

Notions of Relevance for Modeling the Dynamics of Belief

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Abstract

We identify different kinds of relevance relations between formulas that emerge in the process of belief revision. Informal definitions for alternative notions of relevance are suggested and a set of schemas and intuitive postulates for formalizing these notions are proposed. The notions of relevance proposed here are shown to be good candidates for modeling the process of belief revision.

Keywords: Belief Revision, Relevance, Theory Change, Knowledge Representation.

1 Introduction

Belief revision is the process by which an agent changes his previous set of beliefs making a transition from one epistemic state to another. When such an agent learns new information he can realize that this information clashes with his old beliefs. In this case the agent has to revise his belief set and decide which old beliefs need to be eliminated in favor of the new information.

The aim of this work is to characterize the different kinds of relations that may exist between new acquired information and old maintained beliefs. We identify four primitive kinds of relations between a new piece of information α and an old piece of information β :

1. α is *positively relevant* to β if β is incorporated to the belief set whenever the agent learns α .
2. α is *negatively relevant* to β if β is retracted from the belief set whenever the agent learns α .
3. α is *positively irrelevant* to β if β is not incorporated to the belief set when the agent learns α .
4. α is *negatively irrelevant* to β if β is not retracted from the belief set when the agent learns α .

A fifth interesting kind of relation between α and β arises when either α is *positively relevant* to β or α is *negatively irrelevant* to β . This relation holds when β is part of the updated belief set, regardless of whether it belonged to it or not before α was learned.

In [Falappa 99] the notion of negative irrelevance was taken as a primitive notion and a set of intuitive postulates was given to formalize it. The postulates for negative irrelevance were shown to be as powerful as the AGM postulates for contraction. In this work we extend the

above mentioned results by presenting a formal characterization for the fifth kind of relation and by distinguishing three different notions of relevance that naturally emerge and that have their counterpart in three well known contraction functions, namely *maxichoice*, *full meet* and *partial meet*.

2 Belief Revision

One of the most fundamental approaches to the formalization of the dynamics of beliefs is the AGM model [Alchourrón 85], proposed by Carlos Alchourrón, Peter Gärdenfors and David Makinson. In the AGM approach the epistemic states are represented by belief sets. Let \mathbf{K} be a belief set and α a sentence in a propositional language \mathcal{L} . The three main kinds of changes are the following [Gärdenfors 92]:

Expansion: A new sentence is added to an epistemic state regardless of the consequences of the so formed larger set. If $+$ is an expansion operator, then $\mathbf{K}+\alpha$ denotes the belief set \mathbf{K} expanded by α .

Contraction: Some sentence in the epistemic state is retracted without adding any new belief. If $-$ is a contraction operator, then $\mathbf{K}-\alpha$ denotes the belief set \mathbf{K} contracted by α .

Revision: A new sentence is consistently added to an epistemic state. In order to make possible this operation, some sentences may be retracted from the original epistemic state. If $*$ is a revision operator, then $\mathbf{K}*\alpha$ denotes the belief set \mathbf{K} revised by α .

Expansions can simply be defined as the logical closure of \mathbf{K} and α :

$$\mathbf{K}+\alpha = Cn(\mathbf{K} \cup \{\alpha\})$$

It is not possible to give a similar explicit definition of contractions and revisions in logical and set-theoretical notions only. These operations can be defined using logical notions and some selection mechanism. Contractions and Revisions are interdefinable by the following identities:

Levi Identity: $\mathbf{K}*\alpha = (\mathbf{K}-\neg\alpha)+\alpha$.

Harper Identity: $\mathbf{K}-\alpha = \mathbf{K} \cap \mathbf{K}*\neg\alpha$.

By giving a definition of one of these operators we can obtain the other one using the above identities. In this work we will show the relation existing between the notion of (ir)relevance and the contraction operator.

The following Lemma is a straight consequence of Levi Identity.

LEMMA 2.1 Given a belief set \mathbf{K} and a formula $\alpha \in \mathcal{L}$, $\mathbf{K}-\neg\alpha \subseteq \mathbf{K}*\alpha$. ■

2.1 Postulates for Contractions

Gärdenfors [Gärdenfors 88] proposed the following *rationality postulates* for contraction operators:

(K⁻1) **Closure:** For every belief set \mathbf{K} and every sentence α , $\mathbf{K}-\alpha$ is a belief set.

(K⁻2) **Inclusion:** $\mathbf{K}-\alpha \subseteq \mathbf{K}$.

(K⁻3) **Vacuity:** If $\alpha \notin \mathbf{K}$ then $\mathbf{K}-\alpha = \mathbf{K}$.

(K⁻4) **Success:** If $\not\vdash \alpha$ then $\alpha \notin \mathbf{K}-\alpha$.

(K⁻5) **Recovery:** $\mathbf{K} \subseteq (\mathbf{K}-\alpha)+\alpha$.

(K⁻⁶) **Extensionality:** If $\vdash \alpha \leftrightarrow \beta$ then $\mathbf{K}-\alpha = \mathbf{K}-\beta$.

(K⁻⁷) **Conjunctive Overlap:** $\mathbf{K}-\alpha \cap \mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta)$.

(K⁻⁸) **Conjunctive Inclusion:** If $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$ then $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$.

2.2 Contraction functions and their associated postulates

A basic assumption behind most theories of belief revision is that an agent should maintain as many of the earlier beliefs as possible. This means that, in order to eliminate inconsistency, he must do *minimal changes*. But there is not a unified criterion for defining a minimal change.

If we want preserve as much information as possible we should look for a maximal subset of the original state of belief that fails to imply the unwanted formula. The set of all the maximal subsets of \mathbf{K} failing to imply α is denoted $\mathbf{K}^\perp\alpha$.

In general, $\mathbf{K}^\perp\alpha$ contains more than one maximal subset. In order to construct a contraction function it is possible to apply a selection function γ to select one element from $\mathbf{K}^\perp\alpha$. The set that is returned by $\gamma(\mathbf{K}^\perp\alpha)$ can be seen as the “preferred” element from $\mathbf{K}^\perp\alpha$. The contraction of \mathbf{K} by α can be defined as follows:

$$\mathbf{K}-\alpha = \begin{cases} \gamma(\mathbf{K}^\perp\alpha) & \text{if } \mathbf{K}^\perp\alpha \neq \emptyset. \\ \mathbf{K} & \text{otherwise} \end{cases}$$

and is referred to as *maxichoice contraction function*.

