View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by El Servicio de Difusión de la Creación Intelectual

On the Characterization of the Source-to-all-terminal Diameter-constrained reliability domination

Héctor Cancela
Instituto de Computación
Facultad de Ingeniería
Universidad de la República
J. Herrera y Reissig 565
Montevideo, Uruguay
cancela@fing.edu.uy

Louis Petingi
Computer Science Department
College of Staten Island
City University of New York
2800 Victory Boulevard, 1N
Staten Island, N. Y. 10314
USA
petingi@postbox.csi.cuny.edu

Abstract

Let $G = (V, E)$ be a digraph with a distinguished set of terminal vertices $K \subseteq V$ and a vertex $s \in K$. We define the s, K-diameter of G as the maximum distance between s and any of vertices of K. If the arcs fail randomly and independently with known probabilities (vertices are always operational), the *Diameter-constrained* s, K*-terminal reliability* of G, $R_{s,K}(G, D)$, is defined as the probability that surviving arcs span a subgraph whose s, K-diameter does not exceed D [5, 11].

A graph invariant called the domination of a graph G was introduced by Satyanarayana and Prabhakar [13] to generate the non-canceling terms of the classical reliability expression, $R_{s,K}(G)$, based on the same reliability model (i.e. arcs fail randomly and independently and where nodes are perfect), and defined as the probability that the surviving arcs span a subgraph of G with unconstrained finite s, K-diameter. This result allowed the generation of rapid algorithms for the computation of $R_{s,K}(G)$.

In this paper we present a characterization of the diameter-constrained s, K-terminal reliability domination of a digraph $G = (V, E)$ with terminal set $K = V$, and for any diameter bound D, and, as a result, we solve the classical reliability domination, as a specific case. Moreover we also present a rapid algorithm for the evaluation of $R_{s,V}(G, D)$.
KEYWORDS: Reliability, networks, diameter, domination.

1 Introduction

The components of a communication network (e.g. nodes, communication links) may be subject to random failures. Failures may arise from natural catastrophes (e.g. hurricanes), component wearout, or action of intentional enemies.

A communication network can be modeled by a graph (digraph) $G = (V, E)$ where V and E are the set of vertices and edges (arcs) respectively of G. Moreover the probabilities of failure of the network components could be represented by assigning probabilities of failure to the vertices and/or edges (arcs) of its underlying graph (digraph). In this paper we will be only concerned with digraphs without self-loops.

A widely used probabilistic model is the one where the edges (arcs) fail randomly and independently with known probabilities, and where the vertices are always operational; and from this point on, when we mention a probabilistic graph (digraph), we will refer to a graph (digraph) using this model.

Let $G = (V, E)$ be a digraph with a distinguished set $K \subseteq V$, and vertex $s \in K$. For vertices $s, v \in V$ let a s, v-dipath P of G be represented by a sequence of distinct vertices $< s = u_1, u_2, \dots, u_{r+1} = v >$, where $(u_j, u_{j+1}) \in E$, for $1 \leq j \leq r$. Furthermore let the length $L(P)$ of P be r. We also represent a directed cycle (dicycle) as a sequence of vertices $< u_1, u_2, \dots, u_{r+1} = u_1 >$, where $(u_j, u_{j+1}) \in E$, and where all the vertices are distinct with the exception of the first and last vertices of the sequence. The distance from vertex s to vertex u in G, denoted as $dist_G(s, u)$, is the length of the shortest s, u-dipath in G, and the s, K-diameter the maximum distance from s to any vertex $u \in K$.

Let $G = (V, E)$ be a probabilistic digraph, with terminal vertex set K, vertex $s \in K$, and diameter bound D. The Diameter-constrained s, K-terminal reliability is defined as the probability that surviving arcs span a subgraph of G whose s, K-diameter does not exceed D, or equivalently, as the probability that for each vertex $u \in K$, there exists an operating s, u-dipath from s to u of at most D arcs. This reliability measure subsumes the classical reliability $R_{s,K}(G)$ of a probabilistic digraph G, known as the Source-to-K-terminal reliability, defined on the same probabilistic model (arcs fail independently with known probabilities, and vertices are always operational); $R_{s,K}(G)$ is the probability that the surviving arcs span a subgraph where there exists an operational s, u-dipath between s and u, $u \in K$. By noting that the longest s, u-dipath in G is of at most $n - 1$ arcs, where n is the number of nodes of G, then $R_{s,K}(G)$ is equal to $R_{s,K}(G, D)$ for $D = n - 1$.

One particular application of the diameter-constrained reliability measure is when in a directed network (modeled by a digraph $G = (V, E)$), the arcs fail randomly and independently, and the transmissions between a root-vertex s and any vertex in a distinguished subset K are required to experience a maximum delay DT (where T is the delay experienced at single node or edge); then the probability that after random failures of the communication arcs, the surviving network meets the maximum delay requirement is precisely $R_{s,K}(G, D)$.

That is the case in Multicasting-routing with end-to-end delay constraints, where a source node must broadcast messages to a set of destination nodes in a network (e.g. teleconference). This problem can be modeled as a digraph with a source node s, a set M of destination nodes, and where each arc is assigned a weight corresponding to the delay to be experienced by a packet traveling along this arc.

