show only that σ preserves colimits of functors $\mathbb{N} \longrightarrow C^n$. $\square$

Proposition 11. *If s, t are Σ-terms in $\boldsymbol{Tm}_\Sigma(p)$, then $C \models s \preceq t$ iff $C^\omega \models s \preceq t$.* $\square$

We turn now to showing that $\eta : C \longrightarrow C^\omega$ has the universal property stated in Theorem 1.

Suppose that D is an ω-continuous cΣa, and $F : C \longrightarrow D$ is a cΣa-morphism. We want to define $F^\omega : C^\omega \longrightarrow D$. We use Proposition 6.

For each chain $f : \mathbb{N} \longrightarrow C$ be an object of C^ω, choose a colimit cone

$$(\lambda^f_i : f_i F \longrightarrow \kappa(fF))_i \tag{7}$$

in D.

On the object f in C^ω, we define fF^ω as the colimit object $\kappa(fF)$.

Suppose $f, g : \mathbb{N} \longrightarrow C$ are objects in C^ω and $\alpha : f \longrightarrow m_\alpha g$ is any weak map.

Then α determines the weak map $\alpha F : fF \longrightarrow m(gF)$, which in turn determines the map

$$\alpha^\# : \kappa(fF) \longrightarrow \kappa(gF)$$

by the property that for each $i \geq 0$,

$$\lambda^f_i \cdot \alpha^\# = \alpha_i \cdot \lambda^g_{im_\alpha}.$$

Lemma 7. *If $\alpha, \beta : f \longrightarrow g$ are weak maps, and if $\alpha \simeq \beta$, then $\alpha^{\#} = \beta^{\#}$.*

Proof. Since $\alpha \simeq \beta \implies \alpha F \simeq \beta F$. □

Definition 7. *We define $[\alpha]F^\omega = \alpha^{\#}$.*

Proposition 12. *Suppose that $\tau_n^f : f_n\eta \longrightarrow f$ is the monic colimit morphism defined above. Then $[\tau_n^f]F^\omega = \lambda_n^f$.*

Proof. By definition, $[\tau_n^f]F^\omega$ is $\alpha^{\#}$, where $\alpha = \tau_n^f F$. Since $f_n\eta F$ is the constant chain whose object is f_nF, the morphisms $\tau_i^{f_n}$ are all the identity map $\mathbf{1}_{f_nF} : f_nF \longrightarrow f_nF$. Thus, for any $i \geq 0$,

$$\begin{aligned}
\alpha^{\#} &= \tau_i^{f_n} F \cdot \alpha^{\#} \\
&= \tau_n^f(i)F \cdot \lambda_{n+i}^f \\
&= \tau_n^f F(n, n+i) \cdot \lambda_{n+i}^f \\
&= \lambda_n^f.
\end{aligned}$$

Thus, $\tau_n^f F^\omega = \lambda_n^f$, showing that F^ω preserves colimit cocones of functors $f\eta$, for $f : \mathbb{N} \longrightarrow C$. □

Corollary 5. *F^ω is continuous.*

Proof. By Proposition 6, part 3. □

It remains to show F^ω is a cΣa-morphism. When $\sigma \in \Sigma_2$, we want to show that for any objects $f, g \in C^\omega$

$$F^\omega(\sigma(f,g)) = \sigma_D(F^\omega(f), F^\omega(g)),$$

at least up to isomorphism. The method is to show that each side is the colimit object of the same ω-diagram in D. We omit the details. □

7 Conclusion

We have presented a completion theorem for categorical algebras that generalizes the well-known completion of ordered algebras from [Blo76]. We have shown that the completion C^ω is conservative in the sense that it satisfies all (in)equalities that hold in C. In addition to order completion, we have presented two main applications: synchronization trees and words, and thus found concrete descriptions of free continuous categorical algebras satisfying monoid and commutative monoid "equations". We believe that the Completion Theorem will find several more applications in Computer Science. For one example, the collection of countable labeled partial orders over an alphabet, sometimes called pomsets, equipped with the operations of series and parallel composition is a continuous categorical algebra in a natural way, cf. [Pra86, Ren96, LW00]. We expect that this algebra is equivalent to the completion of the categorical algebra determined by the finite pomsets. Further natural sources of applications are *event structures* (cf. [WN95]), or *labeled transition systems* with *bisimulations*, cf. [Mil89].

References

[Blo76] S.L. Bloom. Varieties of ordered algebras. *J. Computer and System Sci.*, vol. 13, no. 2 (1976) 200-212

[BE93] S.L. Bloom and Z. Ésik. *Iteration Theories.* Springer, 1993.

[BE05] S.L. Bloom and Z. Ésik. The equational theory of regular words. *Information and Computation*, 197/1-2 (2005) 55-89.

[BET93] S.L. Bloom, Z. Ésik and D. Taubner. Iteration theories of synchronization trees. *Information and Computation*, 102/1 (1993) 1–55.

[Cou78] B. Courcelle. Frontiers of infinite trees. *RAIRO Inform. Théor.*, 12/4 (1978), 319–337.

[EBT78] C. Elgot, S.L. Bloom, R. Tindell. The algebraic structure of rooted trees. *J. Computer and System Sci.*, vol. 16, no. 3 (1978) 362-399.

[CCL80] G. Gierz, K.H. Hofmann, K.Keimel, J.D. Lawson, M.Mislove, D.S. Scott. *A compendium of continuous lattices.* Springer-Verlag 1980.

[GTWW77] J.A. Goguen, J.W. Thatcher, E.G. Wagner and J.B. Wright. Initial algebra semantics and continuous algebras. *J. Assoc. Comput. Mach.*, 24/1 (1977), 68–95.

[Gue81] I. Guessarian. *Algebraic Semantics.* Lecture Notes in Computer Science, 99. Springer-Verlag, Berlin-New York, 1981.

[Ele02] P.T. Johnstone. *Sketches of an Elephant: A topos theory compendium* Oxford University Press, 2002.

[LW00] K. Lodaya and P. Weil. Series-parallel languages and the bounded-width property. *Theoret. Comput. Sci.* 237 (2000), 347–380.

[Mil89] R. Milner. *Communication and Concurrency.* Prentice-Hall, Englewood Cliffs, NJ., 1989.

[Pra86] V. Pratt. Modeling concurrency with partial orders. *Internat. J. Parallel Programming*, 15/1 (1986), 33–71.

[Ren96] A. Rensink. Algebra and theory of order-deterministic pomsets. Combining logics. *Notre Dame J. Formal Logic* 37/2 (1996), 283–320.

[Win84] G. Winskel. Synchronization trees. *Theoretical Computer Science*, 34 (1984), 33-82.

[WN95] G. Winskel and M. Nielsen. Models for concurrency. in: *Handbook of Logic in Computer Science*, Vol. 4, Oxford University Press, 1995, 1–148.