Wavelet Representation of functions defined on Tetrahedrical Grids

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Abstract. In this paper, a method for representing scalar functions on volumes is presented. The method is based on wavelets and it can be used for representing volumetric data (geometric or scalar) defined on non structured grids. The basic contribution is the extension of wavelets to represent scalar functions on volumetric domains of arbitrary topological type. This extension is made by constructing a wavelet basis defined on any tetrahedrized volume. This basis construction is achieved using multiresolution analysis and the lifting scheme.
Keywords: Volume modeling, multiresolution, wavelets.

1 Introduction

Three dimensional scenes contain highly detailed geometric models that are rapidly emerging as the next frontier requirement for many applications such as Internet-based applications, 3D models for complex virtual environments, collaborative CAD, interactive visualization, and multi-player video games. Then, next generation computer graphics systems are expected to deliver interactive rendering of 3D models for massive use, and a growth of internetworked 3D graphics.

This situation motivates the developing of 3D surface and volume models in order to meet requirements like effective use of disk space and network bandwidth, as well as substantial reduction of network transfer time. Undoubtedly, these models place rigorous demand upon transmission bandwidth, storage capacity and rendering time. Therefore, it is necessary the construction of better 3D surface and volume models, so that it is possible high-quality aproximation of big datasets with good storage space, and better transmission time performance.

The wavelets have proven to be a powerful mathematical tool that allows a hierarchical decomposition of functions and although its origins were different fields, such as signal processing and physics, they have been recently used to solve different problems in computer graphics ([Zorin et al., 1996], [DeVore et al., 1992],[Stollnitz et al., 1996], [Muraki, 1995], [Gross et al., 1997]). Their power lies in the fact that they only require a small number of coefficients to represent general functions and large datasets accurately. This allows compression and efficient computations.

The classical wavelets construction have been made on $\mathbb{R}^1$, but most applications of interest are defined on finite domains. In order to overcome this problem, different techniques for defining wavelets on bounded domain have been developed ([Chui and Quak, 1992], [Cohen et al., 1993]). One extension to bidimensional wavelets was done by means of tensor product of univariate wavelets and, for the case of unbounded domains, the functions so obtained are defined on the whole plane $\mathbb{R}^2$. However, for practical applications to surface design it is necessary to have wavelets defined on bounded domains. In case of rectangular ones, they can be constructed simply doing the tensor product of wavelets defined on an interval. In case of applications in computer graphics, an example of approximation of volumetric data in 3D using wavelets parameterized on a rectangular grid in $\mathbb{R}^3$ was presented in [Muraki, 1992]. Thus, representations with domains contained in $\mathbb{R}^2$ and $\mathbb{R}^3$ are based on tensor product of univariate wavelets. Although this is a simple way of constructing wavelets for surface or volume representation, this method has an important drawback: it can not be used without introducing degeneracies for representing surfaces or volumes defined on general domains of arbitrary topological type, like spherical domains.

