

Computational Implementation of Indices of Power

Aguirre, Jesús Francisco (*)
Oviedo, Jorge Armando (**)
Quintas, Luis Guillermo (***)

(*) Departamento de Informática

e-mail: jaguirre@unsl.edu.ar

(**) Departamento de Matemáticas-IMASL (Instituto de Matemáticas Aplicada San Luis)

e-mail: joviedo@unsl.edu.ar

(***) Departamento de Matemáticas-IMASL (Instituto de Matemáticas Aplicada San Luis)-CONICET

e-mail: lquintas@unsl.edu.ar

Facultad de Ciencias Físico, Matemáticas y Naturales
Universidad Nacional de San Luis
Ejército de los Andes 950 - (5700) San Luis
E-mail: lquintas@unsl.edu.ar

Abstract

In this paper we present algorithms and computational implementation of the Shapley Value and Banzhaf-Coleman Index of Power. Both indices describe the real power of the coalitions involved in strategic interactions. The system allows the study of complex Electoral Applications.

The data input can be done in two different ways: by considering all the possible coalitions, or only the basic coalitions (political parties, sectorial groups, etc.). The system also allows to introduce restrictions (incompatibilities) among some coalitions. We present some applications for computing the Electoral Power in the election of authorities in Universidad Nacional de San Luis.

We describe the client-server design and the implementation of these tools, using the languages C and Tcl/Tk. The server program is written in C and requires Linux. The Client Program is written in Tcl/Tk with namespace mechanism and it supports Linux and Windows.

Keywords: Cooperative Games Theory - Indices of Electoral Power - Computational implementation - Client server paradigm - Scripting - Multi platform

1 Introduction

Game Theory is a mathematical theory which models agents interactions in situations of strategic conflict. A Game is a situation in which two or more players interact. It includes the modelization of the interactions among firms, groups etc. in Economic or Politic Scenarios.

Each player has partial control of the situation, but in general, no player controls it totally. Each player or group of players have certain personal preferences on the set of possible results and tries to obtain the one that favors to him more. These preferences can be you described by some utility functions, in which each player is characterized by a numerical function.

Games can be divided in two categories: "noncooperatives" and "cooperatives". In the first one, only selfishness is assumed. In the second approach, besides this assumption, we consider the possibility of forming coalitions and the groups of players act cooperatively.

In the present article we will work only with cooperative games.

2 Cooperative Games

This games are those where the individuals are free to communicate, to negotiate and to sign contracts to obtain better results. Let us suppose that a game has two or more results and at least two players agree (in preference) in a result, then they (2 or more) will sign an agreement to induce this result as the solution of the game.

In the cooperative games it naturally appears the concept of coalition (players that sign an agreement to induce a result of the game).

A cooperative game is given by: $G = (N; v)$, where $N = \{1; 2; \dots; n\}$ is the set of all the players and v is the characteristic function. This function is defined on the coalitions $S \subseteq N$ and measures the value or utility $v(S)$ that each coalition S has if it forms. Thus $v(S)$ is the utility that the members of S can obtain by themselves.

Definition 1 A cooperative n person game is defined by: $G = (N; v)$, where $N = \{1; 2; \dots; n\}$ is the set of all players and $v : 2^N \rightarrow \mathbb{R}$ is the characteristic function. This is a real function, defined on the subset of N , that fulfill the following properties:

$$v(\emptyset) = 0 \tag{1}$$

$$v(\{i\}) \geq 0 \quad \forall i \in N \tag{2}$$

$$v(S \cup T) \geq v(S) + v(T) \quad \forall (S \cap T) = \emptyset \tag{3}$$

Condition (1) is only for consistency (the empty coalition has no-power). Condition (2) indicates that the security level of each player is zero.

Condition (3) is known as **Superadditivity Property**, and shows the incentives for the players in conforming bigger coalitions.

The cooperative games admit different types the solutions. These are possible forms to distribute the total amount provided by the coalition among the players.

To give a solution or result of the cooperative game is to find a vector (n vector) where each component says how much each player get.

