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Invers Of Tridiagonal Toeplitz Matrix By Adjoin Method

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***Abstract*— Invers of a matrix is important in mathematics, specially in algebra. Invers of matrix applicate in many sectors, either in mathematics or another sector. There are many methods to determine the invers of a matrix, one of them is adjoin method. The adjoin method is a simple method to determine the invers of a matrix. This paper want to determine the general formula to find the invers of a Tridiagonal Toeplitz Matrix by adjoin method. There are three steps to determine invers of a tridiagonal toeplitz matrix. The first one, find general formula of determinant of tridiagonal toeplitz matrix. The second one, find general formula of cofactor matrix of tridiagonal toeplitz matrix. And the last one, find the general formula of invers of tridiagonal toeplitz matrix.**

***Keywords*— adjoin, determinant, invers of matrix, cofactor matrix, tridiagonal toeplitz matrix**

I. Introduction

Howard Anton dan Chris Rorres (2004) define that matrix is square array of numbers. One of kind of matrix is toeplitz matrix. Robert (2005) define toeplitz matrix as a symmetric and circulant matrix, which is every element in the main diagonal are same and every element in corresponding subdiagonal with its main diagonal are also same. General formula of toeplitz matrix is followed.

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$$T_n = (t_{ij}) \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-2)} & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-(n-3)} & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \cdots & t_{-(n-4)} & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{(n-2)} & t_{(n-3)} & t_{(n-4)} & \cdots & t_0 & t_{-1} \\ t_{(n-1)} & t_{(n-2)} & t_{(n-3)} & \cdots & t_1 & t_0 \end{bmatrix} \quad (1)$$

Which is t_{ij} is element in the i-th row and j-th column.

One of kind of toeplitz matrix is tridiagonal toeplitz matrix. Salkuyeh (2006) define this matrix as a matrix with form :

$$A = \begin{bmatrix} b & c & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 & 0 \\ 0 & a & b & c & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & a & b \end{bmatrix} \quad (2)$$

with $a, c \neq 0 \in \mathbb{R}$.

A matrix is invertible if determinant of the matrix is not same with 0. There are many methods to determine invers of a matrix, they are substitution, partition of matrix, adjoin matrix, Gauss Elimination, Gauss-Jordan Elimination, Multiply elementer invers matrix, and matrix decomposition LU. If the order of a matrix bigger then we are more difficult to determine its invers, so we need a better formula to determine invers of a special matrix.

Bakti Siregar et. al. (2014) have found a general formula to determine invers of toeplitz matrix with special form as followed.

$$T_n = \begin{bmatrix} 0 & x & \cdots & x \\ x & 0 & \cdots & x \\ \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & 0 \end{bmatrix} \forall x \in \mathbb{R} \quad (3)$$

The result of their research are followed :

1. Formula to calculate determinant of a toeplitz matrix order n in (3) is

$$\det(T_n) = (-1)^n (n-1) x^n$$

2. Formula to determine cofactor matrix $[K_{ij} T_n]$ order n in (3) is

$$K_{ij} T_n = \begin{cases} \det(T_n) & ; \text{if } i = j \\ (-1)^{n+1} x^{n-1} & ; \text{if } i \neq j \end{cases}$$

3. Formula to determine invers of toeplitz matrix order n in (3) is

$$T_n^{-1} = t_{ij} = \begin{cases} \dfrac{-n-2}{(n-1)x} & ; \text{if } i = j \\ \dfrac{1}{(n-1)x} & ; \text{if } i \neq j \end{cases}$$

From this result, we can see that there is a formula to calculate determinant, to determine cofactor matrix and invers of a toeplitz matrix with form in (3). So, we can easier to find determinant, cofactor matrix, and invers of the matrix, that is only input value of n and x of the matrix to formula of determinant, cofactor matrix, and invers of the matrix which have found.

In this paper, we determine the general formula to determine determinant, cofactor matrix, and invers of tridiagonal toeplitz matriks in (2).

II. TRIDIAGONAL TOEPLITZ MATRIX

Definition 1 (R.M. Gray, 2005) A Toeplitz matrix is an $n\,x\,n$ matrix $T_n = [t_{kj}; k, j = 0,1, \dots, n-1]]$ where $t_{kj} = t_{k-j}$, a matrix of the form

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \cdots & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{(n-1)} & t_{(n-1)} & \cdots & t_1 & t_0 \end{bmatrix}$$

Based Definition 1, then are different kinds Of Toeplitz matrix, within one is Toeplitz tridiagonal matrix.