Mallat [Mallat, 1989], Daubechies [Daubechies, 1992] and Chui [Chui, 1992] wrote the traditional introductory works where they introduced the wavelets from a signal processing point of view. In this case, the wavelets are dilations and translations of a single function, called *mother wavelet*, being the Fourier transform the main tool for this construction. Limitations of Fourier analysis for constructing wavelets on arbitrary domains led researchers to look into other directions. In the particular case of computer graphics, it is necessary to have what is called *hierarchical representation* of functions no matter whether the basis functions are derived from a single function. Lounsbery [Lounsbery, 1994] and Stollnitz *et al.* [Stollnitz et al., 1996] were the first who introduced wavelets from a different point of view, defining them on different bounded domains. In order to do this, they define the scaling function using MRA theory since it is not possible to do it through the Fourier transform as in the classical approach because it is not a valid tool for arbitrary domains. Techniques as those ones used for constructing wavelets from B-splines and tensor product surfaces are extended to arbitrary topological domains. Wavelets defined in that way are a generalization of the standard multiresolution definition since they are defined through scaling refinable functions. Refinability, a key property for MRA, generalizes the notion of both translation and dilation, and it means that a scaling function on a coarser level can be written as a linear combination of scaling functions on a finer level. Within this context, in [Lounsbery, 1994] wavelets defined on arbitrary topological domains on $\mathbb{R}^2$ are built up. For this purpose, he extended MRA to functions defined on surfaces by creating scaling refinable function. This approach was generalized *a posteriori* by Sweldens ([Sweldens, 1995], [Sweldens, 1996]) who recognized that the *lifting scheme* he proposed was a generalization of Lounsbery's methodology. Schröder and Sweldens [Schroeder and Sweldens, 1995] continued their work and proved that subdivision and lifting provide an efficient methodology for costum-design construction of wavelets. They focused their work on wavelet representation of functions defined on a sphere. This construction takes into account the topology of the sphere and its curved geometry as well. Later, Nielson [Nielson et al., 1997a] defined the Haar wavelets over the sphere that have an advantage over the biorthoganl wavelets of Sweldens and Schröder: on planar surfaces of uniform area they converge to one of the two orthogonal triangular wavelet of Haar. Both construction are defined over triangular domains; so it is possible to generate wavelets on arbitrary surfaces based on subdivision schemes, beginning with a triangular net.

Volume modeling using wavelets was restricted to tensor product of univariate wavelets and then, the volumetric data had to be defined on parallelepipeds. In this paper, a method for multiresolution volume representation is presented. While our primary work was to efficiently represent volumes of arbitrary topolgical type, now we examine the case of efficiently representing functions defined on a tetrahedrical domain. This is done by constructing a wavelet basis defined on any tetrahedrized volume. This basis construction is done using the classical MRA and then applying the lifting scheme.

The paper is organized as follows: in Section 2 we give an introduction to volumetric data, in Section 3 we provide a brief description of multiresolution volume modeling and we present the construction of wavelets on arbitrary topological domains. In Section 4 we provide an example and finally, in Section 5 we draw some conclusions and future work.

2 Volume representation

Volumetric data are 3D entities that may have information of the inside of the volume, may or may not consis tof surfaces and have generally a high number of points. Different alternatives have been proposed for modeling volumetric data in Computer Graphics ([Nielson et al., 1994], [Ranjan and Fournier, 1994], [DeFloriani and Puppo, 1995], [Muraki, 1992], [Kaufman et al., 1993]); however, most of them may be classified in two different types

- The ones that represent volumes through its surfaces or union of surfaces and decompose them on simpler constructors like triangles.
- The ones that represent the volume by simpler volume constructors (analog to triangles for surfaces).

We define a *volumetric data* as a set S of samples $\{x,\ y,\ z,\ v\}$ where v represents a certain property of the data at the point x,y,z. Volumetric data can be obtained by sampling, simulation or through modeling techniques. For example, by a tomography or magnetic resonance images a sequence of 2D slices for constructing a 3D volume can be obtained; on other applications, like computational fluid dynamics, simulation results can be viewed as volumetric data. Today, the most widely used techniques for rendering 3D objects are the so called *surface graphics* and are based on patches nets. Recently, many traditional geometric applications like CAD and simulations have exploited the techniques known as *volume graphics* for modeling, manipulation and visualization.

Given a set of volumetric input data, different techniques for obtaining an isosurface or a tetrahedrization of the volume exist. We will not concentrate our work in this problem, but we suppose that we have already a tetrahedrized volume in order to explore its multiresolution representation. Finally, we will present a method for modeling volumes, beginning with its tetrahedral net. This method allows us to represent not only the surface of the volume like the others but also its interior and can also be applied to the representation of a function defined on the volume.