Among the solutions more widely spread for cooperative games we mention: the Shapley Value, the Core, the Banzhaf-Coleman Power Index, the Nucleolus, etc. (see [8] , [9]).

In this work we will study the Shapley Value [10], and the Banzhaf-Coleman Power Index [3], [1], [2].

We will implement both solutions.

3 The Shapley Value

The Shapley Value $\phi(v) = (\phi_1; \phi_2; \dots; \phi_n)$ gives an imputation, it means a way to distribute the total amount obtained by the total coalition N , among the players, giving each one at least the amount each player can obtain by himself and taking into account the average marginal contribution by being (or not) member of each coalition. In symbols we have:

$$\sum_{i=1}^n \phi_i = v(N) \quad \text{y} \quad \phi_i \geq v(\{i\})$$

We give two basic definitions:

Definition 2 A carrier for a game G is a coalition T such that, for any S , $v(S) = v(S \setminus T)$.

Intuitively, definition states that any player who does not belong to a carrier is a dummy—i.e., can contribute nothing to any coalition.

Definition 3 Let G be a n person game, and let π be any permutation of the set N . Then, by πv we mean the game $G^\pi = (N; u)$ such that, for any $S = \{i_1; i_2; \dots; i_n\}$,

$$u(\pi(i_1); \pi(i_2); \dots; \pi(i_n)) = v(S)$$

Effectively, the game G is nothing other than the game G^π , with the roles of the players interchanged by the permutation π . This property is known by anonymity.

With these two definitions, it is possible to give an axiomatic treatment.

By the Shapley Value of a game $G = (N; v)$, we shall mean a n vector, $\phi[v]$. The Shapley Value is characterized by the following axioms:

Axiom S1. If S is any carrier, then

$$\phi_i[v] = v(S).$$

Axiom S2. For any permutation π , and $i \in N$,

$$\phi_{\pi(i)}[\pi v] = \phi_i[v].$$

Axiom S3. If $G = (N; u)$ and $G^\pi = (N; v)$ are any games,

$$\phi_i[u + v] = \phi_i[u] + \phi_i[v].$$

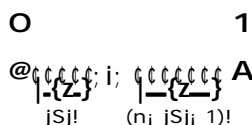
Axiom S1 indicates the close relations of the Shapley Value with the carriers. Axioms S2 says that the Value is anonymous and S3 that it is invariant over linear transformations of the games.

It is a remarkable fact that these axioms are sufficient to determine a value ϕ uniquely, for all games.

Teorema 1 Let $G = (N; v)$ be an game. The Shapley value is a n vector call $\phi(v) = (\phi_1(v); \phi_2(v); \dots; \phi_n(v))$, such that:

$$\phi_i(v) = \sum_{S \in \mathcal{L}_i} \frac{|S|!(n - |S|)!}{n!} [v(S \cup \{i\}) - v(S)]$$

The Shapley value can be given the following heuristic explanation. Suppose the players (the elements of N) agree to meet at a specified place and time. Naturally because of random fluctuations, all will arrive at different times; it is assumed, however, that all orders of arrival (permutations of the players) have the same probability: $\frac{1}{n!}$.



Suppose that, if a player, i , arrives, and finds the members of the coalition $S \cup \{i\}$ (and no others) already there, he receives the amount $v(S \cup \{i\}) - v(S)$, i.e., the marginal amount which he contributes to the coalition, as payoff. Then the Shapley value $\phi_i[v]$ is the expected payoff to player i under this randomization scheme.

The Shapley value (instead of the Core) always exists and it is unique.

4 The Banzhaf–Coleman Power Index

A second index of power has been suggested by Banzhaf [1] and Coleman [3].

This index is defined on simple games and is based on counting for each player the number of coalitions to which it belongs and it is crucial to win.