Contractions defined in this way satisfy postulates (K⁻¹)..(K⁻⁶) together with the following postulate:

(K^{-F}) **Fullness:** If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}-\alpha$ then $(\beta \rightarrow \alpha) \in \mathbf{K}-\alpha$ for any belief set K .

Postulate (K^{-F}) suggests that maxichoice contractions retain too much information. This gives rise to suggest another selection function, one that instead of returning a selected element of $\mathbf{K}^\perp\alpha$, returns the set that results from the intersection of all the elements of $\mathbf{K}^\perp\alpha$. This form of contraction of \mathbf{K} by α can be defined in the following way:

$$\mathbf{K}-\alpha = \begin{cases} \bigcap(\mathbf{K}^\perp\alpha) & \text{if } \mathbf{K}^\perp\alpha \neq \emptyset. \\ \mathbf{K} & \text{otherwise} \end{cases}$$

and is referred to as *full meet contraction function*.

Contractions defined using a full meet contraction function satisfy postulates (K⁻¹)..(K⁻⁶) together with the following postulate:

(K^{-I}) **Intersection:** For all α and β , $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha \cap \mathbf{K}-\beta$.

The above does not seem to be a desirable postulate since by approving it too much information is removed. This is against the desired principle of *minimal change*.

There is a third possibility that results from making a compromise between the maxichoice contraction function and the full meet contraction function. This alternative contraction function is referred to as *partial meet contraction function* and it returns the set that results from the intersection of the “preferred” elements of $\mathbf{K}^\perp\alpha$. According to this construction, the contraction of \mathbf{K} by α can be defined as follows:

$$\mathbf{K}-\alpha = \begin{cases} \bigcap\gamma(\mathbf{K}^\perp\alpha) & \text{if } \mathbf{K}^\perp\alpha \neq \emptyset. \\ \mathbf{K} & \text{otherwise} \end{cases}$$

Contractions defined in this way satisfy postulates (K⁻¹)..(K⁻⁸).

2.3 Postulates for Revision

The following postulates for revision have been proposed by Gärdenfors [Gärdenfors 88]:

- (K*1) **Closure:** $\mathbf{K}*\alpha = \text{Cn}(\mathbf{K}*\alpha)$.
- (K*2) **Success:** $\alpha \in \mathbf{K}*\alpha$.
- (K*3) **Inclusion:** $\mathbf{K}*\alpha \subseteq \mathbf{K}+\alpha$.
- (K*4) **Vacuity:** If $\mathbf{K} \not\vdash \neg\alpha$ then $\mathbf{K}*\alpha = \mathbf{K}+\alpha$.
- (K*5) **Consistency:** If $\not\vdash \neg\alpha$ then $\mathbf{K}*\alpha \neq \mathbf{K}_\perp$.
- (K*6) **Extensionality:** If $\vdash \alpha \leftrightarrow \beta$ then $\mathbf{K}*\alpha = \mathbf{K}*\beta$.
- (K*7) **Superexpansion:** $\mathbf{K}*(\alpha \wedge \beta) \subseteq (\mathbf{K}*\alpha)+\beta$.
- (K*8) **Subexpansion:** If $\neg\beta \notin \mathbf{K}*\alpha$ then $(\mathbf{K}*\alpha)+\beta \subseteq \mathbf{K}*(\alpha \wedge \beta)$.

3 Irrelevance in the context of Belief Revision

The aim of this section is to formally depict the notion of negative irrelevance as presented in [Falappa 99]. In section 3.2 we extend the original results by presenting two additional postulates for negative irrelevance. As previously stated, we expect a formula α to be negatively irrelevant to a formula β when learning α is not a reason to remove β from the belief set.

For representing irrelevance we use a metalinguistic relation between formulas of a propositional language \mathcal{L} . In our approach irrelevance is seen as a “non-interference relation” between two formulas, in the context of a belief set. We will take the notion of negative irrelevance as a primitive relation.

DEFINITION 3.1 *Given a belief set \mathbf{K} and a pair of formulas $\alpha, \beta \in \mathcal{L}$ we say that α is negatively irrelevant to β and we denote it $\alpha \mathbb{I}_{\mathbf{K}}^N \beta$ if and only if the following two conditions are simultaneously satisfied:*

1. $\beta \in \mathbf{K}$, and
2. $\beta \in \mathbf{K}*\alpha$.

The following Lemma is a straight consequence of definition 3.1 and Harper Identity.

LEMMA 3.1 *Given a belief set \mathbf{K} and a pair of formulas $\alpha, \beta \in \mathcal{L}$, $\alpha \mathbb{I}_{\mathbf{K}}^N \beta$ if and only if $\beta \in \mathbf{K} - \neg\alpha$. ■*

3.1 Postulates for Irrelevance Relations

In [Falappa 99] the following set of postulates for negative irrelevance relations together with the rationale for them have been proposed:

- (Irr₁) If $\neg\alpha \mathbb{I}_{\mathbf{K}}^N \alpha$ then $\vdash \alpha$.
- (Irr₂) If $\alpha \mathbb{I}_{\mathbf{K}}^N \beta$ and $\alpha \mathbb{I}_{\mathbf{K}}^N (\beta \rightarrow \delta)$ then $\alpha \mathbb{I}_{\mathbf{K}}^N \delta$.
- (Irr₃) If $\vdash (\alpha \leftrightarrow \beta)$ then $\alpha \mathbb{I}_{\mathbf{K}}^N \delta$ if and only if $\beta \mathbb{I}_{\mathbf{K}}^N \delta$ for all $\delta \in \mathcal{L}$.
- (Irr₄) If $\vdash \beta$ then $\alpha \mathbb{I}_{\mathbf{K}}^N \beta$ for all $\alpha \in \mathcal{L}$.
- (Irr₅) If $\alpha \notin \mathbf{K}$ then $\neg\alpha \mathbb{I}_{\mathbf{K}}^N \beta$ for all $\beta \in \mathbf{K}$.

