Belief Revision and Relevance

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Abstract

The aim of this work is to relate two apparently independent concepts as belief revision and relevance. We propose a formal characterization for the notion of (ir)relevance by means of a set of postulates. Then we define (ir)relevance using change operators and we show that the contraction operations of the AGM model can be modeled through (ir)relevance relations and vice versa.

Keywords: Belief Revision, Relevance, Theory Change, Knowledge Representation.

1 Introduction

Human cognitive process is geared to achieving as much knowledge as possible. On learning new information, a human will try to accommodate it in such a way that this new knowledge does not do damage to the knowledge he already has. However, in many cases, the task of incorporating new information to a knowledge set will require previously maintained beliefs to be discarded. In other words, when an agent learns new information it is possible that this information conflicts with his belief set. When a conflict exists, it is necessary to eliminate old beliefs in order to consistently incorporate the new information. On the other hand, when no conflict exists, the reasoner can simply add the new information to his set of beliefs. This process is known as belief revision.

During the revision process the agent needs to identify the *relevant pieces of information* that should be incorporated or retracted form the old belief set. This means that if a belief set $\mathbf{K}$ is revised by a sentence α and α is a reason to believe in a sentence $\beta \notin \mathbf{K}$, β will be incorporated to the updated belief set. On the other hand, if a belief set $\mathbf{K}$ is revised by a sentence α and α conflicts with a sentence $\beta \in \mathbf{K}$, it will be necessary to retract β. According to this, given a belief set $\mathbf{K}$, there are two main relevance relations in which two formulas, α and β, can be involved:

- α is *positively relevant* to β if and only if β does not belong to $\mathbf{K}$ but β belongs to $\mathbf{K}$ revised by α.
- α is *negatively relevant* to β if and only if β belongs to $\mathbf{K}$ but β does not belong to K revised by α.

Analogously, we can identify two kinds of irrelevance relations between formulas:

- α is *positively irrelevant* to β if and only if β belongs neither to $\mathbf{K}$ nor to $\mathbf{K}$ revised by α.
- α is *negatively irrelevant* to β if and only if β belongs both to $\mathbf{K}$ and to K revised by α.

The above intuitive considerations lead us to propose the following thesis:

The notions of (ir)relevance and belief revision are interdefinable.

In other words:

- It is possible to build the process of belief revision on notions of (ir)relevance.
- It is possible to identify (ir)relevance relations through the process of belief revision.

The driving motivations for this work are, therefore, to analyze the notions of positive and negative (ir)relevance and to study the way in which these notions can be used to develop a constructive model of belief revision. Then we will show the connections existing between belief revision defined in terms of relevance and the AGM postulates.

2 Preliminaries

Depending on how beliefs are represented and what kinds of inputs are used, different kinds of belief changes are possible. In the most common case beliefs are represented by sentences in a logical language. We will adopt a propositional language $\mathcal{L}$ with a complete set of boolean connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$. Formulæ in $\mathcal{L}$ will be denoted by lowercase Greek characters: $\alpha, \beta, \delta, \ldots$. Sets of sentences in $\mathcal{L}$ will be denoted by uppercase Latin characters: $A, B, C, \ldots$. The symbol $\top$ represents a tautology or *truth*. The symbol $\perp$ represents a contradiction or *falsum*. We also use a consequence operator Cn. Cn takes sets of sentences in $\mathcal{L}$ and produces new sets of sentences. The operator Cn satisfies the following three conditions:

Inclusion: $A \subseteq Cn(A)$ for any $A \subseteq \mathcal{L}$.

Iteration: $Cn(A) = Cn(Cn(A))$ for any $A \subseteq \mathcal{L}$.

Monotony: If $A \subseteq B$ then $Cn(A) \subseteq Cn(B)$ for any $A, B \subseteq \mathcal{L}$.

We will assume that the consequence operator includes classical consequences and verifies the standard properties of deduction and compactness:

Supraclassicality: If α can be derived from A by deduction in classical logic, then $\alpha \in Cn(A)$.

Deduction: $\beta \in Cn(A \cup \{\alpha\})$ if and only if $(\alpha \rightarrow \beta) \in Cn(A)$.

