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Three Essays on the Euro-Zone sovereign debt

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TESIS DOCTORAL

Three Essays on the European Sovereign debt market

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Abstract

My PhD thesis consists of three chapters on the Euro-Zone sovereign bond market due to the quick spread of sovereign risk in European countries. In chapter 1, we examine the European bond market efficiency by developing a mathematical programing approach, in order to measure the arbitrage size. Transaction costs may be incorporated. The obtained arbitrage measures have two interesting interpretations: On the one hand they provide the highest available arbitrage profit with respect to the price of the sold (bought) securities. On the other hand they give the minimum relative (per dollar) bid (ask) price modification leading to an arbitrage free market. Moreover, some primal problems lead to optimal arbitrage strategies (if available), while their dual problems generate proxies for the Term Structure of Interest Rates. The developed methodology permits us to implement an empirical test in the Euro-zone during the Euro crisis. Classical literature justifies the relevance of empirical analyses verifying the degree of efficiency during market turmoils. Our empirical study of the German, French and Spanish sovereign bonds markets finds that the main arbitrage opportunities come from the price differences between maturity-matched strips or "On-The-Run Premium" for zero-coupon bonds. When we remove the strips and the zero-coupon bonds the arbitrage still exists in the Spanish market.

Although we cannot reject the existence of the arbitrage in European bond market, in order to provide a general pricing rule we assume that the market is efficient. In chapter 2, we propose a general pricing methodology for completing the European sovereign bond market due to the existence of unreplicable bonds such as a forthcoming joint-guaranteed 'Eurobonds'. To find the optimal EMM, we introduce 'Ambiguity' in our pricing framework to describe the underlying state probability, and we also consider the worst-case Conditional Value-at-Risk to measure the hedging risk. The minimization of the worst-case CVaR of hedging residual risk associated with an uncertain probability set is investigated. We transform the optimization problem into a convex and linear program which gives the robust bid-ask prices, the hedging portfolio and a risk neutral

measure. In the numerical analysis, several synthetic sovereign bonds are created for imitating the performance of Eurobonds since it does not exist.

In chapter 3, we focus on assessing the sovereign risk dependence of European sovereign bonds, based on the worst case analysis. With this analysis, we can also provide a robust optimal portfolio composed of sovereign bonds in the safe and the periphery countries. With uncertain state probability distribution, we adopt a robust Conditional Value at Risk (RCVaR) in the risk-return tradeoff analysis. The empirical results show that a default in a safe country significantly affects the default in periphery countries and the interaction of default risk among periphery countries are strikingly high. Moreover, the robust optimal portfolio performs stably even in the period with the highest risk and the weights in risky countries are significantly greater than zero, which indirectly implies an overpriced-risk in periphery countries.

Resumen

Mi tesis doctoral consta de tres capítulos sobre el mercado de bonos soberanos de la Zona Euro. El interés del tema est á justificado por la rápida propagación del riesgo soberano entre los pa íses europeos.

En el capítulo 1, se analiza la eciencia del mercado europeo de bonos mediante el desarrollo de nuevas medidas del nivel de arbitraje secuencial en los mercados de renta fija. Se pueden incorporar los costes de transacción, y las medidas propuestas son utilizadas para realizar contrastes empíricos.

Las medidas de arbitraje obtenidas tienen dos interpretaciones interesantes: por un lado, proporcionan el mayor benecio de arbitraje disponible con relación al precio de los activos vendidos (comprados). Por otro, proporcionan la variación relativa (o en tanto por uno) m ínima de precios que conduce a un mercado sin arbitraje secuencial.

Las medidas del nivel de arbitraje se definen a través de problemas lineales de optimización. Los problemas primales conducen a estrategias de arbitraje óptimas (si hay arbitraje), mientras que sus problemas duales generan aproximaciones de la estructura temporal de tipos de inter és.

La metodología desarrollada nos permite implementar un contraste empírico en la zona euro durante la crisis reciente. La literatura clásica justica la conveniencia de los análisis empíricos que veriquen el grado de eciencia de los mercados enépocas convulsas. Nuestro estudio empírico de los mercados de bonos soberanos alemanes, franceses y españoles, concluye que las principales oportunidades de arbitraje provienen de las diferencias de precios entre los bonos con cupón y sus réplicas formadas con bonos de cupón cero. Cuando quitamos los bonos de cupón cero sigue habiendo arbitraje en el mercado español.

Aunque no se puede rechazar la existencia del arbitraje en los mercados europeos de bonos, en el capítulo segundo se propone una metodología que permite dar una regla de

valoración en mercados de (hoy por hoy hipotéticos) eurobonos. Consideramos un ambiente de ambigüedad en el que las verdaderas probabilidades de impago (default) son desconocidas con precisión, y por consiguiente las estimaciones incorporan márgenes de error. Tomamos el punto de vista de un intermediario financiero que valora teniendo en cuenta el coste de la cartera óptima de cobertura, que se calcula mediante la minimización del CVaR robusto, o CVaR bajo condiciones de ambigüedad. Éste coincide con el CVaR bajo el sistema de probabilidades más negativo para el intermediario financiero (enfoque del peor escenario).

Transformamos el problema de optimización en un programa convexo (e incluso lineal) que da los precios de oferta y demanda, la cartera de cobertura y una medida de probabilidad neutral al riesgo. En una aplicación numérica/empírica creamos, valoramos y cubrimos distintos tipos de Eurobonos soberanos.

En el capítulo 3 nos centramos en la evaluación de la dependencia del riesgo soberano de los bonos de la Eurozona, basada en el análisis del peor escenario. Con este estudio también podemos ofrecer una cartera de bonos soberanos óptima y robusta. La optimalidad es nuevamente determinada a través de CVaR bajo ambigüedad. El contraste empírico muestra que un problema en un país más solvente afecta signicativamente a los países periféricos. Por otra parte, la cartera óptima es estable incluso en el periodo de mayor riesgo, y los pesos de los países más arriesgados es signicativamente mayor que cero, lo que puede implicar una infravaloración de los bonos con menor solvencia (o una sobreestimación de su prima de riesgo).

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All errors are mine.

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Chapter 1

Sequential arbitrage measurement in bond markets: Theory and empirical applications in the Euro-zone

1.1 Introduction

This paper deals with a mathematical programming approach in order to introduce new measures of the level of sequential arbitrage in bond markets. The approach extends former analyses developed in Balbás and López (2008) for friction-free markets, since market imperfections and other transaction costs may be incorporated. This extension is critical in practical applications, since real markets always reflect frictions. In fact, the second part of the paper is devoted to empirically testing the sovereign bond markets efficiency in the Euro-zone, before and during the Euro crisis started in the late 2009.

In the fundamental theory of finance the absence of arbitrage is a key assumption, often extended in such a manner that markets must be also good deal-free, *i.e.*, strategies with a very large return/risk ratio should not be available to traders either (Cochrane and Saa-Requejo, 2000, Balbás *et al.*, 2013, etc.). However, some researches have empirically evidenced the existence of arbitrage opportunities in practice. For instance, Chen and Knez (1995) examined $NYSE$ and $NASDAQ$ market samples, and they found that these markets did not assign the same price to the same common payoff. Similarly, Balbás *et al.* (2000) pointed out the existence of arbitrage between the Spanish index $IBEX$ and its derivatives. With respect to bond markets, Grinblatt and Longstaff (2000) and Halpern and Rumsey (2000) used data of the $U.S$ and Canada respectively, and they found a significant valuation difference between government bonds and the equivalent packages of strips. Also, the analysis of Armitage *et al.* (2012) showed that in the $U.K.$ sovereign bond market the package of strips was overpriced even when accounting for transaction cost. These works motivated us to investigate the European sovereign bond market efficiency, particularly during the debt crisis, where the sovereign bond prices were persistently volatile.[1] We consider the liquidity effect in our examination because the bid-ask spread for most countries in the Euro zone is significantly wider than before.

There are several papers investigating the arbitrage measurement such as Holden (1995), Chen and Knez (1995) and Balbás and López (2008). We mainly follow the approaches of Ronn (1987) and Balbás and López (2008), because their works are based on a linear programming (LP), which is easy to apply in practice and also provides the size and degree of arbitrage.[2] Hence, in this paper, we use LP with transaction cost analysis to measure the degree of sequential arbitrage. Primal problems maximize the sequential income of a portfolio composed of available bonds, given that the short (long) position is established at the bid (ask) price. Meanwhile, they give optimal strategies for obtaining maximum arbitrage profits (if arbitrage is available). Moreover, these optimal values provide an important clue for investors to modify the bid (ask) price of the bond with highest pricing error. In contrast, the variables in the

[1] Market inefficiencies are more obvious in presence of market turmoils (Balbás *et al.*,2000), and mathematical methods are very effective to verify efficiency when facing or anticipating insatiability and/or crisis (Cheng *et al.*, 2006, Balbás *et al.*, 2008, etc.).

[2] Numerical and computational methods are becoming more and more important in Mathematical and/or Computational Finance (Chiarella *et al.*, 2014, Martín-Vaquero *et al.*, 2014, etc.), but LP may be also a good alternative if it provides us with appropriate investment strategies, pricing rules, risk measure-linked methods etc. (Mansini *et al.*, 2007, among others).

dual problems are closely related to the discount factors, and they provide proxies for the Term Structure of Interest Rate ($TSIR$).

We apply the above methodology in order to empirically test the European sovereign bond market during the period from $2007 - 2012$. We find that the European sovereign bond market reflects inefficiencies, particularly during the debt crisis. Most arbitrage opportunities come from the price difference between old and new-issued zero-coupon bonds, or strips with identical maturities. The former refers to "On-The-Run Premium", which is a popular liquidity measure in treasury bond markets. The latter is consistent with the findings of Daves *et al.* (1993) who investigated with the *U.S.* treasury strips. Although some previous literature indicates that arbitrage resulted from these price discrepancies are not pure and even very risky in a highly volatile market, rich funds from institutional investors in a fair period definitely can induce high arbitrage returns. For instance, the arbitrage income in the German sovereign bond market in 2007 can be easily obtained by rich investors, because German market is highly liquid in a whole Euro-zone sovereign bond market. In addition, we will also remove all the zero coupon bonds and strips which produce main arbitrage opportunities to examine the sequential arbitrage again, but we will still find that the existence of sequential arbitrage cannot be rejected in Spain during the crisis. Hu *et al.* (2013) indicate that financial market liquidity closely relates the amount of arbitrage capitals available, which is crucial for implementing the arbitrage strategy. Specially during liquidity crises, the arbitrage capitals become scare and big investors are not willing to deploy them to supply liquidity. Then the lack of funds hugely limits arbitrageurs trading and even forces them to abandon high return arbitrage. Nevertheless, if there exist institutional investors who have deep pocket and also willing to invest in the Spanish sovereign market, we cannot deny the existence of arbitrage in Spanish market.

It seems that the law of one price does not hold in these markets due to the apparent price differentials between maturity-matched strips and zero-coupon bonds. Nonetheless, Armitage *et al.* (2012) pointed out that the low liquidity in strips market may lead to the difficulty in exploiting the arbitrage opportunities and the transaction costs somehow cannot be fully captured by bid-ask spreads. Also, the short-selling position in principle strips is not completely risk-free, because a margin or collateral is required in this case. Since the sequential arbitrage requires the position to be held for a certain period, the more collateral or margin are likely to be required if price diverges (Huij *et al.*, 2012). That is why Liu and Longstaff (2004) presented an insightful model permitting the price difference of principle and coupon bonds with the same maturity in equilibrium if collateral is required for short position. However, if the price difference exists between the coupon bond and the corresponding package of strips, the arbitrage is definitely risk-free because these two securities are perfect substitute in the market.

The paper is organized as follows. Section 2 presents preliminaries and notations. In section 3 we develop a linear programming approach by considering the liquidity effect, and we explain the relations among the optimal arbitrage strategies, arbitrage profits and proxies for the $TSIR$, which is the main contribution of this paper. Section 4 presents the government bonds information for Germany, France and Spain. In section 5 we report the empirical results of the arbitrage examination, and analyze the degree of arbitrage in details. Section 6 summarizes and concludes.

1.2 Preliminaries and notations

Let us consider n available bonds B_j, $j = 1, 2, ..., n$, and suppose that the bond market is not friction-free (there are transaction costs). As usual in finance (Jouini and Kallal, 1995), in a very general setting one can represent frictions by means of bid-ask spreads. Thus, denote by $P^a = (p_1^a, p_2^a, ..., p_n^a)$ and $P^b = (p_1^b, p_2^b, ..., p_n^b)$ the family of ask and bid prices, and suppose that $p_j^a \geq p_j^b > 0$, $j = 1, 2, ..., n$ holds.

Denote by $t_1 < t_2 < .... < t_m$ the set of future maturities of the cash flows paid by the bonds above. Without loss of generality we will impose the inequality

$$\sum_{i=1}^{m} c_{ij} > p_j^a \geq p_j^b,$$

$j = 1, \ldots, n$, where c_{ij} denotes the cash flow of B_j $(j = 1, \ldots, n)$ at t_i $(i = 1, \ldots, m)$.

In order to simplify some notations, let us introduce the pay-off matrix below

$$C = (c_{i,j})_{i=1,j=1}^{i=m,j=n}.$$

Following usual conventions (Jouini and Kallal, 1995), portfolios will be represented by a couple of matrices (X, Y), $X = (x_1, x_2, ..., x_n)^T$ being the portfolio of long position (purchases) and $Y = (y_1, y_2, \ldots, y_n)^T$ being the portfolio of short ones (sales), and $x_j \geq 0$, $y_j \geq 0$, $j = 1, 2, ..., n$ must hold. The current price of portfolio (X, Y) can be expressed as

$$P(X,Y) = P^a X - P^b Y = \sum_{j=1}^{n} p_i^a x_j - \sum_{j=1}^{n} p_i^b y_j,$$

and its future cash flows can be represented by means of matrix C. Indeed, consider that $C^a = \binom{-P^a}{C}$ denotes a $(m+1) \times n$ matrix combining C with the ask price $-P^a$ and $C^b = \binom{P^b}{-C}$ is obtained by combining A with P^b. Then, $C^a X$ and $C^b Y$ are the whole sets of cash flows of X and Y respectively, $C^a X + C^b Y$ is the set of cash flows of (X, Y), and the future payoff of portfolio (X, Y) equals $C\,(X - Y)$.

Let us introduce the concepts of arbitrage and sequential arbitrage.

Definition 1.1. (X, Y) is said to be an arbitrage portfolio (AP) if $C^a X + C^b Y \neq 0$ and $C^a X + C^b Y \geq 0$. (X, Y) is said to be a sequential arbitrage portfolio (SAP) if $I^*_{m+1}(C^a X + C^b Y) \neq 0$ and $I^*_{m+1}(C^a X + C^b Y) \geq 0$.[3] $\square$

We can see that the arbitrage portfolio requires non-negative cash flows for every date t_i and generates at least a positive amount on some date. The conditions of the sequential arbitrage portfolio are not so restrictive, since negative cash flows are allowed as long as they are compensated by the amount of money previously received.

Additionally, it is known that the absence of (sequential) arbitrage in a frictionless market can be characterized by the existence of discount factors or a Term Structure

[3] Henceforth

$$I_r^* = \begin{pmatrix} 1,0,0,...,0 \\ 1,1,0,...,0 \\ .... \\ 1,1,1,...,1 \end{pmatrix}$$

will be a $r \times r$ square matrix for every $r \in \mathbb{N}$.

of Interest Rate ($TSIR$). But if bond prices are quoted with spreads we will state that there must exist a bundle of discount factors $\{\mu_i\}$ satisfying $p_j^b \leq \sum_{i=1}^{m} c_{ij}\mu_i \leq p_j^a$ for $j = 1, ..., n$. The proof will be showed later.

To measure the level of sequential arbitrage we adopt the concept of strong sequential arbitrage by extending the Definition 1:

Definition 1.2. (X, Y) is said to be a strong sequential arbitrage portfolio (SSA) if $P(X, Y) < 0$ and $I_m^* C(X - Y) \geq 0$. □

Compared with sequential arbitrage, the strong sequential arbitrage is more concerned about current profit, thereby requires a positive initial cash flow (negative price) in the trading strategy which will not be used to compensate negative components in the portfolio payoff.

1.3 Sequential arbitrage measurement and TSIR proxies

Under the notations above, we will measure the level of SSA by means of the following linear optimization problems with decision variables $x_j, y_j,\ h_j,\ k_j,\ j = 0, 1, ..., n$:

$$
\begin{aligned}
& Max \quad -(P^a X - P^b Y) \\
s.t. \quad & I_m^* C(X - Y) \geq 0 \\
& x_j \leq k_j,\ j = 1, 2, ..., n \\
& \sum_{j=1}^{n} k_j p_j^a \leq 1 \\
& x_j \geq 0,\ \ y_j \geq 0,\ \ k_j \geq 0,\ \ j = 1, 2, ..., n
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
& Max \quad -(P^a X - P^b Y) \\
s.t. \quad & I_m^* C(X - Y) \geq 0 \\
& y_j \leq h_j, \; j = 1, 2, ..., n \\
& \sum_{j=1}^{n} h_j p_j^b \leq 1 \\
& x_j \geq 0, \; y_j \geq 0, \; k_j \geq 0, \; j = 1, 2, ..., n
\end{aligned} \tag{1.2}
$$

Both problems attempt to maximize the SSA income $P^b Y - P^a X$, if available (*i.e.*, if the optimal value does not vanish). The unique difference between both optimization problems is given by their constraints, which affect purchases ($x_j \leq k_j$) and sales ($y_j \leq h_j$), respectively. If these constraints are not imposed then their dual problems will easily illustrate that (1.1) and (1.2)) will be unbounded unless their optimal value vanish. In order words, our $SSA-$measures could only reach the values 0 or ∞. Finally note that the common constraint $I_m^* C(X - Y) \geq 0$ guarantees that every solution of (1.1) and (1.2) will be a SSA portfolio or will replicate the null strategy $(X, Y) = (0, 0)$.[4]

Now we move to the dual problems, which are given by:

$$
\begin{aligned}
& Min \quad \theta \\
s.t. \quad & \mu C - \lambda \leq P^a \\
& \mu C \geq P^b \\
& \lambda_j \leq \theta p_j^a, \; j = 1, 2, ..., n \\
& \lambda_j \geq 0, \; j = 1, 2, ..., n \\
& \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0
\end{aligned} \tag{1.3}
$$

[4]Actually, we could integrate (1.1) and (1.2) in a single vector (or multiobjective) optimization problem with two objectives. Then we could apply both the scalarization method or the balance space approach (Galperin and Wiecek, 1999) in order to find Pareto solutions. Nevertheless, we will see that this extension is not interesting in this case because there is a close relationship between the solutions of both (1.1) and (1.2) (see Lemma 4 below).

and

$$
\begin{aligned}
& Min \quad \theta \\
s.t. \quad & \mu C \leq P^a \\
& \mu C + \lambda \geq P^b \\
& \lambda_j \leq \theta p^b_j, \; j = 1, 2, ..., n \\
& \lambda_j \geq 0, \; j = 1, 2, ..., n \\
& \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0
\end{aligned}
\tag{1.4}
$$

where the decision variables are $\theta \in \mathbb{R}$, $\lambda = (\lambda_i)_{i=1}^n$ and $\mu = (\mu_i)_{i=1}^m$.

Both dual Problems (3) and (4) minimize the highest committed error θ for ask and bid prices in percentage. The dual variables μ in (3) and (4) give a proxy for the family of discount factors, but both of them misprice bonds indicated by the respective first four constraints if bond market is not efficient. The first constraint in Problem (3) implies that the difference between market ask price P^a and the theoretical price μC measured by μ is determined by the value of λ. If λ is significantly greater than 0, discount factor μ will overestimate the bonds ask prices and the estimated price μC will be within the interval $[P^a, P^a + \lambda]$. Using a similar argument, a set of discount factors μ given by Problem (4) will underestimate bonds bid prices in portfolio if $\lambda > 0$.

Next, we start to investigate the properties of the solutions of the primal Problems (3) and (4):

Lemma 1.3. *Problems (1), (2), (3) and (1.4) are feasible and bounded. If ℓ^* and ℓ_* are their optimal values, then $0 \leq \ell_* < 1$, $0 \leq \ell^*$ and $\ell_* = 0 \Longleftrightarrow \ell^* = 0 \Longleftrightarrow$ the market is $SSA-$free.*

$0 \leq \ell_*$ and $0 \leq \ell^*$ are clear since $(0, 0, 0)$ is feasible for (1.1) and (1.2). (1.3) and (1.4) will be bounded if (1.3) and (1.4) are feasible. Obviously, $\mu = (1, 1, ..., 1)$, $\lambda = \mu C - P^a$, $\theta = Max \left\{\lambda_j / p^a_j; \; j = 1, ..., n\right\}$ and $\mu = (0, 0, ..., 0)$, $\lambda = P^b$, $\theta = 1$ provide us with feasible elements for (1.3) and (1.4) respectively.

To prove that $\ell_* < 1$ suppose that (X, Y, h) is (1.2)-feasible. Then, $Y \le h \Rightarrow P^bY \le P^bh \le 1 \Rightarrow -P^aX + P^bY \le P^bY \le 1$, so $\ell_* < 1$. Moreover, $\ell_* = 1$ holds if and only if

$$P^aX + P^bY = P^bh = 1. \tag{1.5}$$

Thus, $P^aX = P^bY - P^bh \le P^b(Y - h) \le 0$. Since $P^aX \ge 0$, it follows that $P^aX = 0$ and $X = 0$. Combined with (1.5) we have $P^bY = P^bh$ and $Y = h$, so the first constraint in (1.2) leads to

$$-h_1\left(\sum_{i=1}^{m} a_{i,1}\right) - \dots - h_n\left(\sum_{i=1}^{m} a_{i,n}\right) \ge 0$$

Since $P^b_j \le \sum_{i=1}^m a_{i,j}$ we have $P^bh \le 0$, which contradicts (1.5). Hence $0 \le \ell_* < 1$.