(Irr₆) If $\beta \in \mathbf{K}$ then $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$.

(Irr₇) If $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ and $\beta \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ then $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$.

(Irr₈) If $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ then $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ or $\beta \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$.

According to lemma 3.1 a negative irrelevance relation of the form $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ holds if and only if $\beta \in \mathbf{K} - \neg\alpha$. Based on this correspondence it is possible to establish the following result:

THEOREM 3.1 [Falappa 99]: The postulates for irrelevance relations (Irr₁)..(Irr₈) are satisfied if and only if the postulates for contraction (K⁻¹)..(K⁻⁸) are satisfied. ■

3.2 Two additional postulates for irrelevance

Given the postulates (K^{-F}) and (K⁻¹) it is natural to investigate their corresponding irrelevance relation postulates. We propose two new postulates for irrelevance, namely (Irr₉) and (Irr₁₀). We will show that they can be used to characterize the fullness condition and the intersection condition respectively.

(Irr₉) If $\beta \in \mathbf{K}$ then $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ or $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}(\alpha \rightarrow \neg\beta)$.

(Irr₁₀) $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ if and only if $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ and $\beta \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$.

Postulate (Irr₉) establishes that learning α will never interfere with both β and $(\alpha \rightarrow \neg\beta)$. In this way, belief sets are kept as large as possible while preserving consistency. Postulate (Irr₁₀) subsumes postulate (Irr₇) by also demanding that it is necessary that both α and β do not interfere with δ to guarantee that $(\alpha \vee \beta)$ will not interfere with δ .

LEMMA 3.2 The postulates for irrelevance relations (Irr₁)..(Irr₆) and (Irr₉) are satisfied if and only if the postulates for contraction (K⁻¹)..(K⁻⁶) and (K^{-F}) are satisfied.

PROOF IN THE APPENDIX. ■

LEMMA 3.3 The postulates for irrelevance relations (Irr₁)..(Irr₆) and (Irr₁₀) are satisfied if and only if the postulates for contraction (K⁻¹)..(K⁻⁶) and (K^{-I}) are satisfied.

PROOF IN THE APPENDIX. ■

4 New Relations in the context of Belief Revision

In the previous section the notion of negative irrelevance was defined and characterized. The aim of this section is to introduce other (ir)relevance relations that intuitively emerge in the context of belief revision.

Given a belief set \mathbf{K} , there are two main relevance relations in which two formulas, α and β , can be involved:

- α is *positively relevant* to β if and only if β does not belong to \mathbf{K} but β belongs to \mathbf{K} revised by α . This relation between α and β will be denoted $\alpha \mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta$.
- α is *negatively relevant* to β if and only if β belongs to \mathbf{K} but β does not belong to \mathbf{K} revised by α . This relation between α and β will be denoted $\alpha \mathbb{R}_{\mathbf{K}}^{\mathbf{N}}\beta$.

Analogously, we can identify two kinds of irrelevance relations between formulas. One is the *negative irrelevance* relation presented in definition 3.1. The other one is the following:

- α is *positively irrelevant* to β if and only if β belongs neither to \mathbf{K} nor to \mathbf{K} revised by α . This relation between α and β will be denoted $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{P}}\beta$.

Another relevance relation between α and β emerges when we expect β to belong to \mathbf{K} revised by α regardless of whether β belongs or not to \mathbf{K} . This relation between α and β will be denoted $\alpha\mathbb{R}_{\mathbf{K}}^*\beta$.

Up to this point we have the following definitions for (ir)relevance relations:

$$\text{(Def } \mathbb{I}_{\mathbf{K}}^{\mathbf{P}}) \quad \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{P}}\beta = \{\beta : \beta \notin \mathbf{K} \text{ and } \beta \notin \mathbf{K}*\alpha\}.$$

$$\text{(Def } \mathbb{I}_{\mathbf{K}}^{\mathbf{N}}) \quad \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta = \{\beta : \beta \in \mathbf{K} \text{ and } \beta \in \mathbf{K}*\alpha\}.$$

$$\text{(Def } \mathbb{R}_{\mathbf{K}}^{\mathbf{P}}) \quad \alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta = \{\beta : \beta \notin \mathbf{K} \text{ and } \beta \in \mathbf{K}*\alpha\}.$$

$$\text{(Def } \mathbb{R}_{\mathbf{K}}^{\mathbf{N}}) \quad \alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{N}}\beta = \{\beta : \beta \in \mathbf{K} \text{ and } \beta \notin \mathbf{K}*\alpha\}.$$

$$\text{(Def } \mathbb{R}_{\mathbf{K}}^*) \quad \alpha\mathbb{R}_{\mathbf{K}}^*\beta = \{\beta : \beta \in \mathbf{K}*\alpha\}.$$

The following schemas are consequences of the definitions presented above. According to them, $\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}$, $\mathbb{R}_{\mathbf{K}}^{\mathbf{N}}$, $\mathbb{I}_{\mathbf{K}}^{\mathbf{P}}$ and $\mathbb{R}_{\mathbf{K}}^*$ relations can be described in terms of $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$ relations:

$$\text{(Schema } \mathbb{I}_{\mathbf{K}}^{\mathbf{P}}) \quad \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{P}}\beta \text{ if and only if } \beta \notin \mathbf{K} \text{ and it is not the case that } \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta \text{ holds.}$$

$$\text{(Schema } \mathbb{R}_{\mathbf{K}}^{\mathbf{P}}) \quad \alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta \text{ if and only if } \beta \notin \mathbf{K} \text{ and it is not the case that } \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{P}}\beta \text{ holds.}$$

$$\text{(Schema } \mathbb{R}_{\mathbf{K}}^{\mathbf{N}}) \quad \alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{N}}\beta \text{ if and only if } \beta \in \mathbf{K} \text{ and it is not the case that } \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta \text{ holds.}$$

$$\text{(Schema } \mathbb{R}_{\mathbf{K}}^*) \quad \alpha\mathbb{R}_{\mathbf{K}}^*\beta \text{ if and only if } \alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta \text{ or } \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta \text{ hold.}$$

According to the definitions and schemas presented above, we can give the following definitions for the change operators in terms of (ir)relevance relations:

$$\text{(Def } \mathbf{K}+\alpha) \quad \mathbf{K}+\alpha = \{\beta : \mathbf{K} \cup \{\alpha\} \vdash \beta\}.$$

$$\text{(Def } \mathbf{K}-\alpha) \quad \mathbf{K}-\alpha = \{\beta : \neg\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta\}.$$

$$\text{(Def } \mathbf{K}*\alpha) \quad \mathbf{K}*\alpha = \{\beta : \alpha\mathbb{R}_{\mathbf{K}}^*\beta\}.$$

We can present two alternative definitions for $\mathbf{K}*\alpha$ that are equivalent to the one presented above. The following Lemma presents this result.