3 Multiresolution volume modeling using wavelets

Different methods have been formulated for modeling volumes using tensor product ([Muraki, 1992], [Chui, 1992]). In this case, decomposition and analysis have been applied on the 4-*tupla* and on the 3D volume coordinates, decomposing it on the three spatial directions but limiting the object modeling to an isotropic grid. For modeling an object whose data are sparse, a tetrahedrical net can be used, being this net a general topological domain for the intrinsic representation of the volume. In this case, it is necessary to have wavelets defined over tetrahedra and then to extend MRA to functions defined on them. This extension, the same as the extension case for surfaces [Lounsbery, 1994], is based on refinement.

The net representing the object must store the 3D geometry and the function defined on it; that is, the 3D vertex, its topology and the scalar properties. As a tetrahedrical net can represent a volume of arbitrary topological type, we will see how to construct wavelets beginning with a tetrahedrical net and a function defined on it and using subdivision afterwards.

3.1 Subdivision volumes

A subdivision volume results from successive refinement of a control tetrahedrical 3D net VT (see [Cignoni et al., 1994], [Nielson et al., 1997a], [Nielson et al., 1997b], [Westermann, 1994]). Figure 1 shows one step in recursive subdivision of a tetrahedron which, applied to tetrahedrized volumes leads to a possible collection of refinable spaces that is to a sequence of linear nested spaces as those ones required in a MRA. When doing one tetrahedron subdivision, 8 tetrahedra are generated. Bey [Bey, 1995] presents a refinement algorithm for non structured tetrahedrical grids that generates consistent and stable tetrahedrizations.

3.2 Multiresolution analysis for volumes

The main idea of MRA is the decomposition of a function into a low resolution and a detail. During each step the volume of a fixed resolution will be partitioned in a coarser resolution and a detail. For example, the volume (a) in Figure 2 has finer resolution than (b); the vertex in (b) are computed as a weighted averages of the vertex of (a).

These weighted averages essentially implement a low pass filter which makes possible to represent it as a product of a vector with a matrix A_j, being the columns of A_j the coefficients of the analysis low pass filter. The details consist of a collection of

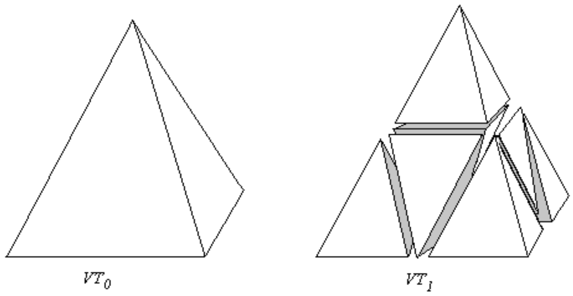

Fig. 1. Recursive subdivision of a tetrahedron into sub tetrahedra

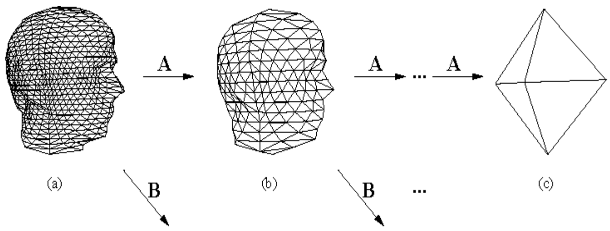

Fig. 2. Surface refinement

wavelet coefficients calculated as a weighted difference of the vertex of (a). These coefficients can be calculated as a product of a vector with matrix B_j, being the columns of B_j the coefficients of the analysis high pass filter. This analysis procedure may be recursively applied to the coarse resolutions until the possible coarsest representation of the volume (c) is obtained.

In case of volumes, the domain of the function could be any polyhedrical net and, in particular, a tetrahedrical net. Beginning with this net, a decomposition in a coarser resolution and its detail is made. The first one corresponds to the representation of the volume at a coarser resolution and the second allows to compute the wavelet coefficients as a weighted difference of the vertex at higher resolution and the vertex at coarser resolution.

Like in the case of surfaces, this decomposition (analysis) procedure can be recursively applied until the coarsest resolution for representing the geometry of a volume or a function (like a texture) based on vertex based schemes is obtained. In the Figure 3 a color function over the tetrahedrical net is presented. Beginning with this net, a decomposition in a coarser resolution (in color) and the details (gray levels) is presented.

