Similarity-Based Graded Modal Logic

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Abstract

Within the approximate reasoning framework, several systems of modal logic have been proposed to formalize several kinds of reasoning models ([MP94], [HM92], [FH91]). In particular, in [EGG95] a Kripke model-like theory for a logic of graded necessity and possibility operators is presented to model similarity-based reasoning. In this paper, we propose an axiomatization for this logic and we show that it is sound and complete with respect to classes of models based where the accesibility relation is defined in terms of fuzzy similarity relations on the set of possible worlds. Finally, we indicate how this logic can be used to characterize several graded entailments proposed in [DEG*95].

Keywords: Similarity Function, Graded Modal Logic and Graded Entailment.

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1 Introduction

One of the goals of a variety of approximate reasoning models is to cope with inference patterns more flexible than those of classical reasoning. For instance, a pattern like this one, if A approximately entails B, and we observe A', then it is plausible, at some extent, to conclude B, whenever A' is close enough to A would be a generalization of the well-known modus ponens rule. This kind of patterns have been the focus of a huge amount of the research done in the field of fuzzy logic, where, in general, the statement "if A approximately entails B" has been modelled as a fuzzy rule whereas A, B and A' are modelled as fuzzy facts (see for instance [ZAD79]). However, terms like "approximately" or "close" above, although fuzzy, denote notions of resemblance or proximity among some propositions which must not be necessarily fuzzy. One way of proceeding is to equip the referential W with a similarity relation S, that is, a reflexive, symmetric and t-norm transitive fuzzy relation ([TV84]). This kind of approach was started out by Ruspini ([RUS91]) by proposing a similarity-based semantics for fuzzy logic, trying to capture inference patterns like the so-called generalized modus ponens. Given a similarity relation S, Ruspini proposes two measures, the implication and consistency measures, to account for the degree with which a proposition B is an approximate consequence from, or is consistent with, another proposition A, respectively. Namely,

$$I_S(B \mid A) = \underset{w_1 \models A}{INF} \; \underset{w_2 \models B}{SUP} \; S(w1, w2) \qquad C_S(B \mid A) = \underset{w_1 \models A}{SUP} \; \underset{w_2 \models B}{SUP} \; S(w1, w2).$$

Based on such measures, Ruspini proposes a formalization of the generalized modus ponens in fuzzy logic. This framework has been recently extended in ([EGG$^+$94c]) and ([EGG94a]) and compared to the possibilistic approach in ([DLP94]) and ([EGG94b]). See also ([KK94]) for another approach to similarity-based reasoning. From a logical point of view, several formalisms can be envisaged to capture a notion of similarity-based reasoning system. In [EGG95a], Esteva, Garcia and Godo describe some of the work done in this direction and to point out some open problems. First, they consider a system of graded consequence relations proposed in ([DEG$^+$95]). Next they turn their attention to a non-standard fuzzy logic approach. Finally, the frameworks of sphere systems and multi-modal logics are also examined at the semantic level, in a similar way Fariñas and Herzig did it for possibility theory ([FH91]). Links among all approaches were also provided. In this paper, we continue in the exploration of the multi-modal logic but now at the syntactic level. Namely, we will present a modal language with graded necessity and possibility operators and a normal system for it. The aximatization of this system is presented and it is proved to be complete and correct with respect to classes of similarity-based models . At the end, we indicate how to capture the different entailment of [DEG$^+$95] in this formalism and mention several open problem.

For our purposes, in the rest of the paper, we will take W as the set of interpretations of a given finite boolean algebra of propositions L. We will identify interpretations w of W with their corresponding maximal elementary conjunctions. The symbol $\otimes$ will denote a t-norm and S will denote an arbitrary $\otimes$-similarity relation on W, i.e. $S : \mathcal{W} \times \mathcal{W} \longmapsto [0,1]$, verifying: reflexivity $S(w,w') = 1$ iff $w = w'$; symmetry $S(w,w') = S(w',w)$; $\otimes$-transitivity $S(w,w'') \geq S(w,w') \otimes S(w',w'')$. The symbol $\models$ will denote the classical entailment. Moreover, given a $\otimes$-similarity relation on W we define a **graded satisfaction relation** between worlds and propositions, written $w \models^\alpha A$ with the follwing intended meaning: although B may be false at world w, B is close to be true at least to the degree α. In particular, when $\alpha = 1$ we want to recover the notion that, at world w, B is true.

Definition 1.1 *$w \models^\alpha B$ iff there exists a B-world w' which is α-similar to w.*

This graded satisfaction relation between worlds and propositions is naturally extended to a consequence relation between propositions as usual.

Definition 1.2 *A proposition A entails a proposition B at degree α, written $A \models^\alpha B$, if for each A-world w, there is a B-world w' such that $S(w,w') \geq \alpha$. In other words, $A \models^\alpha B$ iff $w \models^\alpha B$ for every A-world w.*

This notion of consequence relation is directly related to Ruspini's implication measure in the finite case.

Fact $A \models^{\alpha} B$ iff $I_S(B \mid A) \geq \alpha$.

In [DEG+95] it is also shown how the graded entailement $\models^{\alpha}$ can be extended in several ways to cope with a background knowledge K in the form of a set of propositions. Basically, they correspond to different possibilities of conditionalizing the implication measure I_S as proposed in [EGG94b]. Of particular interest are the consequence relations defined as:

1. $A \models_K^{\alpha} B$ iff $I_{S,K}(B \mid A) \geq \alpha$.
2. $A \models_K^{'\alpha} B$ iff $I_{S,K}(B \wedge A \mid A) \geq \alpha$.

where the conditional implication measure $I_{S,K}$ is defined as:

$$I_{S,K}(B \mid A) = \underset{w \models K}{INF}\ (I_{S,w}(A) \otimes\!\!\longrightarrow I_{S,w}(B)).$$

$\otimes\!\!\longrightarrow$ standing for the residuated many-valued implication generated from the t-norm $\otimes$. Consequence relation (1) is monotonic while (2) is not. It is shown there that they enjoy "good" properties, like $\otimes$-Transitivity or Cut (only (2)), that allow to capture some form of interpolative reasoning very similar to that used in fuzzy control systems.