Extensive research has been done in this area (see [4, 9, 12, 14]) in order to construct Steiner trees containing the source s and the destination nodes (i.e. terminal nodes), in such a way that a packet traveling from the source to a terminal node meets the delay constraints. To our knowledge none of these studies take into account the operational probability of the network components, thus the $R_{s,K}(G, D)$ measure may be applied to determine the suitability of a network to meet end-to-end delay constraints.

A graph invariant called reliability domination of a graph G was originally introduced by Satyanarayana and Prabhakar [13] for the classical reliability, which has since been explored by several researchers in reliability theory [1, 2, 3, 7, 8]. The reliability domination plays an important role, allowing to efficiently implement the principle of Inclusion-Exclusion of probability theory applied to the evaluation of the reliability. This result was further studied and generalized for more general reliability systems.

In Section 2 we introduce some basic notation and definitions that will be used in the following sections. In Section 3 we present a characterization of the Source-to-all-terminal diameter-constrained reliability domination. In Section 4 we present a rapid algorithm for the computation of $R_{s,V}(G,D)$. Finally in Section 5 we present some conclusions and open problems.

The notation in this paper follows that of Harary [6], unless otherwise noted.

2 Preliminaries

Consider a digraph $G = (V, E)$, with terminal set $K \subseteq V$ and vertex $s \in K$. The indegree of a vertex $u \in V$, denoted as $ind_G(u)$, is the number of arcs directed into u, and the outdegree, $out_G(u)$, is the number of arcs directed outward. A tree T of a digraph G is a connected subgraph with no cycles, independently of the direction of the arcs. A rooted tree $T = (V', E')$ of a digraph G, rooted at $s \in V'$, is a tree of G with $ind_T(s) = 0$ and $ind_T(u) = 1$, for any $u \in V' - \{s\}$. A K-tree $T = (V', E')$ of a digraph G is a rooted tree, rooted at s, covering all the vertices of $K \in V'$, such that any pendant vertex u (i.e. $out_T(u) = 0$) of T must belong to K. In addition, a K-tree whose s, K-diameter is at most D, will be called a D, K-tree. From this point on, when refering to a D, K-tree, we will assumed that it is rooted at s.

Before looking at the Diameter-constrained reliability of a digraph, we need additional definitions and notation.

(i). Let $G = (V, E, \mathcal{P}(E))$ be a probabilistic digraph with a distinguished set $K \subseteq V$, vertex $s \in K$, and $D \in Z^+$, with $1 \leq D \leq n-1$, where $n = |V|$, and where $\mathcal{P} : E \mapsto [0,1]$ are the operational probabilities of the arcs in set E. For ease of notation, we represent the operational probability of an arc $x \in E$ as $p(x) = 1 - q(x)$ ($q(x)$ is the probability of failure).

(ii). Let the sample space Ω represent the set of all possible subsets of E, corresponding to sets of operational arcs (i.e. $\Omega = 2^E$).

(iii). Under the assumption of independent arc failures, each $H \in \Omega$ has occurrence probability

$$P(H) = \prod_{x \in H} p(x) \prod_{x \notin H} q(x).$$

(iv). $H \in \Omega$ is a *pathset* or *operating state* if H spans a subgraph whose s, K-diameter is at most D.

(v). Let $\mathcal{O}_K^D(E) = \{H \in \Omega : H \text{ is a pathset }\}$.

(vi). An operating state H of $\mathcal{O}_K^D(E)$ is called a *minpath* if $H - \{x_i\} \notin \mathcal{O}_K^D(E)$, for all $x_i \in H$.

From the definition of $R_{s,K}(G,D)$ and definition (v) one gets

$$R_{s,K}(G,D) = Pr\left(\bigcup_{H \in \mathcal{O}_K^D(E)} H\right) = \sum_{H \in \mathcal{O}_K^D(E)} \prod_{x \in H} p(x) \prod_{x \notin H} q(x). \tag{1}$$

It is clear that a superset H' of a pathset H is also a pathset, by noting that adding arcs to a state does not increase its s, K-diameter. A consequence of this fact is that any operating state must contain a minpath (see definition (vi)), thus $R_{s,K}(G,D)$ is the probability that all the arcs of at least one minpath are operating.

For a digraph $G = (V, E)$, terminal set K, and vertex $s \in K$, let $\mathcal{M} = \{M_1, M_2, \dots, M_l\}$ be the set of minpaths of $\mathcal{O}_K^D(E)$. Define E_i to be the event that all the arcs of M_i operate. By Inclusion-Exclusion we obtain

$$\begin{aligned} R_{s,K}(G,D) &= Pr\left(\bigcup_{i=1}^{l} E_i\right) = \sum_i Pr(E_i) - \sum_{i<j} Pr(E_iE_j) + \dots \\ &\quad +(-1)^{l+1} Pr(E_1E_2 \dots E_l). \end{aligned} \tag{2}$$

where the event $E_iE_j \dots E_m$ is the event that all the arcs of the subgraph obtained by the union of $M_i, M_j, \dots, M_m$ are operating.