Lets assume that $(N; v)$ is a simple game normalized to $(0; 1)$. We say a coalition S is winning when $v(S) = 1$. For each winning coalition S , when $v(S - i) = 0$, is said to be a swing for player i :

For a game $(N; v)$, lets suppose that $\%_i(N; v)$ is the number of swings for i : Lets suppose that $\%_0(N; v) = \sum_{i \in N} \%_i(N; v)$ be the total number of swings of all the players of the game. Then we define the normalized Banzhaf–Coleman index by:

$$b_i(N; v) = \frac{\%_i(N; v)}{\%_0(N; v)}$$

This index can be generalized to general games [Owens 1978] by the following formula:

$$b_i(N; v) = \frac{\sum_{S \subseteq N} \mu_1^{n_i - 1} [v(S) - v(S - i)]}{2^{n-1}}$$

In the following section we will introduce the Tool Command Language (Tcl) and the computational implementation of both indices.

5 The Tool Command Language (Tcl)

For the last fifteen years a fundamental change has been occurring in the way people write computer programs. The change is a transition from system programming languages such as C or C++ to scripting languages such as Perl or Tcl [6].

Scripting languages are designed for different tasks than system programming languages, and this lead to fundamental differences in the languages. System programming languages were designed for building data structures and algorithms from scratch, starting from the most primitive computer elements such as words of memory. In contrast, scripting languages are designed for gluing: they assume the existence of a set of powerful components and are intended primarily for connecting components together.

Tcl is a string-based command language. The language has only a few fundamental constructs and relatively little syntax, which makes it easy to learn. It exists as a C library package that can be used in many different programs [12].

Tk is a toolkit for programming graphical user interfaces. It was designed for the X window system used on UNIX system, and it was ported to the Macintosh and Windows environments. Too provide network sockets, great number of gratuitous extensions and the possibility to incorporating new commands in language C/C++.

Originally Tcl has one global scope for shared variables, local scopes within procedures, and one global name space for procedures. The single global scope for procedures and global variables can become unmanageable as your Tcl application grows. Namespaces help structure large Tcl applications, but they add complexity.[5]

6 The system implementation

This tool uses client-server paradigm [11]. The server program is programming in C and the client in Tcl/Tk.

Client software is a Tcl/Tk script that show graphical user interface, which can run over Unix and Windows platforms. It contacts a server, sends a request, and awaits a response. When the response arrives, the client shows the results. This client application program invoke nonstandard TCP/IP services or locally-defined application services..

Server software creates a socket and binds the socket to the port at which it desires to receive requests. It then enters an infinite loop in which it accepts the next request that arrives from a client, processes the Banzhaf–Coleman and Shapley Value formula, and sends the reply back to the client.

This development allows to work up to 10 players ($N = 10$) for the first method and 50 basic coalitions ($N = 50$) for the second method. These restrictions follow from the fact that, on one hand it is not of practical

use to input data with the ...rst method for more than 10 players ($2^{10} = 1024$ coalitions); and on the other hand 50 basic coalitions is a reasonable number for most applications.

We will describe the implementation of the solutions and we also will show a simple example. We ...rst introduce the size n of the game and the characteristic function. The programme also checks the consistency of the introduced characteristic function (see (1), (2) and (3))

Computational implementation of Indices of Power is realized by two methods:

Method 1: This method allows to input the characteristic function for all the coalitions. Namely:

1- Input of data:

i) $N =$ number of players.

ii) Characteristic function.

2- Verify that ii) is a characteristic function.

3- Computation of Indices of Power.

The following ...gure shows the data input for the ...rst method:

Coalition	Player 1	Player 2	Player 3	V(s)
1:	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	0
2:	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	0
3:	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	0
4:	<input type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	0
5:	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	<input type="checkbox"/>	1
6:	<input checked="" type="checkbox"/>	<input type="checkbox"/>	<input checked="" type="checkbox"/>	1
7:	<input type="checkbox"/>	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	0
8:	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	1

Buttons: Aceptar, Borrar, Cancelar

Fig. 1

The above ...gure shows the input data of the characteristic function of the following game:

$G = (N; v)$, where $N = \{1, 2, 3\}$ and v is the following characteristic function:

$$v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 0$$

$$v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 1$$

The following ...gure shows system interface with the results of the Shapley Value:

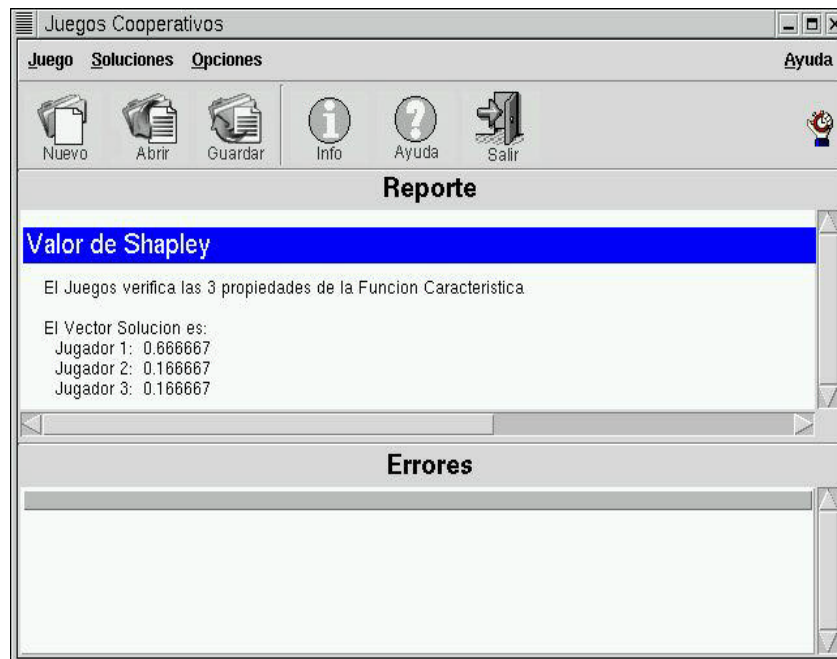


Fig. 2

It indicates that the Shapley Value vector is given by: $v = (0.666670 ; 0.16667 ; 0.16667)$: In a similar way we can compute the Coleman-Banzhaf Index and we obtain: $b(v) = (0.6 ; 0.2 ; 0.2)$.

Even in this simple case both values give different results. It requires the generation of several coalitions and its computation. The possibility of making comparisons of the indices can be useful in practical applications (see section 7).

Method 2: When the number n of players is large, in many cases we can avoid the task of loading the characteristic function for the 2^n coalitions (by the first method). If the characteristic function consists of a sum of votes (as in many Electoral Colleges, Councils, etc.) and there are some basic coalitions (electoral parties or other basic groups) then the system will generate the characteristic function for all the coalitions by summing the weight $q(i)$ (amounts of votes) of these (few) basic coalitions (see fig. 3 and also the Application shown in section 7). Then the winning coalitions S , $v(S) = 1$, are those having a simple majority of votes (the System also allows the use of other majorities: 2/3, etc.); other otherwise $v(S) = 0$. Thus, we have:

1- Input of data:

i) N = number of player

ii) Characteristic function. This function is a formula that depends on the number of players that each coalition contains.

2- Computation of the Indices of Power

The following figure shows the data input for the second method. It corresponds to the Application Problem studied in Section 7:



Fig. 3

7 An Application

A practical actual problem of much interest to Universidad Nacional de San Luis, has to do with the modification of the University Statute [4]. It foresees a direct method of election of the Universities authorities. Besides the good properties of a direct voting system, its implementation also implies changes in the actual electoral power distribution.

In the actual system, for the case of the election of the Dean of a Faculty, it is necessary to obtain a majority of votes in the "Consejo Directivo" of the corresponding Faculty. It is formed by 17 electors consisting of 6 basic groups:

- 1.- Professors (The Majority) = 7 votes. (q(1))
- 2.- Professors (The Minority) = 3 votes. (q(2))
- 3.- Students (The Majority) = 4 votes. (q(3))
- 4.- Students (The Minority) = 1 vote. (q(4))
- 5.- Administrative Employees = 1 vote. (q(5))
- 6.- Graduate = 1 vote.(q(6))

It is necessary to obtain 9 votes in order to decide the election of the Faculty Dean. This is a typical configuration, which was a real result of several recent elections in "Facultad de Ciencias Físico Matemáticas y Naturales". (The other Faculties have similar scenarios but the students have distributions 3 and 2 votes or 3, 1 and 1 votes). It could be other configurations giving alternative scenarios.