Here we will be explarate other natural setting, one of the modal logics which is tailored to account for relations on the set of interpretations. The similarity relation S will be considered as a family of nested accesibility relations R_α over the set of possible worlds W definid as

$$R_\alpha(w, w') \text{ iff } S(w, w') \geq \alpha$$

Therefore, enlarging the logical language we will define, for each α, a usual pair of operators $\Box_\alpha$ and $\Diamond_\alpha$ with the following standard semantics:

$$w \models \Diamond_\alpha B \text{ iff there exists } w' \models B\ R_\alpha(w, w')$$
$$w \models \Box_\alpha B \text{ iff for every } w' \text{ if } R_\alpha(w, w') \text{ then } w' \models B.$$

Combining this definition with the above given of graded satisfaction between a world and a proposition, we will give the first relation between graded entailment and graded possibility operator througt the following equivelence:

$$w \models^{\alpha} B \text{ iff } w \models \Diamond_\alpha B$$

The paper is organized as follows. In section 2 the symilarity-based graded modal language and the its underlying logic are described. In section , possible world semantic for this logic, while in section 4 we present its axiomatization and prove that it is sound and complete respect of given semantic in section 2. Finally, in section 5 some connexions between this similarity-based graded logic and graded entailments are stressed. We end with some concluding remarks and several open problems will be waiting for resolution.

2 Similarity-Based Graded Modal Logic

This section is devoted to a recital of the syntactic based concepts for the language of graded modal logic, many of which are probably known by the reader. The ideas are very simple. The few formal definitions we offer may be helpful, but they are not essential; we state them mainly for the sake of completeness and future reference.
The language of graded modal logic is based on:

1. A denumerable set of atomic formulas: $F_0, F_1, F_2, \ldots..$
2. Two constants: $\top$ and $\bot$.
3. A set of numbers G between 0 and 1 such that 0 and 1 belong to G.
4. One-place operators: $\neg$, $\Box_\alpha$ and $\Diamond_\alpha$ for each α in G.
5. Four two-place operators: $\wedge, \vee, \rightarrow$ and $\leftrightarrow$.

Definition 2.1 *The language of graded modal logic is the minimal set L satisfyting the following conditions:*

1. $F_n \in L$ *for* $n \geq 0$.
2. $\top, \bot \in L$.
3. $\Box_0 \top \in L$.
4. *If* $A \in L$ *then* $\neg A, \Box_\alpha A$ *(with* α *in* G *and* $\alpha \neq 0$*) and* $\Diamond_\alpha A$ *(with* α *in* G*)* $\in L$.
5. *If* $A \in L$ *then* $A \wedge B, A \vee B, A \rightarrow B$ *and* $A \leftrightarrow B \in L$

From now on, we shall use $A, B, C, ...$, sometimes with subcripts and supercripts, for sentences and $\Gamma, \Delta, \Sigma, ...$ for sets of sentences. The sentence $\Box_\alpha A$ is the graded necessitation of A and it represents the grade of necessity of A. By example, we assume that $\Box_1 A$ is true when A is true and that $\Box_0 A$ if and only if A is a tautology.

3 Model Theory

In this section, we define the idea of a model and state the truth and validity conditions for modal sentences in a worlds, in a model and in a class of models.

According to the kripkean idea, a sentence is necessarily true at the actual world if it is true in every world accesible from it. Intuitively, a world is a full interpretation of all atomic formulas $F_0, F_1, F_2, ...$ So, two different worlds determine two different assignment of thuth to $F_0, F_1, F_2, ...$

A similarity-based model for a graded modal logic is a structure $\mathcal{M} = \langle W, R_S, \| \, \| \rangle$, in which W is the set of possible worlds, $\| \, \|$ represents an assignment of possible worlds to atomic formulas, and R_S is a family of nested accesibility relations between possible worlds. Formaly:

Definition 3.1 $\mathcal{M} = \langle W, R_S, \| \, \| \rangle$ *is a similarity-based Kripke's model iff:*

1. W *is a not empty set of possible worlds,*
2. *Given similarity function* $S : W \times W \rightarrow G(\subset [0,1])$, R_S *is a family of nested accesibility relations* $\{R_\alpha\}_{\alpha \in [0,1]}$ *based on the similarity relation* S, *defined by:*

$$R_\alpha(w, w') \text{ iff } S(w, w') \geq \alpha.$$

3. $\| \, \|$ *is a function that given an atomic formula* F *return the set* $W_F \subseteq W$ *where* F *is considered to be true.*

In our case, we consider that possible worlds are interpretations of propositional language and that the accesibility relation R_α represents pairs of worlds with similarity degree greater than α.

We write "$\models_w^{\mathcal{M}} A$" to mean that A is true at the possible world w in the model $\mathcal{M}$. This notion is defined as follows.