Finally, if the market is $SSA-$free, the first constraint in (1.1) or (1.4) will imply $-(P^aX - P^bY) \le 0$ as long as (X, Y) is (1.1) or (1.4) feasible, so $\ell_* = \ell^* = 0$. Conversely, if $\ell_* = \ell^* = 0$, suppose that (X, Y) satisfies $I^*_mC(X - Y) \ge 0$ and $Y = h$. It is clear that (X, Y, h) is (1.1)-feasible. Since $\ell_* = 0$ implies that $-(P^aX - P^bY) \le 0$, it contradicts the definition of SSA, so there are no feasible portfolios generating SSA. With a similar argument, it holds for $\ell^* = 0$. □

Based on Lemma 1, we state that in a SSA free market every bond is priced within the bid-ask spread by a fitted set of discount factors.

Theorem 1.4. *There are no SSA portfolios if and only if there exists μ_* such that $P^b \le \mu_*C \le P^a$ and $\mu_{*1} \ge \mu_{*2} \ge \dots \ge \mu_{*m} \ge 0$.*

Proof. It is clear that the absence of SSA portfolios holds if and only if $\ell_* = \ell^* = 0$ which is equivalent to $\theta_* = \theta^* = 0$. Hence $\lambda_* = \lambda^* = 0$ and μ_*C will be in the range of $[P^b, P^a]$. □

Lemma 1.5. *Suppose that $\ell_* > 0$. if (X^*, Y^*, k^*) solves (1.1) and (X_*, Y_*, h_*) solves (1.2) then $Y_* = h_*$, $P^bh_* = 1$, $X^* = k^*$ and $P^ak^* = 1$.*

Proof. . Since $Y_* \le h_*, P^bY_* \le P^bh_* \le 1$ and P^bY_* is the only strictly positive components in ℓ_*, it is obvious to see that $Y_* = h_*$ and $P^bY_* = P^bh_* = 1$. Clearly,

$X^* \leq k^*, P^aX^* \leq P^ak^* \leq 1$. Suppose that $P^aX^* < 1$ and set $X = X^*/P^aX^*, Y = Y^*/P^aX^*, k = X^*/P^aX^*$, it is obvious that portfolio (X,Y) is feasible, so it gives that

$$-(P^aX - P^bY) = -\frac{-(P^aX^* - P^bY^*)}{P^aX^*} = \frac{\ell^*}{P^aX^*} > \ell^*$$

which obviously has a contradiction. so we have $X^* = k^*$ and $P^ak^* = 1$. □

Lemma 1.6. *(a)* $\ell^* = \dfrac{\ell_*}{1-\ell_*}$, $\ell_* = \dfrac{\ell^*}{1+\ell^*}$ *and* $\ell_* \leq \ell^*$.
(b) $X_* = (1-\ell_*)k^*$ *and* $Y^* = (1+\ell^*)h_*$

$a)$ Consider the functions $\phi_i : \mathcal{A} \longrightarrow \mathrm{I\!R}$, $i = 1, 2$, given by

$$\phi_1(X,Y) = \frac{-P^aX + P^bY}{P^bY}, \qquad \phi_2(X,Y) = \frac{-P^aX + P^bY}{P^aX}$$

where $\mathcal{A} = \{(X,Y) \in \mathrm{I\!R}^n \times \mathrm{I\!R}^n;\ P(X,Y) < 0,\ I_m^*C(X-Y) \geq 0\}$ is non void due to $\ell_* > 0$. Notice that the denominator will never vanish in the definitions above, because $P^bY = 0$ would imply $P(X,Y) = P^aX \geq 0$, contradicting $(X,Y) \in \mathcal{A}$, and $P^aX = 0$ would imply $\ell_* = 1$, contradicting Lemma 1.

Expression $0 < \phi_1(X,Y) < 1$

$$\phi_2(X,Y) = \frac{\phi_1(X,Y)}{1-\phi_1(X,Y)} \tag{1.6}$$

are obvious. Since $[0,1) \ni t \longrightarrow t/(1-t) \in [0,\infty)$ is a one to one increasing function, Problems

$$Max\ \{\phi_i(X,Y);\ (X,Y) \in \mathcal{A}\}, \tag{1.7}$$

$i = 1, 2$, attain the optimal value at the same solutions. It is clear that if $(X,Y) \in \mathcal{A}$ then $\left(X/P^bY, Y/P^bY\right)$ is (1.2)-feasible, and therefore $\left(-P^aX + P^bY\right)/\left(P^bY\right) \leq \ell_*$. Hence Lemma 3 implies that $Y_* = h_*$ and $P^bh_* = 1$. Therefore

$$\phi_1(X_*,Y_*) = \left(-P^aX_* + P^bY_*\right)/\left(P^bY_*\right) = -P^aX_* + P^bY_* = \ell_*,$$

and (X_*, Y_*) solves (1.7). Similarly, (X^*, Y^*) solves (1.7) and $\phi_2(X^*, Y^*) = \ell^*$. Therefore (see (1.6))

$$\ell^* = \phi_2(X^*, Y^*) = \phi_2(X_*, Y_*) = \frac{\phi_1(X_*, Y_*)}{1 - \phi_1(X_*, Y_*)} = \frac{\ell_*}{1 - \ell_*},$$

and the inequality $\ell_* \leq \ell^*$ obviously holds form equation above.

b) Consider a (1.2)-feasible strategy $(\gamma X_*, \gamma Y_*)$ with $\gamma > 0$ such that $\gamma P^a X_* = 1$. Then,

$$1 = \gamma P^a X_* = \gamma\left(-\ell_* + P^b Y_*\right) = \gamma(-\ell_* + 1)$$

and therefore

$$\gamma = \frac{1}{1 - \ell_*} = 1 + \ell^*.$$

Proceeding as in a very parallel proof of Balbás and López (2008), the function $f(X, h)$ equaling the optimal value of (1.2) for every fixed h is increases with h, and the function $f(k, , Y)$ equaling the optimal value of (1.1) for every fixed k is increases with k. Since $P^a k^* = 1$ and $P^a(\gamma X_*) = 1$, it gives $k^* = \gamma X_* = (1/(1 - \ell_*)X_*)$Analogously, $h_* = (1/(1 + \ell^*)X_*)$. □

Now we transfer our attention to the solutions of the dual problems. Assume that $(\ell^*, \lambda^*, \mu^*)$ and $(\ell_*, \lambda_*, \mu_*)$ are the solutions of (1.3) and (1.4). If SSA does not exist in the market, which is indicated by $\ell_* = \ell^* = \lambda_* = 0$, the theoretical prices $P_* = \mu_* C$ and $P^* = \mu^* C$ will be within the interval of $[P^b, P^a]$. However, in a non-efficient market they will satisfy the following relations:

Theorem 1.7. *(a)if* $k_j^* > 0$ *then* $p_j^a = \frac{p_j^*}{1+\ell^*}$. *If* $h_{*j} > 0$ *then* $p_j^b = \frac{p_{*j}}{1-\ell_*}$.

(b) $p_{*j} \leq p_j^b \leq p_j^a \leq p_j^*$, $j = 1, 2, ..., n$.

Proof. The dual optimal values will satisfy $\theta_* = \ell_*, \theta^* = \ell^*$. If $h_* > 0$ previous Lemmas ensure that $Y_* > 0$, so the complementary slackness conditions lead to $\lambda_* = \theta_* P^b = \ell_* P^b$ and $\mu_* A + \lambda_* = P^b$. It gives that $\lambda_* = P^b - \mu_* A = P^b - P_*$, and then $P^b - P_* = \ell_* P^b$. Hence $P^b = \frac{p_*}{1-\ell_*}$. With a similar argument, we can derive that $p_j^a = \frac{p_j^*}{1+\ell^*}$ if $k_{*j} > 0$.

(b) is obvious from the results of (a). $\square$

Measures ℓ_* and ℓ^* appropriately give the level of SSA since they reflect a relative (per dollar) arbitrage gain value. Moreover, according to Theorem 5, the difference between bid (ask) prices and estimated prices p_{*j} (p^*_j) is closely related to the value of ℓ^* and ℓ_*. In fact, based on the value of ℓ^* and ℓ_*,we can modify mispriced prices of some bonds that have large percentage in producing arbitrage opportunities. In addition, let us investigate a new property of ℓ_* and ℓ^* stating that they minimize the maximum relative variation of prices to prevent the existence of SSA, and they also provide a new explanation for the risk premium.

Theorem 1.8. *Let $Q^a = (q^a_1, q^a_2, ..., q^a_n)$ and $Q^b = (q^b_1, q^b_2, ..., q^b_n)$ be vectors of ask and bid prices for bonds $B_1, B_2, ..., B_n$. Suppose that Q^a and Q^b do not generate SSA opportunities. Suppose also that $0 < q^b_j \leq p^b_j$, $p^a_j \leq q^a_j$ for $j = 1, 2, \ldots, n$. Then,*

$$\ell_* = Max\{\frac{p^b_j - p_{*j}}{p^b_j} : j = 1, 2, ..., n\} \leq Max\{\frac{p^b_j - q^b_j}{p^b_j} : j = 1, 2, ..., n\}$$

$$\ell^* = Max\{\frac{p^{*j} - p^a_j}{p^a_j} : j = 1, 2, ..., n\} \leq Max\{\frac{q^a_j - p^a_j}{p^a_j} : j = 1, 2, ..., n\}$$

Proof. Assume that $\ell_* > 0$. The dual constraints lead to $\theta_* = \ell_* \geq \dfrac{\lambda_{*j}}{p^b_j}, j = 1, 2, \ldots, n$. Theorem 5$(a)$ shows that $\ell_* = \dfrac{\lambda_{*j}}{p^b_j}$ if $h_* > 0$. Hence, the arbitrage profit ℓ_* satisfies that

$$\ell_* = Max\{\frac{p^b_j - p_{*j}}{p^b_j} : j = 1, 2, ..., n\}$$

Theorem 2 guarantees the existence of a set of discount factors μ such that $Q^b \leq \mu C \leq Q^a$. Take $\theta = Max\{\dfrac{p^b_j - q^b_j}{p^b_j} : j = 0, 1, ..., N\}$ and $\lambda = P^b - \mu C = P^b - Q^b \geq 0$. (μ, λ, θ) is dual-feasible, so $\theta_* \leq \theta$. The remaining statement can be derived with a similar argument. $\square$

The above theorem indicates that the "authentic" bid (ask) price p_{*j} (p^{*j}) provided by our optimization model can minimize the maximum modification of bond quotes leading to a $SSA-$free market. ℓ_* and ℓ^* play important roles in measuring this minimum difference in percentage. Additionally, they can be understood as a lower bound of the risk premium of a risky bond. To clarify this idea, we assume a portfolio consisted in default free bonds and a risky bond j with bid price p_j^b, and suppose that the SSA disappears ($\ell_* = 0$) when only dealing with default free bonds. But if we include risky bond j, ℓ_* will be greater than zero because bond j should be involved in the buying position h. Although discount factors cannot reflect all the information provided by bond j, we can see that the arbitrage profit ℓ_* will imply a minimum risk premium in percentage to compensate risk for investors. If more riskier bonds are considered, ℓ_* will provide the largest required premium of bonds in portfolio.

The next sections provide an empirical analysis in European sovereign bond markets. We adopt the methodology above to examine wether there exists sequential arbitrage before and during the Euro crisis.

1.4 Data

1.4.1 Data Source

The existence of SSA is tested in the Euro-zone. We will deal with government bond markets due to the European sovereign debt crisis beginning in the late 2009. We will choose Germany, France and Spain, which, along with Italy own the largest government bonds and strips markets in the Euro-zone, and also because they differ significantly in credit ratings.

The main source of our data on daily bond price quotation is *Datastream*. Quoted bid (ask) prices are composite prices calculated by *Datastream* from the average of all the available contributors bid (ask) quotes, excluding the highest and the lowest values. Since bid-ask quotes information are limited, we also take prices without spread measured by "Market Default Prices" (MDP). MPD are reference prices estimated by retrieving the composite bid prices provided from $Datastream's$ or *Thomson*

Reuters′s valuation bid prices, if the prices are liquid.[5] Although there is still a fraction of electronic transactions such as *MTS* for European bond markets, these data are not easily available. Another data source is the *Bank of Spain*, which is the biggest dealer for Spanish Treasury Bonds, and provides daily information of all traded securities in over the counter market. Here, this data source is mainly used to examine the data reliability provided by *Datastream* for the Spanish market. This is because we only find little data about liquidity information as measured by turnover. Our sample ranges from January 2007 to December 2012. This period is particularly suitable for analyzing the Euro-zone Sovereign bonds market efficiency as it covers the stable period before 2008 as well as the chaotic period following a Greek debt crisis.

Our model requires data containing bonds with perfectly predictable cash flows, so only default-free and option-free government bonds are included in the sample. In order to examine the market efficiency for each country, the bid and ask quotation are analyzed during the period from 2010 to 2012, because they are only available in *Datastream* from late 2009. For the remaining years from 2007 to 2009, we use *MDP*. In fact, coupon bearing instruments are traded at their "Gross Prices", which involves calculating accrued interest. So we add it to the quoted bid/ask prices and *MPD* for coupon bonds in the model.

1.4.2 Data Concerns

Although *Datastream* has the largest data information for financial markets, we find some quoted prices in our data keeping constant for more than five trading days in the three-country data-set. To exclude any possibilities of no liquidity problems, we remove these bonds on the day where their quotations are exactly the same as the preceding day when doing daily arbitrage test. After cleaning the data, the daily traded bonds and strips information in the Spanish market during 2007 to 2009 is consistent with the ones provided by *Bank of Spain*. In addition, we also delete some outliers as they appear to be due to obvious data-entry errors.

[5] A price is liquid if it changed in the previous five days.

Table 1.1:

	All bonds		Fixed-income Securities		zero-coupon bonds and Strips	
Year: 2010	Mean	Median	Mean	Median	Mean	Median
Germany	0.4928	0.4100	0.1362	0.0800	0.7471	0.7000
France	0.4354	0.3000	0.2334	0.1900	0.6158	0.4800
Spain	0.4082	0.3800	0.4102	0.3900	0.4105	0.3800
Year: 2011						
Germany	0.4618	0.3500	0.1255	0.0700	0.7702	0.7500
France	0.3847	0.3000	0.2627	0.2100	0.4828	0.4500
Spain	0.5178	0.3100	0.4925	0.4400	0.5234	0.2700
Year: 2012						
Germany	0.5296	0.2900	0.1003	0.0400	0.9709	1.0300
France	0.4811	0.2900	0.2269	0.1800	0.6891	0.5000
Spain	0.6252	0.4800	0.5127	0.4900	0.6859	0.4700

1.4.3 Descriptive Statistics

Table 1 provides descriptive statistics about the mean (median) of bid-ask spreads form 2010 to 2012 for the three countries. In each year, the mean for all the traded securities are presented in the first column. Then we segment the entire sample into coupon bonds and zero coupon bonds consisting of strips and Treasury Bills, shown in the 'Fixed-Income Security' and 'Treasury Bills and Strips' columns.

In the German and French market, bid-ask spread changes of all traded sovereign bonds are small from 2010 to 2012. The average spreads were always around 48 basic points (*bp*), but the mean of the bid-ask spread in the Spanish market kept increasing every year and increased by more than one half in 2012, which potentially reflects investor's lack of confidence in government recovery in debt crisis. For fixed income bonds, then German market showed higher vitality than the other two countries. The average spreads over the three years are less than 13 *bp*. In contrast, the mean spread for less liquid French and Spanish fixed-income securities were approximately 24 *bp* and 50 *bp*, respectively. Surprisingly, overall liquidity performance in strips and zero-coupon bonds for Germany was dismal, the average spread in 2012 is up to 100 *bp*, even worse than in Spain and France which were less than 70 *bp*. In general, Spanish government bond market appears to face higher liquidity risk than Germany and France, based on their high bid-ask spread. However, Germany, who owns the most liquid strips market, shows an apparent liquidity problem.

1.5 Empirical Results

Tables 2, 3 and 4 summarize the days with SSA opportunities from 2007 to 2012 on both aggregate and percentage basis for Germany, France and Spain, respectively. The value of the arbitrage income is divided into eight intervals whose length equals 0.0005, as shown at every row. In each column, "Upper" indicates the maximum profit generated by Problem (1.1). In contrast, "Lower" represents the maximum profit of Problem (1.2). The tables show a pronounced difference in the days with arbitrage for the three countries. In the stable period from 2007 to 2008, Germany who owned one of the largest and most liquid market for sovereign debt, surprisingly, showed quite high frequency of daily SSA. Particularly in 2008, bond pricing errors were more than 1% in approximately 67% working days and they were even above 5% in 30 days at the end of the year. However, sovereign debt markets in France and Spain performed regularly during this period, since more than 97% of the days exhibited low margin close to zero. Although Juji *et al* (2011) showed that price difference between principals and coupon strips with the same maturity from 2002 to 2007 for these three countries may lead to an arbitrage by switching two strips, we could not find any significant riskless profits given by the model in 2007 and 2008, except for Germany. In the following turbulent period from 2009 to 2010 SSA opportunities began to appear in the latter half of 2009 for the French and Spanish sovereign markets. The results show that investors can obtain at most 1% to 5% price differences with the ones provided by arbitrage free market in 36% of trading days. From 2010 to 2012, Germany and Spain showed obvious mispricing problems that persistently existed. German market had maintained a large percentage of arbitrage opportunities over the three years, particularly in 2010. For Spain, the arbitrage profits $\ell^*(\ell_*)$ were lying within the spread $1\% - 5\%$ more than 200 days in 2012, but the days of arbitrage slowly decreased. French market, in contrast, seems to be much more efficient. Arbitrage tended to decrease gradually and almost disappeared in 2011. However, in 2012 it became wrong again.

Table 1.2: Arbitrage days in German sovereign bond market

Germany	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	15	5.8%	15	5.8%	15	5.8%	15	5.8%	9	3.5%	9	3.5%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	14	5.4%	14	5.4%	5	1.9%	5	1.9%	1	0.4%	1	0.4%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	122	46.9%	125	48.1%	32	12.3%	33	12.7%	3	1.2%	3	1.2%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	102	39.2%	99	38.1%	175	67.4%	177	68.1%	144	55.3%	157	60.3%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	6	2.3%	7	2.7%	28	10.7%	25	9.6%	102	39.2%	89	34.2%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	1	0.4%	0	0.0%	5	1.9%	5	1.9%	1	0.4%	1	0.4%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
Year	2010				2011				2012			
Days of Examination	256				252				244			
$\ell_*, \ell^* \leq 0.0005$	42	16.4%	42	16.4%	87	34.5%	87	34.5%	147	60.3%	147	60.3%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	13	5.1%	13	5.1%	5	2.0%	5	2.0%	22	9.0%	22	9.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	57	22.3%	58	22.7%	73	29.0%	74	29.4%	63	25.8%	63	25.8%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	62	24.2%	62	24.2%	50	19.8%	50	19.8%	10	4.1%	10	4.1%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	81	31.6%	80	31.2%	37	14.7%	36	14.3%	2	0.8%	2	0.8%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	1	0.4%	1	0.4%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

Table 1.3: Arbitrage days in French sovereign bond market

France	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	0	0.0%	0	0.0%	148	56.9%	148	56.9%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	1	0.4%	1	0.4%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	5	1.9%	5	1.9%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	3	1.2%	3	1.2%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	94	36.2%	96	36.9%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	9	3.4%	7	2.7%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	258				255				255			
$\ell_*, \ell^* \leq 0.0005$	47	18.2%	47	18.2%	226	88.6%	226	88.6%	29	11.4%	29	11.4%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	47	18.2%	47	18.2%	9	3.5%	9	3.5%	2	0.8%	0	0.8%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	159	61.6%	159	61.6%	5	2.0%	5	2.0%	2	0.8%	2	0.8%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	2	0.8%	2	0.8%	2	0.8%	2	0.8%	9	3.5%	8	3.1%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	3	1.2%	3	1.2%	11	4.3%	12	4.7%	212	83.1%	212	83.1%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	1	0.4%	1	0.4%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	2	0.8%	1	0.4%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

TABLE 1.4: Arbitrage days in Spanish sovereign bond market

Spain	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	260	100.0%	260	100.0%	149	57.2%	149	57.2%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	1	0.4%	1	0.4%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	3	1.2%	3	1.2%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	5	1.9%	5	1.9%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	94	36.2%	94	36.2%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	8	3.1%	8	3.1%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	258				255				256			
$\ell_*, \ell^* \leq 0.0005$	12	4.7%	12	4.7%	18	7.1%	18	7.1%	4	1.6%	4	1.6%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.0%	0	0.0%	4	1.6%	4	1.6%	0	0.0%	0	0.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	7	2.7%	7	2.7%	11	4.3%	11	4.3%	7	2.7%	7	2.7%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	4	1.6%	4	1.6%	0	0.00%	0	0.00%	8	3.1%	8	3.1%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	166	64.3%	175	67.8%	193	75.6%	196	76.8%	215	84.0%	223	87.1%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	69	26.7%	60	23.3%	29	11.4%	26	10.2%	22	8.6%	14	5.5%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

These results are very striking because most of arbitrage profits $\ell_*(\ell^*)$ exceeded 5% , even taking into account a tax effect. This implies that in this three sovereign markets every investor could obtain an arbitrage income in 50% working days on average from 2010 to 2012. Therefore, we examine the corresponding arbitrage strategies to find out the main reason resulting in such a strange result. As presented in Table 5, we summarize arbitrage days in terms of different strategies, which are pricing errors from maturity-matched strips, from On-The-Run Premium and from portfolios composed of fixed-income bonds, strips and zero-coupon bonds. Last two columns show the minimum and maximum arbitrage profits per year. Clearly, in the safe year 2007 for the German sovereign bond market, more than 75% of arbitrage incomes come from complicated strategies by selling or buying a pool of fixed-income bonds, strips and zero-coupon bonds. Although the maximum arbitrage profits are the smallest compared to other years, the arbitrage indeed exist without liquidity risk or capital problem. However, in 2008, the German zero-coupon bond market shows increasing pricing errors due to a strike increase of the arbitrage days of "On-The-Run Premium",[6] which closely relates a liquidity problem in sovereign bond market. In other words, more and more the zero-coupon bonds with shorter maturity are traded at lower prices, compared to recent-issued bonds but with longer maturity. Moreover, a threefold increase in the maximum arbitrage profits also directly implies a lower liquidity in German zero-coupon bond markets than before. By contrast, Spanish and French sovereign bond markets seem quiet and efficient during 2007 and 2008, where the arbitrage profits are close to zero.