An immediate consequence of Equation 2 is that if an arc x of a graph G does not belong to any minpath of $\mathcal{M}$, then $R_K(G, D) = R_K(G - x, D)$.

The next two lemmas give a characterization of the minpaths M of $\mathcal{O}_K^D(E)$; due to space limitations, we do not include the proofs (which are left for the reader).

Lemma 1 *For a digraph $G = (V, E)$, with terminal set K, vertex $s \in K$, and bound D, if M is a minpath of G then every vertex u (terminal or non-terminal) of M can be reached from s by a s, u-dipath.*

Lemma 2 *For a digraph $G = (V, E)$, terminal set K, vertex $s \in K$, and a bound D, then M is a minpath of G if and only if it is a D, K-tree.*

A digraph $G = (V, E)$ with $K \subseteq V$, $s \in K$, and bound D is called a D, K-digraph, if every arc of G lies in some D, K-tree. Let $\mathcal{F}_{D,K}(G)$ be the collection of D, K-trees of G. A *formation* F of G is a collection of D, K-trees of G whose union constitutes the set of arcs E of G. A formation is *odd* or *even* depending on whether F contains an odd or even number of trees, respectively. The *signed domination* of a digraph $G = (V, E)$, denoted $d(E, \mathcal{F}_{D,K}(G))$, with respect to a given subset $K \in V$, $s \in K$, and bound D, is the number of odd minus the even number of formations of G. In Equation (2), we are only concerned with subgraphs obtained by the union of minpaths. As we showed previously, for the Diameter-constrained K-terminal reliability of a digraph G, with terminal set K, vertex $s \in K$, and diameter bound D, the minpaths are D, K-trees, and the subgraphs are D, K-digraphs. The same D, K-digraph can be obtained from different formations; this means that it may appear in Equation (2) more than once, sometimes with positive sign, and others with negative sign, depending if the corresponding formation has an odd or or an even number of D, K-trees. Thus using these facts and the above definitions, we can rewrite Equation (2) as

$$R_{s,K}(G,D) = \sum_{H \in \mathcal{H}} d(E(H), \mathcal{F}_{D,K}(H)) Pr(H) \tag{3}$$

where $\mathcal{H}$ is the class of all D, K-digraphs of G, $V(H)$ and $E(H)$ are the node set and arc set of H respectively, and $Pr(H)$ is the probability that the arcs of H are operative.

The notion of domination was introduced in [13] in the classical reliability context for the specific case $K = \{s, t\}$, and was further studied for general systems. Let E be a finite set, and $P(E)$ be the power set of E. A nonempty subset $\mathcal{C} \subseteq P(E)$ is called a *clutter* of E if for any two elements $C_1, C_2 \in \mathcal{C}$, whenever $C_1 \subseteq C_2$, then $C_1 = C_2$. A pair $(E, \mathcal{C})$ will be referred to as a *system* and a system is *coherent* if each element of E is contained in some element of $\mathcal{C}$. The signed domination of the system $(E, \mathcal{C})$, denoted $d(E, \mathcal{C})$, is also defined as the number of odd formations minus the number of even formations of E, where a formation is a collection of elements of $\mathcal{C}$ whose union yield E. A non-coherent system has no formations, so its signed domination is 0.

The clutters associated with the success and failure of a specific element $x \in E$ are defined as follows: Let $\mathcal{C} - x = \{C - x : C \in \mathcal{C}\}$ and $\mathcal{C}_{-x} = \{C \in \mathcal{C} : x \notin C\}$. Now $\mathcal{C}_{-x}$ is clearly a clutter but $\mathcal{C} - x$ may not be one. We define $\mathcal{C}_{+x}$ to be the collection of elements of $\mathcal{C} - x$ which are not proper supersets of some element of $\mathcal{C} - x$. For an element $x \in E$, $\mathcal{C}_{-x}$ and $\mathcal{C}_{+x}$ are called the minors with respect to x of $\mathcal{C}$. Huseby [7, 8] showed the following result:

Theorem 1 *If $(E, \mathcal{C})$ is a system, with $x \in E$, and minors $\mathcal{C}_{-x}$ and $\mathcal{C}_{+x}$ of $\mathcal{C}$, then $d(E, \mathcal{C}) = d(E - \{x\}, \mathcal{C}_{+x}) - d(E - \{x\}, \mathcal{C}_{-x})$.*

For the case of the diameter-constrained reliability, for a digraph $G = (V, E)$, with vertex $s \in K$, $\mathcal{F}_{D,K}(G)$ is the set of clutter elements, and we denominate this set using the standard notation $\mathcal{C}$; also we refer to $d(E, \mathcal{F}_{D,K}(G))$ as $d_{D,K}(G)$. In addition, let x be an arc of G, then T is a D, K-tree of G such that x is not an arc of T iff T is a D, K-tree of $G - x$. Thus $d_{D,K}(G - x) = d(E - \{x\}, \mathcal{C}_{-x})$.

In the next section we characterize the Source-to-all-terminal diameter-constrained reliability domination.