Definition 2 (Sukluyeh, 2006) A matrix is an $n\,x\,n$ said matrix tridiagonal Toeplitz matrix with orde n if a matrix of the form

$$A = \begin{bmatrix} b & c & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 & 0 \\ 0 & a & b & c & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & a & b \end{bmatrix} \quad \text{dengan } a, c \neq 0 \in \mathbb{R}.$$

There are several methods to determine the determinant of a square matrix, namely Sarrus Method, Minor and Cofactor Method, Chio Method, Gaussian Elimination Method and Matrix Decomposition Method. The author simply using minor and cofactor in finding the determinant of a matrix.

A matrix A has an inverse or invertible can be seen from the determinant of the matrix A. If $\det(A) \neq 0$ means that the matrix A has an inverse. Determining the inverse of a matrix can use adjoin method. Adjoin method obtained from transpose of matrix cofactors and denoted by $adj\,(A)$. So, inverse of matrix A is:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Toeplitz matrix covered by Bakti Siregar, et al in 2014 was T_n Toeplitz matrix as (.3). The matrix will be determined on the determinants, matrix cofactors, and its inverse, with the following steps:

a. Determinant of Toeplitz Matrix

To obtain the value of the determinant formula Toeplitz matrix is done by examining the pattern matrix Toeplitz determinants T_n the order of 2×2 to 7×7 by using elementary row operations. The operation can be seen in the following process:

1. Suppose Toeplitz matrix of order 2×2 is $T_2 = \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$ where $\forall x \in \mathbb{R}$ thus obtained

$$|T_2| = \begin{vmatrix} 0 & x \\ x & 0 \end{vmatrix} (B_1 \leftrightarrow B_2) - \begin{vmatrix} 0 & x \\ x & 0 \end{vmatrix} = -x^2 \text{ then } |T_2| = -x^2$$

2. Suppose Toeplitz matrix of order 3×3 is $T_3 = \begin{bmatrix} 0 & x & x \\ x & 0 & x \\ x & x & 0 \end{bmatrix}$

where $\forall x \in \mathbb{R}$ thus obtained

$$|T_3| = \begin{vmatrix} 0 & x & x \\ x & 0 & x \\ x & x & 0 \end{vmatrix} (B_1 \leftrightarrow B_3) - \begin{vmatrix} x & x & 0 \\ x & 0 & x \\ 0 & x & x \end{vmatrix} (B_1 - B_2)$$
$$- \begin{vmatrix} x & x & x \\ 0 & x & -x \\ 0 & x & x \end{vmatrix} (B_2 - B_3) = - \begin{vmatrix} x & x & 0 \\ 0 & x & x \\ 0 & 0 & -2x \end{vmatrix} = 2x^3,$$

The process for obtaining the determinant of the matrix Toeplitz T_4, T_5, T_6 and T_7 can be searched in the same way in order to get the results in tabular form as shown below

TABLE 1. DETERMINANT TOEPLITZ MATRIX T_n

No	Toeplitz Matrix T_n	Determinant
1	T_2	$-x^2$
2	T_3	$2x^3$
3	T_4	$-3x^4$
4	T_5	$4x^5$
5	T_6	$-5x^6$
6	T_7	$6x^7$

From Table 1 can be obtained that the pattern of the value of the determinant of the matrix Toeplitz T_n seen in Theorem 1.

Theorem 1 (Bakti Siregar, dkk, 2014) Suppose T_n a Toeplitz matrix of order $n \geq 2$ in (3) where the value of the determinant of the matrix $\forall x \in \mathbb{R}$ T_n is

$$|T_n| = (-1)^{n+1}(n-1)x^n$$

b. Matriks kofaktor dari matriks toeplitz

To determine the inverse matrix T_n, necessary cofactors of the matrix T_n. Formula cofactors T_n matrix can be seen in Theorem 2.

Theorem 2 (Bakti Siregar, dkk, 2014) Suppose T_n a Toeplitz Matrix of order $n \geq 2$ in (3) where $\forall x \in \mathbb{R}$ the cofaktors matrix Toeplitz T_n is

$$K_{ij}T_n = \begin{cases} |T_{n-1}|, & untuk\ i = j \\ (-1)^{n+1}x^{n-1}, & untuk\ i \neq j \end{cases}$$

where $K_{ij}T_n$ kofaktors which is the-i th row and j-th column.

Theorem 2 show that cofaktors matrix T_n secara umum, so in Theorem 3 will be show inveres matrix T_n obtained with use method adjoind matrix T_n.