Since 2009 three sovereign markets enter into a turbulent period due to the Greece crisis. More than one third of working days shows arbitrage opportunities for three countries. Particularly in the French and Spanish government bond markets all the maximum profits come from price discrepancies between old and recent-issued zero-coupon, showed in "On-The-Run Premium", which strongly suggests a huge liquidity problem in both markets. Moreover, the maximum income for Spain attains 0.20, which means that the riskless return rate is up to 20% if a trader invests 1 Euro by

[6] "On-The-Run Premium" is a popular liquidity measure used in Treasury bond markets. The just-issued or called on-the-run Treasury bonds are generally more liquid and traded at a premium compared to other old bonds with similar maturity.

implementing the optimal arbitrage strategy provided by our model. From 2010 to 2012, in order to reduce the liquidity effect on arbitrage, we use market bid-ask prices instead of market trading price in the experiment. However, the results presented in Table 6 still reflect significant arbitrage opportunities for three countries. The corresponding strategies mainly focus on the price difference from maturity-matched strips and portfolio strategy, other than “On-The-Run Premium”. More importantly, we observe that 82% of principle strips were sold at higher price than the coupon strips with identical cash flows in our sample period for the three countries, which highlights a strong violation of the law of one price. This phenomenon is consistent with the findings of previous empirical work on the Treasury strips in the U.S. by Jordan *et al.* (2000), who found that bid quoted price of principle strips in U.S. strip market was on average 10.8 basic points higher than matched-maturity coupon strips. Daves and Ehrhardt (1993) claimed that principle strips were more valuable because of a unique role played in reconstitution, which always guarantees market demand.

In general, we cannot deny the existence of SSA opportunities in sovereign bond markets. Although current works indicate that arbitrage is not pure or riskless when arbitrageurs lack capitals to satisfy margin maintenance, or arbitrage is difficult to implement in low liquid market, it is true that the arbitrage opportunities still exist in a safe and high liquid German market in 2007 and 2008 as long as there exist investors or traders with deep pocket.

Finally, we exclude the maturity-matched strips and “On-The-Run Premium” zero-coupon bonds that might lead to risky arbitrage, and re-examine the arbitrage from 2007 to 2012. The results are shown in Tables 7 to 9. Clearly, there are little SSA opportunities from 2007 to 2011 for the three countries. However, in 2012 we cannot reject the existence of SSA in the Spanish sovereign bond market. Investors can obtain price difference $\ell_*(\ell^*)$ greater than 1% in 28 working days without considering the capital requirement in a high liquidity risk period.

TABLE 1.5:

	Number of Arbitrage days				Arbitrage Profits	
Germany	Total	Maturity-matched strips	On-The-Run premium	Others	Min.	Max.
2007	245	32	28	185	0.0022	0.0818
2008	245	15	97	133	0.0011	0.2393
2009	251	63	141	47	0.0025	0.1092
France						
2007	0	0	0	0	0	0
2008	0	0	0	0	0	0
2009	112	2	110	0	0.0046	0.088
Spain						
2007	0	0	0	0	0	0
2008	0	0	0	0	0	0
2009	111	2	109	0	0.0176	0.1931

In this table, we separate the arbitrage days into three groups based on different types of pricing errors. Since bid-ask prices are not available in 2007,2008 and 2009 for above three countries, we use 'MDP' in the arbitrage examination.

TABLE 1.6:

	Number of Arbitrage days				Arbitrage Profits	
Germany	total	Maturity-matched strips	On-The-Run premium	Others	Min.	Max.
2010	213	181	0	32	$5.34e^{-4}$	0.0714
2011	165	132	0	33	$5.18e^{-4}$	0.0273
2012	97	85	0	12	$5.84e^{-4}$	0.0353
France						
2010	86	85	1	0	$5.33e^{-4}$	0.0397
2011	29	29	0	0	0.0005	0.1006
2012	224	224	0	0	$3.6e^{-4}$	0.0505
Spain						
2010	246	231	1	12	0.0016	0.1351
2011	237	112	1	124	$5.34e^{-4}$	0.0822
2012	252	224	0	28	0.0036	0.0733

In this table, we summarize the arbitrage days in 2010, 2011 and 2012. We assume that traders can buy bonds or strips at ask prices, and sell at bid prices. The bid-ask prices used in our arbitrage examination are daily average bid-ask prices obtained from Datastream.

Table 1.7: Arbitrage days in German Fixed-Income sovereign bond market

Germany	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005260$	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	256				252				244			
$\ell_*, \ell^* \leq 0.0005$	256	100.0%	256	100.0%	252	0.0%	252	0.0%	244	0.0%	244	0.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

TABLE 1.8: Arbitrage days in French Fixed-Income sovereign bond market

France	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.0%	0	0.0%	0	0.0%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.0%	0	0.0%	0	0.0%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.0%	0	0.0%	0	0.0%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
Year	2010				2011				2012			
Days of Examination	258				255				255			
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	255	100.0%	255	100.0%	255	100.0%	255	100.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

TABLE 1.9: Arbitrage days in Spanish Fixed-Income sovereign bond market

Spain	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	258				255				256			
$\ell_*, \ell^* \leq 0.0005$	258	100.0%	258	100.0%	255	0.00%	255	0.00%	198	77.2%	198	77.4%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	30	11.8%	31	12.8%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	28	11.0%	27	10.5%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%

1.6 Conclusion

We have presented a mathematical programing approach in order to measure the size of the strong sequential arbitrage of a bond market. Transaction costs may be incorporated. The obtained arbitrage measures ℓ_* and ℓ^* reflect two interesting quantities: On the one hand ℓ_* (ℓ^*) yields the highest available arbitrage profit with respect to the price of the sold (bought) securities. On the other hand ℓ_* (ℓ^*) gives the minimum relative (per dollar) bid (ask) price modification leading to a strong sequential arbitrage free market. The provided primal problems lead to the optimal strong sequential arbitrage strategies (if available), while their dual problems generate proxies for the Term Structure of Interest Rates. Several results have shown the significant analogies between the two provided primal problems and their optimal strategies (X_*, Y_*) and (X^*, Y^*). Similarly, the one to one and increasing relationship $\ell^* = \ell_* / (1 - \ell_*)$ indicates that both arbitrage measures provide analogous information.

The developed theory easily applies in practice. In fact we have empirically studied the existence of strong sequential arbitrage in the European sovereign debt market from 2007 to 2012. The focus has been on sovereign bonds issued by Germany, France and Spain, respectively. During the safe period, from 2007 to 2008, the Spanish and French sovereign bond markets performed efficiently, but the German market reflected strong sequential arbitrage due to the existence of price differences between maturity-matched strips and zero-coupon bonds. In contrast, during the crisis period, from 2009 to 2012, the three bond markets showed market inefficiencies which particularly focused on "On-The-Run Premium" and strips rather than the fix-income bonds. These results are consistent with the findings of Daves and Ehrhardt (1993) and Jordan *et al.* (2000), who claimed that the principle strip price is usually higher than the strip or zero-coupon bonds with the same maturity because of its uniqueness. However, after removing all the zero coupon bonds and strips, we still found a fraction of arbitrage opportunities existing in the Spanish fixed-income bond market, where arbitrage profits were higher than 1%.

Chapter 2

Robust pricing and hedging for 'Eurobonds' under ambiguity

2.1 Introduction

There is a growing call for issuance of common debt (also referred to 'Eurobonds') that would be collectively or partly guaranteed by eurozone countries to alleviate ongoing sovereign debt crisis. Such a potential instrument can directly solve the market distortion and negative externality for most Southern European countries due to a dramatic increase in spreads (Grauwe and Ji, 2012), and it is also linked to a form of fiscal integration and common governance in Eurozone. Although in practice, it raises heated discussions among investors and European institutions, the current pricing methodology is unable to price or hedge these kind of bonds due to the unknown default relations among European countries.

To solve this puzzle, we create a new general pricing methodology for the European sovereign bond market in this paper, specially allows for the Eurobonds which cannot be replicated by trading the available securities. In other words, we extend the pricing rule and provide a valid method for dealers who wants to price or hedge any available bonds in European sovereign bond market, but also for the new sophisticated or unreplicable bonds.

Generally, in financial market, the equivalent martingale measures (EMM) is heavily used in pricing financial derivatives based on no-arbitrage pricing theory (Delbaen and Schachermayer 1997). However, EMM is not unique in incomplete market. Therefore, the challenge in this paper is to select the most appropriate EMM for any European unreplicable sovereign bonds. We follow the main criteria for selecting EMM that extends the market pricing function by minimizing the hedging residual risk (see Schweizer 1996 and Balbás et al. 2010), which can be transformed into a convex or linear program given by Balbás et al. (2010). Moreover, the dual problem in the program directly provides the EMM, and also implies corresponding optimal hedging portfolio weights. Hence, our methodology is closely related to the work of Balbás et al. (2010) which extended the pricing rule by minimizing a general risk function, with the following two new contributions.

First, we introduce the ambiguity (Ellberg 1961) in the pricing framework. The concept of ambiguity aversion is mainly defined via uncertain probability measures (Schmeidler 1989). In our paper, because the underlying payoff distribution of unreplicable sovereign bond is unknown, and the joint or conditional default probability in state members cannot be obtained from market data alone, we use an ambiguity set denoted by κ to contain all possible real state probabilities. In the literature, there are many approaches to model the ambiguous state probabilities, such as Hasselblad (1966) who looked into mixture distributions by estimating their weights; Zhu and Fukushima (2009) who analyzed a domain set of the probabilities with multivariate distribution derived from history returns; and Lucas (2013) who assessed the conditional probabilities of sovereign default based on a Generalized Hyperolic skewed-t coupla approach. However, these methods cannot avoid pinning down the exact distribution of the returns. In this paper, we propose somehow a stochastic approach to think outside this box. The idea of this approach comes from Hull et.al (2004) who claimed that in reality, investors always gain more than risk free rate because the real-world default probabilities are generally less than risk neutral default probabilities. Therefore, we set up a constraint for κ such that the asset price assessed via the probability measures in κ is higher than its market price.

Second, instead of a general risk function, we choose a worst-case Conditional Value at Risk (CVaR, Rockafellar et al., 2006) as the risk measure in the pricing framework. The investor's ambiguity aversion often yields asymmetric payoff distribution (Epstein and Schneider 2008), thus CVaR becomes a better description of loss in the tail. However, Kakouris and Rustem (2014) argued that the calculation of CVaR is also based upon some assumptions on the underlying distribution of assets which is unknown in reality. Therefore, we applied a worst scenario analysis inspired by Zhu and Fukushima (2009) in extending the pricing rule with a CVaR risk measure. Other literatures foscusing on the issue of robustness to reduce the pricing modeling risk arising from the uncertain underlying distribution include Gilboa and Schmeidler (1989), Maccheroni et al. (2006), Goldfarb and Iyengar (2003), and Calafoiore and El Ghaoui (2006), etc.

After introducing the ambiguity, we prove that our new pricing methodology still completely satisfies the main properties of the pricing rule, such as homogeneity, subadditivity. And also it provides a conservative price bound based on the worst CVaR, particularly provides some valuable information for the dealers or policymakers. More importantly, the price and related hedging strategy proposed by our method can perfectly hedge the dealer's global risk who is in a short position.

To test this new pricing method, we choose the 'Eurobonds', which is the focus of policymakers who try to solve the Eurozone sovereign debt crisis in the numerical analysis. There are many types of Eurobonds proposed by researchers, which is summarized by Claessens et al. (2012), but few researchers work on the computation of implied interest rate for Eurobonds under joint or several guarantee, due to the unclear default relations among Eurozone members. Although Mayordomo (2014) firstly proposed a common Eurozone risk free rate, the priced bond is assumed to be replicated by available securities. In our paper, we focus on solving the unreplicable Eurobond pricing dilemma where the liability is joint guaranteed[1]. Since such a bond does not exist yet, we construct them similarly with the proposal by DeGrauwe and Moesen (2009). However, the only difference is that our bond is joint guaranteed, but their are not. First, we fix the interest rate as a weighted average of coupon rate observed in each

[1]Each member state will be responsible not only for its own share of liability but also for any other member that is insolvent.

government bond market, where the weights are given by the government subscribed capital in European investment bank (EIB) Financial reporting in 2012. Second, for simplicity, rather than including all 24 state members in the bonds, we only choose France, Italy and Spain as the risk-sharing countries, because they are among top five in terms of bond market shares and equity shares according to EIB financial reports in 2012. They are also representatives for European countries with and without stressed debt. Throughout the paper, the bond constructed using the above method will be called "synthetic sovereign bond" (SSB).

Although the introduction of joint guarantee in SSB undoubtedly reduces the default risk compared with national government bonds for Spain or Italy, it is not default free due to the unknown default correlations among members. Hence, the focus of the numerical analysis is to provide a robust price bound of SSB and also an optimal hedging strategy for a long or short position. We assume that the issuance of SSB is allowed in any risk-sharing members or European institutions, pooling up to their respective 60% of GDP of national government bond, similarly with the 'Blue bond' proposed by Delpla and Von weizsacker (2011).

In the numerical analysis, we create three 2-year and 10-year SSBs respectively in terms of different risk-sharing members. All the assessed bid and ask prices of SSBs imply a significant reduction in default risk compared with national government bonds issued by countries with stressed debt, which is consistent with the intention of Eurobonds. Moreover, the optimization problem provides a risk neutral measure (The risk neutral probabilities) which minimizes the hedged residual risk in the worst case. And this risk neutral measure can also be used to price other sophisticated derivatives related to joint or conditional default probabilities.

The remaining of the paper is structured as follows. In section 2, we introduce the notations and preliminaries. In section 3, we extend the pricing rule with a Robust CVaR under the ambiguity set κ, and discuss the feasibility in both primal and dual problems. Our main contribution is in section 4 and 5. First, we construct a linear program based on the previous section and derive the hedging portfolio composition. Then, we assess three novel 2-year SSB and 10-year SSBs's prices and corresponding hedging portfolios, and give their implied joint and conditional default probabilities among risk-sharing members. Finally, we conclude and summarize in section 6.

2.2 Notation and Preliminary

Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω denotes a finite set of n states that may arise on a future date T. Let $\mathcal{T} \subset [0, T]$ be a set of trading dates and the filtration, $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$, represents the information available at every t in $\mathcal{T}$. As usual, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Consider a set κ of probabilities on $\mathcal{F}$ reflecting the investor ambiguity (uncertainty). Every element in κ will be called feasible prior, and the lack of ambiguity holds if and only if κ is composed of a single element.

Consider a convex cone $Y \subset \mathbb{R}^n$ composed of reachable payoffs. We assume that $\pi(y)$ represents the ask price of every $y \in Y$, *i.e.*, if S_y denotes the set of price process $\{S_t\}_{t\in\mathcal{T}}$ of self-financing portfolios such that the terminal value satisfies $S_T \geq y$, then

$$\pi(y) = Inf\{S_0; \{S_t\}_{t\in\mathcal{T}} \in S_y\},$$

holds. Moreover, for any $y \in Y$, if there exists a self-financing portfolio such that $S_T = y$, then the market is complete, and incomplete otherwise. Besides, we say that the market is perfect (or friction free) if the pricing rule $\pi : Y \to \mathbf{R}$ is linear.

Suppose that there exists a risk-less asset. Its price can be represented by

$$\pi(1, \cdots, 1) = e^{-r_f T},$$

where r_f denotes a risk free rate.

Let $\mathcal{P}$ be the set of probability measures on Ω, *i.e.*,

$$\mathcal{P} = \left\{ p = (p_1, \cdots, p_n) \in \mathbb{R}^n; \sum_{j=1}^{n} p_j = 1, p_j \geq 0 \right\}.$$

Then $\kappa \subset \mathcal{P}$ obviously holds. The higher level of uncertainty, the wider the set κ. In Financial Economics there are two main approaches to deal with decision making problems under ambiguity (Gilboa and Schmeidler, 1989, and Maccheroni *et al.*, 2006). In this paper we will follow the worst case analysis of Gilboa and Schmeidler (1989) since, according to the results of these authors, it is quite consistent with the existence of ambiguity aversion, empirical finding pointed out by Ellsberg (1961) in his famous paradox.

2.3 Extending the pricing rule

The main purpose of this section is to extend the pricing rule π to the whole space $\mathbb{R}^n$ with a worst-case or robust CVaR (RCVaR).

Given a confidence level μ_0 and a specific set of priors κ, the function of $RCVaR$ is specified as:

$$RCVaR_{(\kappa,\mu_0)}(y) = \max\left\{-\sum_{j=1}^{n} \xi_j y_j; \quad \xi = (\xi_1, \cdots, \xi_n) \in \nabla_{(\kappa,\mu_0)}\right\}$$

Where the $RCVaR$ sub-gradient is given by

$$\nabla_{(\kappa,\mu_0)} = \left\{(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n; \sum_{j=1}^{n} \xi_j = 1, 0 \leq \xi_j \leq \frac{q_j}{1-\mu_0} \quad for j = 1, \cdots, n, q \in \kappa\right\} \tag{2.1}$$

Without ambiguity, the set κ just contains one element, which is the known probability distribution in y. Then, $RCVaR_{(\kappa,\mu_0)}$ is equivalent to the Conditional Value at Risk (CVaR). Otherwise, $RCVaR_{(\kappa,\mu_0)}(y)$ denotes the greatest CVaR incurred by y with respect to the set of priors κ.

It is worth mentioning that even considering an ambiguous probability measure, $RCVaR_{(\kappa,\mu_0)}$ also has been verified as a coherent risk measure[2] like CVaR, seeing the proof in

[2] 'a coherent risk measure' is introduced by Artzner et al. (1999), which aimed at creating a theory that fills the gap between utility maximization and no arbitrage theory.

Zhu&Fukushima (2009), Balbás et al (2012). It satisfies translation invariance, homogeneity, subadditivity and monotonicity which are sufficient conditions in coherent risk measure.

Suppose that $x \in \mathbb{R}^n$ is a new asset's payoff we are interested in hedging and pricing, the states that may happen in future are known. The market is assumed to be incomplete due to frictions or price-jumps in transactions. In this case, x can not be perfectly replicated by traded assets. So if a trader sells x in market, the optimal decision for him is to buy the traded assets $y \in Y$ which can minimize the spanned payoff between y and x with respect to the probability measure p. However, p is always unknown in real market, then we propose a conservative hedging strategy in terms of pervious literature by minimizing $RCVaR$ with respect to (y, P), where P denote the initial endowment for the trader. In other words, no matter how bad the situation is, the strategy given by minimum $RCVaR$ can eliminate the greatest risk measured by CVaR. The related optimization problem is constructed below.

$$\begin{cases} \min \quad RCVaR_{(\kappa,\mu_0)}(y-x)+P \\ \pi(y) \leq Pe^{-r_f T} \\ y \in Y, P \in \mathbb{R} \end{cases} \tag{2.2}$$

Let (y^*, P^*) be the optimal solution, then we say that $RCVaR_{(\kappa,\mu_0)}(y^*-x)+P^*$ provides the worst-case ask price of x, where $RCVaR_{(\kappa,\mu_0)}(y^*-x)$ can be viewed as margins that the trader should keep and P^* gives the cost of hedging strategy. So what happens to the global risk the trader will bear after buying the optimal portfolio y^*. We examine the worst-case risk measured by RCVaR assuming that the optimal ask price of x is represented by P_x^*.

$$\begin{aligned} RCVaR_{(\kappa,\mu_0)}\left(y^*-x+P_x-P^*\right) &= RCVaR_{(\kappa,\mu_0)}\left(y^*-x\right)-P_x+P^* \\ &= RCVaR_{(\kappa,\mu_0)}\left(y^*-x\right)-\left(RCVaR_{(\kappa,\mu_0)}(y^*-x)+P^*\right)+P^* \\ &= 0. \end{aligned} \tag{2.3}$$

We surprisingly find that the optimal hedge strategy (y^*, P^*) provides not only the price information of x, but also drives the global risk to zero even in the worst situation. Although the optimal portfolio y^* cannot fully replicate x, it completely erases the maximum risk measured by CVaR, what exactly the trader holding a short position desires.