3 Characterization of the Source-to-all-terminal diameter-constrained reliability domination

In this section we are concerned with the characterization of Source-to-all-terminal diameter-constrained reliability domination of a probabilistic digraph $G = (V, E)$ with source $s \in V$ and when $K = V$, for a bound D, $1 \leq D \leq |V| - 1$. It is noted that a V-tree is in this case a rooted spanning tree, rooted at s, and a D, V-tree is a rooted spanning tree, rooted at s, whose s, V-diameter is at most D.

For a digraph $G = (V, E)$, with terminal set $K = V$, and distinguished vertex s, we say that G is *s, V-connected* if there exists in G a s, u-dipath for every $u \in V$. If G has $ind_G(s) = 0$, we will denominate this graph s-rooted, and from this point on we will be only concerned with s-rooted digraphs, since if that is not the case, then $d_{D,V}(G) = 0$, as stated in the following claim:

Claim 1 *Suppose that $G = (V, E)$ is a digraph with terminal set $K = V$, and vertex s. If $ind_G(s) > 0$ then $d_{D,K}(G) = 0$.*

In preparation for the main results in this chapter, we define the following operations in a digraph $G = (V, E)$, with distinguished vertex s.

- $\mathcal{SP}(G)$ If there exists a vertex $u \in V - \{s\}$, such that $ind_G(u) > 1$ and $dist_G(s, u) \leq dist_G(s, v)$ for $v \in V - s$, with $ind_G(v) > 1$, this operation returns a s, u-dipath $P_{s,u} =< s = u_1, u_2, \ldots, u_{i-1}, u_i = u >$ of length $L(P_{s,u}) = dist_G(s, u)$; if there is no such vertex, this operation returns the empty set, $\emptyset$.

- $\mathcal{LP}(G)$. If G is s, V connected, this operation returns the length of the longest dipath from s to any vertex $u \in V$; otherwise it returns ∞.

The following lemma plays an important role:

Lemma 3 *Let $G = (V, E)$ be a s-rooted digraph with terminal set $K = V$. Suppose that $P_{s,u} =< s = u_1, u_2, \dots, u_{i-1}, u_i = u >$ is the dipath returned by operation $\mathcal{SP}(G)$, and let $x = (u_{i-1}, u)$, then $d_{D,V}(G) = -d_{D,V}(G - x)$.*

Proof. Let $x' \neq x$ be an arc directed into u, and suppose that $T' = (V, E')$ is a D, V-tree of G such that x' is an arc of T'. Considering Theorem 1, we must show that $d(E - \{x\}, \mathcal{C}_{+x}) = 0$, or equivalently that the system $(E - \{x\}, \mathcal{C}_{+x})$ is not coherent. Consider the following two cases:

1. Suppose that $i - 1 \geq 2$. Since $dist_G(s, u)$ is the smallest possible distance between s and a node v of G for which $ind_G(v) > 1$, then $ind_G(u_j) = 1$, for $2 \leq j \leq i - 1$. Thus every V-tree (not necessarily of s, V-diameter less or equal D) must contain the path $< s = u_1, u_2, \dots, u_{i-1} >$, and $T = T' - x' + x$ is a V-tree of G. Moreover since $P_{s,u}$ is one of the shortest dipaths between s and u in G, then T is a D, V-tree.

2. Suppose $s = u_1 = u_{i-1}$, then clearly $T = T' - x' + x$ is also a D, V-tree of G.

Thus from 1. and 2. one gets $T - x = T' - x' \in \mathcal{C} - x$. Also $T' \in \mathcal{C} - x$, but $T' - x' \subseteq T'$ thus $T' \notin \mathcal{C}_{+x}$. Therefore we conclude that no elements of $\mathcal{C}_{+x}$ contains x', and the system $(E - x, \mathcal{C}_{+x})$ is not coherent. *QED*

We will also characterize s-rooted digraphs for which $\mathcal{SP}(G)$ does not return a (s, u)-dipath (i.e. $\mathcal{SP}(G)$ returns $\emptyset$).

Lemma 4 *Let $G = (V, E)$ be a s-rooted digraph with $K = V$, and $e = |E| \geq n = |V|$. If $\mathcal{SP}(G)$ returns $\emptyset$, then G is not s, V-connected.*

Proof. Suppose that $\mathcal{SP}(G) = \emptyset$ and G is s, V-connected, then there exists a (s, v)-dipath $\forall v \in V$. Since $\mathcal{SP}(G)$ returns $\emptyset$, it must be the case that $ind_G(v) = 1, \forall v \in V$, otherwise $\mathcal{SP}(G)$ will return a dipath. Thus there exists an unique (s, v)-dipath for $v \in V$. Therefore G is a s-rooted spanning tree and $e = n - 1$, a contradiction. *QED*

The following lemma is obtained from the contraposite of Lemma 4.

Lemma 5 *Let $G = (V, E)$ be a s-rooted digraph, with $K = V$, and $e = |E| \geq n = |V|$. If G is s, V-connected then $\mathcal{SP}(G)$ returns a dipath.*

It is also noted that among the digraphs G such that $d_{D,V}(G) = 0$ we can include the following:

Claim 2 *If G is not s, V-connected then $d_{D,V}(G) = 0$.*

Before introducing the following lemma, we say that a digraph G is *cyclic* if G contains a directed cycle (dicycle), and G is *acyclic* if it does not contain one.