Theorem 3 (Bakti Siregar, dkk, 2014) Suppose T_n a Toeplitz matrix of order $n \geq 2$ in (3) where $\forall x \in \mathbb{R}$ and $|T_n| \neq 0$ then inverse topelitz matrix T_n is

$$T_n^{-1} = t_{ij} = \begin{cases} \dfrac{-(n-2)}{(n-1)x}, & untuk\ i = j \\ \dfrac{1}{(n-1)x}, & untuk\ i \neq j \end{cases}$$

t_{ij} are entries which is the-i th row and j-th column

III. MAIN RESULT

To obtain the value of the determinant formula tridiagonal Toeplitz matrix is done by examining the pattern matrix tridiagonal Toeplitz determinants T_n the order of 3x3 to 20x20 and then we'll make a common form of the determinant of the tridiagonal Toeplitz matrix. Seen in theorem 3 below:

Theorem 4. Given A_n a toeplitz tridiagonal matrix of order $n \geq 3$ in (2) where $a, b, c \in \mathbb{R}$ then the value of the determinant of matrix A_n is:

$$\begin{aligned}
|A_n| &= b^n - (n-1)ab^{n-2}c + \sum_{i=1}^{n-3} i a^2 b^{n-4} c^2 - \left(\sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \cdots + \sum_{i=1}^{n-5} i \right) a^3 b^{n-6} c^3 \\
&+ \left[\frac{(n-7)}{1!} \sum_{i=1}^{1} i + \frac{(n-8)}{1!} \sum_{i=1}^{2} i + \frac{(n-9)}{1!} \sum_{i=1}^{3} i + \cdots + 1 \sum_{i=1}^{n-7} i \right] a^4 b^{n-8} c^4 \\
&- \left[\frac{(n-9)(n-8)}{2!} \sum_{i=1}^{1} i + \frac{(n-10)(n-9)}{2!} \sum_{i=1}^{1} i + \cdots + 1 \sum_{i=1}^{n-9} i \right] a^5 b^{n-10} c^5 \\
&+ \left[\frac{(n-11)(n-10)(n-9)}{3!} \sum_{i=1}^{1} i + \frac{(n-12)(n-11)(n-10)}{3!} \sum_{i=1}^{1} i + \cdots + 1 \sum_{i=1}^{n-11} i \right] \\
&a^6 b^{n-12} c^6 \\
&- \left[\frac{(n-13)(n-12)(n-11)(n-10)}{4!} \sum_{i=1}^{1} i + \frac{(n-14)(n-13)(n-12)(n-11)}{4!} \sum_{i=1}^{1} i + \cdots + 1 \sum_{i=1}^{n-13} i \right]^7 b^{n-14} c^7 + \ldots
\end{aligned} \tag{4}$$

Proof of the theorem by using mathematical induction

Proof:

Prove of the theorem by using mathematical induction

1. for $n = 3$ then apply

$$|A| = b^3 - 2abc + 0$$

$$= b^3 - 2abc$$

,true.

2. Assume true for $n = k$ true, ie

$$|A_k| = b^k - (k-1)ab^{k-2}c + \sum_{i=1}^{k-3} ia^2b^{k-4}c^2 - \left(\sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \cdots + \sum_{i=1}^{k-5} i\right)a^3b^{k-6}c^3$$

$$+\left[\frac{(k-7)}{1!}\sum_{i=1}^{1} i + \frac{(k-8)}{1!}\sum_{i=1}^{2} i + \frac{(k-9)}{1!}\sum_{i=1}^{3} i + \cdots + 1\sum_{i=1}^{k-7} i\right]a^4b^{k-8}c^4$$

$$-\left[\frac{(k-9)(k-8)}{2!}\sum_{i=1}^{1} i + \frac{(k-10)(k-9)}{2!}\sum_{i=1}^{1} i + \cdots + 1\sum_{i=1}^{k-9} i\right]a^5b^{k-10}c^5$$

$$+\left[\frac{(k-11)(k-10)(k-9)}{3!}\sum_{i=1}^{1} i + \frac{(k-12)(k-11)(k-10)}{3!}\sum_{i=1}^{1} i + \cdots + 1\sum_{i=1}^{k-11} i\right]$$

$$a^6b^{k-12}c^6$$

$$-\left[\frac{(k-13)(k-12)(k-11)(k-10)}{4!}\sum_{i=1}^{1} i + \frac{(k-14)(k-13)(k-12)(k-11)}{4!}\sum_{i=1}^{1} i + \cdots + 1\sum_{i=1}^{k-13} i\right]$$