Next, we analyze the dual corresponding problem for (2). Following the method proposed by Balbás et al. (2010), we firstly fix P as a concrete number so that there is only one decision variable ($y \in Y$) in (1). Also the market is assumed to be perfect, thereby a unique pricing factor $y_\pi \in Y$ is guaranteed by the Riesz representation theorem, which satisfies that

$$\pi(y) = e^{-r_f T} y_\pi y$$

for every $y \in Y$. By adding a new variable θ, problem (1) is equivalent to

$$\begin{cases} \min \theta + P \\ -\sum_{j=1}^{n} \xi_j (y_j - x_j) \leq \theta, \quad \forall (\xi_j)_{j=1}^{n} \in \nabla_{\kappa,\mu_0} \\ y_\pi y \leq P \\ y \in Y, \theta \in \mathbb{R} \end{cases} \tag{2.4}$$

Notice that the first constraint are valued on the Banach space $\mathcal{C}(\nabla_{\kappa,\mu_0})$ Hence, the Lagrangian function is given by (Balbas *et al.*, 2010b)

$$\begin{aligned} \mathcal{L}(\theta, y, v, \lambda) = \theta \left(1 - \int_{\nabla_{\kappa,\mu_0}} dv(\xi)\right) - \int_{\nabla_{\kappa,\mu_0}} \left(\sum_{j=1}^{n} \xi_j \left(y_j - x_j\right)\right) dv(\xi) \\ + \lambda \left(y_\pi y - P\right) + P \end{aligned} \tag{2.5}$$

for every $(\theta, y, v_1, \lambda) \in \mathbb{R}^n \times Y \times \mathcal{M}(\nabla_{\kappa,\mu_0}) \times \mathbb{R}$. $(v_1, \lambda)$is dual feasible if and only if v_1, λ are non-negative and the infimum of $\mathcal{L}(\theta, y, v_1, \lambda)$ is bounded in $(\theta, y) \in \mathbb{R} \times Y$ which equals to the dual objective on (v_1, λ). Hence, v_1 must become a probability such that $v_1 \in \mathcal{P}(\nabla_{\kappa,\mu_0})$. Then dual variables (v_1, λ) can be replaced by $(\xi, \lambda) \in \nabla_{\kappa,\mu_0} \times \mathbb{R}$ and the Lagrangian function becomes

$$\mathcal{L}(y, v_1, \lambda) = \sum\nolimits_{j=1}^{n} y_j(-\xi_j + \lambda y_{\pi,j}) + \sum\nolimits_{j=1}^{n} \xi_j x_j + P(1-\lambda) \tag{2.6}$$

Therefore, the dual problem is given by

$$\begin{cases} \max \sum\nolimits_{j=1}^{n} \xi_j x_j + P(1-\lambda) \\ (-\xi + \lambda y_\pi)y = 0 \quad \forall y \in Y \\ \lambda \in \mathbb{R}^+, \xi \in \nabla_{(\kappa,\mu_0)} \end{cases} \tag{2.7}$$

Proposition 1. *For every $y \in Y$, the equation $(-\xi + \lambda y_\pi)y = 0$ can only hold for $\lambda = 1$.*

Proof. Consider a risk free asset with payoff $y = 1, \cdots, 1$, the equation leads to

$$(-\xi + \lambda y_\pi)(1, \cdots, 1) = -1 + \lambda = 0$$

Obviously $\lambda = 1$ must hold. □

Hence, we can simplify (5) by plugging into $\lambda = 1$. The problem is equivalent to

$$\begin{cases} \max \sum\nolimits_{j=1}^{n} \xi_j x_j \\ (-\xi + y_\pi)y = 0 \quad \forall y \in Y \\ \xi \in \nabla_{(\kappa,\mu_0)} \end{cases} \tag{2.8}$$

Proposition 2. *(a) There exists $P_0 \in \mathbb{R}^+$ such that Problem (2) is feasible and satisfies the Slater condition, i.e., its two constraints as strict inequalities for some feasible solutions.*

(b) For every $P \in \mathbb{R}^+$, problem (2) is feasible and satisfies the Slater condition.

Proof. We assume there exists a risk-less asset with a constant payoff $y = c(1, \cdots, 1)$ and $c \leq P_0, c \in \mathbb{R}^+$, then its price is given by $ce^{r_f T}$, which implies that the primal constraints in (1) is always feasible, and so is (2). Suppose that(y, θ) is a feasible solution for (2), then consider a strategy $(y-1, \theta+3)$, which implies the strict inequalities

$$\begin{cases} -\sum_{j=1}^{n} \xi_j(y_j - 1 - x_j) = -\sum_{j=1}^{n} \xi_j(y_j - x_j) + 1 \leq \theta + 1 < \theta + 3 \\ y_\pi(y-1) = y_\pi y - 1 \leq P - 1 < P \end{cases} \tag{2.9}$$

hold in (2). Therefore, the problem (2) satisfies the Slater condition.

Based on the proof of (a), It is clear to see that for every $P \in \mathbb{R}^+$, we can always find a feasible solution consists of a risk-less asset and θ, which also satisfies two strict inequalities. □

The feasibility of primal problems implies that the problem (5) or (6) is bounded. However the dual feasible set in (6)

$$D_f = \{\xi \in \nabla_{\kappa,\mu_0}; (y_\pi - \xi)y = 0\}$$

may be void at some point. Here we give an example to illustrate the existence of the duality gap.

Remark 1. Consider a risky asset with payoff $(1,0)$ has a price at 0.5, and $\Omega = \{\omega_1, \omega_2\}$. Assume that there is no friction and ambiguity in the market, then $\kappa = \{(q_{\omega_1}, q_{\omega_2}), \quad q_{\omega_1} = 0.2, q_{\omega_2} = 0.8\}$. Given a confidence level of 0.5, we have

$$\nabla_{\kappa,\mu_0} = \{(\xi_1, \xi_2); \quad \xi_1 + \xi_2 = 1 \quad \text{and} 0 \leq \xi_1 \leq 0.4, 0 \leq \xi_2 \leq 1.6\}$$

In this case the risk measure is actually the Condition Value at Risk, and the conditions gives the set D_f

$$\begin{cases} \xi_1 = 0.5 \\ \xi_1 + \xi_2 = 1 \\ 0 \leq \xi_1 \leq 0.4, 0 \leq \xi_2 \leq 1.6 \end{cases} \tag{2.10}$$

Obviously the set is void.

In fact, previous work such as Jim et al. (2008) and Balbás (2010b) point out that it is not uncommon to exist a duality gap in financial problems. Hence, to get rid of the dual-gap, we give a following assumption.

Assumption 1. *we will assume that the dual set D_f is not empty, and there exists $P_0 \in \mathbb{R}^+$ such that the primal problem (1) or (2) is bounded.*

Theorem 2.1. *The problem (2) and (6) are bounded, the primal infimum equals the attainable dual maximum, and the following Karush-Kuhn-Tucker conditions*

$$\begin{cases} \sum_{j=1}^{n} \xi_j (y_j^* - x_j) \geq \sum_{j=1}^{n} \xi_j^* (y_j^* - x_j) \\ y_\pi y - \xi^* y = 0 \quad \forall y \in Y \\ y^* \in Y, \xi^* \in \nabla_{\kappa,\mu_0} \end{cases} \tag{2.11}$$

must hold, where $y^* \in \mathbb{R}^n$ and $\xi^* \in \nabla_{\kappa,\mu_0}$ solve (2) and (6) respectively.

The proof is similar to the proof in Balbas et al. (2010).

Proposition 3. *The optimal value of (1) or (2) which equals to the optimal value of (6) does not depend on $P \in \mathbb{R}^+$*

Proof. Above theorem implies that dual maximum (1) or (2) equals the primal infimum (5) or (6) for every P if both problems are bounded. Moreover, the optimal value of (6) $\max \sum_{j=1}^{n} \xi_j x_j$ is independent of P in terms of its optimal solution. Hence, the primal maximum does not depend on P neither. □

Next we can check how the optimal hedge strategy y^* changes with P, although the optimal primal value is not affected.

Proposition 4. *Suppose that y^* solves the problem (2) for $P \in \mathbb{R}^+$ and x, then for any $\alpha \in \mathbb{R}$, $y^* + \alpha$ solves (2) for $P + \alpha \in \mathbb{R}^+$ and x.*

Proof. Assume that ξ^* solves (6), then y^* and ξ^* satisfy the Karush-Kuhn-Tucker conditions (8), so it is clear that

$$\begin{cases} \sum_{j=1}^{n} \xi_j(y_j^* + \alpha - x_j) \geq \sum_{j=1}^{n} \xi_j^*(y_j^* + \alpha - x_j) \\ y_\pi(y+\alpha) - \xi^*(y+\alpha) = y_\pi y - \xi^* y + \alpha - \alpha = 0 \quad \forall y \in Y \\ y^* + \alpha \in Y, \xi^* \in \nabla_{\kappa,\mu_0} \end{cases} \tag{2.12}$$

$y^* + \alpha$ also satisfies all the conditions in (8), therefore, $(\theta^* - \alpha, y^* + \alpha)$ solves (2) for $P + \alpha \in \mathbb{R}^+$ and x. □

Based on above propositions, we can define the pricing rule for every $x \in \mathbb{R}^n$ by dealing with a perfect market as follows.

$$\pi_{(\kappa,\mu_0)}(x) = e^{-r_f T} \min\{RCVaR_{(\kappa,\mu_0)}(y - x) + P; \quad y_\pi y \leq P, P \in \mathbb{R}^+, y \in Y\} \tag{2.13}$$

Since the optimal value of (1) equals the optimal value of (2) and (6) for every $P \in \mathbb{R}^+$, the pricing rule $\pi_{(\kappa,\mu_0)}(x)$ is equivalent to

$$\pi_{(\kappa,\mu_0)}(x) = e^{-r_f T} \max\{\sum_{j=1}^{n} \xi_j x_j; \quad \xi \in \nabla_{\kappa,\mu_0}, (-\xi + y_\pi)y = 0, \forall y \in Y\} \tag{2.14}$$

Now we introduce several properties of $\pi_{(\kappa,\mu_0)}$ above.

Theorem 2.2. (a) *If the market is perfect, $\pi_{(\kappa,\mu_0)}$ extends π to the whole space $\in \mathbb{R}^n$.*

(b) *$\pi_{(\kappa,\mu_0)}$ is positive homogenous and sub-additive.*

(c) *Define $\pi_{(\kappa,\mu_0)}(x)$ is the ask price of x. Then $-\pi_{(\kappa,\mu_0)}(-x)$ is the bid price. Moreover, if $x \in Y$, the bid-ask spread is zero.*

Proof. (a) If $x \in Y$, then $(-\xi + y_\pi)x = 0$ must hold in terms of the KKT conditions (8), which leads to $\xi x = y_\pi x$. Therefore, $\pi_{(\kappa,\mu_0)}(x) = e^{-r_f T}\xi x = e^{-r_f T} y_\pi x = \pi(x)$. (b) Let $\xi_{\alpha x} \in D_f$ be the dual optimal solution, where $\alpha > 0$ and $x \in \mathbb{R}^n$, then

$$\pi_{(\kappa,\mu_0)}(\alpha x) = e^{-r_f T} \xi_{\alpha x}(\alpha x) = \alpha e^{-r_f T} \xi_{\alpha x} x$$

Obviously, $\xi_x \in D_f$ and D_f is independent of x. Therefore,

$$\alpha e^{-r_f T}\xi_{\alpha x}x \leq \alpha e^{-r_f T}\xi_x x = \alpha\pi_{(\kappa,\mu_0)}(x)$$

On the other hand, assume that $\xi_{x_1+x_2} \in D_f$ where the maximum is attained, then we have

$$\begin{aligned}\pi_{(\kappa,\mu_0)}(x_1+x_2) &= e^{-r_f T}\xi_{x_1+x_2}(x_1+x_2)\\ &= e^{-r_f T}\xi_{x_1+x_2}x_1 + e^{-r_f T}\xi_{x_1+x_2}x_2\end{aligned}$$

If $\xi_{x1} \in D_f$ and If $\xi_{x2} \in D_f$, analogously

$$\begin{aligned}e^{-r_f T}\xi_{x_1+x_2}x_1 + e^{-r_f T}\xi_{x_1+x_2}x_2 &\leq e^{-r_f T}\xi_{x_1}x_1 + e^{-r_f T}\xi_{x_2}x_2\\ &= \pi_{(\kappa,\mu_0)}(x_1) + \pi_{(\kappa,\mu_0)}(x_2)\end{aligned}$$

(c) Property (b) implies that $\pi_{(\kappa,\mu_0)}(x+(-x)) = \pi_{(\kappa,\mu_0)}(0) \leq \pi_{(\kappa,\mu_0)}(x) + \pi_{(\kappa,\mu_0)}(-x)$. Since $\pi_{(\kappa,\mu_0)}(0) = 0$, then $-\pi_{(\kappa,\mu_0)}(-x) \leq \pi_{(\kappa,\mu_0)}(x)$ holds for every x, which also guarantees that the bid price is lower than the ask price of x. So the bid-ask spread can be represented by

$$\pi_{(\kappa,\mu_0)}(x) - \big(-\pi_{(\kappa,\mu_0)}(-x)\big)$$

If $x \in Y$, then based on (a) above, we have

$$\pi_{(\kappa,\mu_0)}(x) - \big(-\pi_{(\kappa,\mu_0)}(-x)\big) = y_\pi x + y_\pi(-x) = 0 \quad \square$$

(12) shows that the pricing rule $\pi_{(\kappa,\mu_0)}$ equals the maximum dual objective, which is independent of P. Since P is unknown, to obtain the optimal hedge strategy y^* implies from the convex problem (2), we create a linear problem to see if the dual variables of dual problem (8) also gives the information about y^*.

2.4 A linear problem approach

Suppose $\{\mathbf{y}_1, \cdots, \mathbf{y}_k\}$ is a basis of Y, where $k \leq n$. The ambiguity set κ is assumed to be given by a linear constraint such that $Ap \geq b$, where

$$A = \left\{ \begin{matrix} 1, \cdots, 1 \\ a_{11}, \cdots, a_{1n} \\ \vdots \ddots \vdots \\ a_{r1} \cdots a_{rn} \end{matrix} \right\}$$

$a_{ij} \geq 0$ represents the present value of asset i under the state j. $b = (1, b_1, \cdots, b_r), b_i \geq 0$, where b_i denotes an expected return of asset i. Since (1) also shows a linear constraint for $\nabla_{(\kappa,\mu_0)}$, we can construct a linear dual problem as follows,

$$\begin{cases} \max \quad \sum_{j=1}^{n} \xi_j x_j \\ (-\xi + y_\pi)\mathbf{y}_i = 0 \quad \text{for} \quad i = 1, \cdots, k \\ \xi_j \leq \dfrac{q_j}{1-\mu_0} \quad \text{for} \quad j = 1, \cdots, n \\ Aq \geq b \\ \xi_j \geq 0, q_j \geq 0, \quad \text{for} \quad j = 1, \cdots, n \end{cases} \tag{2.15}$$

where $(\xi, q) \in \mathbb{R}^n \times \mathbb{R}^n$ are dual variables. Since the risk-free asset is available, the first equation in (15) implies that $\sum_{j=1}^{n} \xi_j = 1$ can be dropped, which constraints the probability to sum up to unity in the absence of a risk free asset. Then the Lagrangian function

$$\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{r+1}$$

becomes

$$
\begin{aligned}
\mathcal{L}(\xi, q, w, u, v, \tau) = &-\sum\nolimits_{j=1}^{n} \xi_j x_j + \sum\nolimits_{i=1}^{k} w_i \sum\nolimits_{j=1}^{n} (\xi_j - y_{\pi,j}) y_{i,j} + \sum\nolimits_{j=1}^{n} u_j (\xi_j - \frac{q_j}{1-\mu_0}) \\
&+ \sum\nolimits_{h=1}^{r+1} v_h (\sum\nolimits_{j=1}^{n} a_{hj} q_j - b_h) \\
&= \sum\nolimits_{j=1}^{n} \xi_j (-x_j + \sum\nolimits_{i=1}^{k} w_i y_{i,j} + u_j) + \sum\nolimits_{j=1}^{n} q_j (-\frac{1}{1-\mu_0} u_j + \sum\nolimits_{h=1}^{r+1} a_{hj} v_h) \\
&- \sum\nolimits_{i=1}^{k} \sum\nolimits_{j=1}^{n} w_i y_{\pi,j} y_{i,j} - \sum\nolimits_{h=1}^{r+1} v_h b_h
\end{aligned}
$$

where $(w, u, v) \in \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{r+1}$ are Lagrangian multipliers. Then the corresponding bi-dual problem can be expressed as

$$
\begin{cases}
\max \quad -\sum\nolimits_{i=1}^{k} \sum\nolimits_{j=1}^{n} w_i y_{\pi,j} y_{i,j} - \sum\nolimits_{h=1}^{r+1} v_h b_h \\
- x_j + \sum\nolimits_{i=1}^{k} w_i y_{i,j} + u_j \geq 0 \quad \text{for} \quad j = 1, \cdots, n \\
- \frac{1}{1-\mu_0} u_j + \sum\nolimits_{h=1}^{r+1} a_{hj} v_h \geq 0 \quad \text{for} \quad j = 1, \cdots, n \\
u_j \geq 0, \quad \text{for} \quad j = 1, \cdots, n \\
v_h \geq 0, \quad \text{for} \quad h = 1, \cdots, k+1 \\
w_j \in \mathbb{R}
\end{cases}
\tag{2.16}
$$

Since (15) is assumed to be feasible (see assumption 1), the absence of duality gap between (15) and (16) can be guaranteed. Then the complementary slackness

$$
u_j (\xi_j - \frac{q_j}{1-\mu_0}) = 0 \quad \text{for} \quad j = 1, \cdots, n \tag{2.17}
$$

$$
\xi_j (-x_j + \sum\nolimits_{i=1}^{k} w_i y_{i,j} + u_j) = 0 \quad \text{for} \quad j = 1, \cdots, n \tag{2.18}
$$

$$
q_j (-\frac{1}{1-\mu_0} u_j + \sum\nolimits_{h=1}^{r+1} a_{hj} v_h) = 0 \quad \text{for} \quad j = 1, \cdots, n \tag{2.19}
$$

must hold.

Theorem 2.3. *Suppose that* $(\mathbf{w}^*, \mathbf{u}^*, \mathbf{v}^*)$ *solves (16) and* $(\xi^*, q^*) \in D_f$. *Let* $\mathbf{y}^* = \sum_{i=1}^{k} w_i^* \mathbf{y_i}$, *then* (ξ^*, q^*) *solves (15) if and only if there exists a partition* $\Omega = \Omega_{q^0} \cup \Omega_0 \cup \Omega^* \cup \Omega_{\mu_0}$ *such that* *(i)* $q_j^* = 0$ *if and only if* $j \in \Omega_{q^0}$.

(ii) $y_j^* \geq x_j$ *and* $\xi_j^* = 0, q_j^* \neq 0,\ j \in \Omega_0$.

(iii) $y_j^* = x_j$ *and* $0 < \frac{\xi^*}{q^*} < \frac{1}{1-\mu_0},\ j \in \Omega^*$.

(iv) $y_j^* \leq x_j$ *and* $\frac{\xi^*}{q^*} = \frac{1}{1-\mu_0},\ j \in \Omega_{\mu_0}$.

(v) $\xi^* \mathbf{y}^* = y_\pi \mathbf{y}^*$

In such a case $(\mathbf{y}^*, 0)$ *solves (2).*

Proof. If (ξ^*, q^*) solves (15), the third equation in the constraints of (15) implies that $\xi^* \mathbf{y}^* = \xi^*(\sum_{i=1}^k w_i^* \mathbf{y_i}) = \sum_{i=1}^k w_i^* \xi^* \mathbf{y_i} = \sum_{i=1}^k w_i^* y_\pi \mathbf{y_i} = y_\pi \mathbf{y}^*$, thereby (v) holds. Bearing in mind that (ξ^*, q^*), $(\mathbf{w}^*, \mathbf{u}^*, \mathbf{v}^*)$ must satisfy the optimality condition (17), (18) and (19), along with $0 \leq \xi^* \leq \frac{q^*}{1-\mu_0}$ we have:

(i) Let Ω_{q^0} denote a set where q_j^* vanishes, then $q_j^* = 0$ leads to $\xi_j^* = 0$, and the relations of y_j^* and x_j is unrestricted.

In the following cases, we only consider sets getting rid of $q^* = 0$:

(ii) Let Ω_0 denote a set where $\frac{\xi_j^*}{q_j^*} = 0$, then $u_j = 0$ must hold implying from (17). Combining with the first constraint in (16), they lead to

$$\sum_{i=1}^k w_i y_{i,j} \geq x_j - u_j = x_j \Rightarrow y_j^* \geq x_j \tag{2.20}$$

(iii) Denote Ω_{μ_0} by a set where $\frac{\xi_j}{q_j} = \frac{1}{1-\mu_0}$, then conditions (17), (18) gives that $u_j \geq 0$ and

$$-x_j + \sum_{i=1}^k w_i y_{i,j} + u_j = 0 \Rightarrow y_j^* \leq x_j$$

therefore,

$$\sum_{i=1}^{k} w_i y_{i,j} = x_j - u_j \leq x_j \tag{2.21}$$

(iv) If $\Omega^* = \Omega \setminus (\Omega_{q^0} \cup \Omega_0 \cup \Omega_{\mu_0})$, trivially $0 < \frac{\xi_j}{q_j} < \frac{1}{1-\mu_0}$ holds in the set Ω^*. Then $u_j = 0$ showed in (17) and $\xi \neq 0$ imply that

$$\sum_{i=1}^{k} w_i y_{i,j} = x_j - u_j = x_j \Rightarrow y_j^* = x_j \tag{2.22}$$

Above four cases show that for every state j, $\sum_{i=1}^{k} w_i^* y_{i,j}$ is closely related to the payoff of asset x_j. Hence, $\mathbf{y}^*$ or $\sum_{i=1}^{k} w_i^* y_{i,j}$ can be understood as the payoff of the hedging portfolio composed of w_i^* units of available security i. Moreover, for a fixed P, $\mathbf{y}^*$ is equivalent to the optimal hedging solution in (2).

Conversely, suppose that there exists a partition $\Omega = \Omega_{q^0} \cup \Omega_0 \cup \Omega^* \cup \Omega_{\mu_0}$ and $\mathbf{y}^* = \sum_{i=1}^{k} w_i^* \mathbf{y_i} \in Y$. Take

$$\begin{cases} a_1 = y_j^* - x_j & \text{on} \quad \Omega_0 \\ a_1 = 0 & \text{otherwise} \end{cases}$$

and

$$\begin{cases} a_2 = -y_j^* + x_j & \text{on} \quad \Omega_{\mu_0} \\ a_2 = 0 & \text{otherwise} \end{cases}$$

where a_1 a_2 are non-negative constant. Since (ξ^*, q^*) on Ω_{q^0} and Ω^* always guarantees that

$$\xi_j (y_j^* - x_j) = \xi_j^* (y_j^* - x_j)$$

it is clear that on the partition Ω the first constraint in (11) that

$$\sum_{j=1}^{n} \xi_j (y_j^* - x_j) \geq \sum_{j=1}^{n} \xi_j^* (y_j^* - x_j)$$

always hold. Besides, $(\xi^*, q^*) \in D_f$ and (v) shows that all the conditions in (11) hold. Therefore, (ξ^*, q^*) solves (15) and $\mathbf{y}^*$ solves (2). □

we can see that all the results are consistent with the Theorem 13 in Balbas (2010) even considering the investor is averse to ambiguity.