The *analysis* in case of volumes is also characterized by the matrices A_j and B_j whose columns correspond to the analysis filters. These filters can be inverted to obtain the *synthesis* filters P_j and Q_j. The synthesis, that is the construction of the original polyhedron from the coarsest resolution and the details, have two steps

– **Partitioning**: each low resolution tetrahedron is partitioned on eight tetrahedra by introducing new vertex on the original one, according to the chosen subdivision method.

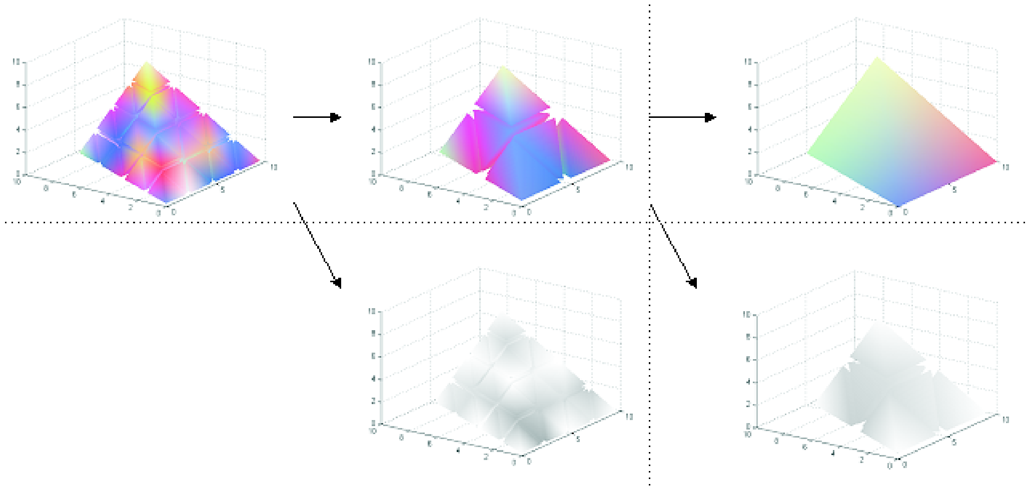

Fig. 3. Function over volume refinement

– **Perturbation**: each outcoming vertex is perturbated according to the wavelet coefficients.

In order to formulate the multiresolution analysis for arbitrary topological type volume, the four filters A_j, B_j, P_j and Q_j must be designed. As in the case of wavelets in the real line, there is often no analytic expression for the scaling and dual scaling functions and they are defined as the limit of an iterative procedure called the *cascade algorithm*. To do so, a set of *partitionings* and a *filter* are needed [Sweldens, 1996]. In our work the set of partitionings is obtained using refinement strategies for *simple grids* and the scaling functions are *interpolating* (see for example [Schroeder and Sweldens, 1995] and [Sweldens, 1996]). These grids produce consistent and stable tetrahedrizations.

In order to define the sequence of nested spaces V_j necessary for any MRA, we use a sequence of tetrahedra and, in this case, the *basic net* is the tetrahedrization VT_0. The simplest basic net possible is a single tetrahedron T. The net VT_1 is created from VT_0 by subdividing the tetrahedron T into eight tetrahedra of equal volume such that each vertex $T_{i_1} \in VT_1, 1 \leq i \leq 8$ is coincident either with a vertex or a midpoint of $T \in VT_0$. In this way, we first connect the edges of each triangular face of T and obtain the 4 vertex tetrahedra of the original tetrahedron that are congruent with T (T_1, T_2, T_3, T_4); from the inside tetrahedron Oct we take the other four tetrahedra an so on (see Figure 4). This tetrahedrical net is the partitionong set mentioned above.