LEMMA 4.1 Definition (Def $\mathbf{K}*\alpha$) is equivalent to the following ones:

$$\text{(Def' } \mathbf{K}*\alpha) \quad \mathbf{K}*\alpha = \{\beta : \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta \text{ or } \alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta\}.$$

$$\text{(Def'' } \mathbf{K}*\alpha) \quad \mathbf{K}*\alpha = \{\beta : \text{exists } \delta \text{ such that } \alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta \text{ and } \{\alpha\} \cup \{\delta\} \vdash \beta\}.$$

PROOF IN THE APPENDIX. ■

4.1 Postulates for $\mathbb{R}_{\mathbf{K}}^*$ Relations

The following postulates characterize $\mathbb{R}_{\mathbf{K}}^*$ relations. It is interesting to note their likeness with the postulates that characterize $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$ relations. Note, however, that a correspondence of this form between the AGM postulates for contraction and the AGM postulates for revision is not so evident:

$$\text{(Rel}_1^*) \quad \text{If } \neg\alpha\mathbb{R}_{\mathbf{K}}^*\alpha \text{ then } \vdash \alpha.$$

$$\text{(Rel}_2^*) \quad \text{If } \alpha\mathbb{R}_{\mathbf{K}}^*\beta \text{ and } \alpha\mathbb{R}_{\mathbf{K}}^*(\beta \rightarrow \delta) \text{ then } \alpha\mathbb{R}_{\mathbf{K}}^*\delta.$$

$$\text{(Rel}_3^*) \quad \text{If } \vdash (\alpha \leftrightarrow \beta) \text{ then } \alpha\mathbb{R}_{\mathbf{K}}^*\delta \text{ if and only if } \beta\mathbb{R}_{\mathbf{K}}^*\delta \text{ for all } \delta \in \mathcal{L}.$$

- (Rel₄^{*}) If $\vdash \beta$ then $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$ for all $\alpha \in \mathcal{L}$.
- (Rel₅^{*}) If $\alpha \notin \mathbf{K}$ then $\neg \alpha \mathbb{R}_{\mathbf{K}}^* \beta$ for all $\beta \in \mathbf{K}$.
- (Rel₆^{*}) If $\beta \in \mathbf{K}$ then $\alpha \mathbb{R}_{\mathbf{K}}^* (\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$.
- (Rel₇^{*}) If $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\beta \mathbb{R}_{\mathbf{K}}^* \delta$ then $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$ for all $\delta \in \mathbf{K}$.
- (Rel₈^{*}) If $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$ then $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ or $\beta \mathbb{R}_{\mathbf{K}}^* \delta$.

Postulate (Rel₁^{*}) establishes that if the negation of a formula α preserves α , then α is logically true. Postulate (Rel₂^{*}) stands for the condition of modus ponens in the consequent. The irrelevance of the syntax condition is valid for the $\mathbb{R}_{\mathbf{K}}^*$ relation and this is represented by postulate (Rel₃^{*}). According to postulate (Rel₄^{*}) theorems are always preserved. Postulate (Rel₅^{*}) establishes that if a formula α is not in the belief set, the negation of α preserves any formula that belongs to the belief set. We will assume that a formula α preserves or introduces to the revised belief set the formula that results from the disjunction of α and any other formula of the belief set. This last condition is represented by postulate (Rel₆^{*}). According to postulate (Rel₇^{*}), if α and β preserves δ , then the formula that results from the disjunction of α and β preserves δ . Finally, postulate (Rel₈^{*}) establishes that if the formula $(\alpha \vee \beta)$ preserves δ or introduces δ to the revised belief set then, either α preserves or introduces δ or β preserves or introduces δ .

LEMMA 4.2 The postulates for irrelevance relations (Irr₁)..(Irr₈) are satisfied if and only if the postulates for $\mathbb{R}_{\mathbf{K}}^*$ relations (Rel₁^{*})..(Rel₈^{*}) are satisfied.

PROOF IN THE APPENDIX. ■

THEOREM 4.1 The postulates for $\mathbb{R}_{\mathbf{K}}^*$ relations (Rel₁^{*})..(Rel₈^{*}) are satisfied if and only if the postulates for revision (K*1).. (K^*8) are satisfied.

PROOF IN THE APPENDIX. ■

4.2 Two additional postulates for $\mathbb{R}_{\mathbf{K}}^*$

As an extension to the set of postulates for $\mathbb{R}_{\mathbf{K}}^*$ presented above, we propose two new postulates for $\mathbb{R}_{\mathbf{K}}^*$ that are the natural counterpart of postulates (Irr₉) and (Irr₁₀).

- (Rel₉^{*}) If $\beta \in \mathbf{K}$ then $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$ or $\alpha \mathbb{R}_{\mathbf{K}}^* \neg \beta$.
- (Rel₁₀^{*}) $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$ if and only if $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\beta \mathbb{R}_{\mathbf{K}}^* \delta$ for all $\delta \in \mathbf{K}$.

Postulate (Rel₉^{*}) establishes that either β or its negation will be preserved or introduced when new information is learned. Postulate (Rel₁₀^{*}) establishes that the formula $(\alpha \vee \beta)$ preserves δ exactly in those cases in which α preserves δ and β preserves δ .

LEMMA 4.3 The postulates for irrelevance relations (Irr₁)..(Irr₆) and (Irr₉) are satisfied if and only if the postulates for $\mathbb{R}_{\mathbf{K}}^*$ (Rel₁^{*})..(Rel₆^{*}) and (Rel₉^{*}) are satisfied.

