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Petar Bojanić
Sanja Todorović

Realism's Understanding of Negative Numbers

Abstract Our topic is the understanding of the nature of negative numbers – the entities to which expressions such as '-1' refer. Following Frege, we view positive whole numbers as providing the answer to the question „how many?" In this light, how are we to view negative numbers? Both positive and negative numbers can be ordered through the relation of larger or smaller. It is then true of all negative numbers that they are entities which are (somehow) smaller than zero. For many, this has been understood as an ontological paradox: how can something be „less than nothing?"

Some propose to avoid the paradox by treating negative numbers as mere *façons de parler*. In this paper, we propose a more realist account, taking as our starting point the thesis that there is at least one familiar type of object, the magnitude of which can be expressed with negative numbers, namely, debt. How can the sense of an expression be ontologically paradoxical, yet the expression itself still plausibly refer to a social object such as a debt? Or, put differently, how is it possible to be, at the same time, a realist in financial theory and a nominalist in mathematical theory? The paper first shows that the paradox arises when the two distinct ways in which negative numbers are connected to real objects are run together. The first of the two refers to debt only, whereas the second could refer to debt, as well as to physical objects. Finally, we claim that a debt is at once a specifically social object and part of reality as described by physics.

Keywords: Negative Numbers; Debt; Realism; Magnitude; Object

Introduction

Why should a realist consider the problem of reality of negative numbers? Surely, the widespread application of negative numbers in physics ought to be enough to justify their reality? Indeed, the indispensability argument, first formulated by Euler, is used to show that by being applied in a real context (physics), negative numbers have reality themselves. Putnam says as much in *What is Mathematical Truth?*: „Mathematical experience teaches us that mathematics is true in a certain interpretation; physical experience teaches us that the given interpretation is realistic." (Putnam 1975:74)

However, the indispensability argument rests on a specific interpretation of negative numbers. Our contention is that there is at least one application of negative numbers that is not covered by this interpretation. This application is the social object of debt. As a familiar quotidian object, debt provides a specific application of negative numbers. However, if we are correct, it also

Petar Bojanić: Principal Research Fellow, Institute for Philosophy and Social Theory, University of Belgrade, bojanic@instifdt.bg.ac.rs.

Sanja Todorović: Centre for Ethics, Law and Applied Philosophy, Belgrade.

rests on an interpretation of negative numbers that potentially runs into ontological problems.

Negative numbers

Negative numbers simply do not have as universal an application as natural numbers. Understood naively, natural numbers are understood as answers to the question „how much"? Their main property is to denote a quantity of something. Zero, here, would denote an absence of something. In other words, zero would mean the same as nothing. In this view, negative numbers would appear paradoxical, as it would be meaningless to speak of a quantity smaller than zero.

Set up this way, as answers to the question „how much", numbers could be added infinitely; subtraction, on the other hand, comes with limits, i.e. prohibitions. If numbers denote an amount of something, and negative numbers are paradoxical because denoting something that is „less than nothing", then there has to be a prohibition of subtracting a greater number from a smaller.

However, negative numbers afforded mathematicians the opportunity to perform a more universal algebraic operation. They allowed mathematicians to perform both addition and subtraction to infinity. Negative numbers was for a long time seen as syncategorematic: necessary for the language of mathematics, but without real meaning.

In order to avoid an ontological paradox of „less than nothing" and give reality to negative numbers, beginning in the 13th century, mathematicians resorted to the concept of debt. Indeed, the history of mathematics from the Renaissance to the end of the 19th century is partially the history of the attempt of geometry and physics to arrive at new interpretations of negative magnitudes. Around the same time that Euler offers his indispensability argument, a new, geometric interpretation also replaces the „unscientific" example of debt as negative magnitude: the oriented line segment. (Euler 1822: 324)

Such interpretation included direction as an additional determination of a given magnitude (force, trajectory, etc.). This meant that negative numbers were understood as a quantity with an additional determination (e.g. Bolzano). As Carl Friedrich Gauss (1831) puts it in a sentence quoted by Frege in §162 of *Grundgesetze*: „Positive and negative numbers can find application only where that which is counted has an opposite, so that thinking them in union amounts to annihilation." (Frege 2013: 159)

What made this interpretation of negative numbers significant was that it allowed for a framework in which they could stand alongside positive whole numbers and refer to something existing. The domain in which whole numbers (positive, negative, zero) could be applicable, had to be composed of two mutually opposed parts. This has been the way physics has understood negative numbers and applied them to a wide range of phenomena: coordinates, electric charge, force vectors, temperature scales, etc.