Lemma 6 *Suppose that $G = (V, E)$ is a s-rooted cyclic digraph.*
Suppose that $P_{s,u} =< s = u_1, u_2, \dots, u_{i-1}, u_i = u >$ is the dipath returned from operation $\mathcal{SP}(G)$, and let $x = (u_{i-1}, u)$, then $G - x$ is also cyclic.

Proof. Let $\mathcal{U} = \{s = u_1, u_2, \dots, u_{i-1}\}$. Since $ind_G(v) = 1$, for $v \in \mathcal{U} - \{s\}$, then the vertex u_{i-1} can only be reached by a dipath in G from a vertex in $\mathcal{U}$. If $x = (u_{i-1}, u)$ belongs to a dicycle in G, this dicycle must be of the form $< u_{i-1}, u, v_1, v_2, \dots, v_j = u_{i-1} >$, which implies that either $ind_G(s) > 0$ or some vertex v in $U - \{s\}$ must have $ind_G(v) > 1$; a contradiction. Therefore $G - x$ is also cyclic. *QED*

Lemma 7 *Suppose that $G = (V, E)$ is a s-rooted, s, V-connected acyclic digraph. Suppose that $P_{s,u} =< s = u_1, u_2, \ldots, u_{i-1}, u_i = u >$ is the dipath retuned from operation $\mathcal{SP}(G)$, and let $x = (u_{i-1}, u)$, then*

(a) $G - x$ is also s rooted, s, V-connected and acyclic, and

(b) $\mathcal{LP}(G) = \mathcal{LP}(G - x)$.

Proof. We first note that $\mathcal{LP}(G)$ takes a finite value because we are assuming that G is s, V-connected.
To prove (a) consider an arc $x' \neq x$, such that $x' = (w, u)$. If $G - x$ is not s, V-connected, then there is not a (s, u)-dipath in $G - x$. Also there is not a (s, w)-dipath in $G - x$, otherwise there would exist a dipath between s and u. Thus any (s, w)-dipath in G must be of the form $< s = u_1, u_2, \ldots, u_{i-1}, u, v_1, \ldots, w >$, that is, this path must include the path $P_{s,u}$, because the arc x can be only be reached using this path. Since (w, u) is an arc then $< u, v_1, \ldots, w, u >$ forms a dicycle in G; a contradiction since G is acyclic.

To show (b), let $P_{s,t}$ be a dipath in G of length $\mathcal{LP}(G)$ and let $x' \neq x$ be an arc such that $x' = (w, u)$. If x is not an arc of the path $P_{s,t}$, then $\mathcal{LP}(G) = \mathcal{LP}(G - x)$, since $G - x$ contains $P_{s,t}$.
Next suppose that x is contained in every dipath of G of length $\mathcal{LP}(G)$. Let $P_{s,t} =< s = u, u_1, \ldots, u_{i-1}, u_i = u, v_1, \ldots, v_k = t >$ (i.e. this dipath must contain the path $P_{s,u}$, because the arc x can only be reached using this path) be a dipath of length $\mathcal{LP}(G)$. Also let $P_{s,w} =< s = w_1, w_2, \ldots, w_r = w >$ be a s, w-dipath in G. We first note that u is not a vertex of the dipath $P_{s,w}$, otherwise G will contain a dicycle (since $x' = (w, u)$ is an arc), thus $w \neq t$. Suppose that in G the set of vertices of the dipath $P_{s,w}$, $\{s = w_1, w_2, \ldots, w_r = w\}$ intercepts the subset of vertices of the dipath $P_{s,t}$, $\{u, v_1, \ldots, v_k = t\}$. Let j be the largest index in the set $\{s = w_1, w_2, \ldots, w_r = w\}$ such that w_j is also a vertex v_m of the subset $\{u, v_1, \ldots, v_k = t\}$ (i.e. the last vertex of $P_{s,w}$ to intercept the dipath $P_{s,t}$). But then $< w_j, w_{j+1}, \ldots, w, u, v_1, \ldots, w_j = v_m >$ is a dicycle, a contradiction since G is acyclic. Thus $\{s = w_1, w_2, \ldots, w_r = w\} \cap \{u, v_1, \ldots, v_k = t\} = \emptyset$. Therefore $P =< s = w_1, w_2, \ldots, w_r = w, u, v_1, \ldots, v_k = t >$ is a dipath of G not containing the arc x. Let $P' =< s = w_1, w_2, \ldots, w_k = w, u >$ be a dipath of G. But $L(P_{s,u}) \leq L(P')$ (since $P_{s,u}$ is the shortest path between s and u), thus $L(P_{s,t}) \leq L(P)$. This contradicts the hypothesis that every dipath of length $\mathcal{LP}(G)$ must contain the arc x. Thus there exists a dipath of G of length $\mathcal{LP}(G)$ not containing x, allowing us to conclude that $\mathcal{LP}(G) = \mathcal{LP}(G - x)$. *QED*

We are ready to show the main results:

Theorem 2 *Let $G = (V, E)$ be a s-rooted cyclic digraph with terminal set $K = V$, $e = |E|$ arcs, $n = |V|$ vertices, $n > 2$, and let D be the diameter bound. Then $d_{D,V}(G) = 0$.*