$$a^7b^{k-14}c^7 + \cdots$$

3. Will be shown for $n = k + 1$ is also true,
Note that

$$|A_{k+1}| = b|A_k| - ac|A_{k-1}|$$

$$= b^{k+1} - (k-1)\,ab^{k-1}c + \sum_{i=1}^{k-3} i.\ a^2b^{k-3}c^2 - \left(\sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \cdots + \sum_{i=1}^{k-5} i\right)a^3b^{k-5}c^3$$

$$+\left[\frac{(k-7)}{1!}\sum_{i=1}^{i}\ i + \frac{(k-8)}{1!}\sum_{i=1}^{2}\ i + \frac{(k-9)}{1!}\sum_{i=1}^{3}\ i + \cdots + 1\cdot\sum_{i=1}^{k-7}\ i\right]a^4b^{k-7}c^4$$

$$-\left[\frac{(k-9)(k-8)}{2!}\sum_{i=1}^{1} i + \frac{(k-10)(k-9)}{2!}\sum_{i=1}^{2} i + \cdots + 1\cdot\sum_{i=1}^{k-9} i\right]a^5b^{k-9}c^5$$

$$+\left[\frac{(k-11)\ (k-10)\ (k-9)}{3!}\sum_{i=1}^{1} i + \frac{(k-12)\ (k-11)\ (k-10)}{3!}\sum_{i=1}^{2} i + \cdots + 1.\sum_{i=1}^{k-11} i\right]a^6b^{k-11}c^6$$

$$+\cdots+\left[\frac{(k-13)(k-12)(k-11)(k-10)}{4!}\sum_{i=1}^{1} i + \cdots + \frac{(k-14)(k-13)(k-12)(k-11)}{4!}\sum_{i=1}^{2} i + \cdots + 1.\sum_{i=1}^{k-13} i\right]a^7b^{k-13}c^8$$

$$+\ldots - ac\left\{b^{k-1} - (k-2)\ ab^{k-3}c + \sum_{i=1}^{k-4} i\,.a^2b^{k-5}c^2 - \left(\sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \cdots + \sum_{i=1}^{k-6} i\right)a^3b^{k-7}c^3\right.$$

$$+\left[\frac{k-8}{1!}\sum_{i=1}^{1} i + \frac{k-9}{1!}\sum_{i=1}^{2} i + (k-10)\sum_{i=1}^{3} i + \cdots\sum_{i=1}^{k-8} i\right]a^4b^{k-9}c^4$$

$$-\left[\frac{(k-10)(k-9)}{2!}\sum_{i=1}^{1} i + \frac{(k-11)(k-10)}{2!}\sum_{i=1}^{2} i \cdots + 1.\sum_{i=1}^{k-10} i\right]a^5b^{k-11}c^5$$

$$\left.+\left[\frac{(k-12)(k-11)(k-10)}{3!}\sum_{i=1}^{1} i + \frac{(k-13)(k-12)(k-11)}{3!}\sum_{i=1}^{2} i + \cdots + 1.\sum_{i=1}^{k-12} i\right]a^7b^{k-15}c^7 + \cdots\right\}$$

It means we get

$$= b^{k+1} - k\,ab^{k-1}c + \sum_{i=1}^{k-2} i\,.\,a^2b^{k-3}c^2 - \left(\sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \cdots + \sum_{i=1}^{k-4} i\right) a^3b^{k-5}c^3$$

$$+\left[(k-6)\sum_{i=1}^{1} i + (k-7)\sum_{i=1}^{2} i + (k-8)\sum_{i=1}^{3} i + \cdots + 1.\sum_{i=1}^{k-6} i\right] a^4b^{k-7}c^4$$

$$-\left[\frac{(k-8)(k-7)}{2!}\sum_{i=1}^{1} i + \frac{(k-9)(k-8)}{2!}\sum_{i=1}^{2} i + \frac{(k-10)(k-9)}{2!}\sum_{i=1}^{3} i + \cdots + 1.\sum_{i=1}^{k-8} i\right] a^5b^{k-9}c^5$$

$$+\left[\frac{(k-10)(k-9)(k-8)}{3!}\sum_{i=1}^{1} i + \frac{(k-11)(k-10)(k-9)}{3!}\sum_{i=1}^{2} i + \cdots 1.\sum_{i=1}^{k-10} i\right] a^6b^{k-11}c^6$$

$$-\left[\frac{(k-12)(k-11)(k-10)(k-9)}{4!}\sum_{i=1}^{1} i + \frac{(k-13)(k-12)(k-11)(k-10)}{4!}\sum_{i=1}^{2} i + \cdots + \sum_{i=1}^{k-12} i\right] a^8b^{k-13}c^8 + \cdots$$

Proof ends.