Definition 3.2 *Let* w *be a world in a model* $\mathcal{M} = \langle W, R_S, \| \, \| \rangle$ *then:*

1. $\models_w^{\mathcal{M}} F_n$ *iff* $w \in \|F_n\|$, *for any* $n \geq 0$.
2. $\models_w^{\mathcal{M}} \top$
3. *Not* $\models_w^{\mathcal{M}} \bot$
4. $\models_w^{\mathcal{M}} \neg A$ *iff not* $\models_w^{\mathcal{M}} A$
5. $\models_w^{\mathcal{M}} A \wedge B$ *iff both* $\models_w^{\mathcal{M}} A$ *and* $\models_w^{\mathcal{M}} B$.
6. $\models_w^{\mathcal{M}} A \vee B$ *iff either* $\models_w^{\mathcal{M}} A$ *or* $\models_w^{\mathcal{M}} B$, *or both* .
7. $\models_w^{\mathcal{M}} A \rightarrow B$ *iff* $\models_w^{\mathcal{M}} A$ *implies* $\models_w^{\mathcal{M}} B$.
8. $\models_w^{\mathcal{M}} A \leftrightarrow B$ *iff* $\models_w^{\mathcal{M}} A$ *if and only if* $\models_w^{\mathcal{M}} B$.

9. $\models_w^{\mathcal{M}} \Box_\alpha A$ *iff for every* w' *in* $\mathcal{M}$ *such that* $wR_\alpha w'$, $\models_{w'}^{\mathcal{M}} A$.

10. $\models_w^{\mathcal{M}} \Diamond_\alpha A$ *iff for some* w' *in* $\mathcal{M}$ *such that* $wR_\alpha w'$, $\models_{w'}^{\mathcal{M}} A$.

We write "$\models^{\mathcal{M}} A$" to mean that A is valid in the model $\mathcal{M}$, and "$\models_C A$" to mean that A is valid in the class C of models. We recall these definitions formally.

Definition 3.3 $\models^{\mathcal{M}} A$ *iff for every world* w *in* $\mathcal{M}$, $\models_w^{\mathcal{M}} A$

Definition 3.4 $\models_C A$ *iff for every model* $\mathcal{M}$ *in* C, $\models^{\mathcal{M}} A$

Definition 3.5 *Given the family of min-transitive similarity functions* $\mathcal{S}$, *we define* C_m *as the class of models defined by this family.*

Notice that if the similarity function is min-transitive then for each α, R_α is a equivalence relation. Moreover, in this case, $\Box_\alpha$ and $\Diamond_\alpha$ are a pair of dual S5 "classical" modal operators.

4 Axiomatic System

Here, we are going to present the $GS5$ system of the graded modal logic which is very related to that presented by L.F. Goble in [GLO70].

Definition 4.1 *The graded modal logic* $GS5$ *is the smallest set of sentences containing every instance of the following axiom schemes and closed under the two last rules:*

PL. A *is a tautology in propositional logic.*

GDF. $\Box_\alpha A \leftrightarrow \neg\Diamond_\alpha\neg A$ *(with* α *in* G *and* $\alpha \neq 0$*).*

GK. $\Box_\alpha(A \rightarrow B) \rightarrow (\Box_\alpha A \rightarrow \Box_\alpha B)$.

GT. $\Box_\alpha A \rightarrow A$.

G5. $\Diamond_\alpha A \rightarrow \Box_\alpha \Diamond_\alpha A$ *: with* $\alpha \neq 0$.

GC. $A \rightarrow \Box_1 A$.

GN. $\Box_\alpha A \rightarrow \Box_\beta A$ *: with* $\beta \geq \alpha$.

GRN. $\frac{A}{\Box_\alpha A}$ *: for all* α *in* G *and* $\alpha \neq 0$.

MP. $\frac{A, A \rightarrow B}{B}$.

Notice that the first five axioms are the $S5$ classical modal logic for the graded necessity and possibility operators. GC and GN axioms provide the centered and nested properties, respectively.

Since the theorems of a system are just the sentences in it, we usually write "$\vdash_{GS5} A$" to mean that A is a theorem of $GS5$, this is $\vdash_{GS5} A$ iff $A \in GS5$.

Theorem 4.1 *The system* $GS5$ *has the following rules of inference and theorems.*

GRM. $\frac{A \rightarrow B}{\Box_\alpha A \rightarrow \Box_\alpha B}$ *: for all* α *in* G *and* $\alpha \neq 0$. .

GT$\Diamond$. $A \rightarrow \Diamond_\alpha A$.

GD. $\Box_\alpha A \rightarrow \Diamond_\alpha A$.

GT'. $\Box_\alpha \Box_\beta A \rightarrow \Box_\alpha A$.

GE. $\Box_1 A \leftrightarrow \Diamond_1 A$.

GB. $A \rightarrow \Box_\beta \Diamond_\alpha A$ *: with* $\beta \geq \alpha$ *and* $\beta \neq 0$.

G4. $\Box_\alpha A \rightarrow \Box_\beta \Box_\alpha A$ *: with* $\beta \geq \alpha$.

Proof:

GRM. If $A \rightarrow B \in GS5$ then, by appling the *GRN* rule, it results that $\Box_\alpha(A \rightarrow B) \in GS5$ (for all α in G and $\alpha \neq 0$) and combining it with the *GK* scheme via the *MP* rule, we obtain that $\Box_\alpha A \rightarrow \Box_\alpha B \in GS5$.

GT◇. Since $\Box_\alpha \neg A \rightarrow \neg A \in GS5$ (*GT* scheme), then $A \rightarrow \neg\Box_\alpha\neg A \in GS5$, and using the *GDF* schema, we get the proof.

GD. Trivial. By combination of the *GT* scheme and the previous theorem.

GT'. For all $\beta \in G$, the *GN* schema gives the following result: $\Box_\beta A \rightarrow \Box_1 A$. Now, applying the *RM* rule, we obtain $\Box_\alpha\Box_\beta A \rightarrow \Box_\alpha\Box_1 A$. Furthermore, by combination of the *GT* axiom as $\Box_1 A \rightarrow A$, and the *RM* rule, it results $\Box_\alpha\Box_1 \rightarrow \Box_\alpha A$. Finally, by chaining of implications we obtain the desired result.