Proof. If G is rooted at s, and $n <= 2$, then G is not cyclic; we will consider all s-rooted cyclic digraphs with $n > 2$ vertices. We proceed by induction on $e = |E|$.
Basis. Let $e = n - 1$. In this case since G is cyclic, then it is not s, V-connected, because the only s, V-connected digraph with this number of arcs is a s-rooted spanning tree. Thus from Claim 2, $d_{D,V}(G) = 0$.
Inductive Step. Suppose the hypothesis is true for digraphs with n vertices and $e = m - 1$ arcs, with $m - 1 \geq n - 1$.
Suppose G is a s-rooted digraph with n vertices and $e = m > n - 1$ arcs.
If $\mathcal{SP}(G)$ does not return a dipath, then by Lemma 4, G is not s, V-connected, thus from

Claim 2, one gets $d_{D,V}(G) = 0$.
Next suppose that $\mathcal{SP}(G)$ returns a dipath $P_{s,u}$. By Lemma 3, $d_{D,V}(G) = -d_{D,V}(G - x)$ where $x = (u_{i-1}, u)$ is the last arc of the dipath $P_{s,u}$. But from Lemma 6, $G - x$ is also cyclic. Moreover $G-x$ has n vertices and $m-1$ arcs, thus from the inductive step, $G-x$ has domination $d_{D,V}(G - x) = 0$, allowing us to conclude that $d_{D,V}(G) = 0$. *QED*

Theorem 3 *Let $G = (V, E)$ be a s-rooted, s, V-connected acyclic digraph with terminal set $K = V$, $e = |E|$ arcs, $n = |V|$ vertices, and let D be the diameter bound, then*

$$d_{D,V}(G) = \begin{cases} (-1)^{e-n+1} & : \quad \mathcal{LP}(G) \leq D \\ 0 & : \quad \textit{otherwise} \end{cases}$$

Proof. We will consider all s-rooted, s, V-connected acyclic digraphs with n vertices. We proceed by induction on $e = |E|$.
Basis. Let $e = n - 1$. In this case since G is acyclic, the only s-rooted, s, V-connected digraph with this number of arcs is a s-rooted spanning tree. Thus if $\mathcal{LP}(G) \leq D$, then G is a D, V-tree, thus $d_{D,V}(G) = 1$. If $\mathcal{LP}(G) > D$, then G is not a D, V-tree, thus $d_{D,V} = 0$.
Inductive Step. Suppose the hypothesis is true for digraphs with n vertices and $e = m-1$ arcs, with $m - 1 \geq n - 1$.
Suppose G is a s-rooted, s, V-connected acyclic digraph with n vertices and $e = m > n - 1$ arcs. But from Lemma 5, $\mathcal{SP}(G) \neq \emptyset$ (i.e. there will be a $P_{s,u}$ dipath returned by operation $\mathcal{SP}(G)$). By Lemma 3, $d_{D,V}(G) = -d_{D,V}(G - x)$ where $x = (u_{i-1}, u)$ is the last arc of the dipath $P_{s,u}$. Moreover from Lemma 7, $G - x$ is also s, V-connected acyclic, and the longest dipath has length $\mathcal{LP}(G - x) = \mathcal{LP}(G)$. Moreover $G - x$ has n vertices and $m - 1$ arcs. From the inductive step, if $\mathcal{LP}(G - x) = \mathcal{LP}(G) > D$, $G - x$ has domination $d_{D,V}(G - x) = 0$, therefore $d_{D,V}(G) = 0$.
If $\mathcal{LP}(G-x) = \mathcal{LP}(G) \leq D$, $d_{D,V}(G-x) = (-1)^{(e-1)-n+1}$, and since $d_{D,V}(G) = -d_{D,V}(G-x)$, then we conclude that $d_{D,V}(G) = (-1)^{e-n+1}$. *QED*

By employing the results of Theorem 2 and Theorem 3 in Equation (3), there follows a very significant simplification of the D, V-subgraphs of a digraph G that will contribute to the reliability; we should only be concerned with s-rooted, s, V-connected acyclic digraphs of the original digraph G, whose longest s, u-dipath length do not exceed D. In the next section we present an algorithm that generates these digraphs.

4 The algorithm

This section gives an algorithm for efficiently generating all s-rooted, s, V-connected acyclic D, V-digraphs of an original digraph G, where the length of their longest s, u-dipaths do not exceed D.

As a first step, we assume that G is s-rooted. If this is not the case we can simply delete any arc directed into s, obtaining a s-rooted digraph.

The algorithm has four stages.

(a) Determine if G is s, V connected. If G is not s, V-connected, then we do not generate any subgraphs from G.

(b) If G contains a dicycle, generate acyclic subgraphs of G.

(c) If G is acyclic, determine if $\mathcal{LP}(G) > D$. If that is the case, generate all possible acyclic subgraphs G' of G such that $\mathcal{LP}(G') \leq D$.

(d) If G is acyclic and $\mathcal{LP}(G) \leq D$, then generate all possible subgraphs of G.