Compactness: If $\alpha \in Cn(A)$ then $\alpha \in Cn(A')$ for some finite subset A' of A.

To simplify notation, we write $Cn(\alpha)$ for $Cn(\{\alpha\})$ where α is any sentence in $\mathfrak{L}$. We also write $\alpha \in Cn(A)$ as $A \vdash \alpha$. The epistemic states will be represented by belief set, *i.e.*, set of sentences closed under logical consequence. This means that $\mathbf{K}$ is a belief set if and only if $\mathbf{K} = Cn(\mathbf{K})$. Usually, belief sets will be denoted by boldface uppercase Latin characters.

3 Recalling Belief Revision

In the AGM approach the epistemic states are represented by belief sets. Let $\mathbf{K}$ be a belief set and α a sentence in $\mathfrak{L}$. The three main kinds of changes are the following [Ga92]:

Expansion: A new sentence is added to an epistemic state regardless of the consequences of the so formed larger set. If $+$ is an expansion operator, then $\mathbf{K}+\alpha$ denotes the belief set $\mathbf{K}$ expanded by α.

Contraction: Some sentence in the epistemic state is retracted without adding any new belief. If $-$ is a contraction operator, then $\mathbf{K}-\alpha$ denotes the belief set $\mathbf{K}$ contracted by α.

Revision: A new sentence is consistently added to an epistemic state. In order to make possible this operation, some sentences may be retracted from the original epistemic state. If $*$ is a revision operator, then $\mathbf{K}*\alpha$ denotes the belief set $\mathbf{K}$ revised by α.

Expansions can simply be defined as the logical closure of $\mathbf{K}$ and α:

$$\mathbf{K}+\alpha = Cn(\mathbf{K} \cup \{\alpha\})$$

It is not possible to give a similar explicit definition of contractions and revisions in logical and set-theoretical notions only. These operations can be defined using logical notions and some selection mechanism. Contractions and Revisions are interdefinable by the following identities:

Levi Identity: $\mathbf{K}*\alpha = (\mathbf{K}-\neg\alpha)+\alpha$.

Harper Identity: $\mathbf{K}-\alpha = \mathbf{K} \cap \mathbf{K}*\neg\alpha$.

By giving a definition of one of these operators we can obtain the other one using the above identities. In this work we will show the relation existing between the notion of (ir)relevance and the contraction operator.

3.1 Postulates for Contractions

Gärdenfors [Ga88] proposed the following *rationality postulates* for contraction operators:

(K^-1) **Closure:** For every belief set $\mathbf{K}$ and every sentence α, $\mathbf{K}-\alpha$ is a belief set.

(K^-2) **Inclusion:** $\mathbf{K}-\alpha \subseteq \mathbf{K}$.

(K^-3) **Vacuity:** If $\alpha \notin \mathbf{K}$ then $\mathbf{K}-\alpha = \mathbf{K}$.

(K^-4) **Success:** If $\nvdash \alpha$ then $\alpha \notin \mathbf{K}-\alpha$.

(K^-5) **Recovery:** $\mathbf{K} \subseteq (\mathbf{K}-\alpha)+\alpha$.

(K^-6) **Extensionality:** If $\vdash \alpha \leftrightarrow \beta$ then $\mathbf{K}-\alpha = \mathbf{K}-\beta$.

(K^-7) **Conjunctive Overlap:** $\mathbf{K}-\alpha \cap \mathbf{K}-\beta \subseteq \mathbf{K}-(\alpha \wedge \beta)$.

(K^-8) **Conjunctive Inclusion:** If $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$ then $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$.

The following Lemma is a straight consequence of Levi Identity.

LEMMA 3.1 Given a belief set $\mathbf{K}$ and a formula $\alpha \in \mathcal{L}$, $\mathbf{K}-\neg\alpha \subseteq \mathbf{K}*\alpha$. ■

Consider the following postulate for contraction:

(K^-F) **Conjunctive Factoring:** $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$, or $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$, or $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha \cap \mathbf{K}-\beta$.

The following Lemma states that given postulates (K^-1) to (K^-6), postulate (K^-F) is equivalent to the conjunction of postulates (K^-7) and (K^-8).