PROOF IN THE APPENDIX. ■

LEMMA 4.4 The postulates for irrelevance relations (Irr₁)..(Irr₆) and (Irr₁₀) are satisfied if and only if the postulates for $\mathbb{R}_{\mathbf{K}}^*$ (Rel₁^{*})..(Rel₆^{*}) and (Rel₁₀^{*}) are satisfied.

PROOF IN THE APPENDIX. ■

5 Conclusion

We have presented different kinds of relevance relations that can be used for modeling the process of theory change. We have proposed a formal characterization for the $\mathbb{R}_{\mathbf{K}}^*$ relation. The postulates that depict this relation naturally follow from the ones that depict the negative irrelevance relation. While negative irrelevance can be used to model the contraction operator, the new relation $\mathbb{R}_{\mathbf{K}}^*$ can be used to model the process of belief revision. Extensions for the initial set of irrelevance relation postulates and for the initial set of $\mathbb{R}_{\mathbf{K}}^*$ relation postulates were proposed. We have shown that this extensions are the natural counterpart of existing extensions to the AGM basic postulates.

6 Appendix: Proofs

LEMMA 3.2. The postulates for irrelevance relations (Irr₁)..(Irr₆) and (Irr₉) are satisfied if and only if the postulates for contraction (K⁻1)..(K⁻6) and (K⁻F) are satisfied.

Proof. The correspondence between the postulates (Irr₁)..(Irr₆) and (K⁻1)..(K⁻6) has been proved in [Falappa 99]. It rests to show that (Irr₉) is satisfied if the mentioned postulates for contraction are satisfied and that (K⁻F) is satisfied if the mentioned postulates for irrelevance are satisfied.

(Irr₉) Suppose $\beta \in \mathbf{K}$. We want to prove that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \beta$ or $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} (\alpha \rightarrow \neg \beta)$. If $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \beta$ we are done. If $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \beta$ is not the case, by Lemma 3.1, we have $\beta \notin \mathbf{K} - \neg \alpha$. Then, it follows from $\beta \in \mathbf{K}$, by (K⁻F) that $(\beta \rightarrow \neg \alpha) \in \mathbf{K} - \neg \alpha$. This is equivalent to $(\alpha \rightarrow \neg \beta) \in \mathbf{K} - \neg \alpha$, which by Lemma 3.1 is equivalent to $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} (\alpha \rightarrow \neg \beta)$. This concludes the proof.

(K⁻F) Assume $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} - \alpha$. We want to show that $(\beta \rightarrow \alpha) \in \mathbf{K} - \alpha$. It follows from $\beta \notin \mathbf{K} - \alpha$ by Lemma 3.1 that it is not the case that $\neg \alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \beta$. Then, by (Irr₉), $\neg \alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} (\neg \alpha \rightarrow \neg \beta)$ must be the case. It follows by (Irr₂) that $\neg \alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} (\beta \rightarrow \alpha)$, which by Lemma 3.1, is equivalent to $(\beta \rightarrow \alpha) \in \mathbf{K} - \alpha$. This finishes our proof. ■

LEMMA 3.3. The postulates for irrelevance relations (Irr₁)..(Irr₆) and (Irr₁₀) are satisfied if and only if the postulates for contraction (K⁻1)..(K⁻6) and (K⁻I) are satisfied.

Proof. The correspondence between the postulates (Irr₁)..(Irr₆) and (K⁻1)..(K⁻6) has been proved in [Falappa 99]. It rests to show that (Irr₁₀) is satisfied if the mentioned postulates for contraction are satisfied and that (K⁻I) is satisfied if the mentioned postulates for irrelevance are satisfied.

According to Lemma 3.1 postulates:

(Irr₁₀) $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \delta$ if and only if $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \delta$ and $\beta \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \delta$, and

(K⁻I) **Intersection:** For all α and β , $\mathbf{K} - (\alpha \wedge \beta) = \mathbf{K} - \alpha \cap \mathbf{K} - \beta$

are equivalent. This finishes the proof. ■

LEMMA 4.1. Definition (Def $\mathbf{K} * \alpha$) is equivalent to the following ones:

(Def' $\mathbf{K} * \alpha$) $\mathbf{K} * \alpha = \{\beta : \alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \beta \text{ or } \alpha \mathbb{R}_{\mathbf{K}}^{\mathbf{P}} \beta\}$.

(Def'' $\mathbf{K} * \alpha$) $\mathbf{K} * \alpha = \{\beta : \text{exists } \delta \text{ such that } \alpha \mathbb{I}_{\mathbf{K}}^{\mathbf{N}} \delta \text{ and } \{\alpha\} \cup \{\delta\} \vdash \beta\}$.

Proof. The equivalence of (Def $\mathbf{K}*\alpha$) and (Def' $\mathbf{K}*\alpha$) is a straight consequence of (Schema $\mathbb{R}_{\mathbf{K}}^*$). Let us prove the equivalence of (Def' $\mathbf{K}*\alpha$) and (Def'' $\mathbf{K}*\alpha$).

Let us begin by showing that (Def' $\mathbf{K}*\alpha$) implies (Def'' $\mathbf{K}*\alpha$). Assume there is some β such that $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ or $\alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta$. Let us show that there is some δ such that $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ and $\{\alpha\} \cup \{\delta\} \vdash \beta$. If $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ is the case, it is clear that by taking $\delta = \beta$ we obtain the desired result. If $\alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta$ is the case, we have by (Def $\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}$) that $\beta \in \mathbf{K}*\alpha$ and by Levi Identity we have $\beta \in (\mathbf{K}-\neg\alpha)+\alpha$. Then $\alpha \rightarrow \beta \in \mathbf{K}-\neg\alpha$, which by Lemma 3.1 is equivalent to $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\alpha \rightarrow \beta$. Since $\{\alpha\} \cup \{\alpha \rightarrow \beta\} \vdash \beta$, it is clear that by taking $\delta = \alpha \rightarrow \beta$ we obtain the desired result.