These 8 tetrahedra have all the same volume, but the inside ones are, in general, not congruent with T. In [Bey, 1995] it is proved that a wrong election of sub tetrahedra can lead to degenerate elements. However, in that article is introduced a simple algorithm that allows to generate sub tetrahedra without obtaining those degenerate elements. Then, it is always possible to subdivide the tetrahedron VT_j in order to produce a grid VT_{j+1}. These sub tetrahedra may divided again to produce VT_2 and so on (see Figure 4). With each net VT_j we define the space V_j as the set of all real continuous functions that are linear on each tetrahedron of VT_j. These are nested spaces since

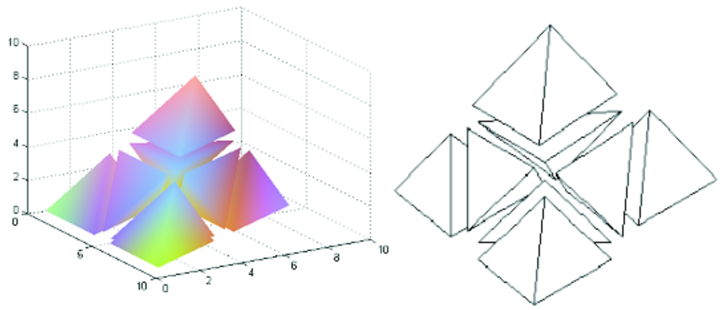

Fig. 4. Function over volume refinement

any function that is linear on the tetrahedra of VT_j is also linear on the tetrahedra of VT_{j+1}. Formally,

$$V_j \doteq \{f | f : VT_j \rightarrow \mathbb{R}, f \text{ linear on the tetrahedra of } VT_j\}$$

As the space V_j only contains piecewise linear functions, any member of V_j will be uniquely determined by its values on the vertex of VT_j. The scaling functions $\varphi_{i,j}$ that span V_j must be integrable functions since the interior product defined over the spaces V_j is based on integration. The wavelets $\psi_{i,j}$ are the basis functions of the complements spaces W_j, with support on $\mathbb{R}^3$. Specifically, we can say that a polyhedrical net with vertex $c_{J,i} = (x_{J,i}, y_{J,i}, z_{J,i})$ topologically equivalent to the basic net VT_0 is defined by a function

$$Vol(\mathbf{x}) = \sum_{i \in v(VT_J)} c_{J,i} \varphi_{J,i}(\mathbf{x}) \tag{1}$$

being

$$\begin{cases} \mathbf{x} \in VT_0 & \\ v(VT_J) & \text{set of indices that indexes the } VT_J \text{ vertices.} \end{cases}$$

Then it is possible to write the wavelet decomposition of $Vol(\mathbf{x})$ in the following way

$$\begin{aligned} Vol(\mathbf{x}) = & \sum_{i \in v(VT_J)} c_{J,i} \varphi_{J,i}(\mathbf{x}) + \\ & \sum_{j=0}^{J-1} \sum_{i \in (v(VT_{j+1}) - v(VT_j))} d_{j,i} \psi_{j,i}(\mathbf{x}), \end{aligned} \tag{2}$$

for a proper choice of wavelets $\psi_{j,i}$.

3.3 Interior product on subdivision volumes

In order to define the wavelets and then to complete de MRA definition, it is necessary to define the interior product for functions with domain on the volume; in this way, the orthogonality is then characterized. Given two functions $f,\ g \in V_j(VT_0)$, $j < \infty$, we define the *interior product* of f and g by

$$\langle f, g \rangle \doteq \sum_{\tau \in \Delta(VT_0)} \frac{1}{Volume(\tau)} \int_{\mathbf{x} \in \tau} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} \tag{3}$$

being

$$\begin{cases} d(\mathbf{x}) & \text{the differential of volume in } \mathbb{R}^3 \\ \Delta(VT_0) & \text{the set of tetrahedra of } VT_0 \\ \tau & \text{a tetrahedron of } \Delta(VT_0). \end{cases}$$

As this interior product does not depend on the geometric position of the vertex of VT_0, this allows to pre-compute them.