LEMMA 3.2 [AGM85] If a contraction operator satisfies (K^-1)..(K^-6) then it satisfies (K^-7) and (K^-8) if and only if it satisfies (K^-F). ■

4 Our approach to (Ir)relevance

Relevance is a conceptual relation. A piece of information is said to be relevant if it is "of importance, of worth, of consequence", "bearing upon, connected with, pertinent to the matter in hand."[1] There are many different proposals aimed to characterize the notion of relevance in logic systems. For a detailed analysis of different approaches to relevance that appear in the literature see [Ma97].

We propose a characterization of the notion of relevance as a metalinguistic relation between formulas of a propositional language $\mathcal{L}$. In our approach relevance is seen as an "interference relation" between two formulas, in the context of a belief set. We will take the notion of negative irrelevance as a primitive relation. This notion is formalized by the following set of postulates.

[1] The Oxford English Dictionary.

4.1 Postulates for Irrelevance Relations

(Irr_1) If $\neg\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\alpha$ then $\vdash \alpha$.

(Irr_2) If $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta$ and $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}(\beta \rightarrow \delta)$ then $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$.

(Irr_3) If $\vdash (\alpha \leftrightarrow \beta)$ then $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ if and only if $\beta\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ for all $\delta \in \mathcal{L}$.

(Irr_4) If $\vdash \beta$ then $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta$ for all $\alpha \in \mathcal{L}$.

(Irr_5) If $\alpha \notin \mathbf{K}$ then $\neg\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta$ for all $\beta \in \mathbf{K}$.

(Irr_6) If $\beta \in \mathbf{K}$ then $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$.

(Irr_7) If $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ and $\beta\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ then $(\alpha \vee \beta)\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$.

(Irr_8) If $(\alpha \vee \beta)\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ then $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ or $\beta\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$.

Postulate (Irr_1) establishes that if the negation of a formula α does not interfere with α, then α is logically true. Postulate (Irr_2) stands for the condition of modus ponens in the consequent. The irrelevance of the syntax condition is valid for negative irrelevance and this is represented by postulate (Irr_3). According to postulate (Irr_4) no formula of the language can interfere with a theorem. Postulate (Irr_5) establishes that if a formula α is not in the belief set, the negation of α cannot interfere with any formula that belongs to the belief set. We will assume that a formula α cannot interfere with the formula that results from the disjunction of α and any other formula of the belief set. This last condition is represented by postulate (Irr_6). According to postulate (Irr_7), if neither α nor β interferes with δ, then the formula that results from the disjunction of α and β cannot interfere with δ. Finally, postulate (Irr_8) establishes that if the formula $\alpha \vee \beta$ does not interfere with δ then, either α does not interfere with δ or β does not interfere with δ.

5 From Belief Revision to (Ir)relevance

We have already proposed informal definitions of (ir)relevance in terms of belief revision. The following formalizes those definitions:

(Def $\mathbb{I}^{\mathrm{P}}_{\mathbf{K}}$) $\alpha\mathbb{I}^{\mathrm{P}}_{\mathbf{K}}\beta = \{\beta \quad \beta \notin \mathbf{K} \text{ and } \beta \notin \mathbf{K}{*}\alpha\}$.

(Def $\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}$) $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta = \{\beta \quad \beta \in \mathbf{K} \text{ and } \beta \in \mathbf{K}{*}\alpha\}$.

(Def $\mathbb{R}^{\mathrm{P}}_{\mathbf{K}}$) $\alpha\mathbb{R}^{\mathrm{P}}_{\mathbf{K}}\beta = \{\beta \quad \beta \notin \mathbf{K} \text{ and } \beta \in \mathbf{K}{*}\alpha\}$.

(Def $\mathbb{R}^{\mathrm{N}}_{\mathbf{K}}$) $\alpha\mathbb{R}^{\mathrm{N}}_{\mathbf{K}}\beta = \{\beta \quad \beta \in \mathbf{K} \text{ and } \beta \notin \mathbf{K}{*}\alpha\}$.

The following schemas are consequences of the definitions presented above. According to them, positive irrelevance, positive relevance and negative relevance can be described in terms of negative irrelevance:

(Schema $\mathbb{I}^{\mathrm{P}}_{\mathbf{K}}$) $\alpha\mathbb{I}^{\mathrm{P}}_{\mathbf{K}}\beta$ if and only if $\beta \notin \mathbf{K}$ and it is not the case that $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta$ holds.