Specifies known cofactor matrix using the equation $C_{ij} = (-1)^{i+j}M_{ij}$. Then get the matrix of cofactors for tridiagonal Toeplitz matrix. So we can formulate the general form of the cofactor matrix to tridiagonal Toeplitz matrix of order $n \, x \, n$. The following general formulation of cofactor matrix of Toeplitz tridiagonal matrix of order $n \, x \, n$ in the theorem 5, as well as his proof using mathematical induction.

Theorem 5 Given A_n a toeplitz tridiagonal matrix of order $n \geq 3$ in (2) where $a, b, c \in \mathbb{R}$ then cofactor matrix from A_n is:

$$C_n = \begin{bmatrix} (-1)^2|A_{n-1}| & (-1)^3 a\,|A_{n-2}| & (-1)^4 a\,|A_{n-3}| & \cdots & (-1)^n a^{n-2}\,|A_1| & (-1)^{n+1} a^{n-1} \\ (-1)^3 c\,|A_{n-2}| & (-1)^4 |A_1||A_{n-2}| & (-1)^5 a\,|A_1||A_{n-3}| & \cdots & (-1)^{n+1} a^{n-3}\,|A_1||A_1| & (-1)^{n+2} a^{n-2}\,|A_1| \\ (-1)^4 c^2\,|A_{n-3}| & (-1)^5 c\,|A_1||A_{n-3}| & (-1)^6 |A_2||A_{n-3}| & \cdots & (-1)^{n+2} a^{n-4}\,|A_1||A_2| & (-1)^{n+3} a^{n-3}\,|A_2| \\ (-1)^5 c^3\,|A_{n-4}| & (-1)^6 c^2\,|A_2||A_{n-4}| & (-1)^7 c\,|A_2||A_{n-4}| & \cdots & (-1)^{n+3} a^{n-5}\,|A_1||A_3| & (-1)^{n+4} a^{n-4}\,|A_3| \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n c^{n-2}\,|A_1| & (-1)^{n+1} c^{n-3}\,|A_1||A_1| & (-1)^{n+2} c^{n-4}\,|A_1||A_2| & \cdots & (-1)^{n+n-2}\,|A_{n-2}||A_1| & (-1)^{n+n-2} a\,|A_{n-2}| \\ (-1)^{n+1} c^{n-1} & (-1)^{n+2} c^{n-2}\,|A_1| & (-1)^{n+3} c^{n-3}\,|A_2| & \cdots & (-1)^{n+n-1} c\,|A_{n-2}| & (-1)^{2n} c\,|A_{n-1}| \end{bmatrix}$$

Proof:

Consider the following matrix

$$A_n = \begin{bmatrix} b & c & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & b & c & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & b & c & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & b \end{bmatrix} \quad where \quad a, b, c \in R$$

Furthermore, we will verify every entry of the matrix cofactor. Starting from the entry cofactor matrix first row and the first column to the first row to the n-column . Furthermore, the second row and the first column to the second row and n - column. Onwards to do the same thing to the n- row and n - column . The process is given as follows:

a. Entries first line of the matrix of cofactors as follows:

$$C_{11} = (-1)^2 \begin{vmatrix} b & c & 0 & \cdots & 0 & 0 & 0 \\ a & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix}_{n-1} = (-1)^2|A_{n-1}|$$

$$C_{12} = (-1)^3 \begin{vmatrix} a & c & 0 & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix}_{n-1} = (-1)^3 a\,|A_{n-2}|$$

The same thing is done up C_{1n}, so that we can shape public for the first line entry matrix cofactors, namely:

$$C_{1n} = (-1)^{n+1} \begin{vmatrix} a & b & c & 0 & \cdots & 0 & 0 \\ 0 & a & b & c & \cdots & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & b \end{vmatrix}_{n-1} = (-1)^{n+1} a^{n-1}$$

b. Entries second line of the matrix of cofactors as follows

$$C_{21} = (-1)^3 \begin{vmatrix} c & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix}_{n-1} = (-1)^3 c \left|A_{n-2}\right|$$

$$C_{22} = (-1)^4 \begin{vmatrix} b & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & c & 0 & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix}_{n-1} = (-1)^4 \left|A_{n-2}\right| . b = (-1) \left|A_{n-2}\right| \left|A\right.$$