3.4 Wavelets construction

In what follows, we construct an interpolating scheme based on vertex. The chosen tetrahedron subdivision causes the new vertex to be on the midpoint of the tetrahedron edges; this means that, when passing from a resolution level j to a resolution level $j+1$, 6 new vertex will be added on one tetrahedron. In the resolution level j we will have an index set $\mathcal{K}_j$ that allows to index the vertex of the tetrahedron. In level $j+1$, we will have the previous set and a new set of indices $\mathcal{M}_j$ that corresponds to the introduced vertices. Then the set of vertices $\mathcal{K}_{j+1}$ in the resolution $j+1$ is $\mathcal{K}_{j+1} = \mathcal{K}_j \cup\ \mathcal{M}_j$ (see Figure 5).

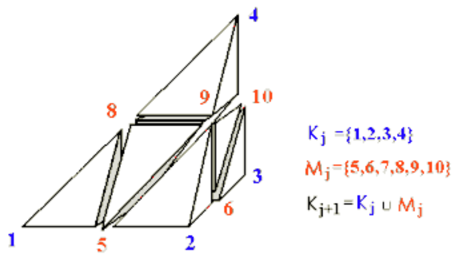

Fig. 5. Tetrahedron indexation

In what follows, we note by $c_{j,k}$ the scaling coefficients of a given function f at the resolution j and $d_{j,k}$ the wavelet or detail coefficients of that function at the same resolution level. As always, the coefficients $c_{0,k}$ are the coarsest approximation to the underlying function and the process begins with a given set of coefficients $c_{N,k}$ where

N is some finest resolution level. As we shall write an interpolating scheme for deriving a vertex basis, the unlifted scaling coefficients are just subsampled in the analysis and upsampled in the synthesis; meanwhile for computing the wavelet coefficients it is necessary to make some calculations.

Analysis

$$\begin{array}{ll} \forall k \in \mathcal{K}_j, & c_{j,k} \doteq c_{j+1,k}, \\ \forall m \in \mathcal{M}_j, & d_{j,m} \doteq c_{j+1,m} - \sum_{k \in \mathcal{K}_m} s_{j,k,m} c_{j,k}. \end{array} \tag{4}$$

Synthesis

$$\begin{array}{ll} \forall m \in \mathcal{K}_j, & c_{j+1,k} \doteq c_{j,k} \\ \forall m \in \mathcal{M}_j, & c_{j+1,m} \doteq d_{j,m} - \sum_{k \in \mathcal{K}_m} s_{j,k,m} c_{j,k}. \end{array} \tag{5}$$

Following the above scheme, we begin with a basic interpolatory form for analysis and synthesis

$$\begin{array}{l} d_{j,m} \doteq c_{j,m} - \frac{1}{2}\left(c_{j+1,u} + c_{j+1,v}\right) \\ c_{j,m} \doteq d_{j,m} - \frac{1}{2}\left(c_{j+1,u} + c_{j+1,v}\right), \end{array}$$

that is $s_{j,k,m} = \frac{1}{2}$ in equations (4) and (5).

We now proceed to use the lifting scheme in order to obtain wavelets with one vanishing moment. We start with the wavelets proposed by Schröder and Sweldens [Schroeder and Sweldens, 1995] given by

$$\psi_{j,m} = \varphi_{j+1,m} - s_{j,u,m}\varphi_{j,u} - s_{j,v,m}\varphi_{j,v} \tag{6}$$

that is, the wavelet on a midpoint of an edge is defined as a linear combination of the scaling function on the midpoint $(j+1, m)$ and two scaling functions defined on a coarser resolution computed on the extrema value (u, v) of the segment whose midpoint is m.