Remark 2. (a) Notice that the first constraint in (15) is equivalent to

$$\xi \mathbf{y} = y_\pi \mathbf{y} \tag{2.23}$$

Therefore, in a perfect market the price of $y \in Y$ can also be expressed as

$$\pi(y) = e^{-r_f T} y_\pi \mathbf{y} = e^{-r_f T} \xi \mathbf{y} \tag{2.24}$$

Since $\xi \in \mathbb{R}^n \subset \mathcal{P}$, ξ may be interpreted as a risk neutral probability measure in this market. Hence, for the security x, we also have $\pi(x) = e^{-r_f T} \xi x$.

(b) Suppose (ξ^*, q^*), $(\mathbf{w}^*, \mathbf{u}^*, \mathbf{v}^*)$ solves (15) (16) respectively, and the ask price of x is reasonable connected to the market (y, π). Then under the ambiguity assumption, ξ^* provides the maximum risk the investor must face if he sells x, measured by CVaR, and the robust probability distribution is denoted by q^*. Furthermore, in order to hedge the global position, the optimal strategy $\sum_{i=1}^{k} w_i^* \mathbf{y_i}$ implied in (16) guarantees zero risk (see in (3)) even in the worst case. In other words, if the real probability belongs to the set of priors κ, the investor at worst will face zero risk after hedging. But if not, it cannot get worse than previous situation, and the global risk would be negative.

2.5 Numerical results

2.5.1 Assumption

In this section, we make several assumptions before the pricing and hedging for SSB. We assume that the hedging strategy is composed of national government bonds issued by risk sharing members (France, Italy and Spain) in SSB. To provide a specific ambiguity set κ, we simplify the third constraint in (15) by setting

$$\mathbf{y_i} q \geq price(i)(1+\alpha) \quad \text{for} \quad i = F, I, S$$

TABLE 2.1: Subscribed capital in billions

Germany	France	Italy	Spain
39.195	39.195	39.195	23.517

* Based on the subscribed capital of France, Italy and Spain in EIB, the corresponding weights used to calculate the coupon rate of SSB are 38.46%, 38.46%, 23.08% respectively.

where α denotes national government bond returns that an investor at least expects from the market. The lower the value, the wider the set κ and higher degree of ambiguity aversion. We conservatively set $\alpha = 0.0005$ for short-term bonds and $\alpha = 0.005$ for long-term bonds in terms of market short and long-term sovereign yield of France, Italy and Spain. If the investor is extremely ambiguity averse, α will be close to zero. Then, the risk neutral probabilities will be included in the set κ, which implies that the market is risk neutral for this investor, and he has no incentives to hold any risky asset (Cao et al. 2005 and Bossaerts et al. 2010).

Additionally, for simplicity, we assume that for all SSBs, the states that may happen in future are known, which is related to the event of default of risk-sharing members. Once default occurs, in other words, all the members cannot meet their obligations. The bondholders will receive the compensation on the next redemption date. Other parameter values are given by: recovery rate $R = 70\%$, confidence level $\mu_0 = 80\%$.

2.5.2 Data description

We construct three short-term (2-year) and long-term (10-year) synthetic sovereign bonds (SSB) to illustrate the robust pricing analysis and hedging strategy. The coupon rate is fixed at the weighted average of coupon rate observed in national government bonds issued in France, Italy and Spain, where the weights are given by the subscribed capital share reported in EIB in 2012 (see in table 1).

We assume that a trader will sell (or buy) any SSB mentioned above in the market on 28 May 2013. The national government bonds used in hedging strategy are described

Table 2.2: Bond information

Bonds	coupon	Face value	Price	yield
Germany 2-year bond	0.035	1	1.069	0.05%
10-year bond	0.035	1	1.177	0.98%
France 2-year bond	0.0375	1	1.072	0.15%
10-year bond	0.0375	1	1.184	1.13%
Italy 2-year bond	0.0425	1	1.064	1.60%
10-year bond	0.0425	1	1.069	3.30%
Spain 2-year bond	0.043	1	1.059	1.81%
10-year bond	0.043	1	1.066	3.57%

* Table 2 reports the market price, coupon rate, face value, time to maturity (TTM) and bond yield on May 28, 2013 for 2-year and 10-year sovereign bonds issued by Germany, France, Italy and Spain, respectively. The10-year market price, TTM and yield are provided by (MDP) in Datastream. But for 2-year bonds, their prices are the present value of coupon and principal discounted by the average 2-year bond yield provided by market maker on May 28th, 2013.

in table 2. Since there exists a sequential arbitrage due to the zero-coupon bonds and strips, only fixed-income sovereign bonds are preferred in hedging portfolio. Also, to reasonably control the numbers of states and time difference of cash flows in the experiments, the bonds with similar payment date are allowed. Therefore, we select three 10-years sovereign bonds issued by Germany, France, Italy and Spain from active market as hedging candidates for 10-year SSBs, where the bond prices and yields are provided by Datastream. However, the 2-year sovereign bonds with similar payment date are not available in market, we have to create hedging bonds based on 2-year national government bond yields provided by market maker on May 28th, with the same coupon and payment date as in the long-term hedging bonds. Additionally, the discount rate is provided by a term structure of interest rate of Triple-A bonds on 28 May 2013 from European Central Bank.

Obviously, either in long or short maturity, German government bonds shown in Table 2 have the lowest internal rate of return as compared to France, Italy and Spain, thereby are regarded as default free bonds throughout the paper. Italy and Spain bonds yields are similar because both have the worst credit rating in the sample. The French IRR for a 2-year or 10-year bond always sits between German IRR and Italian or Spain IRR.

2.5.3 Two-years Synthetic sovereign bonds

2.5.3.1 Robust price and Hedging strategy

We first start with three 2-year SSB created based on different groups of risk sharing members which are France&Spain, France&Italy and France&Spain&Italy, respectively. All the synthetic bonds are assumed to have the same coupon rate of 4%[3], face value of 1 and the same payment date as in the Spanish 2-year bond.

Table 3 shows the worst case ask and bid prices for three 2-year SSB at the confidence level of $\mu_0 = 80\%$ and with an ambiguity set indicated by $\alpha = 0.05\%$. The corresponding hedging portfolio units allocated in each sovereign bonds are reported in the last column. Surprisingly, all three SSBs with different risk sharing countries show the same bid or ask prices. These prices stay constant all the time, at any confidence level where $\mu_0 \in (0, 0.99]$ with a fixed ambiguity set indicated by $\alpha = 0.05\%$, or for any reasonable feasible ambiguity set where $\alpha \in (0, 0.07\%)$ with a fixed confidence level of $\mu_0 = 80\%$ implied in Figure 1. Also, we check the internal rate of return (IRR) for these three SSBs. Based on ask price, all the IRRs in column 2 are equal to 0.046%, and this is rather similar to the Germany 2-year bond. Moreover, the optimal hedging strategy for a short selling position in column 3 suggests that the trader will only buy 1.009 unit of a risk free bond (the two-year Germany bond in our case) to hedge the position at any confidence level. However, when the trader is involved in a long position of SSB, he will choose a strategy of selling 1.013 units of France 2-year bond and buying 0.008 units of Germany bond. See column 4. In this case, the IRR suddenly rises to 0.208%, which is higher than a 2-year French bond but lower than a 2-year Spanish or Italian bonds.

To find out why the prices and trader's optimal decision do not change with the confidence level and the degree of ambiguity, we randomly investigate the discounted payoff of hedging portfolio and hedged SSB guaranteed by France and Spain for every state reported in figure 2. Obviously, the difference of discounted payoff between

[3]The coupon rate of each 2-year SSB is given by weighted average interest rate of 2-year nation government bonds issued by risk sharing members. Since all the results based on different risk-sharing group are quite close to 4, we use 4% as the coupon rate for all 2-year SSBs.

Table 2.3: Bid-Ask Robust price and Hedging strategy for 2-year SSBs

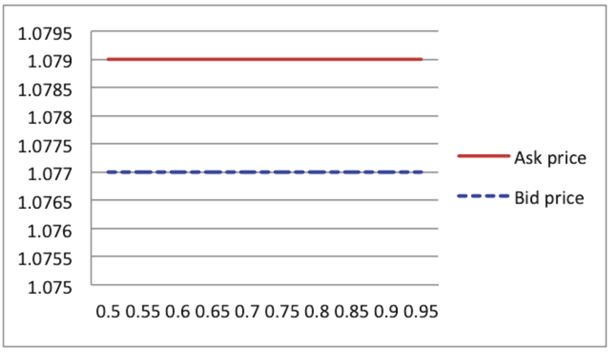
Sovereign bonds	IRR	R. Hedging units (sell)	R. Hedging units (buy)
France&Spain:			
Germany	0.048%	1.009	0.009
France	0.146%	0.000	-1.013
Spain	1.809%	0.000	0.000
$Price_{ask} = 1.079$,	$IRR = 0.046\%$;	$Price_{bid} = 1.077$,	$IRR = 0.208\%$.
France&Italy:			
Germany	0.048%	1.009	0.009
France	0.146%	0.000	-1.013
Italy	1.598%	0.000	0.000
$Price_{ask} = 1.079$,	$IRR = 0.046\%$;	$Price_{bid} = 1.077$,	$IRR = 0.208\%$.
France&Spain&Italy:			
Germany	0.048%	1.009	0.009
France	0.146%	0.000	-1.013
Italy	1.598%	0.000	0.000
Spain	1.809%	0.000	0.000
$Price_{ask} = 1.079$,	$IRR = 0.046\%$;	$Price_{bid} = 1.077$,	$IRR = 0.208\%$.

* The robust bid prices are estimated by substituting the payoff matrix x as $-x$. The value of parameters are given by: $\mu_0 = 80\%, \alpha = 0.0005$.

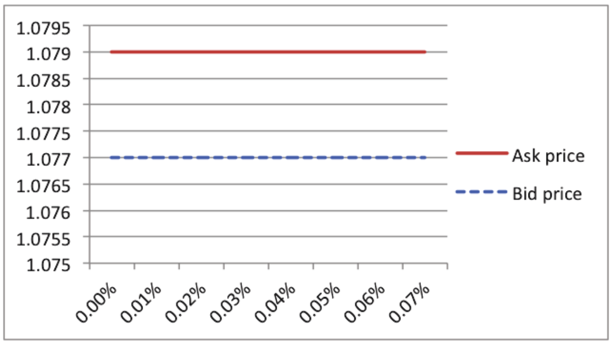
hedging portfolio and hedged SSB is non-negative for every state, no matter in a short or long position in SSB, which definitely implies the existence of the first order dominance, thereby leading to a zero CVaR.

2.5.3.2 Physical Probabilites

Table 4, 5 and 6 report the default probabilities for France, Italy and Spain based on the estimated state physical probabilities q given by the optimization problem (15). All the marginal and conditional probabilities in the three tables are similar, whether the SSB are guaranteed by two or three countries. The marginal and conditional default probabilities for Spain and Italy are much higher than France due to their different credit risk. Especially, Spain has the largest default probability because of its highest credit spread compared with the other two countries. For a long position of SSB, the joint and conditional probabilities between the countries in the last column are on average 20% higher than the probabilities for a short position reported in the second column. As we know, the SSB will default only if all the risk sharing members default. Therefore, it is obvious that the greater the default relations between countries, the more possibility that a SSB buyer incurs a loss. Hence, the buyer will conservatively



(a) Bid-ask prices change with the confidence level



(b) Bid-ask prices change with the ambiguity level

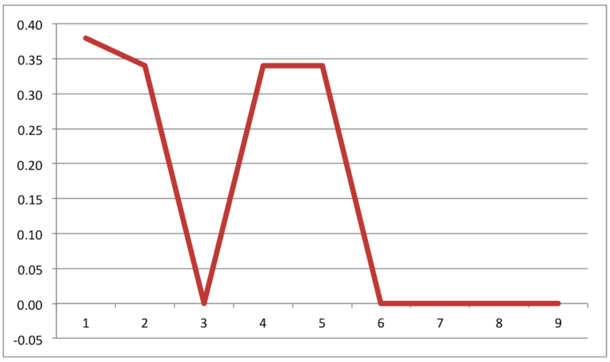
FIGURE 2.1: Comparison of cash flows

TABLE 2.4: France&Spain

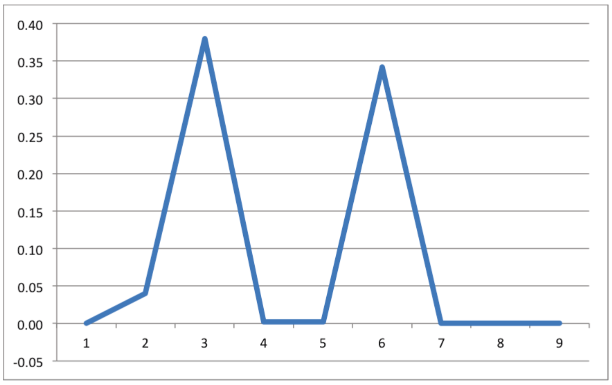
Default Probabilities	Short selling	Buy
$P_r\{$France defaults$\}$	0.593%	0.586%
$P_r\{$Spain defaults$\}$	5.747%	5.753%
$P_r\{$France $\cap$ Spain$\}$	0.326%	0.448%
$P_r\{$France defaults$\|$Spain defaults$\}$	5.672%	7.792%
$P_r\{$Spain defaults$\|$France defaults$\}$	54.97%	76.49%

* All the probabilities reported above are computed based on the state joint probabilities q^*estimated from a SSB guaranteed by a 2-year French bond and a 2-year Spanish bond.

bid a price by predicting the worst case joint or conditional default probability as high as possible. In contrast, from a standpoint of a SSB seller, he will foresee a "negative" relation in the worst case by taking into account lower joint or conditional default probabilities.



(a) Difference of discounted payoff between hedging portfolio and hedged SSB (sell)



(b) Difference of discounted payoff between hedging portfolio and hedged SSB (sell)

FIGURE 2.2: Discounted payoff difference between hedging portfolio and hedged SSB

TABLE 2.5: France&Italy

Default Probabilities	Short selling	Buy
P_r{France defaults}	0.593%	0.586%
P_r{Italy defaults}	5.135%	4.922%
P_r{France $\cap$ Italy}	0.325%	0.451%
P_r{France defaults|Italy defaults}	6.334%	9.163%
P_r{Italy defaults|France defaults}	54.83%	76.99%

* All the probabilities reported above are computed based on the state joint probabilities q^*estimated from a SSB guaranteed by a 2-year French bond and a 2-year Italian bond.

Table 2.6: France&Italy&Spain

Default Probabilities	Short Selling	Buy
$P_r\{\text{France defaults}\}$	0.60%	0.59%
$P_r\{\text{Italy defaults}\}$	5.62%	5.58%
$P_r\{\text{Spain defaults}\}$	6.25%	6.20%
$P_r\{\text{France} \cap \text{Spain} \cap \text{Italy}\}$	0.25%	0.35%
$P_r\{\text{France defaults}\vert\text{Italy defaults}\}$	7.05%	8.09%
$P_r\{\text{Italy defaults}\vert\text{France defaults}\}$	66.22%	76.37%
$P_r\{\text{France defaults}\vert\text{Spain defaults}\}$	6.32%	7.27%
$P_r\{\text{Spain defaults}\vert\text{France defaults}\}$	65.96%	76.25%

* All the probabilities reported above are computed based on the state joint probabilities q^* estimated from a SSB guaranteed by a 2-year French bond, a 2-year Italian bond and a 2-year Spanish bond.

2.5.4 10-year synthetic sovereign bonds

Next we focus on the long-term SSBs. We create three 10-year synthetic sovereign bonds (SSB) in the same manner as the 2-year SSBs, that is, with a coupon rate of 4%, face value of 1. As can be seen, the results in Table 7 are similar to those reported in table 2. The estimated robust bid and ask prices of three 10-year SSBs are exactly the same, where $p_{\text{ask}} = 1.211, p_{\text{bid}} = 1.199$, respectively. Meanwhile, the IRR based on a short position also reports the same return with the Germany 10-year bond. However, for a long position, IRR is similar with French 10-year bond due to the increased risk resulting from short selling the France bond in hedging portfolio. Also, we find that the level of ambiguity aversion and confidence still do not affect the estimated prices. So we check the present value of hedging portfolio and hedged SSBs future cash flows for all the states. Figure 3 and 4 report the differences between the expected payoff of hedging portfolio and hedged SSB guaranteed by three countries for state 1 to 512. It is clear that whether a long or short position, the difference curve is always above or overlap the horizontal axis with a value of 0 for every state. Hence, we consider the expected payoff of the SSB as being dominated by the payoff of the corresponding hedging portfolio, which strongly suggests the existence of the first order dominance.

To test the reliability of the above results, we analyze the default relations between countries reported in table 8, 9, 10. In the worst case, the estimated marginal probabilities for the three countries perform consistently in the three tables. The joint and

TABLE 2.7: Bid-Ask robust prices and Hedging portfolio for 10-year SSBs

Sovereign bonds	IRR	R. Hedging units (sell)	R. Hedging units (buy)
France&Spain:			
Germany	0.98%	1.0288	0.020
France	1.13%	0.000	-1.034
Spain	3.57%	0.000	0.000
$Price_{ask} = 1.211$,	$IRR = 0.966\%$;	$Price_{bid} = 1.199$,	$IRR = 1.14\%$.
France&Italy:			
Germany	0.98%	1.0288	0.020
France	1.13%	0.000	-1.034
Italy	3.30%	0.000	0.000
$Price_{ask} = 1.211$,	$IRR = 0.966\%$;	$Price_{bid} = 1.199$,	$IRR = 1.14\%$.
France&Spain&Italy:			
Germany	0.98%	1.0288	0.020
France	1.13%	0.000	-1.034
Italy	3.30%	0.000	0.000
Spain	3.57%	0.000	0.000
$Price_{ask} = 1.211$,	$IRR = 0.966\%$;	$Price_{bid} = 1.199$,	$IRR = 1.14\%$.

* The robust bid prices above are estimated by substituting the payoff vector x as $-x$. In this case, x contains 512 elements. The value of parameters are given by: $\mu_0 = 80\%, \alpha = 0.0005$.

TABLE 2.8: France&Spain

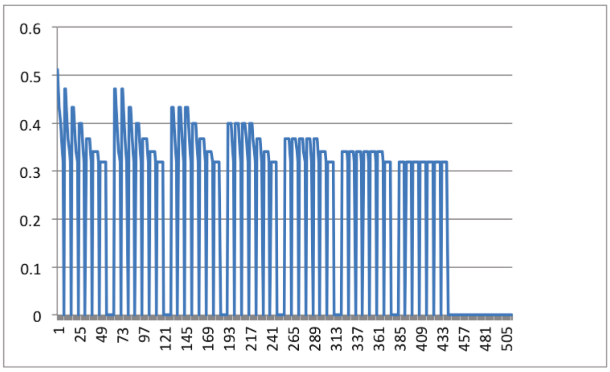
Default Probabilities	Short sell	Buy
$P_r\{$France defaults$\}$	1.15%	1.04%
$P_r\{$Spain defaults$\}$	39.04%	38.93%
$P_r\{$France $\cap$ Spain$\}$	0.57%	0.94%
$P_r\{$France defaults$\|$Spain defaults$\}$	1.45%	2.41%
$P_r\{$Spain defaults$\|$France defaults$\}$	49.37%	90.32%

* All the probabilities reported above are computed based on the state joint probabilities q^* estimated from a SSB guaranteed by a 10-year French and a 10-year Spanish bond.

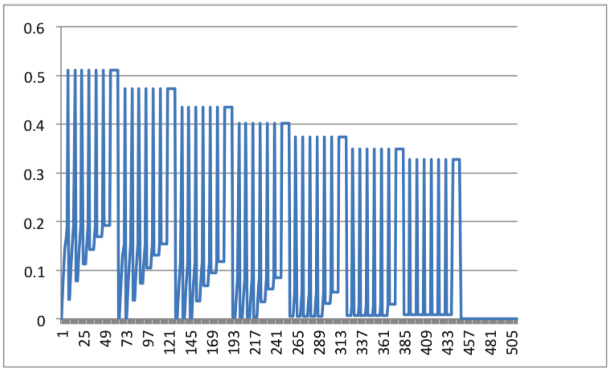
conditional default probabilities in a long position of SSB are obviously higher than the probabilities in a short position, especially for the default probabilities conditional on France defaults, which even exceed 90% for a buy position. Hence, the results are consistent with our worst case assumption.