Let us prove now that that (Def'' $\mathbf{K}*\alpha$) implies (Def' $\mathbf{K}*\alpha$). Suppose that there is some β for which there exists some δ such that $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ and $\{\alpha\} \cup \{\delta\} \vdash \beta$. Let us prove that $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ or $\alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta$. It follows from $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ that

$$\delta \in \mathbf{K}-\neg\alpha. \quad (1)$$

It follows from $\{\alpha\} \cup \{\delta\} \vdash \beta$ and 1 that $\beta \in (\mathbf{K}-\neg\alpha)+\alpha$. Then, by Levi Identity we have $\beta \in \mathbf{K}*\alpha$ which has been shown to be equivalent to $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ or $\alpha\mathbb{R}_{\mathbf{K}}^{\mathbf{P}}\beta$. This concludes the proof. ■

LEMMA 4.2. The postulates for irrelevance relations (Irr_1)..(Irr_8) are satisfied if and only if the postulates for $\mathbb{R}_{\mathbf{K}}^*$ relations (Rel_1^*)..(Rel_8^*) are satisfied.

Proof. Suppose that the set of postulates for irrelevance are satisfied. We have to show that the set of postulates for $\mathbb{R}_{\mathbf{K}}^*$ relations are satisfied.

(Rel_1^*) Assume $\neg\alpha\mathbb{R}_{\mathbf{K}}^*\alpha$. We have to prove $\vdash \alpha$. It follows from $\neg\alpha\mathbb{R}_{\mathbf{K}}^*\alpha$ and Lemma 4.1 that there exists some β such that

$$\neg\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta, \quad (2)$$

and

$$\{\neg\alpha\} \cup \{\beta\} \vdash \alpha. \quad (3)$$

It follows from 3 that $\{\beta\} \vdash (\neg\alpha \rightarrow \alpha)$ which is equivalent to

$$\vdash (\beta \rightarrow \alpha). \quad (4)$$

It follows from 4 by (Irr_4) that

$$\neg\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}(\beta \rightarrow \alpha). \quad (5)$$

It follows from 2 and 5 that $\neg\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\alpha$. Then by (Irr_1) we can conclude $\vdash \alpha$ that is the desired result.

(Rel_2^*) Assume $\alpha\mathbb{R}_{\mathbf{K}}^*\beta$ and $\alpha\mathbb{R}_{\mathbf{K}}^*(\beta \rightarrow \delta)$. We want to prove $\alpha\mathbb{R}_{\mathbf{K}}^*\delta$. It follows from $\alpha\mathbb{R}_{\mathbf{K}}^*\beta$ that there exists ϕ such that

$$\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\phi, \quad (6)$$

and

$$\{\alpha\} \cup \{\phi\} \vdash \beta. \quad (7)$$

It follows from $\alpha\mathbb{R}_{\mathbf{K}}^*(\beta \rightarrow \delta)$ that there exists φ such that

$$\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\varphi, \quad (8)$$

and

$$\{\alpha\} \cup \{\varphi\} \vdash (\beta \rightarrow \delta). \quad (9)$$

By (Irr₄) we have $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}}(\varphi \rightarrow (\phi \rightarrow \varphi))$. Then it follows from 8, by (Irr₂) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}}(\phi \rightarrow \varphi)$. It follows from $\vdash ((\phi \rightarrow \varphi) \leftrightarrow (\phi \rightarrow (\phi \wedge \varphi)))$, by (Irr₃) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}}(\phi \rightarrow (\phi \wedge \varphi))$. Finally, from 6 by (Irr₂), we can conclude

$$\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}}(\phi \wedge \varphi) \quad (10)$$

From 7 and 9 we can obtain $\{\alpha\} \cup \{\phi \wedge \varphi\} \vdash \delta$ which, together with 10, by Lemma 4.1, leads to $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$. This concludes our proof.

(Rel₃^{*}) Suppose that $\vdash (\alpha \leftrightarrow \beta)$. We have to prove $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ if and only if $\beta \mathbb{R}_{\mathbf{K}}^* \delta$ for all $\delta \in \mathcal{L}$. The desired result follows directly by (Irr₃) and Lemma 4.1.

(Rel₄^{*}) We have to prove that if $\vdash \beta$ then $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$ for all $\alpha \in \mathcal{L}$. It follows from $\vdash \beta$ by (Irr₄) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \beta$ for all $\alpha \in \mathcal{L}$. Since $\{\alpha\} \cup \{\beta\} \vdash \beta$ we can conclude by Lemma 4.1 that $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$.

(Rel₅^{*}) This postulate states that if $\alpha \notin \mathbf{K}$ then $\neg \alpha \mathbb{R}_{\mathbf{K}}^* \beta$ for all $\beta \in \mathbf{K}$. From $\alpha \notin \mathbf{K}$ by (Irr₅) we can conclude $\neg \alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \beta$ for all $\beta \in \mathbf{K}$. Then, since $\{\alpha\} \cup \{\beta\} \vdash \beta$ we have, by Lemma 4.1, $\neg \alpha \mathbb{R}_{\mathbf{K}}^* \beta$ for all $\beta \in \mathbf{K}$ that is the desired result.

(Rel₆^{*}) Suppose $\beta \in \mathbf{K}$. We have to prove $\alpha \mathbb{R}_{\mathbf{K}}^*(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$. It follows from $\beta \in \mathbf{K}$ by (Irr₆) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}}(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$. Since $\{\alpha\} \cup \{\alpha \vee \beta\} \vdash (\alpha \vee \beta)$ we can conclude by Lemma 4.1 that $\alpha \mathbb{R}_{\mathbf{K}}^*(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$.

(Rel₇^{*}) Suppose that $\delta \in \mathbf{K}$, $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\beta \mathbb{R}_{\mathbf{K}}^* \delta$. We have to prove $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$. It follows from $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\delta \in \mathbf{K}$, by Lemma 4.1, that

$$\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta. \quad (11)$$

Similarly, it follows from $\beta \mathbb{R}_{\mathbf{K}}^* \delta$ and $\delta \in \mathbf{K}$, by Lemma 4.1, that

$$\beta \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta. \quad (12)$$

From 11 and 12, by (Irr₇) we can conclude $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$. Then, since $\{\alpha \vee \beta\} \cup \{\delta\} \vdash \delta$ we can conclude by Lemma 4.1, $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$ which is the desired result.