GE. Left to right is obtained by combining the *GT* scheme and the *GT*◇ theorem. For the reverse, consider the dual of the *GC* scheme and then chaining $\Diamond_1 A \rightarrow A$ and $A \rightarrow \Box_1 A$.

GB. Trivial. By combination of above the *GT*◇ theorem and the *G*5 scheme.

G4. Since $\Diamond_\alpha\Box_\alpha A \rightarrow \Box_\alpha A$ is the contrareciprocal of the *G*5 scheme, then applying the *RM* rule it results $\Box_\alpha\Diamond_\alpha\Box_\alpha A \rightarrow \Box_\alpha\Box_\alpha A$. Furthermore, by the *GB* theorem, $\Box_\alpha A \rightarrow \Box_\alpha\Diamond_\alpha\Box_\alpha A$. Thus, combining the last two formulas it results $\Box_\alpha A \rightarrow \Box_\alpha\Box_\alpha A$ and using the *GN* axiom and *MP* rule.

∇

In terms of theoremhood we can characterize notions of deducibility and consistency. A sentence A is deducible from a set of sentences Γ in the system $GS5$, written $\Gamma \vdash_{GS5} A$, if and only if $GS5$ contains a theorem of the form

$$A_1 \wedge ... \wedge A_n \rightarrow A$$

where the conjuncts A_i ($i = 1, ..., n$) of the antecedent are sentences in Γ. A set of sentences Γ is consistent in $GS5$, written $Con_{GS5}\Gamma$, just in case the sentence $\perp$ is not $GS5$-deductible from Γ. Thus Γ is inconsistent in $GS5$ just when $\Gamma \vdash_{GS5} \perp$. We recall these definitions formally.

Definition 4.2 $\Gamma \vdash_{GS5} A$ *iff there are* $A_1, ..., A_n \in \Gamma (n \geq 0)$ *such that* $\vdash_{GS5} A_1 \wedge ... \wedge A_n \rightarrow A$

Definition 4.3 $Con_{GS5}\Gamma$ *iff not* $\Gamma \vdash_{GS5} \perp$

Before continuing, we comment about the modal degree and the $GS5$ reduction theorems. The modal degree of a modal formula is the number of nested modal operators (do not confuse nested with iterated). For instance, in the classical S5 system of modal logic there exists a disjunctive normal form. However in this logic the reduction laws are only valid when the grade of the left-hand side operator is greater or equal to the grade of the right-hand side operator. Therefore, there does not exist a normal form. Morever, in the worst case, the modal degree of one formula can be equal to the cardinality of G.

Now, we prove the soundness of $GS5$ system with respect to the class of models C_m already defined.

Theorem 4.2 *The schemes* $PL, GDF, GK, GT, G5, GC$ *and* GN *are valid in the class* C_m *and the rules preserve validity in this class.*

Proof:

PL. Trivial.

GDF. By definition $\models^{\mathcal{M}}_w \Box_\alpha A$ if and only if $\forall w'(S(w,w') \geq \alpha$ then $\models^{\mathcal{M}}_{w'} A)$ and equivalently $\forall w'(w' \not\models A$ then $S(w,w') < \alpha)$. Thus $\forall w'(w' \models \neg A$ then $S(w,w') < \alpha)$ if and only if $\neg(\exists w'(w' \models \neg A$ and $S(w,w') \geq \alpha)$, it is equal to $\models^{\mathcal{M}}_w \neg\Diamond_\alpha\neg A$.

GK. Suppose $\models^{\mathcal{M}}_w \Box_\alpha(A \rightarrow B)$. Thus, for all w' if $S(w,w') \geq \alpha$ and $\models^{\mathcal{M}}_{w'} A$ then it results that $\models^{\mathcal{M}}_{w'} B$. On the other hand, if $\models^{\mathcal{M}}_w \Box_\alpha A$ occurs then $\forall w'(S(w,w') \geq \alpha$ then $\models^{\mathcal{M}}_{w'} A)$ and combining the two implications results that $\models^{\mathcal{M}}_w \Box_\alpha B$.

GT. If $\models^{\mathcal{M}}_w \Box_\alpha A$ then for every world w' such that "$S(w,w') \geq \alpha$ implies $\models^{\mathcal{M}}_{w'} A$". In particular, if w' $= w$ then $S(w,w) = 1 \geq \alpha$ and thus $\models^{\mathcal{M}}_w A$.

G5. We know that $\models_w^{\mathcal{M}} \Box_\alpha \Diamond_\alpha A$ iff for every world w' such that $S(w,w') \geq \alpha$ then there exists a world w" with $S(w,w") \geq \alpha$ and such that $\models_{w"}^{\mathcal{M}} A$. Now, suppose that $\models_w^{\mathcal{M}} \Diamond_\alpha A$ and hence there exists a world $w^\#$ such that $S(w,w^\#) \geq \alpha$ and $\models_{w'}^{\mathcal{M}} A$. For any world w', if $S(w,w') \geq \alpha$, then $S(w',w^\#) \geq \alpha$ (by min-transitivity of S), and since $\models_{w^\#}^{\mathcal{M}} A$ we have $\models_w^{\mathcal{M}} \Box_\alpha \Diamond_\alpha A$.

GC. If $\models_w^{\mathcal{M}} A$ then, for every world w' such that $S(w,w') \geq 1$, it holds $\models_{w'}^{\mathcal{M}} \Box_1 A$. As by definition $S(w,w') \geq 1$ if and only if $w = w'$, then $\models_{w'}^{\mathcal{M}} \Box_1 A$ holds.

GN. If $\models_w^{\mathcal{M}} \Box_\alpha A$ then, for every world w' such that $S(w,w') \geq \alpha$, it holds $\models_{w'}^{\mathcal{M}} A$. So if $\beta \geq \alpha$ then, for every world w" such that $S(w,w") \geq \beta$, we have $S(w,w") \geq \alpha$ too. Thus $\models_{w"}^{\mathcal{M}} A$, as desired.