In order to know the real electoral power of each group, we will compute the Shapley Value and the Coleman-Banzhaf Index.

Figure 4. shows the Coleman-Banzhaf Index computed by using the system. It is worth to note that making the computation without this application would be difficult because there are $2^6 = 64$ coalitions to deal with.

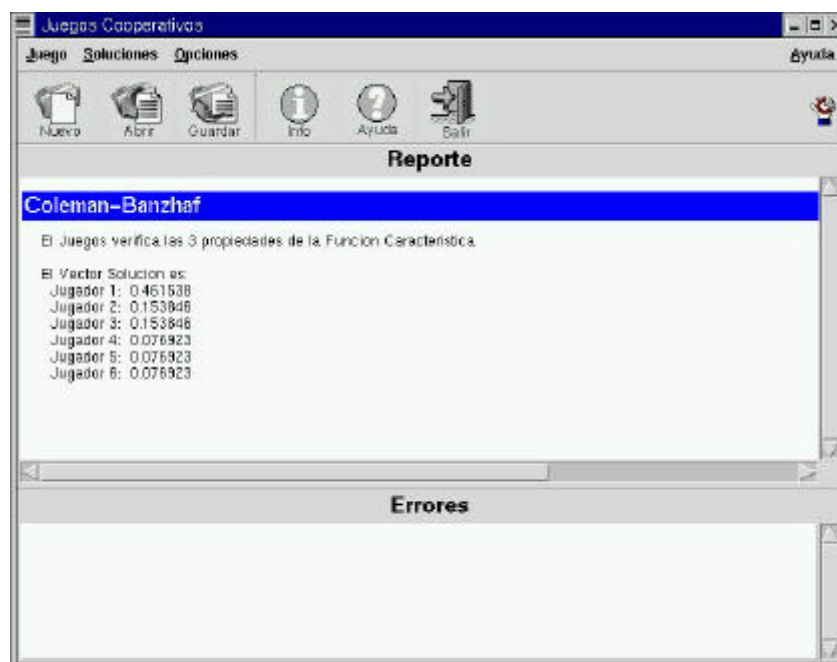


Fig. 4

It indicates that the Coleman-Banzhaf Index vector is given by:

$$b(v) = (0:46153 ; 0:15384 ; 0:15384 ; 0:07692 ; 0:07692 ; 0:07692)$$

In a similar way we compute the Shapley Value and we obtain:

$$'(v) = (0:46667 ; 0:16667 ; 0:16667 ; 0:06667 ; 0:06667 ; 0:06667):$$

In this case both indices give similar values. We note that they do not exactly correspond to the actual size of the group share in the "Consejo Directivo " (for instance, the Professors Majority group has a share of $7=17 = 0:41176$ of the votes, but the power indices are still higher: 0:46667 and 0:46153). We also note that groups with different shares could have the same Power Index: thus the group 2 and 3 have 3 and 4 votes respectively, but they have the same electoral power because they have the same possibilities of conforming a simple majority including them.

It is worth noticing that in practice some groups could belong to opposite sectors and thus they do not form a coalition. This is the typical situation between the Professors Majority and Professors Minority. Thus, we can recalculate the Indices and we obtain the following values.

The Shapley Value is given by:

$$'(v) = (0:38333 ; 0:08333 ; 0:23333 ; 0:1 ; 0:1 ; 0:1):$$

and the Coleman-Banzhaf Index is given by:

$$b(v) = (0:310345 ; 0:08333 ; 0:23333 ; 0:1 ; 0:1 ; 0:1):$$

This seems to be closer to the real situation. In this case the Students Majority has a higher power and the Professors Minority has very few power. It is consistent with the fact that in practice usually both, the Professors Majority and the Students Majority, have been able to form the winning coalition and chose the Faculty Dean.