The same thing is done up, so that we can shape common to the second line entry matrix cofactors, namely:

$$C_{2n} = (-1)^{n+2} \begin{vmatrix} b & c & 0 & 0 & \cdots & 0 & 0 \\ 0 & a & b & c & \cdots & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & a \end{vmatrix} = (-1)^{n+2}\, b\, a^{n-2} = (-1)^{n+2} a^{n-2} \left|A_1\right|$$

c. The third line entry Cofactor matrix as follows

$$C_{31} = (-1)^4 \begin{vmatrix} c & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & c & 0 & \cdots & 0 & 0 & 0 \\ a & a & b & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix}_{n-1} = (-1)^4 c \left|A_{n-3}\right|$$

$$C_{32} = (-1)^5 \begin{vmatrix} b & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & c & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & a & b & c \\ 0 & 0 & 0 & \cdots & 0 & a & b \end{vmatrix}_{n-1}$$

$$= (-1)^5\, b\, .\, c \left|A_{n-3}\right| = (-1)^5 c \left|A_1\right| \left|A_{n-3}\right|$$

The same thing is done up, so that we can shape common to the second line entry matrix cofactors, namely:

$$C_{3n} = (-1)^{n+3} \begin{vmatrix} b & c & 0 & 0 & \cdots & 0 & 0 \\ a & b & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & a \end{vmatrix}_{n-1} = (-1)^{n+3} a^{n-3} \left|A_2\right|$$

Entries row 4,5,6 and so on is done in the same way, so that the obtained entry for the n-th row as follows:

$$C_{n1} = (-1)^{n+1} \begin{vmatrix} c & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & c & 0 & \cdots & 0 & 0 & 0 \\ a & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & a & b & c \end{vmatrix}_{n-1} = (-1)^{n+1} c^{n-1}$$

$$C_{n2} = (-1)^{n+2} \begin{vmatrix} b & 0 & 0 & \cdots & 0 & 0 & 0 \\ a & c & 0 & \cdots & 0 & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & a & b & c \end{vmatrix}_{n-1}$$

$$= (-1)^{n+2}\, b\, .\, c^{n-1} = (-1)^{n+2} c^{n-2} \left|A_1\right|$$

The same thing is done up, so that we can shape common to the second line entry matrix cofactors, namely:

$$C_{nn} = (-1)^{2n} \begin{vmatrix} b & c & 0 & 0 & \cdots & 0 & 0 \\ a & b & c & 0 & \cdots & 0 & 0 \\ 0 & a & b & c & \cdots & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & a & b \end{vmatrix}_{n-1} = (-1)^{2n} \left|A_{n-1}\right|$$

From the above calculation, obtained:

$$C_n = \begin{bmatrix} (-1)^2\left|A_{n-1}\right| & (-1)^3 a\left|A_{n-2}\right| & (-1)^4 a\left|A_{n-3}\right| & \cdots & (-1)^n a^{n-2}\left|A_1\right| & (-1)^{n+1} a^{n-1} \\ (-1)^3 c\left|A_{n-2}\right| & (-1)^4 \left|A_1\right|\left|A_{n-2}\right| & (-1)^5 a\left|A_1\right|\left|A_{n-3}\right| & \cdots & (-1)^{n+1} a^{n-3}\left|A_1\right|\left|A_1\right| & (-1)^{n+2} a^{n-2}\left|A_1\right| \\ (-1)^4 c^2\left|A_{n-3}\right| & (-1)^5 c\left|A_1\right|\left|A_{n-3}\right| & (-1)^6 \left|A_2\right|\left|A_{n-3}\right| & \cdots & (-1)^{n+2} a^{n-4}\left|A_1\right|\left|A_2\right| & (-1)^{n+3} a^{n-3}\left|A_2\right| \\ (-1)^5 c^3\left|A_{n-4}\right| & (-1)^6 c^2\left|A_2\right|\left|A_{n-4}\right| & (-1)^7 c\left|A_2\right|\left|A_{n-4}\right| & \cdots & (-1)^{n+3} a^{n-5}\left|A_1\right|\left|A_3\right| & (-1)^{n+4} a^{n-4}\left|A_3\right| \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n c^{n-2}\left|A_1\right| & (-1)^{n+1} c^{n-3}\left|A_1\right|\left|A_1\right| & (-1)^{n+2} c^{n-4}\left|A_1\right|\left|A_2\right| & \cdots & (-1)^{n+n-2}\left|A_{n-2}\right|\left|A_1\right| & (-1)^{n+n-2} a\left|A_{n-2}\right| \\ (-1)^{n+1} c^{n-1} & (-1)^{n+2} c^{n-2}\left|A_1\right| & (-1)^{n+3} c^{n-3}\left|A_2\right| & \cdots & (-1)^{n+n-1} c\left|A_{n-2}\right| & (-1)^{2n} c\left|A_{n-1}\right| \end{bmatrix}$$

Proof complete.