(Schema $\mathbb{R}^{\mathrm{P}}_{\mathbf{K}}$) $\alpha\mathbb{R}^{\mathrm{P}}_{\mathbf{K}}\beta$ if and only if $\beta \notin \mathbf{K}$ and it is not the case that $\alpha\mathbb{I}^{\mathrm{P}}_{\mathbf{K}}\beta$ holds.

(Schema $\mathbb{R}^{\mathrm{N}}_{\mathbf{K}}$) $\alpha\mathbb{R}^{\mathrm{N}}_{\mathbf{K}}\beta$ if and only if $\beta \in \mathbf{K}$ and it is not the case that $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta$ holds.

The following Lemma is a straight consequence of (Def $\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}$) and Harper Identity.

LEMMA 5.1 Given a belief set $\mathbf{K}$ and a pair of formulas $\alpha, \beta \in \mathcal{L}$, $\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta$ if and only if $\beta \in \mathbf{K}-\neg\alpha$. ■

LEMMA 5.2 If the postulates for contraction are satisfied then the postulates for irrelevance relations are satisfied.
PROOF IN THE APPENDIX. ■

6 From (Ir)relevance to Belief Revision

Now we define AGM change operators in terms of relevance and irrelevance notions:

(Def $\mathbf{K}+\alpha$) $\mathbf{K}+\alpha = \{\beta \quad \mathbf{K} \cup \{\alpha\} \vdash \beta\}$.

(Def $\mathbf{K}-\alpha$) $\mathbf{K}-\alpha = \{\beta \quad \neg\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta\}$.

(Def $\mathbf{K}*\alpha$) $\mathbf{K}*\alpha = \{\beta \quad \alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\beta \text{ or } \alpha\mathbb{R}^{\mathrm{P}}_{\mathbf{K}}\beta\}$.

LEMMA 6.1 If the postulates for irrelevance relations are satisfied then the postulates for contraction are satisfied.
PROOF IN THE APPENDIX. ■

7 Concluding Result

The following theorem establishes the interrelation between the notion of irrelevance and the AGM contraction operators.

THEOREM 7.1 The postulates for irrelevance relations $(\mathsf{Irr}_1)..(\mathsf{Irr}_8)$ are satisfied if and only if the postulates for contraction $(\mathsf{K}^{-}1)..(\mathsf{K}^{-}8)$ are satisfied.
PROOF IN THE APPENDIX. ■

8 Conclusion

We have proposed a formal characterization for the notion of (ir)relevance. An intuitive relation between the process of contraction and the notion of negative irrelevance has been identified. This led us to show that the notions of (ir)relevance and belief revision are interdefinable.

9 Appendix: Proofs

LEMMA 5.2. If the postulates for contraction are satisfied then the postulates for irrelevance relations are satisfied.
Proof. Suppose that the set of postulates for contraction are satisfied. We have to show that the set of postulates for irrelevance relations are satisfied.

(Irr_1) Assume $\neg\alpha\mathbb{I}^{N}_{\mathbf{K}}\alpha$. We have to prove $\vdash \alpha$. According to Lemma 5.1 $\neg\alpha\mathbb{I}^{N}_{\mathbf{K}}\alpha$ is equivalent to $\alpha \in \mathbf{K}-\alpha$ which, by ($\mathsf{K}^{-}4$), implies $\vdash \alpha$. This concludes our proof.

(Irr_2) Assume $\alpha\mathbb{I}^{N}_{\mathbf{K}}\beta$ and $\alpha\mathbb{I}^{N}_{\mathbf{K}}(\beta \to \delta)$. We want to prove $\alpha\mathbb{I}^{N}_{\mathbf{K}}\delta$. By Lemma 5.1, from $\alpha\mathbb{I}^{N}_{\mathbf{K}}\beta$ we have $\beta \in \mathbf{K}-\neg\alpha$, and from $\alpha\mathbb{I}^{N}_{\mathbf{K}}(\beta \to \delta)$ we have $\beta \to \delta \in \mathbf{K}-\neg\alpha$. Then, by ($\mathsf{K}^{-}1$), we have $\delta \in \mathbf{K}-\neg\alpha$ which, by Lemma 5.1, is equivalent to $\alpha\mathbb{I}^{N}_{\mathbf{K}}\delta$.