GRN. Suppose that for every world $w \models_w^{\mathcal{M}} A$. So for any world w' and any α in [0,1], if $S(w',w) \geq \alpha$ then $\models_{w'}^{\mathcal{M}} A$, that is equivalent to say that for every world w, $\models_w^{\mathcal{M}} \Box_\alpha A$.

MP. Trivial.

∇

Corollary 4.3 *The $GS5$ system is sound with respect to the class of models C_m.*

Next, we define the idea of a canonical model for a $GS5$ system and prove some fundamental theorems about completeness. Before of introducing the concept of canonical model, we need to define the concept of maximality. Intuitively, a set of formulas is maximal if it is consistent and contains as many formulas as it can without becoming inconsistent. We write $Max_{GS5}\Gamma$ to mean that Γ is $GS5$-maximal, and we state the definition as follows.

Definition 4.4 $Max_{GS5}\Gamma$ *iff (i)* $Cons_{GS5}\Gamma$, *and (ii) for every A, if* $Cons_{GS5}(\Gamma \bigcup \{A\})$ *then* $A \in \Gamma$.

Theorem 4.4 *Let Γ be a $GS5$-maximal set of formulas. Then:*

1. $A \in \Gamma$ *iff* $\Gamma \vdash_{GS5} A$.
2. $GS5 \subseteq \Gamma$.
3. $\neg A \in \Gamma$ *iff* $A \notin \Gamma$.

Proof: As usual. ∇

In terms of maximality we can define what we shall call the proof set of a formula. Relative to system $GS5$, the proof set of a formula A (denoted by $\mid A \mid_{GS5}$) is the set of $GS5$-maximal sets of formulas containing A:

Definition 4.5 $\mid A \mid_{GS5} = \{Max_{GS5}\Gamma : A \in \Gamma\}$.

We can state that a sentence is deducible from a set of sentences if and only if it belongs to every maximal extension of the set.

Theorem 4.5 *Let Γ and A be a set of sentences and a sentence repectively, then*

$$\Gamma \vdash_{GS5} A \text{ iff } A \in \Delta \text{ for every } \Delta \in \mid \Gamma \mid_{GS5}$$

Proof: It follows from the Lindenbaum's Lemma. ∇

Definition 4.6 *Let* $\mathcal{M}^* = \langle W^*, R^*, \| \ \|^* \rangle$ *be a structure of model for $GS5$. $\mathcal{M}^*$ is the proper canonical standard model for $GS5$ iff:*

1. $W^* = \{\Gamma : Max_{GS5}\Gamma\}$.
2. *For every w and w' in W,* $wR_\alpha^* w'$ *iff* $\{A : \Box_\alpha A \in w\} \subseteq w'$.
3. $\|F_i\|^* = \mid F_i \mid_{GS5}$ *for* $i \geq 0$.

Next lemma shows that the proper canonical model for $GS5$ is like another model $\mathcal{M}^\#$ with $W^\#$ and $\|F_i\|^\#$ equal above, and $R^\#$ defined so that a world collects all the possibilitations of sentences ocurring in its accessibles.

Lemma 4.6 *$\mathcal{M}^{\#} = \langle W^{\#}, R^{\#}, \| \|^{\#} \rangle$ is the proper canonical standard model for $GS5$ iff $W^{\#}$ and $\| \|^{\#}$ are as in definition 4.7, and for every w and w' in $\mathcal{M}^{\#}$,*

$$wR^{\#}_{\alpha}w' \text{ iff } \{\Diamond_{\alpha}A : A \in w'\} \subseteq w.$$

Proof: From left to right, suppose $wR^{\#}_{\alpha}w'$ and $\Box_{\alpha}A$ in w, then A in w' and so $\neg A$ is not in w', implying that $\neg\Box_{\alpha}\neg A$ is not in w and then by maximality $\Diamond_{\alpha}A$ in w.
In the other direction, suppose that $\Diamond_{\alpha}A$ is in w, since w is consistent, $\Box_{\alpha}\neg A$ is not in w, hence $\neg A$ is not in w' and then A belongs to w'. ∇

Theorem 4.7 *Let $\mathcal{M}^*$ be the proper canonical standard model for $GS5$. Then for every w in $\mathcal{M}^*$ and for every formula A in $GS5$:*