Of the matrix of cofactors above we will determine the adjoint of the matrix of cofactors. Adjoin determine the matrix of cofactors in a way mentransposkan the matrix cofactors, ie mengubag rows into columns and change the column into line, as follows:

$$Adj(A_n) = \begin{bmatrix} (-1)^2\left|A_{n-1}\right| & (-1)^3 c\left|A_{n-2}\right| & (-1)^4 c^2\left|A_{n-3}\right| & (-1)^5 c^3\left|A_{n-4}\right| & \cdots & (-1)^n c^{n-2}\left|A_1\right| & (-1)^{n+1} c^{n-1} \\ (-1)^3 a\left|A_{n-2}\right| & (-1)^4 \left|A_1\right|\left|A_{n-2}\right| & (-1)^5 c\left|A_1\right|\left|A_{n-3}\right| & (-1)^6 c^2\left|A_2\right|\left|A_{n-4}\right| & \cdots & (-1)^{n+1} c^{n-3}\left|A_1\right|\left|A_1\right| & (-1)^{n+2} c^{n-2}\left|A_1\right| \\ (-1)^4 a\left|A_{n-3}\right| & (-1)^5 a\left|A_1\right|\left|A_{n-3}\right| & (-1)^6 \left|A_2\right|\left|A_{n-3}\right| & (-1)^7 c\left|A_2\right|\left|A_{n-4}\right| & \cdots & (-1)^{n+2} c^{n-4}\left|A_1\right|\left|A_2\right| & (-1)^{n+3} c^{n-3}\left|A_2\right| \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n a^{n-2}\left|A_1\right| & (-1)^{n+1} a^{n-3}\left|A_1\right|\left|A_1\right| & (-1)^{n+2} a^{n-4}\left|A_1\right|\left|A_2\right| & (-1)^{n+3} a^{n-5}\left|A_1\right|\left|A_3\right| & \cdots & (-1)^{n+n-2}\left|A_{n-2}\right|\left|A_1\right| & (-1)^{n+n-1} c\left|A_{n-2}\right| \\ (-1)^{n+1} a^{n-1} & (-1)^{n+2} a^{n-2}\left|A_1\right| & (-1)^{n+3} a^{n-3}\left|A_2\right| & (-1)^{n+4} a^{n-4}\left|A_3\right| & \cdots & (-1)^{n+n-2} a\left|A_{n-2}\right| & (-1)^{2n} c\left|A_{n-1}\right| \end{bmatrix} \quad (5)$$

The third general formulation is a general formulation of the inverse matrix tridiagonal topelitz. Next we will substitute (4) and (5) the common form of all of the above equation into the general equation inverse matrix, namely

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

or

$$(A_n)^{-1} = \frac{1}{|\mathrm{A_n}|} adj(A_n)$$

IV. Conclusion

Before we find invers of tridiagonal toeplitz matrix by adjoin method, we must determine general formula for determinant and cofactor matrix of tridiagonal toeplitz matrix. After that, we get invers of tridiagonal toeplitz matrix by substitution determinant and cofactor matrix to equation $(A_n)^{-1} = \frac{1}{|\mathrm{A_n}|} adj(A_n)$. The general formula of determinant, cofactor matrix, and invers of tridiagonal toeplitz matrix are as followed.

1. The general formula of determinant of tridiagonal toeplitz matrix.

$$|A_n| = b^n - (n-1)ab^{n-2}c + \sum_{i=1}^{n-3} i a^2 b^{n-4} c^2 - \left(\sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \cdots + \sum_{i=1}^{n-5} i \right) a^3 b^{n-6} c^3$$

$$+ \left[\frac{(n-7)}{1!} \sum_{i=1}^{1} i + \frac{(n-8)}{1!} \sum_{i=1}^{2} i + \frac{(n-9)}{1!} \sum_{i=1}^{3} i + \cdots + 1 \sum_{i=1}^{n-7} i \right] a^4 b^{n-8} c^4$$

$$- \left[\frac{(n-9)(n-8)}{2!} \sum_{i=1}^{1} i + \frac{(n-10)(n-9)}{2!} \sum_{i=1}^{1} i + \cdots + 1 \sum_{i=1}^{n-9} i \right] a^5 b^{n-10} c^5$$

$$+ \left[\frac{(n-11)(n-10)(n-9)}{3!} \sum_{i=1}^{1} i + \frac{(n-12)(n-11)(n-10)}{3!} \sum_{i=1}^{1} i + \cdots + 1 \sum_{i=1}^{n-11} i \right]$$

$$a^6 b^{n-12} c^6$$

$$- \left[\frac{(n-13)(n-12)(n-11)(n-10)}{4!} \sum_{i=1}^{1} i + \frac{(n-14)(n-13)(n-12)(n-11)}{4!} \sum_{i=1}^{1} i + \cdots + 1 \sum_{i=1}^{n-13} i \right]^7 b^{n-14} c^7 + \ldots$$