2.5.5 Discussion

Overall, under the worst case analysis, the assessed bid and ask prices imply a striking reduction in default risk, whether it is a 2-year SSB or a 10-year SSB. The robust bid and ask prices of 2-year SSBs are 1.077 and 1.079, respectively. The three10-year SSB



(a) Payoff difference for a seller



(b) Payoff difference for a buyer

FIGURE 2.3: Payoff difference between hedging portfolio and hedged SSB

TABLE 2.9: France&Italy

Default Probabilities	Short sell	Buy
$P_r\{\text{France defaults}\}$	1.15%	1.04%
$P_r\{\text{Italy defaults}\}$	38.37%	38.24%
$P_r\{\text{France defaults} \cap \text{Italy defaults}\}$	0.57%	0.94%
$P_r\{\text{France defaults} \mid \text{Italy defaults}\}$	1.48%	2.45%
$P_r\{\text{Italy defaults} \mid \text{France defaults}\}$	49.38%	90.34%

* All the probabilities reported above are computed based on the state joint probabilities q^* estimated from a SSB guaranteed by a 10-year French and a 10-year Italian bond.

bid ask prices are 1.199 and 1.211, respectively. They keep constant even with different risk sharing members. Based on ask prices of 2-year and 10-year SSBs, we surprisingly find that all of them perform like default-free bond, whose IRR is close to or even lower than the Germany bond. Therefore, the corresponding optimal hedging strategy for a short position is only to hold 1.099 units of 2-year Germany bond and 1.199 units of 10-year Germany bond, respectively. However, from a standpoint of a buyer, his beliefs in default events are pretty higher than a seller in the worst case situation, thereby

TABLE 2.10: France&Italy&Spain

Default Probabilities	Short Selling	Buy
$P_r\{\text{France defaults}\}$	1.16%	1.06%
$P_r\{\text{Italy defaults}\}$	38.32%	38.20%
$P_r\{\text{Spain defaults}\}$	38.70%	38.58%
$P_r\{\text{France} \cap \text{Spain} \cap \text{Italy}\}$	0.53%	0.92%
$P_r\{\text{France defaults}\vert\text{Italy defaults}\}$	2.15%	2.34%
$P_r\{\text{Italy defaults}\vert\text{France defaults}\}$	70.95%	93.34%
$P_r\{\text{France defaults}\vert\text{Spain defaults}\}$	2.13%	2.55%
$P_r\{\text{Spain defaults}\vert\text{France defaults}\}$	70.93%	93.33%

* All the probabilities reported above are computed based on the state joint probabilities q^* estimated from a SSB guaranteed by a 10-year French bond, a 10-year Italian bond and a 10-year Spanish bond.

the IRR based on bid prices implies risk-rewards, which is between the IRR of France bonds and Italy bonds. Additionally, the corresponding hedging portfolio consists of short sale in 1.013 units of France 2-year bond and buying 0.009 units of Germany 2-year bond for hedging 2-year SSBs, and short sale in 1.034 units of France 10-year bond and buying 0.02 units of Germany bond for hedging 10-year SSBs. Furthermore, in the worst case, the trader who holds a short position in SSB will predict lower default relations among risk sharing members compared with default relations in a long position, which is also confirmed by our physical probabilities analysis. We find that the joint or conditional probabilities implied from ask prices are significantly lower than the probabilities implied from bid prices.

However, the performance of the bid ask prices and optimal hedging portfolio, which do not change with the degree of ambiguity aversion or the confidence level, is beyond our expectations?. After comparing the discounted payoff of hedging portfolio y_j^* with the hedged SSB x_j for all states, we find that y_j^* is always higher than or equal to x_j for any state j, which means that SSB can be perfectly hedged either sell or buy. This result strongly implies the existence of the first order dominance and thereby leading to zero CVaR. Also, only holding German bonds in hedging portfolio for a short position would likely cause a sequential arbitrage. For example, a trader sells a 2-year SSB in European bond market and simultaneously hedges the position by buying 1.009 units of a 2-year Germany bond. Assume that the SSB defaults in year 1 due to the serious financial crisis in Euro area, the seller can short sell the Germany bonds immediately so as to pay the remaining value of the bond to the buyer. In this case, the total cash flows at the initial period are 0.04% and the cash flows in year 1 are definitely positive, subsequently leading to a sequential arbitrage. In our future work, we plan to impose some constraints to overcome the existence of the first order dominance and the sequential arbitrage to improve the results.

2.6 Conclusion

This paper focuses on providing a general pricing and hedging methodology for completing the European sovereign bond market where the forthcoming products such as the Eurobonds cannot be replicated by trading available securities. The challenge of our paper is related to the selection of EMM in incomplete sovereign market under the unknown joint or conditional default probabilities among European state members, which lead to an ambiguous payoff distribution of Eurobonds. Therefore, we use a robust analysis for pricing and hedging these unreplicable securities by minimizing the worst-case CVaR of the hedging residual risk. Our methodology is somehow conservative, but the proposed prices and hedging portfolio can make the CVaR of global portfolio risk virtually vanish even in the worst situation. Particularly, the uncertainty set specified in this analysis is more general and contains more possible physical probabilities. Hence, it provides some fresh ideas for a trader who is more concerned with hedging the risk during the financial crisis than with a high reward. Additionally, the optimization problem provides a risk neutral measure for joint or conditional sovereign default probabilities, which will become a valuable tool for pricing more sophisticated derivatives.

In the numerical analysis, we introduce a new product, named 'Eurobonds' that are used to solve debt crisis in the European sovereign bond market. Although it does not exist in reality, we construct three novel 2-year and 10-year bonds called SSB with the same characteristics as the 'Eurobonds'. The numerical results are consistent with the arguments of Claessens et al (2012) who claimed that the introduction of joint guarantees makes these Eurobonds virtually default free. The assessed prices imply a significant reduction in default risk of SSB compared with national government debt in Spain and Italy. However, the hedging portfolio provided in the program can completely replicate hedged SSB, which contradicts our assumption of an incomplete market. We find that this abnormal result is mainly induced by the existence of the sequential arbitrage. This is left for the further investigation.

Chapter 3

The worst-case Sovereign risk dependence in the European bond market

3.1 Introduction

The ongoing sovereign debt crisis in the Euro zone since the late 2009 has led to a growing concern about the interacting sovereign risk among investors and researchers. This uncertainty of underlying interdependence of sovereign risk results in highly conservative portfolio choices of investors, who shift more assets to the bonds deemed safe such as German bonds. Consequently, it increases more overpriced-risk in periphery European countries. In this paper we contribute to this discussion and attempt to provide a worst-case optimal portfolio composed of sovereign bonds from safe and periphery countries in the Euro zone, by estimating a joint probability of sovereign default from the observed bond prices. Furthermore, we try to examine whether or not there exits the mispricing of sovereign bonds in heavily indebted countries.

Currently, the research in portfolio optimization with ambiguous (uncertain) asset probability distribution is mainly applied to the stock markets. Because historical stock market data is generally transparent and comprehensive, the uncertain asset correlations are estimated accurately. Recent papers include Calafiore (2007), Zhu and Fukushima (2009), Kakouris and Rustem (2014). By contrast, in bond markets there are not enough time-series data to accurately assess the default dependence because default events are rare, especially for sovereign bonds (Zhou 2001). Therefore, our methodology tries to avoid this issue by recovering joint default probability based on a forward-looking approach. We develop a flexible structure for the interdependence between sovereign risk similar to the procedure of pricing credit default swap (CDS) (Hull, 2006). First, we assume that for an individual sovereign bond, the default can take place at any point in the coupon or principal interval[1]. Second, for every bond, the state that may happen in the future is a joint event including all other bond-state information. Thus, the state probability is a joint probability instead of a marginal probability. Finally, we introduce an ambiguous or (uncertain) probability set which includes all possible real state joint probability distribution in order to obtain a robust empirical result. The ambiguity set can be traced back to Ellsberg (1961). This term is also called an investor's set of priors. Motivated by the work of Garlappi et al (2007) who found that the optimal portfolio is more stable when dealing with ambiguous expected returns, compared to portfolios from classical and Bayesian models, we impose this set. Likewise, in our paper, the bond price assessed via the probabilities in such a set is assumed to be no less than its market price. This assumption is based on a claim by Hull (2004), who pointed out that a real default probability is always lower than a risk neutral probability in reality because investors always overstate the default risk.

Indeed, there are two other forward-looking approaches for estimating the dependence of sovereign risk. One approach focuses on a parametric modeling assumption such as the works of Lucas et al. (2013), who proposed a dynamic Generalized Hyperbolic skewed-t distribution to fit the changes of sovereign CDS spreads; as well as Giacometti and Pianeti (2012), who first estimated the marginal probability generally implied from CDS (Hull and White 2000, O'kane 2008), then using copula to derive the default probability dependence. These work are closely related to the statistical literature for multiple defaults (Hull and White 2004, Avesani et al. 2006). Another approach is also based on the marginal default information derived from CDS spread,

[1]Coupon or principal interval represents a period of the time between two redemption date.

but the intuition of building a multivariate distribution highly relies on the structure credit model by Merton (1974). See for example Radev (2012). However, these two approaches typically build on assumptions of a specific distribution, and may suffer from some extreme events in the market. By contrast, our approach described above provides a more general form for dependence assessment, because all possible real default joint probability distribution are included in our ambiguity set. This is thus more conductive to a robust result in portfolio optimization as compared to the above approaches.

With ambiguous probability distribution, we adopt the robust (or worst-case) conditional value at risk (RCVaR) as a risk measure in optimal risk-return tradeoff analysis. RCVaR is actually an extension of CVaR. Recently, RCVaR has been commonly used in robust portfolio optimization because it is a coherent risk measure (Artzner et al,1999) like CVaR as shown by Zhu and Fukushima (2009), Balbás et al. (2012), and somehow is closely related to the utility theory.[2] Moreover, since RCVaR describes the worst-case loss on the tail of any distributions, it overcomes the weakness of CVaR that relies exclusively on a specific distribution assumption (Giglio, 2011). It is thereby an appropriate sovereign risk measure under our ambiguous probability assumption. Hence, in this paper, we extend CVaR through a specific ambiguous probability set. We follow Balbás et al (2012)'s work because they derived RCVaR with a general form of ambiguity set, and also provided a robust optimal portfolio.

Our results show that the joint or conditional default probability for safe and risky countries performs consistently with the change of bond yield in sample period. Particularly, safe country has the highest individual contribution to the joint sovereign risk in the case of default, and the influence of the default interaction in periphery countries is much stronger than the mutual effect between safe and periphery countries. With respect to the robust optimal portfolio, we find that the worst-case optimal weight allocated in bonds of safe and periphery countries are fairly stable even in 2012 where Spanish and Italian bonds yield show peaks. Also, the weights of Spanish and Italian bonds are significantly different from zero. Therefore, these results provide a powerful

[2]Pflug (2000) and Ogryczak and RUszczynski (2002) showed that CVaR is highly related to the stochastic dominance principles, which is closely related to the utility theory.

evidence that sovereign risk in periphery countries are overstated and are consistent with the findings of Grauwe and Ji (2012), Beirne and Fratzscher (2013).

The remaining of the paper is organized as follows. In section 2, we introduce the RCVaR, the ambiguous probability set and give a brief description of Balbás et al (2012) optimization method and their main findings. Section 3 presents the application to the European sovereign bonds and the specification of uncertainty set. In section 4, we present the results on default probabilities and the robust optimal solutions. Section 5 summarizes and concludes.

3.2 Methodology

3.2.1 Notations and Preliminaries

Let Ω represent a finite set of states that may happen on future date T, $y = (y_1, \cdots, y_m)^T \in \mathbb{R}^m$ denote the reachable payoffs of the m risky assets, $\mathbf{w}$=$(w_1, \cdots, w_m)^T \in \mathbb{R}^m$ represent the weight of the investment in m risky assets decided by the investor. Thus, the payoff of a portfolio is defined as

$$y_p = w^T y$$

Let κ be a set of probabilities measure an investor's ambiguity (or uncertainty). Then, any element in κ will be called as a feasible prior and the absence of ambiguity holds if and only if κ is composed of one element. Finally, consider a probability measure on Ω such that,

$$\mathcal{P} = \left\{ p = (p_1, \cdots, p_n) \in \mathbb{R}^n; \sum_{j=1}^{n} p_j = 1, p_j \geq 0 \right\}.$$

Then, $\kappa \subset \mathcal{P}$ obviously holds.

Under the ambiguity aversion assumption, portfolio optimization by the classical return-risk trade-off analysis (Markowitz 1952) is not appropriate. In this paper we follow the worst case analysis proposed by Gilboa and Schmeidler (1989), Zhu and Fukushima

(2009), Balbás et al. (2012) to solve the portfolio selection problem. The results of these authors are quite consistent with the existence of ambiguity aversion and the empirical finding pointed out by Ellsberg (1961). We also use the same risk measure, which is a robust (or worst-case) $CVaR$ ($RCVaR$), as in Zhu and Fukushima (2009), Balbás et al. (2012). Then, the robust portfolio selection problem can be formulated as

$$\min_{\mathbf{w}\in\mathbf{W}} RCVaR \tag{3.1}$$

where W represents the constraint in portfolio selection, which usually requires the minimum expect return in the worst case. Moreover, it includes information of the investor's uncertainty denoted by $p \in \kappa$. Let r be the minimum expected return required by an investor, thus W can be specified in a discrete form:

$$W = \{w : \sum\nolimits_{j=1}^{n} p_j y_{pj} \geq r; p \in \kappa\}$$

For a given confidence level μ_0 and a specific ambiguity set κ, the calculation of $RCVaR$ suggested by Balbás et al (2012) can be represented as

$$RCVaR_{(\kappa,\mu_0)}(y) = \max\left\{-\sum\nolimits_{j=1}^{n} \xi_j y_j; \quad \xi = (\xi_1, \cdots, \xi_n) \in \nabla_{(\kappa,\mu_0)}\right\}$$

Where the $RCVaR$ sub-gradient is given by

$$\nabla_{(\kappa,\mu_0)} = \left\{(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n; \sum_{j=1}^{n} \xi_j = 1, 0 \leq \xi_j \leq \frac{q_j}{1-\mu_0} \quad forj = 1, \cdots, n, q \in \kappa\right\}$$

Note that $RCVaR_{(\kappa,\mu_0)}$ has been verified as a coherent risk measure (Artzner et al.1999), which is consistent with the utility maximization problem. Without ambiguity assumption, $RCVaR_{(\kappa,\mu_0)}$ is equivalent to the CVaR due to a unique underlying probability.

Under our ambiguity framework, the most proper risk measure is CVaR. The investors with ambiguity averse always overestimate the bad things and underestimate good payoff probability, which implies a asymmetric payoff distribution (Epstein and Schneider

2008). Thus CVaR is a good descriptor for a loss in tails. Additionally, it is a coherent risk measure. Although there are better risk measures called spectral risk measures that can accurately measure the tail risk better, it is too complicated to apply them in an empirical work. If we choose other risk measures such as standard deviation, VaR, they will not satisfy our ambiguity aversion assumption. The standard deviation is not consistent with the second order stochastic dominance if there are asymmetries; and the VaR is not sub-additive beyond the normal distribution assumption. It does not facilitate the diversification of risks. Therefore, under our ambiguity framework, the most proper risk measure is CVaR. In addition, the robust CVaR is actually an extension of CVaR. Since the calculation of CVaR also needs to fix the probability distribution, we try to avoid this specification by imposing a set of priors for including all possible underlying probability distributions. Hence, the RCVaR means the smallest CVaR (or biggest risk) under the set of priors.

To solve the above problem (1), we follow the methodology suggested by Balbás et al. (2012) because they provide an optimal portfolio under a more general form of ambiguity set than other researchers.

3.2.2 Robust Portfolio optimization problem suggested by Balbás et al.

In Balbás et al. (2012)'s portfolio choice model, the uncertainty set κ is assumed to be fixed. Given the confidence level μ_0, they formulate the following portfolio selection problem with respect to y_p as follows,

$$\begin{cases} \min RCVaR_{(\kappa,\mu_0)} \\ y_\pi y_p \leq e^{r_f T} \\ \sum_{j=1}^{N} p_j y_{pj} \geq r \quad \forall p \in \kappa \\ y_p \in Y \end{cases} \tag{3.2}$$

and the corresponding dual problem with respect to (ξ, λ, p) is given by

$$\begin{cases} \max \lambda \\ \phi_Y\left(\frac{1}{\lambda}\xi + (1-\frac{1}{\lambda})p\right) = y_\pi \\ \xi \in \nabla_{(\kappa,\mu_0)}, p \in \kappa, \lambda \in \mathbb{R}, \lambda \geq 1 \end{cases} \tag{3.3}$$

In the primal problem, Y denotes a set of all possible portfolios composed of m risky bonds, and y_π denotes a unique and known pricing tool guaranteed by The Riesz representation theorem, which can be represented as

$$e^{-r_f T} y_\pi^T y_i = 1 \quad i = 1, \cdots, m \tag{3.4}$$

The known y_π indicates that an investor always knows the pricing rule in the market, and his uncertainty is only related to the underlying probability other than the price. Furthermore, if the market is complete, y_π would be a unique risk neutral probability measure in the market.

In the dual problem, the function $\phi_Y()$ represents an orthogonal relation to the portfolio set Y, which leads that for any assets $i = 1, \cdots, m$

$$\left(\frac{1}{\lambda}\xi + (1-\frac{1}{\lambda})p - y_\pi\right)^T y_i = 0 \tag{3.5}$$

must hold..

Combining (3) and (4), it is clear that

$$e^{-r_f T}\left(\frac{1}{\lambda}\xi + (1-\frac{1}{\lambda})p\right) y_i = 0 \tag{3.6}$$

where the term $\left(\frac{1}{\lambda}\xi + (1-\frac{1}{\lambda})p\right)$ is a linear combination of ξ and p, so it also belongs to the probability set $\mathcal{P}$ and thereby can be viewed as a risk neutral measure of market.

Let y_p^*, (λ^*, ξ^*, p^*) be the optimal solutions to (2) and (3). Balbás et al. (2012) has shown that, first, the feasible set of the dual problem (3) does not depend on the expected return r. Second, under an ambiguity assumption, for any given r, the optimal portfolio proportion W^* invested in risky bonds are stable even in the worst case. In other words, r only depends on the investor's preference in risky or risk-free bonds, not the proportion in risky portfolio. The less risk averse an investor, the more r he expects and the more weights allocated to risky assets. By contrast, investors with more risk averse would prefer more investment in risk-free assets. Furthermore, if there is no ambiguity in the states of nature but only for the probability distribution, they can even construct a CAPM-like formula with a robust risk measure $RCVaR$, and the optimal proportion in y_p^* will reflect conservative market portfolio. In addition, they also provide a new method to price securities and derivatives with risk neutral probability measure $\left(\frac{1}{\lambda^*}\xi^* + (1 - \frac{1}{\lambda^*})p*\right)$ estimated in the worst case. Therefore, these important findings provide strong theoretical evidences for us to analyze the optimal portfolio choice in the European sovereign bond markets.

3.3 Application

Following the model and main findings described above, we investigate the European sovereign debt market due to the serious debt crisis since 2009. We mainly focus on the government debts issued by Germany, France, Italy and Spain which own the first four largest debt markets in Europe. These countries are representatives of the safe and periphery countries (Germany and France are considered as safe countries, while Spain and Italy are periphery countries). Throughout the paper, German bonds are assumed to be unambiguous and credit-free, which means that their payoffs are identical across states.

3.3.1 Notations

Consider four Sovereign coupon bonds B_G, B_F, B_I, B_S issued by the above four countries respectively, with current market prices $\{P_G, P_F, P_I, P_S\}$ and the time to maturity $T = \{T_G, T_F, T_I, T_S\}$. Their cash flows are generated, respectively in N_G, N_F, N_I, N_S periods , where one period is defined as the time to the next coupon date since the previous coupon date. We use the first capital letter to denote the country name.

Let $\Omega = \{\omega_{1,1,1}, \omega_{1,1,2}, \cdots, \omega_{n,n,n}\}$ denote a finite set of states of nature for a portfolio composed by these four bonds. The state describes a joint default condition that are likely to occur in future periods for the three risky countries (France, Italy, Spain). For instance, $\omega_{1,1,2}$ represents a state that France and Italy default in the first period and Spain defaults in the second period; $\omega_{n,n,n}$ denotes no default at all. Therefore, we have $N = (N_F + 1) \times (N_I + 1) \times (N_S + 1)$ states in this case.

Suppose that the expected recovery rate is R in case of a default, then the bondholder is assumed to receive a proportion R of the face value at the end of the occurring period. For each sovereign bond, we can obtain a payoff vector y_i, where $i = G, F, I, S$ which represents a set of present value of future cash flows in each state. Therefore, the payoff vectors have the same dimensions with the set of states.

3.3.2 Specification of an ambiguous state probability set

Instead of assuming a specified state probability distribution, we provide a general form such as an ambiguity set denoted by κ including all possible real state probability distribution. The specification of such a set is motivated from Hull (2004) who claimed that investors gained more in holding corporate or sovereign bonds because the real default probability was always lower than a risk neutral default probability. Based on a perspective that a real default probability brings higher return, we specify the set such that:

$$y_i q \geq P_i(1+\alpha) \quad \text{for} \quad i = F, I, S$$

where α denotes the gain that an investor can obtain based on a real probability compared to a risk-neutral probability. However, the value of α mainly depends on the level of uncertainty. An investor with high ambiguity (uncertainty) aversion is more likely to overstate the "bad things" in the market (Bossaerts et al. 2010), the real probability in his belief may bring less return than an investor with lower aversion in ambiguity, thereby α is smaller. Hence, we say that the higher degree of ambiguity aversion, the wider the set κ. Particularly in an extreme case, the risk-neutral probabilities will be included in the set indicated by $\alpha = 0$, which implies that an investor has no incentives to hold risky bonds because the market is risk neutral for him. This assumption is also consistent with the empirical findings by Cao et al. (2005).

3.3.3 Optimization problems

To obtain the robust optimal portfolio y^*, we first try to estimate a worst-case state probability based on the dual problem (3) by solving a nonlinear optimization problem with variables $\lambda, \{\xi_j\}_{j=1}^N, \{p_j\}_{j=1}^N, \{q_j\}_{j=1}^N$. More importantly, the dual solution has been proven to be independent of the value of r (Balbás et al. 2012).

$$\max \lambda \tag{3.7}$$

$$\text{s.t.} \quad \frac{1}{\lambda}\xi + (1 - \frac{1}{\lambda})p - y_\pi \in y_i^\perp, \quad i = F, I, S; \tag{3.8}$$

$$E_p(y_i) \geq P_i(1 + \alpha), \quad i = F, I, S; \tag{3.9}$$

$$E_q(y_i) \geq P_i(1 + \alpha), \quad i = F, I, S; \tag{3.10}$$

$$\sum\nolimits_{j=1}^{N} p_j = 1, \sum\nolimits_{j=1}^{N} \xi_j = 1, \sum\nolimits_{j=1}^{N} q_j = 1; \tag{3.11}$$

$$\xi_j \leq \frac{q_j}{1 - \mu_0}, \quad j = 1, \ldots, N; \tag{3.12}$$

$$\lambda \geq 1; \tag{3.13}$$

$$\xi_j \geq 0, p_j \geq 0, q_j \geq 0, \quad j = 1, \ldots, N. \tag{3.14}$$

Where the variables ξ, p, q are N dimensional vectors. ξ plays a key role in the function of robust $CVaR$ ($RCVaR$) and must satisfy a condition that $\xi \in \nabla_{(\kappa,\mu_0)}$. To make this condition hold, we impose q denoting a random probability measure belonging to

κ, which has a relation with ξ such that $0 \leq \xi_j \leq \frac{q_j}{1-\mu_0}$. p represents a series of real state joint probabilities. Under our assumption, $p \in \kappa$ must hold.