Generation of duplicate subgraphs at all stages is completely avoided by a simple check. The algorithm grows a rooted directed tree with the following properties:

1. Vertices represent nonempty subgraphs of G, the root vertex being G itself. Any vertex, say k, corresponds one-to-one with the subgraph G_k which is of one of the following four types: a) No s, V-connected, b) s, V-connected and cyclic, c) s, V-connected, acyclic, and $\mathcal{LP}(G) > D$, d) s, V-connected, acyclic, and $\mathcal{LP}(G) \leq D$.

2. A arc directed from vertex i to vertex j of the tree is labeled with the arc deleted from G_i to obtain G_j.

Additional Definitions (directed tree generation):
Father (Child): Vertex $i(j)$ is the father (child) of $j(i)$ when there exists an arc directed from i to j.

Ancestor: Vertex i is the ancestor to j when i is contained in the path from the root vertex to j $(i \neq j)$.

Brother: Vertices having the same father are termed brothers.

Younger (Elder) Brother: A vertex i is the younger (elder) brother of vertex j, if the algorithm generates the children of vertex i later (earlier) than the children of vertex j.

Rooted Directed Tree Generation:
Starting from the root vertex, the algorithm grows the tree progressively generating children, if any, of every vertex. There are four rules for generating the children of vertex k, depending on the nature of G_k.

Rule 1 G_k *is not* s, V*-connected*. In this case G_k does not generate any children.

Rule 2 G_k *is* s, V*-connected and cyclic*. Consider a dicycle C in G_k containing the arcs $e_1, e_2, \ldots, e_c$. Then $G_{k_j} = G_k - e_j$, $(j = 1, 2, \ldots, c)$, is a child of G_k, provided $e_j \cap X = \emptyset$, where X is the label of the arc incident into any elder brother of k or elder brother of an ancestor of k. Determination of a dicycle is determined by application of Depth First Search (applied in Rule 1). Clearly a state $G_k - e_j$ where e_j does not belong to the dicycle C, contains also C, thus by Theorem 2, $d_{D,V}(G_k - e_j) = 0$, so it is not necessary to generate this state.

Rule 3 G_k *is* s, V*-connected, acyclic, and* $\mathcal{LP}(G_k) > D$. Consider a longest s, u-dipath L in G_k containing the arcs $e_1, e_2, \ldots, e_l$. Then $G_{k_j} = G_k - e_j$, $(j = 1, 2, \ldots, l)$, is a child of G_k, provided $e_j \cap X = \emptyset$, where X is the label of the arc incident into any elder brother of k or elder brother of an ancestor of k. Determination of a longest s, u-dipath is determined by application of the PERT algorithm (see for example [10]) which is of time complexity $O(|V| + |E|)$ for digraphs. It is noted that is not necessary to consider a state $G_k - e_j$ where e_j does not belong to the dipath L, because $G_k - e_j$ is either not s, V-connected and its domination is 0 (see Claim 2), or it is s, V-connected and contains the path L of length greater than D, and by Theorem 3 its domination is also 0.

Rule 4 *G_k is s, V-connected, acyclic, and $\mathcal{LP}(G_k) \leq D$.* Let $G_k = (V, E_k)$, then $G_{k_j} = G_k - e_j$, $e_j \in E_k$ is a child of G_k, provided $e_j \cap X = \emptyset$, where X is the label of the arc incident into any elder brother of k or elder brother of an ancestor of k.

The algorithm will use recursion and an auxiliary rooted tree *Auxt* that is built during the execution of the program. This auxilary tree will be also used at any particular time of the recursive algorithm to determine if a particular arc to be considered, is the arc incident into any brother or elder brother of an ancestor of a vertex k, in order to avoid states duplication. Each vertex label k of *Auxt* will contain all the information regarding the digraph G_k, such as the label of the father of k, the labels of the children of k, and the label of the corresponding arcs.

Algorithm

Input: Original s-rooted digraph G, and diameter bound D.
Output: Source-to-all-terminal reliability R of G.

Data structures:

$\mathcal{P}(E)$. Represents the operational probabilities of the set of arcs E of the original digraph G, and the operational probability of an arc $e \in E$ is denoted as $p(e)$.

R. Global variable to represent the diameter-constrained reliability. Originally $R = 0$.

n. Number of vertices of original digraph G.

e. Number of arcs of the original graph G.

k. Current vertex being considered. This is a global variable and originally $k = 0$.

G_k. Current digraph under consideration. Originally $G_0 = G$.

e_k. Number of arcs of G_k.

Auxt. Rooted tree auxiliary data structure. Originally *Auxt* contains only the vertex $k = 0$, that represents the original graph $G_0 = G$.

Auxiliary Procedures:

1. *AddAuxt* (vertex l, vertex m, arc e_j). This procedure will add an arc from vertex l into a new vertex m of *Auxt*, whose label is e_j.

2. *bool CheckAuxt* (vertex l, arc e_j). This procedure will backtrack from vertex l to find if the arc e_j is incident into any elder brother or ancestor's elder brother of a vertex l (we assume that each vertex contains the label of its father). If that is the case will return *true*, otherwise will return *false*. This routine is very efficient, since the longest possible path from the root is of at most e arcs.