2. The general formula of cofactor matrix of tridiagonal toeplitz matrix.

$$C_n = \begin{bmatrix} (-1)^2 |A_{n-1}| & (-1)^3 a |A_{n-2}| & (-1)^4 a |A_{n-3}| & \cdots & (-1)^n a^{n-2} |A_1| & (-1)^{n+1} a^{n-1} \\ (-1)^3 c |A_{n-2}| & (-1)^4 |A_1||A_{n-2}| & (-1)^5 a |A_1||A_{n-3}| & \cdots & (-1)^{n+1} a^{n-3} |A_1||A_1| & (-1)^{n+2} a^{n-2} |A_1| \\ (-1)^4 c^2 |A_{n-3}| & (-1)^5 c |A_1||A_{n-3}| & (-1)^6 |A_2||A_{n-3}| & \cdots & (-1)^{n+2} a^{n-4} |A_1||A_2| & (-1)^{n+3} a^{n-3} |A_2| \\ (-1)^5 c^3 |A_{n-4}| & (-1)^6 c^2 |A_2||A_{n-4}| & (-1)^7 c |A_2||A_{n-4}| & \cdots & (-1)^{n+3} a^{n-5} |A_1||A_3| & (-1)^{n+4} a^{n-4} |A_3| \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n c^{n-2} |A_1| & (-1)^{n+1} c^{n-3} |A_1||A_1| & (-1)^{n+2} c^{n-4} |A_1||A_2| & \cdots & (-1)^{n+n-2} |A_{n-2}||A_1| & (-1)^{n+n-2} a |A_{n-2}| \\ (-1)^{n+1} c^{n-1} & (-1)^{n+2} c^{n-2} |A_1| & (-1)^{n+3} c^{n-3} |A_2| & \cdots & (-1)^{n+n-1} c |A_{n-2}| & (-1)^{2n} c |A_{n-1}| \end{bmatrix}$$

3. The general formula of cofactor matrix of tridiagonal toeplitz matrix is transposed to get the general formula of adjoin matrix of tridiagonal toeplitz matrix.

$$Adj(A_n) = \begin{bmatrix} (-1)^2 |A_{n-1}| & (-1)^3 c |A_{n-2}| & (-1)^4 c^2 |A_{n-3}| & (-1)^5 c^3 |A_{n-4}| & \cdots & (-1)^n c^{n-2} |A_1| & (-1)^{n+1} c^{n-1} \\ (-1)^3 a |A_{n-2}| & (-1)^4 |A_1||A_{n-2}| & (-1)^5 c |A_1||A_{n-3}| & (-1)^6 c^2 |A_2||A_{n-4}| & \cdots & (-1)^{n+1} c^{n-3} |A_1||A_1| & (-1)^{n+2} c^{n-2} |A_1| \\ (-1)^4 a |A_{n-3}| & (-1)^5 a |A_1||A_{n-3}| & (-1)^6 |A_2||A_{n-3}| & (-1)^7 c |A_2||A_{n-4}| & \cdots & (-1)^{n+2} c^{n-4} |A_1||A_2| & (-1)^{n+3} c^{n-3} |A_2| \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n a^{n-2} |A_1| & (-1)^{n+1} a^{n-3} |A_1||A_1| & (-1)^{n+2} a^{n-4} |A_1||A_2| & (-1)^{n+3} a^{n-5} |A_1||A_3| & \cdots & (-1)^{n+n-2} |A_{n-2}||A_1| & (-1)^{n+n-1} c |A_{n-2}| \\ (-1)^{n+1} a^{n-1} & (-1)^{n+2} a^{n-2} |A_1| & (-1)^{n+3} a^{n-3} |A_2| & (-1)^{n+4} a^{n-4} |A_3| & \cdots & (-1)^{n+n-2} a |A_{n-2}| & (-1)^{2n} c |A_{n-1}| \end{bmatrix}$$

4. The general formula of invers of tridiagonal toeplitz matrix.

$$(A_n)^{-1} = \frac{1}{|A_n|} adj(A_n)$$

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