(Rel₈^{*}) We have to show that from $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$ we can conclude $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ or $\beta \mathbb{R}_{\mathbf{K}}^* \delta$. It follows from $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$, by Lemma 4.1 that there is some ϕ such that

$$(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \phi, \quad (13)$$

and

$$\{\alpha \vee \beta\} \cup \{\phi\} \vdash \delta. \quad (14)$$

It follows from 14 by (Irr₈) that

$$\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \phi \text{ or } \beta \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \phi. \quad (15)$$

It follows from 14 that

$$\{\alpha\} \cup \{\phi\} \vdash \delta \text{ and } \{\beta\} \cup \{\phi\} \vdash \delta. \quad (16)$$

It follows from 15 and 16, by Lemma 4.1 that $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ or $\beta \mathbb{R}_{\mathbf{K}}^* \delta$. This finishes the first part of our proof.

Now suppose that the set of postulates for $\mathbb{R}_{\mathbf{K}}^*$ relations are satisfied. We have to show that the set of postulates for irrelevance are satisfied.

(Irr₁) Assume $\neg\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\alpha$. We have to prove $\vdash \alpha$. It follows from $\neg\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\alpha$, by (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$), that $\alpha \in \mathbf{K}$ and $\neg\alpha\mathbb{R}_{\mathbf{K}}^*\alpha$. Then, by (Rel₁^{*}) we can conclude $\vdash \alpha$.

(Irr₂) Assume $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ and $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}(\beta \rightarrow \delta)$. We want to prove $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$. It follows from $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$, by (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$), that

$$\beta \in \mathbf{K} \text{ and } \alpha\mathbb{R}_{\mathbf{K}}^*\beta \quad (17)$$

It follows from $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}(\beta \rightarrow \delta)$, by (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$), that

$$(\beta \rightarrow \delta) \in K \text{ and } \alpha\mathbb{R}_{\mathbf{K}}^*(\beta \rightarrow \delta) \quad (18)$$

It follows from 17 and 18 by (Rel₂^{*})

$$\delta \in K \text{ and } \alpha\mathbb{R}_{\mathbf{K}}^*\delta,$$

which by (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$), are equivalent to $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$. This concludes the proof.

(Irr₃) Suppose that $\vdash (\alpha \leftrightarrow \beta)$. We have to prove $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ if and only if $\beta\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ for all $\delta \in \mathcal{L}$. The desired result follows directly by (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$), (Def $\mathbf{K}*\alpha$) and (Rel₃^{*}).

(Irr₄) We have to prove that if $\vdash \beta$ then $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ for all $\alpha \in \mathcal{L}$. By (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$), this is equivalent to prove that if $\vdash \beta$ then $\alpha\mathbb{R}_{\mathbf{K}}^*\beta$ and $\beta \in \mathbf{K}$ for all $\alpha \in \mathcal{L}$. Condition $\alpha\mathbb{R}_{\mathbf{K}}^*\beta$ follows from $\vdash \beta$ by (Rel₄^{*}), while $\beta \in \mathbf{K}$ follows trivially from $\vdash \beta$.

(Irr₅) This postulate states that if $\alpha \notin \mathbf{K}$ then $\neg\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\beta$ for all $\beta \in \mathbf{K}$. By (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$), proving this postulate is equivalent to prove that if $\alpha \notin \mathbf{K}$ then $\neg\alpha\mathbb{R}_{\mathbf{K}}^*\beta$ for all $\beta \in \mathbf{K}$. But this is exactly what (Rel₅^{*}) states.

(Irr₆) Suppose $\beta \in \mathbf{K}$. We have to prove $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$. If $\beta \in \mathbf{K}$ then, by (Rel₆^{*}), $\alpha\mathbb{R}_{\mathbf{K}}^*(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$ and $(\beta \vee \beta) \in \mathbf{K}$. It follows by (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$) that this result is equivalent to $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$, which is the desired result.

(Irr₇) Suppose that $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ and $\beta\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$. We have to prove $(\alpha \vee \beta)\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$. According to (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$) this is equivalent to prove that if $\delta \in \mathbf{K}$ and $\alpha\mathbb{R}_{\mathbf{K}}^*\delta$ and $\beta\mathbb{R}_{\mathbf{K}}^*\delta$ we can conclude $(\alpha \vee \beta)\mathbb{R}_{\mathbf{K}}^*\delta$. This holds by (Rel₇^{*}).

(Irr₈) We have to show that from $(\alpha \vee \beta)\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ we can conclude $\alpha\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$ or $\beta\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}\delta$. According to (Def $\mathbb{I}_{\mathbf{K}}^{\mathbf{N}}$) and (Def $\mathbf{K}*\alpha$) this is equivalent to prove that from $\delta \in \mathbf{K}$ and $(\alpha \vee \beta)\mathbb{R}_{\mathbf{K}}^*\delta$ we can conclude $\alpha\mathbb{R}_{\mathbf{K}}^*\delta$ or $\beta\mathbb{R}_{\mathbf{K}}^*\delta$. This result is valid by (Rel₈^{*}). The proof is complete. ■

THEOREM 4.1. The postulates for $\mathbb{R}_{\mathbf{K}}^*$ relations (Rel₁^{*})..(Rel₈^{*}) are satisfied if and only if the postulates for revision (K*1).. (\mathbf{K}^*8) are satisfied.

Proof. It follows from Lemma 4.2 that postulates (Rel₁^{*})..(Rel₈^{*}) are satisfied if and only if postulates (Irr₁)..(Irr₈) are satisfied. It follows from Theorem 3.1 that postulates (Irr₁)..(Irr₈) are satisfied if and only if postulates (K⁻1).. (\mathbf{K}^-8) are satisfied. Finally, in [Alchourrón 85] it is shown that postulates (K⁻1).. (\mathbf{K}^-8) are satisfied if and only if postulates (K*1).. (\mathbf{K}^*8) are satisfied. ■

LEMMA 4.3. The postulates for irrelevance relations $(\text{Irr}_1)..(\text{Irr}_6)$ and (Irr_9) are satisfied if and only if the postulates for $\mathbb{R}_{\mathbf{K}}^*$ $(\text{Rel}_1^*)..(\text{Rel}_6^*)$ and (Rel_9^*) are satisfied.