$$\models^{\mathcal{M}^*}_w A \text{ iff } A \in w$$

Proof: The proof is by induction on the complexity of A. It suffices to give it for the case in which A is (1) an atom F_n,(2) a negation $\neg B$, (3) a conditional $B \to C$ and (4) a necessitation, $\Box_{\alpha}B$. For (1). By the definition 3.2, $\models^{\mathcal{M}^*}_w F_n$ iff $w \in \|F_n\|^*$ and by definition 4.6 this occurs iff $w \in| F_n |_{GS5}$. But by definition 4.6 this holds iff $F_n \in w$. So the result holds when A is atomic. For the inductive cases we make the hypothesis that the result holds for all formula shorter than A.
For(2), $\models^{\mathcal{M}^*}_w \neg B$ iff (def.3.2) not $\models^{\mathcal{M}}_w B$ iff (inductive hypothesis) $B \notin w$ iff $\neg B \in w$. So the result holds when A is a negation.
For (3). $\models^{\mathcal{M}^*}_w B \to C$ iff (def. 3.2) if $\models^{\mathcal{M}^*}_w B$ then $\models^{\mathcal{M}^*}_w C$ and this iff (inductive hypothesis) if $B \in w$ then $C \in w$ iff $B \to C \in w$. So the result holds when A is a conditional.
For (4). $\models^{\mathcal{M}^*}_w \Box_{\alpha}B$ iff (def.3.2), for every world w' in $\mathcal{M}^*$ such that $wR^*_{\alpha}w'$, $\models^{\mathcal{M}^*}_{w'} B$, iff (inductive hypothesis), for every world w' in $\mathcal{M}^*$ such that $wR^*_{\alpha}w'$, $B \in w'$, iff (def.4.7) for every world w' such that $\{A : \Box_{\alpha}A \in w\} \subseteq w'$ then $B \in w'$. Now consider separataly both implicatoions:
i) If $\Box_{\alpha}B \in w$, then it is easy since by definition 4.6 B belongs to every $GS5$-maximal extension of the set $\{A : \Box_{\alpha}A \in w\}$.
ii) The other direction is the most interesting. If B belongs to every $GS5$-maximal extension of the set $\{A : \Box_{\alpha}A \in w\}$ then, by theorem 4.4 $\{A : \Box_{\alpha}A \in w\} \vdash_{GS5} B$. This in turn means that there are sentences $A_1 \wedge ... \wedge A_n$ in this set, such that $\vdash_{GS5} (A_1 \wedge ... \wedge A_n) \to B$. Because $GS5$ contains the necesitation rule, we may infer, by the GK scheme, that $\vdash_{GS5} (\Box_{\alpha}A_1 \wedge ... \wedge \Box_{\alpha}A_n) \to \Box_{\alpha}B$. But w contains each $\Box_{\alpha}A_1, ..., \Box_{\alpha}A_n$ so $\Box_{\alpha}B \in w$. ∇

Corollary 4.8 *Let $\mathcal{M}^*$ be the proper canonical standard model for $GS5$. Then for every w in $\mathcal{M}^*$ and for every formula A in $GS5$:*

$$\models^{\mathcal{M}^*} A \text{ iff } \vdash_{GS5} A$$

Proof: This is easy to prove from last theorem. ∇

Before the completeness theorem, it is necessary to present the following result. It is similar to the classical case.

Proposition 4.9 *The accesibility relation R^*_{α} in the family R^* of the proper canonical standard model for $GS5$, $\mathcal{M}^*$, is an equivalence relation for each α in [0,1].*

Proof:
1) R^*_{α} is reflexive. Trivial by definition and the fact that axiom GT is in $GS5$.
2) R^*_{α} is symmetric. Already, we proved that Theorem GB is in $GS5$. Let us prove that for all $GS5$-maximal set of sentences w_1 and w_2,

$$\text{if } \{A : \Box_{\alpha}A \in w\} \subseteq w' \text{ then } \{\Diamond_{\alpha}A : A \in w\} \subseteq w'.$$

By definition 4.6 and lemma 4.6, this means that R^*_{α} is symmetric, i.e. that for every w and w' in $\mathcal{M}^*$,

$$\text{if } wR^*_{\alpha}w' \text{ then } w'R^*_{\alpha}w$$

Assume that $\{A : \Box_{\alpha}A \in w\} \subseteq w'$, and also that $A \in w$. It remains only to be shown that $\Diamond A \in w'$. But if w contains A and the theorem $A \to \Box_{\alpha}\Diamond_{\alpha}A$ too, then $\Box_{\alpha}\Diamond_{\alpha}A$ is in w. Hence $\Diamond_{\alpha}A \in w'$.
3) R^*_{α} is transitive. We know that $GS5$ contains the theorem G4: $\Box_{\alpha}A \to \Box_{\alpha}\Box_{\alpha}A$. We express the transivity of R^*_{α} by saying that for every $GS5$-maximal set of sentences w_1,w_2 and w_3,

$$\text{If } \{A : \Box_\alpha A \in w_1\} \subseteq w_2 \text{ and } \{B : \Box_\alpha B \in w_2\} \subseteq w_3 \text{ then } \{C : \Box_\alpha C \in w_1\} \subseteq w_3$$

Suppose that $\{A : \Box_\alpha A \in w_1\} \subseteq w_2$, $\{B : \Box_\alpha B \in w_2\} \subseteq w_3$ and that $\Box_\alpha A \in w_1$. The presence of $G4$ in w_1 and the last assumption imply that $\Box_\alpha \Box_\alpha A$ is in w_1. By the first assumption, then, $\Box A$ is in w_2, and by the second, A is in w_3. ∇

Notice that the proper canonical standard model for $GS5$, above defined, is not a model in the sense presented at section 3, because W^* is not a set of interpretations of propositional language. We obtain the appropriate model for our goal via the collapsation through Γ_F of the model $\mathcal{M}^*$ and where Γ_F is the set of atomic formulas.

Definition 4.7 *Let $\mathcal{M}^*$ be the proper canonical standard model for $GS5$ and let Γ_F be the set of atomic formulas $(F_0, F_1, F_2, \ldots..)$. Then a collapsation of $\mathcal{M}^*$ through Γ_F is the model $\mathcal{M}^*{}_c = \langle W_c^*, R_c^*, \| \|_c^* \rangle$ such that:*

1. *$W_c^* = [W^*]_{\Gamma_F}$.*
2. *For every $[w1]$ and $[w2]$ in W_c^*:*
   $[w_1] R^*_{c\alpha} [w_2]$ iff there exists w in $[w_1]$ and w' in $[w_2]$ such that $w R^*_\alpha w'$.
3. *$\|F_n\|_c^* = [\|F_n\|^*]_{\Gamma_F}$ for any $n \geq 0$.*

where $[X]_{\Gamma_F}$ denote the set of equivalence class of worlds in X with respect to Γ_F.

Since collapsation is a particular case of filtration, we are able to apply the theorems 3.19 and 3.20 in [CHE80]. Thus the following result is immediate.