Main Procedure:

CalcRel (int e_k **)**

1. Let $crntvrtx = k$; current vertex of the rooted tree.
2. Apply Depth First Search to determine s, V-connectedness or detect dicycles.
3. If ($G_k = (V, E_k)$ is not s, V-connected) Return;
4. If ($G_k = (V, E_k)$ is cyclic)
   - 4.1. Let $C = \{e_1, e_2, \dots, e_c\}$ be the arcs of a dicycle of G_k.
   - 4.2. For ($e_i \in C$) do
     - 4.2.1. If (*CheckAuxt*(crntvrtx,e_i)==*false*)
       - 4.2.1.1. Let $k = k + 1$;
       - 4.2.1.2. *AddAuxt*($crntvrtx$,k,e_i);
       - 4.2.1.3. *CalcRel*($e_k - 1$);
   - 4.3. Return;
5. Apply PERT to determine $\mathcal{LP}(G_k)$;
6. If ($G_k = (V, E_k)$ is acyclic and $\mathcal{LP}(G_k) > D$).
   - 6.1. Let $L = \{e_1, e_2, \dots, e_L\}$ be the arcs of a longest s, u-dipath of G_k.
   - 6.2. For ($e_i \in L$) do
     - 6.2.1. If (*CheckAuxt*(crntvrtx,e_i)==*false*)
       - 6.2.1.1. Let $k = k + 1$;
       - 6.2.1.2. *AddAuxt*($crntvrtx$,k,e_i);
       - 6.2.1.3. *CalcRel*($e_k - 1$);
   - 6.3. Return;
7. If ($G_k = (V, E_k)$ is acyclic and $\mathcal{LP}(G_k) \leq D$).
   - 7.1. Let $R = R + (-1)^{e_k - n + 1} \times \prod_{e \in E_k} p(e)$;
   - 7.2. For ($e \in E_k$) do
     - 7.2.1. If (*CheckAuxt*(crntvrtx,e_i)==*false*)
       - 7.2.1.1. Let $k = k + 1$;
       - 7.2.1.2. *AddAuxt*($crntvrtx$,k,e_i);
       - 7.2.1.3. *CalcRel*($e_k - 1$);
   - 7.3. Return;

Steps 4. and 6. represent a significant reduction on the number of executable steps performed by the above algorithm since many states are avoided, especially when the original digraph contains several dicycles or D is small.

5 Conclusions and future work

In this paper we gave a characterization of the source-to-all-terminal diameter-constrained reliability domination for digraphs. In addition we presented a rapid algorithm for the computation of $R_{D,V}(G)$. These results represent a generalization of the classical all-terminal reliability domination for digraphs.

The general case for arbitrary number of terminal vertices remains open, but empirical results lead to conjecture that for a digraph $G = (V, E)$, $d_{D,K}(G) = (-1)^{|E|-|V|-1}$ if G is an acyclic D, K-digraph whose longest s, u-dipath is of length D or less; and that if that is not the case, then $d_{D,K}(G) = 0$.

Acknowledgement

This work was funded by Grant #64407-00-33 of the City University of New York Research Foundation.

References

[1] Agrawal A., Barlow R.E.: A survey of network reliability and domination theory. Operations Research **32** (1984) 478-492.

[2] Barlow R.E.: Set-theoretic signed domination of coherent systems. Technical Report ORC **82-1**, Operations Research Center, University of California, Berkeley (1982).

[3] Barlow R.E., Iyer S.: Computational complexity of coherent systems and the reliability polynomial, Probability in the Enginnering and Information Sciences **2** (1988(461-469.

[4] Bharath-Kumar, K., Jaffe, J. M.: Routing to multiple destinations in computer networks, IEEE transactions on Communications, COM-31(3) (1983) 343-351.

[5] Cancela H., Petingi L.: Diameter constrained network reliability: exact evaluation by factorization and bounds. Proceedings of the International Conference on Industrial Logistics, Okinawa, Japan (2001) 359-366.

[6] Harary, F.: Graph Theory, Addison-Wesley, Reading, Mass (1971).

[7] Huseby, A.B.: A unified theory of domination and signed with applications to exact reliability domination, Statistical Research Report, Institute of Mathematics, University of Oslo, Norway (1984).

[8] Huseby, A.B.: Domination Theory and the crapo-β-invariant, Networks **19** (1989) 135-149.

[9] Kortsarz, G., Peleg, D.: Approximating the weight of shallow Steiner trees, Discrete Applied Mathematics **93**:2-3 (1999) 265-285.

[10] Lawler, E.: Combinatorial Optimization: Networks and Matroids, Sauders College Publishing (1976).

[11] Petingi L., Rodriguez J.: Reliability of Networks with Delay Contraints. Congressus Numerantium **152** (2001) 117-123.

[12] Rouskas, G. N., Baldine, I.: Multicast Routing with End-to-End Delay and Delay Variation Constraints,IEEE Journal on Selected Areas in Communications, **15**:3 (1997) 346-356.

[13] Satyanarayana, A., Prabhakar A.: New topological formula and rapid algorithm for reliability analysis of complex networks, IEEE Transactions on Reliability **R-27**-2 (1978) 82-97.

[14] Waxman, B. W.: Routing of multipoint connections, IEEE Journal on selected areas in Communications **6**(9) (1988) 1617:1622.