This framework requiring two opposing parts has proved incredibly useful for the domain of physics.

In constructing this framework, however, science relegated debt to a secondary, unexplained application of negative numbers. Thus, in *The Road to Reality*, Roger Penrose asks: „Negative integers certainly have an extremely valuable organizational role, such as with bank balances and other financial transactions. But do they have direct relevance to the physical world?“ (Penrose 2007: 63) The application of negative numbers in financial theory does not conform to the established scientific framework. In debt, negative numbers are not understood as a positive quantity with an additional determination. Rather they describe a real lack or deficiency. Debt represents a very specific example of negative numbers that does not fit the concept of negative magnitude as it appears in physics.

The application of negative numbers as it appears in debt is not some speculative model in mathematical logic. Rather, it is a model emerging from a social application of negative numbers. Debt is a real application that no one doubts. Yet, as an example of negative magnitudes, the interpretation applied in debt still seems somehow less real than the one in physics.

Frege, Dedekind

Both physics and financial theory use the same negative magnitudes. The difference between the interpretation used in physics, and the one that would have to be used for debt is not to be found on the algebraic structure. What, then is this difference? The difference is in the way negative numbers are founded in order to avoid the paradox of less than nothing. Now, if numbers are founded axiomatically, then the entire set of real numbers – positive, negative, ration, irrational – is simply given, thus avoiding a paradox. However, such founding also tells us nothing about the application of numbers to reality.

On the other hand, if we follow the genetic method of founding negative numbers, we encounter two paths. (The genetic method was so called by David Hilbert, wishing to distinguish it from his own axiomatic method of founding numbers.) Although many mathematicians worked on this problem, for the purposes of this paper we will refer to the two genetic methods of founding numbers as the Frege method and the Dedekind method, because they are the two methods' most prominent and influential thinkers.

For both Frege and Dedekind what determines the nature of numbers is their application. The most basic application of natural numbers is counting. Each begins with this application of numbers and then extends the set of natural numbers to reach the set R. Frege begins with children's, mercantile numbers, while Dedekind begins with the scientific application. Frege considers numbers as the answer to the question 'how much', while what is important for Dedekind is their order. For Frege, natural numbers are finite cardinal numbers, while for Dedekind, they are finite ordinal numbers. (Dedekind 1969: 2)[1]

This, we reckon, is the clearest way of showing this specific difference in interpretation of negative numbers. Each interpretation will have a corresponding application in reality. One application is particular to physics; the other to social practice (financial and banking transactions). If we follow the Dedekind method, the appearance of negative numbers presents no problem. Dedekind obtained numbers by identifying them with points on a real line such as it is given to perception, and then sought logical emancipation from geometric intuition. If the nature of numbers is to be found in recursive progression, then going from 1 to 0 to -1 conforms to their nature. This construction of negative numbers as ordered pairs of natural numbers[2] can be found in mathematical literature from van der Vaerden's *Modern Algebra*, to, for example, Birkhoff and MacLane's *Algebra*. In this case, both addition and subtraction can be performed without any prohibition.

However, should we follow the Frege method, we run into problems. His approach to natural numbers as cardinal (*Anzahlen*) makes the prohibition of subtraction ontological in nature. If numbers are children's, mercantile numbers, and are the answer to the question 'how much', then the subtraction of greater number from smaller lapses into ontological paradox. Frege himself had to split natural numbers from positive whole numbers in order to avoid the paradox of 'less than nothing'. Once split, positive numbers can be regarded as inextricably tied to negative numbers. However, this entirely transforms the initial set of natural numbers.

It is important to note that the potential paradox, 'less than nothing', is not simply a matter of naïve understanding. The paradox will present itself every time one begins from natural numbers in the way they are understood in everyday life, i.e. as the answer to the question 'how much'. In *Nachlaß,*

1 Dedekind's letter to H. Weber of 24 January 1888 (Dedekind 1969: 488–490). We cite this letter in particular because in it Dedekind explicitly points out that he considers the ordinal number more original than the cardinal. Considering that we are here interested in Dedekind only to the extent that he represents the position that departs form the ordinal number, we are not citing all the places in the more canonical *Continuity and Irrational Numbers* and *What Are Numbers and What Should They Be?*, although we are of course referring to those.

2 The same is true as in the previous footnote. This is an important text for our argument because in it Dedekind explicitly constructs the extension of the term number on the basis of sequence of natural numbers in the modern way: $NxN/\sim$, where $\sim$ is the corresponding relation of equivalence between pairs of natural numbers.