Proposition 4.10 *Each accesibility relation $R^*_{c\alpha}$ in the family R_c^* of the collapsation of proper canonical standard model for $GS5$, $\mathcal{M}^*{}_c$, is an equivalence relation for any α in [0,1].*

Now, we must prove that $\mathcal{M}^*{}_c$ belongs to the class of models $\mathcal{C}_m$. For this, first we analyse the case where the set of atomic formulas and the set of numbers G are finite and then the set of $GS5$-maximal sets in $\mathcal{M}^*$ and the set of possible worlds of each $\mathcal{M}$ in $\mathcal{C}_m$ too. This will serve to guaratee that all subsets of the family R_c^* have maximum. Before, we present a lemma from [ZAD71] that will be useful in the posterior theorem.

Lemma 4.11 *A function $S : \mathcal{W} \times \mathcal{W} \longmapsto [0,1]$ is a min-transitive similarity function if and only if there exists a nested family of equivelence relations R_α on the set of posible worlds such that:*

$$\forall w, w' \in W(S(w, w') = max\{\alpha \mid R_\alpha(w, w')\})$$

Thus we are just prepared to present our main result of completeness.

Theorem 4.12 *If the set of atomic formulas of L and the set of numbers G are finite, then the collapsation of the proper canonical standard model for $GS5$ belongs to the class of models $\mathcal{C}_m$.*

Proof: Let $\mathcal{M}^*{}_c = \langle W_c^*, R_c^*, \| \|_c^* \rangle$ be the collapsation of the proper canonical standard model for $GS5$. First, we define a binary relation S^* via the family of accesibility relations R_c^* as follows:

$$\forall [w], [w'] \in W_c^*(S^*([w], [w']) = max\{\alpha \mid [w] R^*_{c\alpha} [w']\}).$$

By proposition 4.10, $R^*_{c\alpha}$ are equivalence relations, therefore, by lemma 4.11,S^* to defined a min-transitive similarity relation. Furthermore, let us see that S^* is discriminality and universal.
1) S^* is discriminality if and only if for all $[w][w'] \in W_c^*$, $S^*([w], [w']) = 1$ implies $[w] = [w']$. This holds if and only if $[w] R^*_{c1} [w']$. But is so, since by definition 4.6 $[w] R^*_{c1} [w']$ iff there exists w in $[w]$ and w' in $[w']$ such that $[w] R^*{}_1 [w']$. Since the axiom GC is in w then it must be w=w'.
2) S^* is universal if and only if for all $[w]$ and $[w']$ in W_c^* results that $S^*([w], [w']) \geq 0$. It is equivalent to require that for all $[w]$ and $[w']$ in W_c^*, $[w] R^*_{c0} [w']$. By definition 4.6, this occurs if and only if there exists w_1 in $[w]$ and w_2 in $[w']$ such that $\{A : \Box_0 A \in w_1\} \subseteq w_2$ (for all$[w], [w']$). But sinse $\Box_0 A$ iff $A = \top$ by definition 2.1, and since $\top$ is in all worlds, we may consider the proof completed. ∇

Corollary 4.13 *The system $GS5$ is complete with respect to the class of models $\mathcal{C}_m$.*

5 Relations with Graded Entailment

As we already mentioned, different consequence relations that make sense in similarity-based reasoning are presented in [DEG$^+$95]. Here, we characterize them in the framework of similarity-based graded modal logic. Before, we give some useful definitions and propositions. We have also to notice that the different graded entailments are defined from a similarity relation and not from a class of them. Thus, given a similarity relation S we extend the modal system $GS5$, in such a way that the extension represents the class of models that has as a unique element, $\mathcal{M}_S = \langle W, R_S, \| \ \| \rangle$. Since we are considering a finite set of atomic formula, then for each world w we can build the formula $\hat{w}$ such that $w \models \hat{w}$ and $w' \not\models \hat{w}$ for all $w' \neq w$. Furthermore since G is also taken finite, we can enumerate its elements. Let E be the mapping from G to [0,n]. So on, we write R_i instead of R_α with $i = E(\alpha)$.
Now, we can consider a set of formulas corresponding a similarity S:

Definition 5.1 *Given a similarity function S the set of formulas that it represent is:*

$$\mathcal{F}_S = \{\hat{w} \rightarrow \Box_{i-1}\hat{w}' \wedge \Diamond_i \hat{w}' \mid w \text{ and } w' \in W \text{ and } E(S(w,w')) = i\}.$$

This set of formula capture the similarity S in the sense that o formula $\hat{w} \rightarrow \Box_{i-1}\hat{w}' \wedge \Diamond_i\hat{w}'$ is valid in $\mathcal{M}_S$ if and only if $E(S(w,w')) = i\}$
Now, we can extend the system $GS5$ adding the formulas of $\mathcal{F}_S$. Formally:

Definition 5.2 *Let S be a similarity function, the extension of $GS5$ by S, noted S_{GS5}, is the smallest set of formulas containing the axioms of $SG5$ plus those of $\mathcal{F}_S$ and closed under GRN and MP rules.*

For our purpose, we want to see that when the similarity function is back built through the system S_{GS5} via the collapsation of the proper canonical standard model for S_{GS5}, this function is recovered. This is established by the following theorem.