(Irr_3) Suppose that $\vdash (\alpha \leftrightarrow \beta)$. We have to prove $\alpha\mathbb{I}^{N}_{\mathbf{K}}\delta$ if and only if $\beta\mathbb{I}^{N}_{\mathbf{K}}\delta$ for all $\delta \in \mathcal{L}$. From $\vdash (\alpha \leftrightarrow \beta)$ we have $\vdash (\neg\alpha \leftrightarrow \neg\beta)$. Therefore, by ($\mathsf{K}^{-}6$), we have $\mathbf{K}-\neg\alpha = \mathbf{K}-\neg\beta$, which according to Lemma 5.1 is equivalent to $\alpha\mathbb{I}^{N}_{\mathbf{K}}\delta$ if and only if $\beta\mathbb{I}^{N}_{\mathbf{K}}\delta$, and we are done.

(Irr_4) We have to prove that if $\vdash \beta$ then $\alpha\mathbb{I}^{N}_{\mathbf{K}}\beta$ for all $\alpha \in \mathcal{L}$. From $\vdash \beta$, by ($\mathsf{K}^{-}1$), we have $\beta \in \mathbf{K}-\delta$ for all $\delta \in \mathcal{L}$. This implies, by Lemma 5.1, $\neg\delta\mathbb{I}^{N}_{\mathbf{K}}\beta$ for all $\delta \in \mathcal{L}$, which is equivalent to $\alpha\mathbb{I}^{N}_{\mathbf{K}}\beta$ for all $\alpha \in \mathcal{L}$, and we are done.

(Irr_5) This postulate states that if $\alpha \notin \mathbf{K}$ then $\neg\alpha\mathbb{I}^{N}_{\mathbf{K}}\beta$ for all $\beta \in \mathbf{K}$. From $\alpha \notin \mathbf{K}$ it follows by ($\mathsf{K}^{-}3$) that $\mathbf{K}-\alpha = \mathbf{K}$. Let $\beta \in \mathbf{K}$. Then $\beta \in \mathbf{K}-\alpha$. According to Lemma 5.1 this results in $\neg\alpha\mathbb{I}^{N}_{\mathbf{K}}\beta$.

(Irr_6) Suppose $\beta \in \mathbf{K}$. We have to prove $\alpha\mathbb{I}^{N}_{\mathbf{K}}(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$. If $\beta \in \mathbf{K}$, it follows by ($\mathsf{K}^{-}5$) that for any $\delta \in \mathcal{L}$ we have $\beta \in (K-\delta)+\delta$. Then $\beta \in Cn(K-\delta \cup \{\delta\})$ if and only if $(\delta \to \beta) \in Cn(\mathbf{K}-\delta)$ By ($\mathsf{K}^{-}1$) this is equivalent to $(\delta \to \beta) \in \mathbf{K}-\delta$. Again, by ($\mathsf{K}^{-}1$), $(\neg\delta \vee \beta) \in \mathbf{K}-\delta$. Hence, according to Lemma 5.1, $\neg\delta\mathbb{I}^{N}_{\mathbf{K}}(\neg\delta \vee \beta)$ for all $\delta \in \mathcal{L}$. In particular for $\delta = \neg\alpha$ we have $\alpha\mathbb{I}^{N}_{\mathbf{K}}(\alpha \vee \beta)$ for all $\alpha \in \mathcal{L}$.

(Irr_7) Suppose that $\alpha\mathbb{I}^{N}_{\mathbf{K}}\delta$ and $\beta\mathbb{I}^{N}_{\mathbf{K}}\delta$. We have to prove $(\alpha \vee \beta)\mathbb{I}^{N}_{\mathbf{K}}\delta$. It follows from $\alpha\mathbb{I}^{N}_{\mathbf{K}}\delta$ and $\beta\mathbb{I}^{N}_{\mathbf{K}}\delta$, by Lemma 5.1, that $\delta \in \mathbf{K}-\neg\alpha$ and $\delta \in \mathbf{K}-\neg\beta$. Then, by ($\mathsf{K}^{-}7$), $\delta \in \mathbf{K}-(\neg\alpha \wedge \neg\beta)$. Since $\vdash (\neg\alpha \wedge \neg\beta) \leftrightarrow \neg(\alpha \vee \beta)$, by ($\mathsf{K}^{-}6$), $\delta \in \mathbf{K}-\neg(\alpha \vee \beta)$. But according to Lemma 5.1 this is equivalent to $(\alpha \vee \beta)\mathbb{I}^{N}_{\mathbf{K}}\delta$, which is the desired result.