Numbers and Arithmetic (1924–1925), Frege links the failure of his project to ground arithmetic with this very starting point – the quotidian use of numbers. A better starting point, continues Frege, perhaps with Dedekind in mind, is geometry.

Dedekind's, „geometric" founding of numbers is indeed the interpretation of negative numbers used in physics. But the use of negative numbers in physics does not express any actual lack or deficiency. Negative charge, negative direction on the coordinate system, negative particles – none of these has anything to do with insufficiency. They are equally extant as their positive counterparts, but with an additional designation of the negative sign.

A negative bank balance, on the other hand, expresses something different. In accounting (as a type of financial formal language), a negative magnitude is an expression of actual lack. In this, in order for debt to be an expression of real absence, it must rely on a different concept of negative magnitude. Debt is the only application of negative numbers where they appear as truly negative magnitudes. Only in this case is it possible to realize the subtraction of greater number from smaller, understood as amount with no further determination, without lapsing into ontological paradox. The question of the interpretative framework that allows for such application is certainly interesting, but beyond the scope of the present article.

Conclusion

At present, it seems to us that the indispensability argument, used to great effect in physics, does not cover the interpretation of negative numbers found in their application in debt. However, we fail to see why this interpretation of negative numbers could not have its reality recognized precisely by its successful application in financial theory. It seems to us inconsistent to claim that the interpretation of negative numbers as debt is ontologically paradoxical, while also holding trust in one's bank and financial transactions. A realist would demand to acknowledge the reality of the successful application of negative numbers in financial theory or bank transactions. This would require the argument of indispensability to broaden its scope to include certain kinds of social objects.

The indispensability argument need not be applied exclusively to mathematics, nor be particularly bound to a given historical moment. Fibonacci introduced negative numbers into mathematics implicitly applying the indispensability argument. Had it been explicitly applied in the 13th century, the indispensability argument, as the only grounding for the reality of negative numbers, would undoubtedly have had the reality of debt. Euler supported the reality of space and time based on their indispensability to Newton's laws. He also attempted to justify the reality of negative magnitudes and rescue them from the seeming paradox of being less than nothing. In so doing, he interpreted negative magnitudes as amount of debt. It could be easily shown that since then, the basic meaning of debt has not changed. Nor indeed the meaning in which negative numbers refer to debt. If there were no other way today to show the reality of negative numbers, debt would still be a sound basis for the argument of indispensability of negative numbers. Such a position would be entirely compatible with realism.

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Petar Bojanić
Sanja Todorović

Razumevanje negativnih brojeva u realizmu

Apstrakt

Naša tema je razumevanje prirode negativnih brojeva – entiteta na koje referišu izrazi kao što je '-1'. Sledeći Fregea, mi razmatramo cele pozitivne brojeve kao one koji daju odgovor na pitanje: „koliko"? U svetlu toga, kako bi trebalo da razmatramo negativne brojeve? I pozitivni, i negativni brojevi mogu biti ponizani kroz relaciju većeg ili manjeg. U tom je slučaju istinito za sve negativne brojeve da su entiteti koji su (nekako) manji od nule. Za mnoge, ovo se smatralo za ontološki paradoks: kako nešto može biti „manje od ničega"?
Da bi se izbegao paradoks, neki predlažu da se negativni brojevi tretiraju kao puki *façons de parler*. U ovom članku predlažemo objašnjenje koje je više realističko, pri čemu kao početnu tačku uzimamo tezu da postoji najmanje jedan poznati tip objekta čija veličina može da se izrazi negativnim brojevima, a to je dug. Kako može da smisao izraza bude ontološki paradoksalan, a da ipak izraz kao takav plauzibilno referiše na socijalni objekt kao što je dug? Ili, drugačije rečeno, kako je moguće da se bude istovremeno realista u finansijskoj teoriji i nominalista u matematičkoj teoriji? Ovaj članak pokazuje, najpre, da paradoks nastaje kada se dva distinktna načina na koji su negativni brojevi povezani sa stvarnim objektima uzimaju kao objedinjeni. Prvi od načina referiše samo na dug, dok drugi može da referiše kako na dug, tako i na fizičke objekte. Na kraju, tvrdimo da je dug ujedno i specifično društveni objekt, i deo stvarnosti koji opisuje fizika.

Ključne reči: Negativni brojevi, dug, realizam, magnituda, objekt