The objective function λ is closely related to market price of risk, represented by $\frac{1}{\lambda-1}$. In the constraints, y_π in (7) satisfies a pricing rule. The inequalities (9) and (10) measure the ambiguity set of bondholders with respect to p and q such that the expected payoff denoted by $E_p(y_i)$ and $E_q(y_i)$ for each sovereign bond must be no less than its market price P_i, given that $r_f = 0$. The remaining constraints from (11) to (14) for ξ, p and q guarantee the fundamental properties of a probability measure and CVaR.

If (7) is bounded and feasible, the optimal dual solution $(\lambda^*, \xi^*, p^*, q^*)$ will hold for any level of required return r. In terms of the conclusion of risk neutral probability (rp) provided by Balbás et al. (2012), we have the optimal risk neutral probability as follows:

$$rp^* = \frac{1}{\lambda^*}\xi^* + (1 - \frac{1}{\lambda^*})p^*$$

Remark 3. Notice that in a risk neutral world, the pricing rule y_π can be represented as a linear combination of three risky bonds' payoff in $y_i, i = F, G, I$, thereby y_π. However, the real market is incomplete due to many frictions such as transaction cost, liquidity and price jumps, so $y_\pi \in \mathcal{P}$ does not necessarily hold.

We assume that the optimal portfolio is denoted by y^*. Following Karush-Kuhn-Tucker-like conditions, y^* and $(\lambda^*, \xi^*, p^*, q^*)$ must satisfy:

$$\begin{cases} \sum_{j=1}^{N} \xi_j y_j^* \geq \sum_{j=1}^{N} \xi_j^* y_j^* \qquad \forall (\xi_j)_{j=1}^{N} \in \nabla_{(\kappa,\mu_0)} \\ (\lambda^* - 1)(\sum_{j=1}^{N} p_j^* y_j^* - r) = 0 \\ y_\pi y^* = 1 \\ \sum_{j=1}^{N} p_j y_j^* \geq r \\ \lambda^* \geq 1 \end{cases} \tag{3.15}$$

As long as all the conditions above hold, we can also infer some other propositions for the relations of $\sum_{j=1}^{N} \xi_j^* y$ and $\sum_{j=1}^{N} \xi_j^* y^*$, $\sum_{j=1}^{N} p_j^* y_j$ and $\sum_{j=1}^{N} p_j^* y_j^*$.

Proposition 5. *Suppose that $\lambda^* > 1$, if y and (λ, ξ, p) are feasible solutions for the implied pair of optimization problem* (1) *and* (2) *respectively, they must meet the inequality such that* $\sum_{j=1}^{N} p_j y_j^* \geq \sum_{j=1}^{N} p_j^* y_j^*$.

Proof. The assumption $\lambda^* > 1$ gives that $\sum_{j=1}^{N} p_j^* y_j^* = r$. Since the inequality $\sum_{j=1}^{N} p_j y_j^* \geq r$ holds for any $p \in \kappa$, it is clear that $\sum_{j=1}^{N} p_j y_j^* \geq \sum_{j=1}^{N} p_j^* y_j^*$. □

Remark 4. The above proposition and equation (15) show that if (7) is solvable, the optimal solution ξ^* provides a worst-case CVaR for a given level of y^*, compared to other $\xi \in \nabla_{(\kappa,\mu_0)}$. With a similar argument, the minimum expected return r is attained by p^*, keeping y^* fixed, and other $p \in \kappa$ will produce a return no less than r. Hence, both inequalities guarantee a robust results as we expect. Based on that, we further analyze the composition of y^* by constructing two optimization problems below.

Corollary 1. The first optimization problem is created by minimizing ξy^* such that :

$$\begin{cases} \min \xi y^* \\ \displaystyle\sum_{j=1}^{N} q_j y_{ij} \geq P_i(1+\alpha) \quad i = F, I, S; \\ \xi_j \leq \dfrac{q_j}{1-\mu_0}, \quad j = 1, \dots, N; \\ \sum_{j=1}^{N} \xi_j = 1, \sum_{j=1}^{N} q_j = 1; \\ \xi_j \geq 0, q_j \geq 0, \quad j = 1, \dots, N. \end{cases} \tag{3.16}$$

The Lagrangian function is expressed as:

$$\begin{aligned} L(W,U,\xi,q) = \sum_{j=1}^{N} \xi_j y_j^* + \sum_{i=1}^{3} w_i \left(P_i(1+\alpha) - \sum_{j=1}^{N} q_j y_{ij}^* \right) + w_4 (\sum_{j=1}^{N} \xi_j - 1) \\ + w_5 (\sum_{j=1}^{N} q_j - 1) + \sum_{j=1}^{N} u_j (\xi_j - \frac{q_j}{1-\mu_0}) \end{aligned} \tag{3.17}$$

Which is equivalent to

$$
\begin{aligned}
L(W,U,\xi,q) = {} & \sum\nolimits_{j=1}^{N} \xi_j (y_j^* + w_4 + u_j) + \sum\nolimits_{j=1}^{N} q_j \left(-w_1 y_{1j} - w_2 y_{2j} - w_3 y_{3j} - \frac{u_j}{1-\mu_0} + w_5 \right) \\
& + \sum\nolimits_{i=1}^{3} w_i P_i (1+\alpha) - w_4 - w_5
\end{aligned}
\tag{3.18}
$$

Where $W = (w_1, \ldots, w_5), U = (u_1, \ldots, u_N)$ are dual variables, and w must be non-negative due to Slater's condition. Let (ξ', q') be the solution of (15). If $\xi_j' > 0$ and $q_j' > 0$, we can derive a linear relation between y^* and dual variables based on complementary slackness conditions :

$$
\begin{cases}
y_j^* + w_4 + u_j = 0, \quad j = 1, \ldots, N \\
- w_1 y_{1j} - w_2 y_{2j} - w_3 y_{3j} - \dfrac{u_j}{1-\mu_0} + w_5 = 0. \quad j = 1, \ldots, N
\end{cases}
\tag{3.19}
$$

The other optimization problem is constructed by minimizing py^*, then we have

$$
\begin{cases}
\min py^* \\
\displaystyle\sum_{j=1}^{N} p_j y_{ij}^* \geq P_i(1+\alpha) \quad i = F, I, S; \\
\sum\nolimits_{j=1}^{N} p_j = 1; \\
p_j \geq 0, \quad j = 1, \ldots, N.
\end{cases}
\tag{3.20}
$$

With a similar argument, the Lagrangian function can be expressed as

$$
\begin{aligned}
L(L,p) = {} & \sum\nolimits_{j=1}^{N} p_j \left(y_j^* - l_1 y_{1j} - l_2 y_{2j} - l_3 y_{3j} + l_4 \right) \\
& + \sum\nolimits_{i=1}^{3} l_i P_i (1+\alpha) - l_4.
\end{aligned}
\tag{3.21}
$$

Where $L = (l_1, \ldots, l_4)$ are dual variables. Assume that p' solves the above problem and $p' > 0$, then we also have

$$
y_j^* = l1 y_{1j} + l_2 y_{2j} + l_3 y_{3j} - l_4 \tag{3.22}
$$

Remark 5. In both optimization problems, the optimal portfolio y^* can be represented by a linear combination of payoffs in risky sovereign bonds and a credit-free bond, see (19) and (20). The weights allocated in risky bonds are closely related to the dual variables, which provide important information for estimating an optimal portfolio without a credit-free bond such as German bonds. Furthermore, the optimal solutions of (16) and (20) also must solve (7). Indeed, $\xi^* = \xi'$, $q^* = q'$ and $p^* = p'$, respectively.

3.3.4 Approximation of λ

Although problem (7) provides a robust way to estimate a risk-neutral and a real probability associated with an uncertainty set κ, a nonlinear constraint in (8) hinders the process. Hence, in this section we introduce a vector $\{s_j\}_{j=1}^N$ with N unknown variables to transform a nonlinear problem into a linear problem, where $s_j = \lambda p_j$ must hold for all states. Combining with (11), we have that $\lambda = \sum_{j=1}^N s_j$. Thereby the constraint in (8) can be converted into a linear equation such that $\frac{1}{\sum_{j=1}^N s_j}(\xi + s - p)y_i = y_\pi y_i$ for $i = F, I, S$. In this case, λ is completely replaced in problem (7). However, a new nonlinear problem comes out due to the condition $s_j = \lambda p_j$. So we have to impose a loose constraint that $s_j \geq p_j$ based on the condition that $\lambda \geq 1$. Then problem (7) is transformed as a linear problem specified as:

$$\max \sum_{j=1}^{N} s_j \tag{3.23}$$

$$\text{s.t.} \quad \frac{1}{\sum_{j=1}^N s_j}(\xi + s - p)y_i = y_\pi y_i, \quad i = F, I, S; \tag{3.24}$$

$$E_p(y_i) \geq P_i(1+\alpha), \quad i = F, I, S; \tag{3.25}$$

$$E_q(y_i) \geq P_i(1+\alpha), \quad i = F, I, S; \tag{3.26}$$

$$s_j \geq p_j, \quad j = 1, \ldots, N. \tag{3.27}$$

$$\sum\nolimits_{j=1}^N \xi_j = 1, \sum\nolimits_{j=1}^N q_j = 1; \tag{3.28}$$

$$\xi_j \leq \frac{q_j}{1-\mu_0}, \quad j = 1, \ldots, N; \tag{3.29}$$

$$\xi_j \geq 0, p_j \geq 0, q_j \geq 0, \quad j = 1, \ldots, N. \tag{3.30}$$

The optimization above gives a preliminary estimator denoted by $\hat{\lambda} = \sum_{j=1}^{N} \hat{s_j}$ due to the loose constraint (27). If we are lucky that $\frac{\hat{s_j}}{\hat{p_j}}$ keeps constant for all states, $\frac{\hat{s_j}}{\hat{p_j}}$ or $\sum_{j=1}^{N} \hat{s_j}$ just equals the real λ. Otherwise, $\sum_{j=1}^{N} \hat{s_j}$ only provides us the upper bound for the real λ. We also justify that as we infinitely repeat the process by changing the constraint $s_j \geq p_j$ as $s_j \geq \hat{\lambda} p_j$ for $j = 1, \ldots, N$, where $\hat{\lambda}$ is the optimal objective value obtained from the last optimization, till the problem is no longer feasible or bounded, $\sum_{j=1}^{N} \hat{s_j}$ finally converges to the underlying value of λ.

In this case, the risk neutral probability can be modified as $rnp^* = \frac{1}{\sum_{j=1}^{N} s_j}(\xi + s - p)$, as implied from (24).

3.3.5 Optimal portfolio choice

Based on Corollary 1, we estimate the robust optimal portfolio by minimizing a sum of square errors with respect to W and L such that

$$
\begin{aligned}
\min \sum_{j}^{N} & (\hat{y}_{W,j} - \hat{y}_{L,j})^2 + \sum_{i}^{3} w_i \Big(\sum_{j=1}^{N} q_j^* y_{ij} - P_i(1+\alpha)\Big)^2 \\
& + \sum_{i}^{3} l_i \Big(\sum_{j=1}^{N} p_j^* y_{ij} - P_i(1+\alpha)\Big)^2 \\
\text{s.t.} \quad & w_1 P_F + w_2 * P_I + w_3 P_S = 1 \\
& l_1 P_F + l_2 * P_I + l_3 P_S = 1 \\
& w_1, w_2, w_3 \geq 0, w_4, w_5 \in R \\
& l_1, l_2, l_3 \geq 0, l_4 \in R
\end{aligned}
\tag{3.31}
$$

where p^* and q^* are optimal solutions in (23). Since r in the primal problem can vary with the weight invested in risk-free bond indicated by Balbas et al. (2012)'s model, we choose to estimate y^* from the implications of the dual problem, rather than the primal problem directly. In the above objective function, the first term represents the difference between the estimated portfolios from (19) and (22) with respect to W and L. The remaining square terms are used to guarantee the slack conditions of (16) and (20). To reduce the calculation errors, we fix the investment in risky bonds as one Euro, as shown in the constraints.

In the next section, we apply our methodology with the European sovereign bonds to estimate the sovereign risk dependence and the robust optimal portfolio.

3.4 Empirical results

3.4.1 Data

We choose four 10-year government coupon bonds issued by Germany, France, Italy and Spain to construct an optimal portfolio, where the corresponding coupons are 3.75%, 4.24% and 4.3% respectively. To examine how the robust portfolio and risk perform in different periods, we select 36 monthly dates from June 2010 to May 2013, which covers the chaotic periods since the Greek debt crisis in European bond market. Similar to the second chapter, the German[3] bonds are created based on the term structure of Triple A interest rate provided by the European central bank with a coupon rate of 3.75% and the same maturity as the French bond.

We take the bond market prices from Datastream. The prices used in the optimization problems already include accrued interest generated by annual fixed coupon payment. Figure 1 plots the bond yield for four countries from June 2010 to May 2013 provided by Datastream and ECB. Clearly, the first big jump in four sovereign yields came at the beginning of 2011, which is consistent with the report from the European central bank (ECB 2012) that the Euro area sovereign bond market experienced severe tensions in 2011. As a result of loosing investor's confidence in a whole European sovereign bond market, the bond yields of larger countries such as Germany and France which gain from the "safe haven" effects even had big fluctuations in 2011. German and French 10-year bond yield peaked on March 2011 at the point of 3.36% and 3.71%, respectively. Also, Italian 10-year bond yield continued to rise and attained the highest point in the end of 2011 since the political rumor in August 2011, and the Spanish 10-year bond yield tops 6% since Rajoy took office in December 2011. However, these

[3]Since we can not find any 10-year German bond whose redemption date can match three other bonds, we have to create it base on a Triple-A yield.

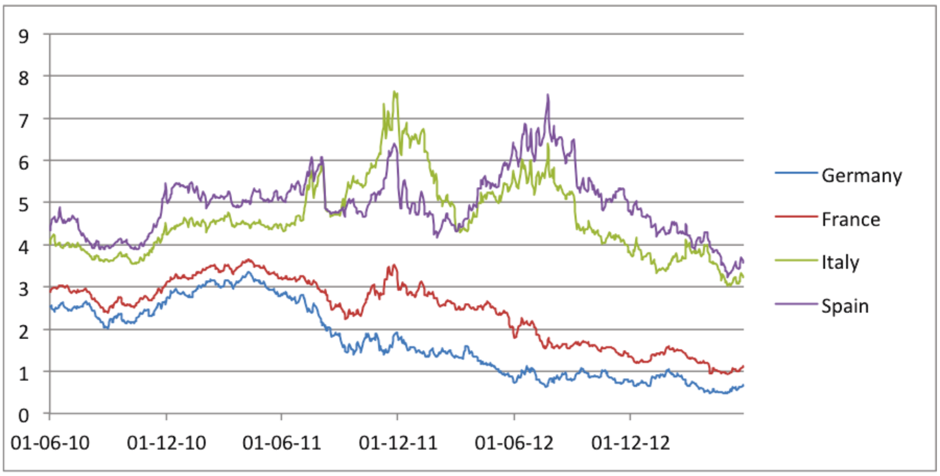


FIGURE 3.1: Sovereign bond yields from June 2010 to May 2013

yields started to diverge since 2012. The French and Italian yields rose persistently in a series of high reach in 2011. After a Spanish government bond auction falls short of its fund-raising target in April, the 10-years bond yield continued to rise and hoverd around 6.5%. By contrast, the German yield declined at the begining in 2012 , which is more likely to be related with the credible announcement adequate fiscal adjustment or 'Flight to safety'.

We summarize the three 10-year bond yield spreads relative to "Germany" for 36 dates in figure 2. Clearly, the rise of Germany bond yield in 2011 lead to a fair move for three risky sovereign bonds yield spread. Italian and Spanish yield spreads fairly swift in 2010 and 2011, and reached the highest in 2012, at 5.01% and 4.61%, respectively. French bond has a similar performance which might be less crisis-hit in 2010 as compared to Spain and Italy.

To examine whether our robust portfolio performs well in the periods of 'jumps' or not, we select two dates, September 2nd, 2011 and February 1st, 2012 where the Italian and Spanish bond yields are in the lows before their first jumps to estimate the robust optimal portfolio. Then we compare the portfolio returns with a weighted average portfolio in the periods of jumps, which are 'out of sample'. The discount rates used in the calculation are derived from the term structure of interest rate of Triple-A bonds provided by ECB. In addition, we assume that the recovery rate $= 60\%$, $\mu_0 = 0.8$, $\alpha = 0.005$.

3.4.2 Estimation of λ

We report estimates of λ denoted by $\hat{\lambda}$ in table 1. For every date, $\hat{\lambda} > 1$, which suggests that there does not exist a good deal[4]. In other words, a trader can never construct a portfolio composed of the four sovereign bonds above to attain a desired expected return (as much as possible) but with a fixed risk level at -1, assuming that the risk free rate is zero. To verify the validity of $\hat{\lambda}$, we first examine the ratio of s_j^* to p_j^* under each state j for all the dates. Generally, we find that more than 90% of $\frac{s_j^*}{p_j^*}$ on seven dates fall in the interval with an error of 1% of $\hat{\lambda}$ in the last column. Second, we compare $\hat{\lambda}$ to each sovereign bond spread against Triple-A benchmark. As shown in figure 2, the performance of $\hat{\lambda}$ is similar with the whole spread movements and consistently attaining the highest values in 2012. Particularly as the Italian and Spanish yield spreads increase by more than 4% during 2012, $\hat{\lambda}$ dramatically shifts from 5.84 to 31 or so. This is consistent with the argument of Balbas et al. (2012), where λ^* is positively correlated with market risk measured by optimal robust CVaR (RCVaR). As the sovereign yield spreads move upward, the market risk increases. Then, λ^* also rises and the market price of risk denoted by $\frac{1}{\lambda^*-1}$ deceases. Hence, based on the above two points, we will say that $\hat{\lambda}$ is a valid estimator, and the optimal solution of (23) (s^*, p^* and q^*) where $\hat{\lambda}$ is attained also provides a reliable information for estimating probabilities.

3.4.3 Joint and conditional sovereign default probabilities

This section discusses the worst-case joint and conditional default probabilities among France, Italy and Spain. Such probabilities are derived based on the risk-neutral and real state probabilities provided by (23) as well as the marginal sovereign default probabilities for each country. In (23), the risk neutral state probability mainly depends on s^*, ξ^* and λ^*, so we can calculate them based on the solutions of (23) where the optimal λ is found.

[4] A 'Good deal' was first introduced by Gochrane and Saa-Requejo (2000). It was mainly described as an investment strategy that provides a extremely high sharp ratio for a trader. In Balbas et al. (2012) model, this caveat was specifically indicated by $\lambda = 1$.

Table 3.1: Monthly λ estimates

Month	$\hat{\lambda}$	% of $\frac{s_j^*}{p_j^*}$	λ Interval
06/2010	3.539	94%	[3.53, 3.54]
07/2010	6.434	92%	[6.43, 6.44]
08/2010	1.546	98%	[1.54, 1.55]
09/2010	2.734	96%	[2.73, 2.74]
10/2010	7.182	96%	[7.10, 7.20]
11/2010	5.211	97%	[5.20, 5.22]
12/2010	8.204	95%	[8.19, 8.21]
01/2011	5.229	94%	[5.22, 5.24]
02/2011	4.896	92%	[4.89,4.91]
03/2011	5.372	98%	[5.37, 5.38]
04/2011	4.919	96%	[4.90, 4.92]
05/2011	3.319	96%	[3.31, 3.32]
06/2011	5.833	94%	[5.83, 5.84]
07/2011	3.832	92%	[3.83, 3.84]
08/2011	8.621	90%	[8.62, 8.63]
09/2011	11.046	96%	[11.04, 11.05]
10/2011	10.318	96%	[10.31, 10.32]
11/2011	27.191	97%	[27.19, 27.20]
12/2011	18.652	95%	[18.65, 18.67]
01/2012	32.465	94%	[32.46, 32.47]
02/2012	29.089	92%	[29.08, 29.09]
03/2012	18.962	91%	[18.96, 18.97]
04/2012	21.587	96%	[21.58, 21.59]
05/2012	30.144	96%	[30.14, 30.15]
06/2012	24.369	94%	[24.36, 24.37]
07/2012	19.191	92%	[19.19, 19.20]
08/2012	12.646	98%	[12.64, 12.65]
09/2012	15.676	96%	[15.67,15.68]
10/2012	13.446	96%	[13.44, 13.45]
11/2012	12.510	97%	[12.50, 12.51]
12/2012	8.338	95%	[8.33, 8.34]
01/2012	7.369	94%	[7.36,7.37]
02/2012	7.969	92%	[7.96, 7.97]
03/2012	8.514	98%	[8.51, 8.52]
04/2012	5.967	96%	[5.96, 5.97]
05/2012	2.643	96%	[2.64, 2.65]

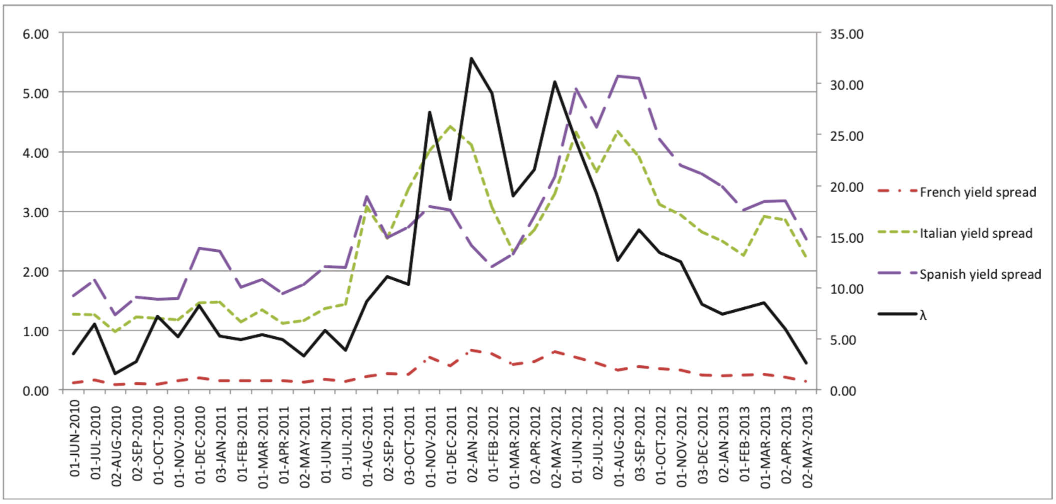


FIGURE 3.2: Comparison of $\hat{\lambda}$ and sovereign yield spread

Figure 3 plots the estimates of the worst-case risk-neutral and physical marginal default probabilities from June 2010 to May 2013. These probabilities are also computed based on the risk neutral (rnp^*) and physical state probability p^*, respectively. In figure 3 we use dotted lines to represent a physical marginal default probability. Clearly, the risk neutral marginal probability is slightly higher than physical probability for France, Italy and Spain over the sample period, which is consistent with our assumption that investors always overestimate default risk so that the real (physical) default probability is lower than risk neutral default probability. Additionally, both marginal probability measures perform consistently with the movement of corresponding sovereign yield spread in the sample. Particularly in December 2011 and June 2012, the estimated Italian and Spanish marginal default probability reached around 80%, which happens to the period of credit spread jumps for both countries.