We often observe that the Professor Majority impose the Graduate representative. In this case we have only ...ve basic coalitions:

- 1.- Professors (The Majority) = 8 votes. (q(1))
- 2.- Professors (The Minority) = 3 votes. (q(2))
- 3.- Students (The Majority) = 4 votes. (q(3))
- 4.- Students (The Minority) = 1 vote. (q(4))
- 5.- Administrative Employees = 1 vote. (q(5))

Now if we do not impose any further condition on the coalitions (in this case we allow any group to joint each other, including both Professors groups), we have the following power indices:

The Shapley Value is given by:

$v = (0.6 ; 0.1 ; 0.1 ; 0.1 ; 0.1)$:

and the Coleman-Banzhaf Index is given by:

$b(v) = (0.636364 ; 0.090909 ; 0.090909 ; 0.090909 ; 0.090909)$:

If we again impose the restriction that both Professor groups belong to opposite sectors, and thus they do not form a coalition, then we have the following indices:

The Shapley Value is given by:

$v = (0.55 ; 0.05 ; 0.133333 ; 0.133333 ; 0.133333)$:

and the Coleman-Banzhaf Index is given by:

$b(v) = (0.565217 ; 0.043478 ; 0.130435 ; 0.130435 ; 0.130435)$:

As a consequence of our study we can get a better understanding of the Real Power of each group. It is not just a matter of counting how many votes each group has in the Council.

It is interesting for instance to compare the share of votes of the Professors Minority : 3/17 (approximately the 18%) with the corresponding indices of power. These indices (the second numbers of v and $b(v)$) are much smaller, they range between 0.043478 (approximately 4%) and 0.1 (10%). These results are consistent with the following observation: Taking into account the last 24 Dean's Elections (6 elections in each of the 4 Faculties) only in 2 cases (approximately the 8% of the times) the Dean was elected with the support by a Professor Minority and without the full support of the Professors Majority.

The system we have developed can also be used to compute the groups power after the Dean is elected (the total number of members in the Council now grows to 18 and in case of a tie the vote of the Dean counts double). We can also use the system for the election of the Rector of the University.

It is also useful for computing electoral power under alternative electoral systems.

8 Conclusions

A goal that is obtained with the development of this system is that by using it we can offer support to game theory researchers, as well as to teachers and students. It is particularly useful in the specific area of Game Theory to have this tool because the indices are often used in practical applications (Measures of the different power of political groups in the Parliaments, Power of each country in International Organizations, etc.). It also allows to study a great number of cases in order to foresee regularities leading to general results.

An extension that will be added to the system in the near future is a facility that allows to deal with a family of predefined characteristic functions. We expect to merge several computing systems that we have developed for computing cooperative solutions, including this one, into a single system. It will allow the comparison of the different solutions.

References

- [1] Weighted Voting Doesn't Work: A Mathematical Analysis. Banzhaf, J. F., Rutgers Law Review 19, 1965
- [2] One Man, 3.312 Votes: A Mathematical Analysis of the Electoral College. Banzhaf, J. F., Villanova Law Review 13, 1968
- [3] Control of Collectivities and the Power of a Collectivity to Act. Coleman, J. S., Social Choice (B. Lieberman, ed), London (Gordon and Breach), 1971
- [4] Estatuto Universitario, 1990, Universidad Nacional de San Luis.
- [5] Tcl Style Guide. Johnson, Ray. Sun Microsystems Inc. August 1997.
- [6] Tcl: An Embeddable Command Language. Ousterhout, J., Proc. USENIX Winter Conference. January 1990.
- [7] Scripting: Higher Level Programming for the 21st Century. Ousterhout, J., IEEE Computer magazine, march 1998.
- [8] Characterization of the Banzhaf-Coleman Index. SIAM Journal of Applied Mathematics 35.

- [9] Game Theory (Third Edition). Owen, G., Academic Press, 1995.
- [10] A Value for n-person games . Annals 28, 1953.
- [11] Internetworking with TCP/IP. Vol. III: Client-Server Programming and Applications. Douglas E. Comer and David L. Stevens, Prentice Hall 1993.
- [12] Practical Programming in Tcl and Tk -Second Edition. Welch, Brent B. Prentice Hall, 1997.