The weights $s_{j,*,m}$ are chosen in such a way that the resulting wavelet has null integral

$$\begin{aligned} \int_V \psi_{j,m} &= \int_V \psi_{j+1,m} dV - \int_V s_{j,u,m}\varphi_{j,u} dV - \\ &\quad \int_V s_{j,v,m}\varphi_{j,v} dV \\ &= \int_V \psi_{j+1,m} dV - s_{j,u,m}\int_V \varphi_{j,u} dV - \\ &\quad s_{j,v,m}\int_V \varphi_{j,v} dV. \end{aligned} \tag{7}$$

$$s_{j,*,m} = \frac{\int_V \varphi_{j+1,m} dv}{2\int_V \varphi_{j,*}} = \frac{I_{j+1,m}}{2I_{j+1,*}}. \quad (8)$$

The integral $\int_V \varphi dV$ can be approximated at the finest resolution level using a quadrature method and then recursively computed on the coarser levels using the refinement relationships. Then it is possible to express $\psi_{j,m}$ as follows

$$\psi_{j,m} = \varphi_{j+1,m} - \frac{I_{j+1,m}}{2I_{j+1,*}} \varphi_{j,k}, \quad (9)$$

so the analysis and synthesis algorithms for computing the fast wavelet transform that give the lifted set of coefficients are the following:

Analysis

- Compute the detail coefficients

$$\forall m \in \mathcal{M}_j d_{j,m} \doteq c_{j+1,m} - \frac{1}{2}\left(c_{j+1,u} + c_{j+1,v}\right) \quad (10)$$

- Calculate the coefficients $c_{j,k}$

$$\begin{array}{c} \forall k \in \mathcal{K}_j, \quad c_{j,k} = c_{j+1,k} \\ \forall m \in \mathcal{M}_j, u, v \in \mathcal{K}_j \begin{cases} c_{j,u} = c_{j,u} + s_{j,u,m} d_{j,m} \\ c_{j,v} = c_{j,v} + s_{j,v,m} d_{j,m} \end{cases} \end{array} \quad (11)$$

Synthesis

- Compute the $c_{j+1,k}$

$$\begin{array}{c} \forall k \in \mathcal{K}_j, \quad c_{j+1,k} = c_{j,k} \\ \forall m \in \mathcal{M}_j, u, v \in \mathcal{K}_j \begin{cases} c_{j,u} = c_{j,u} - s_{j,u,m} d_{j,m} \\ c_{j,v} = c_{j,v} - s_{j,v,m} d_{j,m} \end{cases} \end{array} \quad (12)$$

- Use the $c_{j+1,k}$ already computed in order to compute the $c_{j+1,m}$

$$\forall m \in \mathcal{M}_j, \; c_{j+1,m} \doteq d_{j,m} + \frac{1}{2}\left(c_{j+1,u} + c_{j+1,v}\right) \quad (13)$$

4 Example

In order to show how the defined wavelets are used, we choose to represent a density function on a tetrahedon using the scaling functions representation. This density is mapped on a color on each vertex of a tetrahedra. The idea is to treat each color

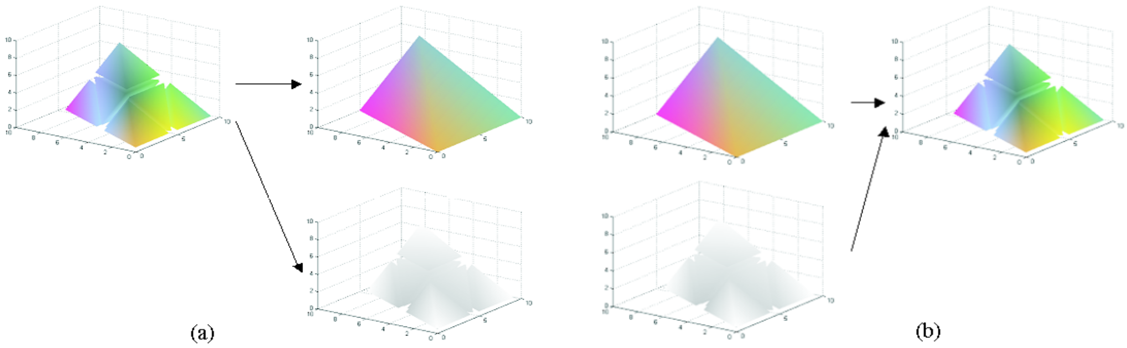

Fig. 6. One step in Analysis and in Synthesis

component (red, green and blue) as a scalar function defined on the base mesh VT_0. Each color function can be converted to multiresolution form using the filter bank analysis and the wavelet coefficients. In Figure 6 is shown one step in the analysis (a) and one step in the synthesis (b).