Proof. The correspondence between $(\text{Irr}_1)..(\text{Irr}_6)$ and $(\text{Rel}_1^*)..(\text{Rel}_6^*)$ has been proved in Lemma 4.2. It rests to show that (Irr_9) is satisfied if the mentioned postulates for $\mathbb{R}_{\mathbf{K}}^*$ are satisfied and that (Rel_9^*) is satisfied if the mentioned postulates for irrelevance are satisfied.

To show that (Irr_9) is valid assume $\beta \in \mathbf{K}$. We want to show $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \beta$ or $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} (\alpha \rightarrow \neg \beta)$. It follows from $\beta \in \mathbf{K}$ by (Rel_9^*) that $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$ or $\alpha \mathbb{R}_{\mathbf{K}}^* \neg \beta$. If $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$ is the case we can conclude by (Schema $\mathbb{R}_{\mathbf{K}}^*$) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \beta$ holds, and we are done. If $\alpha \mathbb{R}_{\mathbf{K}}^* \neg \beta$ is the case then it follows by (Def $\mathbb{R}_{\mathbf{K}}^*$) and (Def' $\mathbf{K} * \alpha$) that there exists δ such that

$$\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta, \quad (19)$$

and $\{\alpha\} \cup \{\delta\} \vdash \neg \beta$. The last relation is equivalent to $\vdash (\delta \rightarrow (\alpha \rightarrow \neg \beta))$. Then, it follows by (Irr_4) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} (\delta \rightarrow (\alpha \rightarrow \neg \beta))$ and then from 19 by (Irr_2) we conclude $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} (\alpha \rightarrow \neg \beta)$ that is the desired result.

To show that (Rel_9^*) follows from the postulates of irrelevance, assume $\beta \in \mathbf{K}$. Let us show that either $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$ or $\alpha \mathbb{R}_{\mathbf{K}}^* \neg \beta$ holds. It follows from $\beta \in \mathbf{K}$ by (Irr_9) that either $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \beta$ or $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} (\alpha \rightarrow \neg \beta)$ holds. Suppose $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \beta$ is the case. Then, it follows by (Def $\mathbb{R}_{\mathbf{K}}^*$) and (Def' $\mathbf{K} * \alpha$) that $\alpha \mathbb{R}_{\mathbf{K}}^* \beta$ holds. If $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} (\alpha \rightarrow \neg \beta)$ is the case, by (Def $\mathbb{R}_{\mathbf{K}}^*$) and (Def' $\mathbf{K} * \alpha$), we have

$$\alpha \mathbb{R}_{\mathbf{K}}^* (\alpha \rightarrow \neg \beta). \quad (20)$$

It follows by (Def $\mathbb{R}_{\mathbf{K}}^*$) and Levi Identity that $\alpha \mathbb{R}_{\mathbf{K}}^* \alpha$. Then from 20, by (Rel_2^*) we can conclude $\alpha \mathbb{R}_{\mathbf{K}}^* \neg \beta$, that is the desired result. ■

LEMMA 4.4. The postulates for irrelevance relations $(\text{Irr}_1)..(\text{Irr}_6)$ and (Irr_{10}) are satisfied if and only if the postulates for $\mathbb{R}_{\mathbf{K}}^*$ $(\text{Rel}_1^*)..(\text{Rel}_6^*)$ and (Rel_{10}^*) are satisfied.

Proof. The correspondence between $(\text{Irr}_1)..(\text{Irr}_6)$ and $(\text{Rel}_1^*)..(\text{Rel}_6^*)$ has been proved in Lemma 4.2. It rests to show that (Irr_{10}) is satisfied if the mentioned postulates for $\mathbb{R}_{\mathbf{K}}^*$ are satisfied and that (Rel_{10}^*) is satisfied if the mentioned postulates for irrelevance are satisfied.

In order to show that (Irr_{10}) holds we have to show:

Part 1: From $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ we can conclude $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ and $\beta \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$.

Part 2: From $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ and $\beta \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ we can conclude $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$.

Part 2 has already been shown to be valid in Lemma 4.2 $((\text{Irr}_7))$. To prove part 1, suppose $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$, then it follows from (Def $\mathbb{I}_{\mathbf{K}}^{\mathbb{N}}$), (Def $\mathbb{R}_{\mathbf{K}}^*$) and (Def' $\mathbf{K} * \alpha$) that $\delta \in \mathbf{K}$ and $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$ holds. Then, it follows from (Rel_{10}^*) that $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\beta \mathbb{R}_{\mathbf{K}}^* \delta$ hold. Hence, since $\delta \in \mathbf{K}$, we can conclude by (Schema $\mathbb{R}_{\mathbf{K}}^*$) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ and $\beta \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ are valid.

To show that (Rel_{10}^*) follows from the postulates of irrelevance we have to show:

Part 1: From $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$ we can conclude $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\beta \mathbb{R}_{\mathbf{K}}^* \delta$.

Part 2: From $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\beta \mathbb{R}_{\mathbf{K}}^* \delta$ we can conclude $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$.

Part 2 has already been shown to be valid in Lemma 4.2 $((\text{Rel}_7^*))$. To prove part 1, suppose $\delta \in \mathbf{K}$ and $(\alpha \vee \beta) \mathbb{R}_{\mathbf{K}}^* \delta$, then it follows from (Schema $\mathbb{R}_{\mathbf{K}}^*$) that $(\alpha \vee \beta) \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ holds. Then, it follows from (Irr_{10}) that $\alpha \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ and $\beta \mathbb{I}_{\mathbf{K}}^{\mathbb{N}} \delta$ hold. Hence, since $\delta \in \mathbf{K}$, we can conclude by (Schema $\mathbb{R}_{\mathbf{K}}^*$) that $\alpha \mathbb{R}_{\mathbf{K}}^* \delta$ and $\beta \mathbb{R}_{\mathbf{K}}^* \delta$ are valid. ■

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