Figure 4 and 5 report the worst-case implied joint and conditional default probabilities. The joint default probability is defined as a probability of two or more credit event among France, Italy and Spain, and the conditional probability measures a default probability given that Italy, France or Spain defaults. Since the joint probability that both Italian and Spanish default are remarkably higher than the others, we present it on the right side axis shown in figure 4. Overall, the joint default probabilities related to French credit event and the conditional default probability of France given that Spain or Italy defaults are roughly similar in the sample period. These probabilities

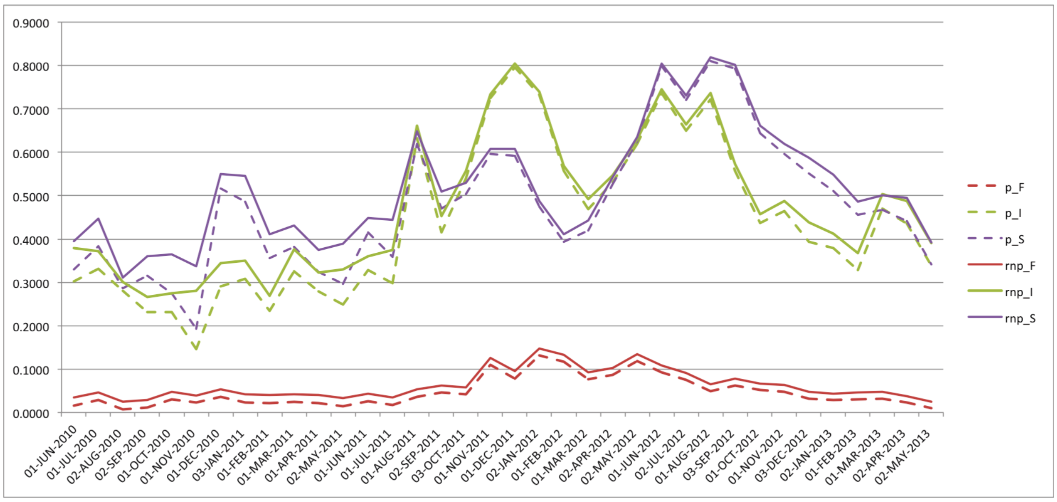


FIGURE 3.3: Marginal sovereign default probabilities

are significantly smaller than the probabilities that only related to the Spanish and Italian default events, but they perform consistently with the change of market risk implied by λ^*. Moreover, these small conditional probabilities imply that there is no clear evidence that the default in the riskier countries have obvious effects on the safer country's credit. By contrast, the performance of joint or conditional default probability related to Italian and Spanish credit event is distinctly different. The joint default probabilities exceeded 45% on average and even reached up to 60% on June 1st 2011. The conditional default probability was almost above 60%. Therefore, both probabilities imply a high default relation between Spain and Italy, which implies a high interaction of sovereign risk among periphery countries. Moreover, the Spanish or Italian default probability conditional on France defaults was strikingly high, and even exceeded 90% in 2012, which is also consistent with the empirical findings of Radev (2013), who found that a default in a safe country significantly affected the default in periphery countries.

3.4.4 Dynamic optimal portfolio

Based on the worst-case state probability p^* and q^* implied from the approximation of λ, we obtain the optimal portfolio from (31). Table 2 reports the optimal weights

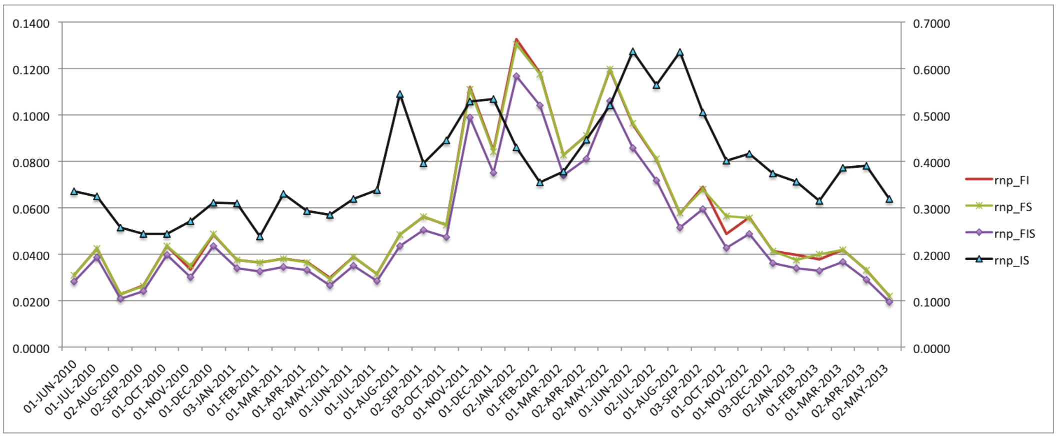
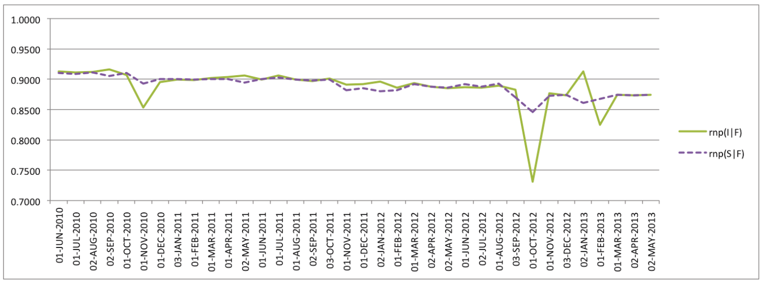
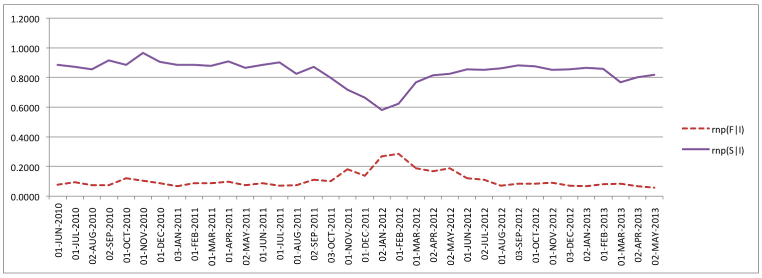


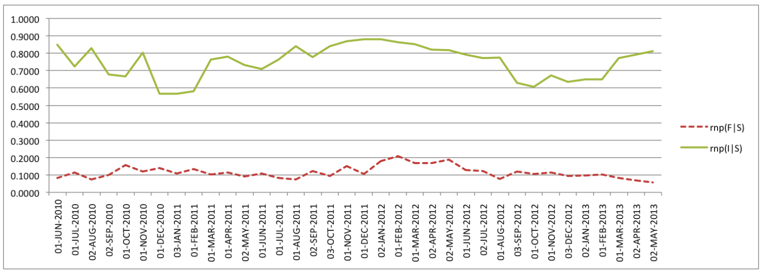
FIGURE 3.4: Joint sovereign default probabilities



(a) Conditional Sovereign default probabilities given France defaults



(b) Conditional Sovereign default probabilities given Italy defaults



(c) Conditional Sovereign default probabilities given Spain defaults

FIGURE 3.5: Conditional sovereign default probabilities

allocated in French, Italian and Spanish 10-year bonds under the worst case situation from June 2010 to May 2013.

In general, the performance of the optimal portfolios in table 2 is remarkably stable, which empirically supports the theoretical conclusion of Balbas et al. (2012) who claimed that under the worst-case analysis, the optimal portfolio works more stably compared to other portfolio choice methodology. Also, the robust weights allocated in three countries are significantly different from zero, indirectly implying a over-evaluate credit risk for Spain and Italy in sovereign bond market. The overall weights in the French bond basically stayed at the level of 0.768 except the dates in September and October for each year, where they are above 0.8. Such an obvious difference results from the different coupon dates for three 10-year bonds. The coupon date for Italian bond is on September 1st and the coupon dates for France and Spain are at the end of October, which lead the times of receiving the Italian bond's future cash flows during these two month are one less than the times from French and Spanish bonds. Therefore, the robust optimal weight for Italian bond in September and October dramatically decreases. Excluding these two special months, the changes of the weights in Italian and Spanish bond are slightly bigger than French bond because of high fluctuations in bond yields. Particularly in the end of 2011 and in June 2012, the weights in both Italian and Spanish bond obviously declined 2% due to a striking increase in the yield from 4.7% to 6.95% and 4.7% to 6.3%, respectively .

Additionally, we follow Kakouris and Rustem (2014)'s empirical analysis, including an equally weighted portfolio (EWP) for comparison purposes. This simple portfolio is not based on the optimization problem but has equal allocations in three sovereign risky bonds. To examine how the robust optimal portfolio works in the extremely bad situation indicated by the jumps in yield, we estimate the portfolio on two dates, September 2nd, 2011 and February 1st, 2012, which are in the fair period before the jumps. Then we keep the weights unchanged for the remaining days. Hence, in an out of sample period from September 5th, 2011to May 31st, 2013 and February 2nd, 2012 to May 31st, 2013, the performances of worst case portfolio (WCP) and EWP are compared two times.

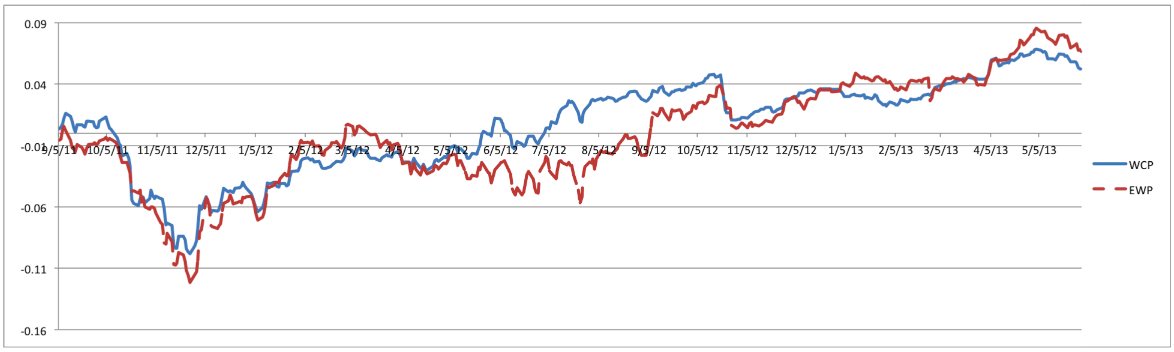


FIGURE 3.6: Portfolio comparison on Sept 2nd, 2011

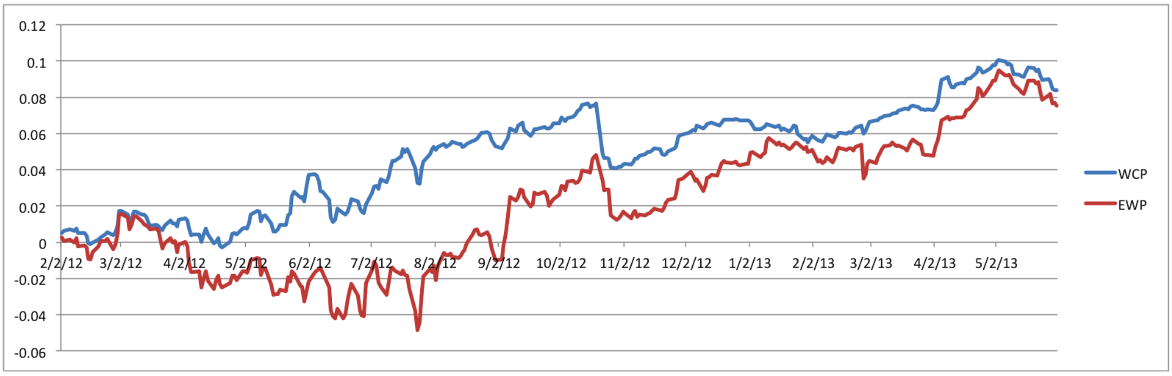


FIGURE 3.7: Portfolio comparison on Feb 1st, 2012

Figure 6 and 7 report the daily returns for WCP and EWP based on the portfolio price on Sep 2, 2011 and Feb 1, 2012 respectively . As shown in Figure 6, during the first jump period in December 2011, the WCP daily returns are slightly above the returns of EWP. However, in June 2012, when both Italian and Spanish bond yield rise up around 6%, WCP obviously performs better than EWP. Moreover, WCP estimated on February 1, 2012 in figure 7 demonstrates a great advantage compared to EWP. The evolution of returns for WCP is completely above the returns of EWP.

3.4.5 Price Comparison

In this paper, we also provide a new pricing instrument, which is the implied worst-case risk neutral state probability (rnp^*), for more sophisticated financial derivatives. This probability measures a state joint probability of France, Italy and Spain in the worst-case scenario. Thereby it can be viewed as an EMM in no-arbitrage pricing model. For the pricing comparison purpose, we include a 10-year 'Eurobond' analyzed in Chapter

TABLE 3.2: Monthly Optimal weights in the Robust efficient portfolio

Date	France	Italy	Spain
06/2010	0.7689	0.1159	0.1152
07/2010	0.7695	0.1180	0.1126
08/2010	0.7641	0.1181	0.1178
09/2010	0.8167	0.0321	0.1512
10/2010	0.8163	0.0322	0.1515
11/2010	0.7629	0.1221	0.1151
12/2010	0.7696	0.1229	0.1075
01/2011	0.7702	0.1220	0.1078
02/2011	0.7633	0.1235	0.1131
03/2011	0.7673	0.1184	0.1143
04/2011	0.7640	0.1200	0.1160
05/2011	0.7655	0.1202	0.1143
06/2011	0.7675	0.1196	0.1129
07/2011	0.7684	0.1182	0.1134
08/2011	0.7845	0.1071	0.1085
09/2011	0.8246	0.0292	0.1462
10/2011	0.8267	0.0280	0.1453
11/2011	0.7806	0.1038	0.1156
12/2011	0.7843	0.0990	0.1167
01/2012	0.7745	0.1014	0.1240
02/2012	0.7668	0.1095	0.1237
03/2012	0.7678	0.1135	0.1187
04/2012	0.7728	0.1130	0.1142
05/2012	0.7777	0.1110	0.1113
06/2012	0.7926	0.1065	0.1008
07/2012	0.7859	0.1098	0.1043
08/2012	0.7954	0.1061	0.0985
09/2012	0.8423	0.0331	0.1246
10/2012	0.8314	0.0347	0.1338
11/2012	0.7753	0.1182	0.1065
12/2012	0.7734	0.1199	0.1067
01/2012	0.7710	0.1208	0.1082
02/2012	0.7670	0.1218	0.1113
03/2012	0.7727	0.1132	0.1140
04/2012	0.7727	0.1137	0.1136
05/2012	0.7661	0.1169	0.1170

* We consider that the sum of the weights allocated in three risky sovereign bonds is one. Then, the optimal units w^* and l^* are obtained by solving problem (31).
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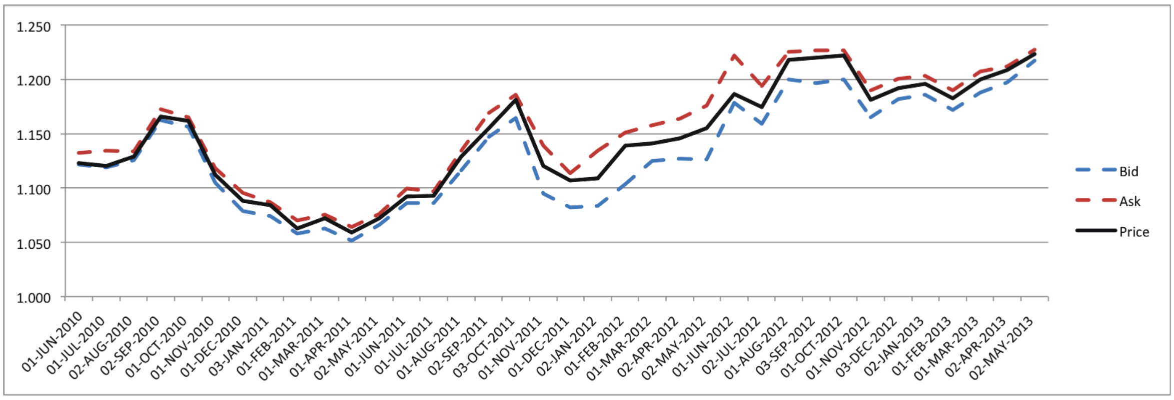

FIGURE 3.8: Portfolio comparison to chapter 2

2. The Eurobond is guaranteed by the above three countries with a coupon of 4 and the same maturity as the French 10-year bond. Figure 8 shows the estimated monthly Eurobond prices from June 2010 to May 2013 . The price based on rnp^* is reported in the middle line, and the bid-ask price bound depicted with dot line is obtained based on the methodology in chapter 2. Clearly, over the whole period, the former price always falls in the bid-ask bound, which suggests a higher Eurobond yield compared to the yield implied from the ask price in chapter 2. This result makes sense because the risk neutral measure in this paper is inferred under the risk-return tradeoff analysis that produces a highest sharp ratio in the worst case. However, chapter 2 only focuses on minimizing the worst-case risk regardless of the return. Therefore, (rnp^*) gives a lower price of the Eurobond than the ask price in chapter 2. We do not compare the prices with lower bound (or bid prices) here because the optimal portfolio[5] weights given in this paper are positive, which implies that an investor holds a short position in the Eurobond. Hence, the prices estimated in this chapter also can be viewed somehow as the ask prices of the Eurobond.

3.5 Conclusion

In this paper, we introduce a broad ambiguous joint probability set in a sovereign bond portfolio optimization framework where the worst scenario is considered. These joint probabilities are used to measure the interdependence of sovereign risk in the Euro

[5]The bonds in the portfolio is exactly the hedging portfolio candidates in chapter 2

zone. To avoid pinning down a specific joint probability distribution, we impose that the probability in the set must produce a higher individual bond price based on Hull (2004), who claimed that the investors alway obtained a higher return because the real default probability was lower than the risk-neutral default probability. Therefore, our method is more flexible and allows for more possible multivariate dependence than other specified multivariate distributions.

With this wide set, we follow the methodology proposed by Balbás et al. (2012) to study the optimal portfolio problem in the European sovereign bond market. The minimization of the RCVaR with a minimum expected return is analyzed, where RCVaR is extended from CVaR through the uncertainty set. The primal problem gives a robust optimal portfolio and the dual problem provides the worst-case real and risk neutral state joint probability. Although there exists a nonlinear problem in calculation, we solve it by adopting a linear approximation.

We apply the methodology with four European sovereign bonds (Spain, Italy, France and Germany) from June 1st, 2010 to May 31th, 2013. The results show a significant time variation in sovereign risk dependence measured by joint or conditional default probability. France has the highest individual contribution to the joint sovereign risk in case of default, which empirically verifies the argument of Radev (2013) that the default of safe countries has a significant effect on the default risk of risky countries. We also find that the default of Italy sustainably affects the default likelihood of Spain, as the market perception are concerned. With respect to the optimal portfolio, the robust weights are particularly stable in the sample period even in the risky year (2011, 2012). This is a highly significant finding because first, it provides a powerful evidence for the CAPM[6]-like model proposed by Balbas et al (2012), where the model implies a steady market portfolio with a coherent risk measure in the worst case. Second, the robust weights allocated in Spanish and Italian bonds are significantly positive throughout the sample period, which gives an indirect evidence to the overpriced sovereign risk in periphery European countries. Additionally, we compare the performance of the robust portfolio to EWP. In an out-of sample analysis, EWP suffers a higher loss especially

[6]the capital asset pricing model (CAPM) was introduced by Jack Treynor (1961, 1962), William Sharpe (1964), John Lintner (1965a,b) and Jan Mossin (1966).

during 2011 and 2012. Although the advantage of the robust portfolio estimated in September 2nd is not obvious during the first jump period, the whole performance is much better than EWP.

Hence, we conclude that introducing an uncertainty set provides a more flexible description of sovereign dependencies compared to other specified distributions. Therefore, the optimal solutions are robust. The empirical results are meaningful for policy-makers who make efforts to control the spread of sovereign risk in Europe, and also for essentially sidelined investors. Additionally, in future work, we will try a continuous state probability distribution instead of a discrete distribution to measure the dependence. The relations between the size of ambiguity set and robust portfolio choice also require a deeper analysis.

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