In the Figure 7, it is possible to visualize the density representation over diferent steps of resolutions, going from a fine resolution to the coarsest one. We only present here three different resolutions to make possible a good visualization.

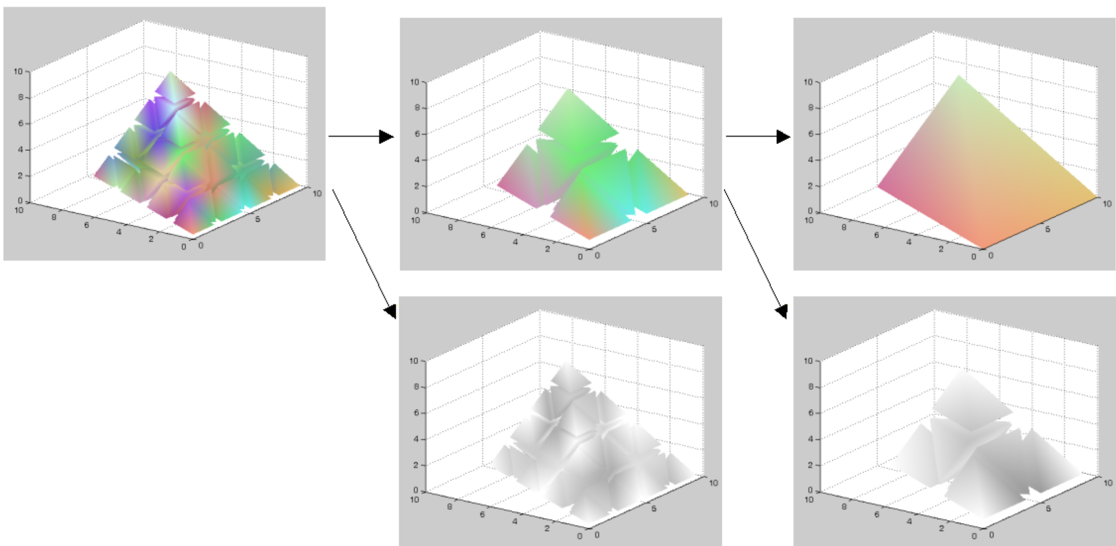

Fig. 7. Decomposition of density from finest to coarsest resolution

5 Conclusions

Nowadays, surfaces are the supporting framework for modeling objects in Computer Graphics. There exist many algorithms to deal with surfaces; in fact many workstations and PCs are specially designed to both process and render surfaces and its polygonal approximation. During the last years, algorithms and hardware systems to produce volume rendering have been intensively studied. Nevertheless, only a short time ago volumetric models for that kind of data have been developed. In this case, the methods have mainly pointed to the representation and modeling the attributes of the 3D objects and their interior. In this case, the main emphasis is given to the inside, contrary to the surface methods, which assume that the interior of the volume is homogeneous.

The wavelets have been proved to be a powerful tool to represent general functions and large datasets accurately. In this paper, a method, based on wavelets, for representing functions defined over tetrahedrical grids has been presented. This is achieved by a generalization of MRA to arbitrarily topological type and using then the lifting scheme. These hierarchical representations are of great importance since they provide the opportunity for compression, multiresolution editing and net progressive transmission of volumes.

Now, based on this scheme we are working on new alternatives for the compression of volume models. Actually, is very important to obtain new approaches to 3D data models allowing interactive visualization and exploration of large datasets on PCs and across networks. This pretends to make visualization technology accesible to a much wider range of users.

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