(Irr_8) We have to show that from $(\alpha \vee \beta)\mathbb{I}^{N}_{\mathbf{K}}\delta$ we can conclude $\alpha\mathbb{I}^{N}_{\mathbf{K}}\delta$ or $\beta\mathbb{I}^{N}_{\mathbf{K}}\delta$. It follows from $(\alpha \vee \beta)\mathbb{I}^{N}_{\mathbf{K}}\delta$, by Lemma 5.1, that $\delta \in \mathbf{K}-\neg(\alpha \vee \beta) = \mathbf{K}-(\neg\alpha \wedge \neg\beta)$. It follows, by ($\mathsf{K}^{-}\mathsf{F}$) that at least one of the following cases holds:

- Case 1: $\delta \in \mathbf{K}-\neg\alpha$, or
- Case 2: $\delta \in \mathbf{K}-\neg\beta$, or
- Case 3: $\delta \in \mathbf{K}-\neg\alpha \cap \mathbf{K}-\neg\beta$.

According to Lemma 5.1, if case 1 holds we can conclude that $\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$. Analogously, if case 2 holds we conclude $\beta \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$. Finally, if case 3 holds both $\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$ and $\beta \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$, hold. This finishes our proof.

■

LEMMA 6.1. If the postulates for irrelevance relations are satisfied then the postulates for contraction are satisfied.

Proof. Suppose that the set of postulates for irrelevance relations are satisfied. We have to show that the set of postulates for contraction are satisfied.

(K^-1) We have to prove that for every belief set $\mathbf{K}$ and every sentence α, $\mathbf{K}-\alpha$ is a belief set, *i.e.*, we have to prove

1. if $\vdash \beta$ then $\beta \in \mathbf{K}-\alpha$, and
2. $\mathbf{K}-\alpha$ is closed under implications.

For part 1, if $\vdash \beta$ we have by (Irr_4) and Lemma 5.1, that $\beta \in \mathbf{K}-\neg\delta$ for all $\delta \in \mathcal{L}$. In particular, for $\alpha = \neg\delta$ we have $\beta \in \mathbf{K}-\alpha$, which is the desired result.

For part 2 we need to show that if $\beta \in \mathbf{K}-\alpha$ and $\beta \rightarrow \delta \in \mathbf{K}-\alpha$ then $\delta \in \mathbf{K}-\alpha$. By Lemma 5.1, from $\beta \in \mathbf{K}-\alpha$ we have $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \beta$, and from $\beta \rightarrow \delta \in \mathbf{K}-\alpha$ we have $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \beta \rightarrow \delta$. Then , by postulate ($\mathsf{Irr}_2$) we can conclude $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$. Hence, by Lemma 5.1, $\delta \in \mathbf{K}-\alpha$, and we are done.

(K^-2) Suppose that $\beta \in \mathbf{K}-\alpha$. We want to prove $\beta \in \mathbf{K}$. But from $\beta \in \mathbf{K}-\alpha$, by Lemma 5.1, we have that $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \beta$, and by (Def $\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}$), we have $\beta \in \mathbf{K}$.

(K^-3) Suppose that $\alpha \notin \mathbf{K}$. Then, by (Irr_5), we have $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \beta$ for all $\beta \in \mathbf{K}$, and by Lemma 5.1, we can conclude $\mathbf{K}-\alpha = \mathbf{K}$.

(K^-4) Suppose that $\nvdash \alpha$. It follows by (Irr_1) that it is not the case that $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \alpha$ holds. Then, according to Lemma 5.1, we can conclude $\alpha \notin \mathbf{K}-\alpha$, which is the desired result.