Theorem 5.1 *Let S and S_{GS5} be a similarity function and the extension of $GS5$ from S respectively, and let $\mathcal{M}^*{}_c = \langle W_c^*, R_c^*, \| \ \|_c^* \rangle$ be the collapsation of the proper canonical standard model for S_{GS5}. Then*

$$\forall [w], [w'] \in W_c^* (S([w],[w']) = max\{\alpha \mid [w] R^*_{c\alpha} [w']\}).$$

Proof: Trivial from theorem 4.11. ∇

Having defined the system S_{GS5} from a similarity S, we can present the relation between this system and the different entailments with respect to S. Following the presentation from [DEG$^+$95], where $w \models_S^\alpha B$ means there exists a B-world w' such that $S(w, w') \geq \alpha$, it is easy to check that this entailment is directy related to the possibility operator $\Diamond_\alpha$ in the following sense:

$$w \models_S^\alpha B \text{ iff } \vdash_{S_{GS5}} \hat{w} \rightarrow \Diamond_\alpha B$$

It is easy to see that the right-hand side of the equivelence represents that any world w satisfying $\hat{w}$ must also satisfy $\Diamond_\alpha B$, *i.e.* there must exist a B-world w' such that $S(w, w') \geq \alpha$.
The idea of the above graded satisfaction in a world can be extended over to a more general graded semantic entailment relation. According to [DEG$^+$95], a proposition A entails a proposition B to the degree α, written $A \models_S^\alpha B$, if and only if each A-world makes B at least α-true, *i.e.* $\forall w \exists w' (w \models A \rightarrow (w' \models B \wedge S(w, w') \geq \alpha))$.
Using the modal logic setting this can be expressed as:

$$\vdash_{S_{GS5}} A \rightarrow \Diamond_\alpha B$$

According the notion of deducibility, this means that $\Diamond_\alpha B$ follows from A in S_{GS5}.

As it is pointed out in [DEG$^+$95], a natural question about the entailment is how to deal with some prior information which is available under the form of a set K of formulas. In this paper, the authors propose three extensions of $A \models_S^\alpha B$ that take into account the background information K.
The first and direct option is just to take the set K as a restriction on the set of A-worlds, and thus considering the entailment $\models_{S,K}^\alpha$ defined as follows:

$$A \models_{S,K}^\alpha B \text{ iff } K \wedge A \models_S^\alpha B$$

In our modal opproach, this can also be expressed as:

$$K \wedge A \vdash_{S_{GS5}} \Diamond_\alpha B$$

A second option (in [DEG+95]) to take into account prior knowledge into the entailment relation, and related to Ruspini's proposal of what he calls "conditional necessity functions" ([RUS91]), is to define the following entailment:

$$A \models^{\alpha}_{S,K} B \text{ iff } I_{S,K}(B \mid A) \geq \alpha$$

where $I_{S,K}(B \mid A) = \underset{w \models K}{INF}\ (I_{S,w}(A) \otimes\!\!\longrightarrow I_{S,w}(B))$.

To represent the last entailment in our logical system S_{GS5}, first we need to define a new family of modal operators $\bigcirc_\alpha$ with α in G. A sentence $\bigcirc_\alpha A$ is to be true at a world w if and only if the A-worlds that are closest to w are, at most, α-similar to w. At the syntactic level $\bigcirc_\alpha$ is defined as follows:

$$\bigcirc_i A \equiv \Box_{i-1} \neg A \wedge \Diamond_i A \text{ with } i = E(\alpha)$$

Now, the previous entailment can be writen as:

$$K \vdash_{S_{GS5}} ((\bigwedge_{\beta<\alpha} (\bigcirc_\beta A \rightarrow \Diamond_\beta B)) \vee \Diamond_\alpha B)$$

Finally, the third consequence operator that is presented in [DEG+95] corresponds to a modified version of the conditional measure $I_{S,K}(B \mid A)$, that is proposed in [EGG94b], and it is defined as:

$$A \models^{\alpha}_{S,K} B \text{ iff } I^2_{S,K}(B \mid A) = I_{S,K}(B \wedge A \mid A)$$

Again, we can characterize this entailment via the following expresion:

$$K \vdash_{S_{GS5}} ((\bigwedge_{\beta<\alpha} (\bigcirc_\beta A \rightarrow \Diamond_\beta (B \wedge A))) \vee \Diamond_\alpha (B \wedge A))$$

6 Conclusions and Open Problems

We have presented the axiomatization of a similarity-based graded multi-modal system $GS5$, indicating how several entailments proposed in [DEG+95] can be characterized inside this system whenever the propositional language, as well as the ranges of grades of similarity, are finite. Furthermore, the axiomatization has been proved to be correct and complete with respect to the classes of Kripke models whose accesibility relations are based in finite min-transitive similarity relations.

The extension to the infinite case is not direct, because we can not be sure about the fact that the nested family of accesibility relations in theorem 4.11 can be recovered by the α-cut of similarity relations which is constructed there, using supreme instead of maximun. We think that a solution is to incorporate the following rule in $GS5$:

$$\frac{\Diamond_{\alpha_i} A}{\Diamond_\alpha A} \text{ with } \alpha_i \uparrow \alpha$$

where $\alpha_i \uparrow \alpha$ express as an ascendent succession with limit α. The proof of the usefulness of this rule remains.

Another remark is the limitation of the language in definition 2.1 due to the requerement that any pair of worlds have to be at least 0-similar. For this we impose the restriction that the formula $\Box_0 A$ does not belong to the language if $A \not\equiv \top$. Other conditions should be studied too.

In [EGG95b] a semantical interpretation of possibilistic deduction via another similarity-based graded modal logic is established, called it "multi-modal logic". This logic has four kind of graded operators, namely, $\Diamond_{o\alpha}$, $\Box_{o\alpha}$, $\Diamond_{c\alpha}$ and $\Box_{c\alpha}$. The difference between $\Diamond_{o\alpha} q$ and $\Diamond_{c\alpha} q$ is that the first is true in a world w if there exists a q-world w' such that $S(w, w') > \alpha$, and the second is true in w if $S(w, w') \geq \alpha$. An extension of our axiomatization with this operator is being studied.

Further interesting topics for future research are, among others:

1) To provide an aximatization for the case when the similarity relations are transitive with respect to a t-norm different from *minimum*,

2) To define inference mechanisms for S_{GS5}.

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