(K^-5) Suppose that $\delta \in \mathbf{K}$. Then, by (Irr_6), we have $\beta \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} (\beta \vee \delta)$ for all $\beta \in \mathcal{L}$. In particular, for $\beta = \neg\alpha$ we have $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} (\neg\alpha \vee \delta)$. Then, by Lemma 5.1, we have that $\mathbf{K}-\alpha \vdash \neg\alpha \vee \delta$, which is equivalent to $\mathbf{K}-\alpha \vdash \alpha \rightarrow \delta$. Therefore, $(\mathbf{K}-\alpha) \cup \{\alpha\} \vdash \delta$, and then, by (Def $\mathbf{K}+\alpha$), we can conclude $\beta \in (\mathbf{K}-\alpha)+\alpha$.

(K^-6) We have to show that from $\vdash \alpha \leftrightarrow \beta$ we can conclude $\mathbf{K}-\alpha = \mathbf{K}-\beta$. From $\vdash \alpha \leftrightarrow \beta$ we have, $\vdash \neg\alpha \leftrightarrow \neg\beta$. Then, by ($\mathsf{Irr}_3$), we have $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$ if and only if $\neg\beta \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$ for all $\delta \in \mathcal{L}$, which by Lemma 5.1 is equivalent to $\mathbf{K}-\alpha = \mathbf{K}-\beta$.

(K^-7) We have to show that if $\delta \in \mathbf{K}-\alpha$ and $\delta \in \mathbf{K}-\beta$ then $\delta \in \mathbf{K}-(\alpha \wedge \beta)$. By Lemma 5.1, we have from $\delta \in \mathbf{K}-\alpha$ and $\delta \in \mathbf{K}-\beta$, $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$ and $\neg\alpha \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$, respectively. Then, by (Irr_7), we have that $(\neg\alpha \vee \neg\beta) \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$, Since $\vdash (\neg\alpha \vee \neg\beta) \leftrightarrow \neg(\alpha \wedge \beta)$, by ($\mathsf{Irr}_3$), $\neg(\alpha \wedge \beta) \mathbb{I}^{\mathrm{N}}_{\mathbf{K}} \delta$. Hence, by Lemma 5.1, we obtain $\delta \in \mathbf{K}-(\alpha \wedge \beta)$, which is the desired result.

(K^-8) We know by Lemma 3.2 that if postulates (K^-1) to (K^-6) are valid and if ($\mathsf{K}^-\mathsf{F}$) is also valid, we can conclude (K^-8) is valid. Therefore, it is enough to show that ($\mathsf{K}^-\mathsf{F}$) is valid. Postulate ($\mathsf{K}^-\mathsf{F}$) states that at least one of the following cases holds:

- Case 1: $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha$, or
- Case 2: $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\beta$, or
- Case 3: $\mathbf{K}-(\alpha \wedge \beta) = \mathbf{K}-\alpha \cap \mathbf{K}-\beta$.

According to Lemma 5.1 to prove that at least one of the above cases holds is equivalent to prove that at least one of the following cases holds:

- Case 1: $(\neg\alpha \vee \neg\beta)\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ if and only if $\neg\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$, or
- Case 2: $(\neg\alpha \vee \neg\beta)\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ if and only if $\neg\beta\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$, or
- Case 3: $(\neg\alpha \vee \neg\beta)\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ if and only if $\neg\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ and $\neg\beta\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$.

After some algebraic manipulations we can conclude that proving that at least one of the above cases holds is equivalent to prove that the following two cases simultaneously hold:

- Case a: $\neg\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ and $\neg\beta\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ implies $(\neg\alpha \vee \neg\beta)\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$.
- Case b: $(\neg\alpha \vee \neg\beta)\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ implies $\neg\alpha\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$ or $\neg\beta\mathbb{I}^{\mathrm{N}}_{\mathbf{K}}\delta$.

It is easy to see that case a is equivalent to (Irr_7), while case b is equivalent to (Irr_8). This concludes our proof.

■

THEOREM 7.1. The postulates for irrelevance relations (Irr_1)..(Irr_8) are satisfied if and only if the postulates for contraction (K^-1)..(K^-8) are satisfied.
Proof. Straightforward from Lemma 5.2 and Lemma 6.1.

■

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