Universidad Carlos III de Madrid

Sampling Theory in Shift-Invariant Spaces: Generalizations

PhD THESIS

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A mamá, papá, el pollo y la conchi,...

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Contents

Resumen

A grandes rasgos la teoría de muestreo estudia el problema de recuperar una función continua a partir de un conjunto discreto de sus valores. El resultado más importante y pilar fundamental de esta teoría es el conocido teorema de muestreo de Shannon que afirma que:

Si una señal $f(t)$ no contiene frecuencias mayores que $1/2$ ciclos por segundo entonces está completamente determinada por sus ordenadas en una sucesión de puntos espaciados en un segundo. Además puede ser reconstruida mediante la fórmula

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\frac{\sin \pi(t-k)}{\pi(t-k)}\,, \quad t \in \mathbb{R}\,.$$

En otras palabras, la formula anterior es válida para funciones bandalimitadas (al intervalo $[-\pi, \pi]$ en este caso), i.e., funciones para las cuales la transformada de Fourier se anula en el exterior de cierto intervalo ($[-\pi, \pi]$ en este caso). Este resultado, a pesar de su impacto en teoría de la señal, presenta varios problemas que muchos investigadores (matemáticos, físicos e ingenieros) han tratado de solucionar. Es por esto que el estudio de la teoría de muestreo en espacios invariantes por traslación ha cobrado gran importancia en la comunidad científica que trabaja en problemas relacionados con el procesado de señales.

Una herramienta importante que usaremos recurrentemente es la conocida transformada de Fourier para funciones en $L^2(\mathbb{R})$. Esta está definida en $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ como

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}wt}\,dt$$

o

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-2\pi\mathrm{i}\xi t}\,dt$$

y luego se extiende, mediante un argumento de densidad, a todo $L^2(\mathbb{R})$. Sus respectivas fórmulas inversas son

$$f(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(t)\mathrm{e}^{\mathrm{i}wt}\,dt$$

o

$$f(\xi) = \int_{-\infty}^{\infty} \widehat{f}(t) e^{2\pi i \xi t}\, dt$$

A lo largo de esta memoria usaremos ambas definiciones indiferentemente. La primera mide la frecuencia angular en radianes por segundo mientras que en la segunda ξ representa la frecuencia en hertzios, o ciclos por segundo. Si seleccionamos, por ejemplo la primera, podemos definir el espacio de Paley-Wiener de la siguiente forma:

$$PW_\pi = \left\{ f \in L^2(\mathbb{R}) \ : \ \operatorname{supp} \widehat{f} \subseteq [-\pi, \pi] \right\}.$$

En otras palabras

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(w) e^{iwt}\, dw = \left\langle \widehat{f}, \frac{e^{-iwt}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]}, \quad t \in \mathbb{R}.$$

En general, se definiría análogamente el espacio $PW_{\sigma\pi}$ con $\sigma > 0$. Con la segunda definición de transformada de Fourier, el espacio anterior estaría definido por la condición $\operatorname{supp} \widehat{f} \subseteq [-1/2, 1/2]$.

Las generalizaciones más comunes de los espacios Paley-Wiener son las siguientes:

- La primera consiste en sustituir el espacio de Hilbert $L^2[-\pi, \pi]$ y el núcleo de Fourier en la expresión anterior por un espacio de Hilbert arbitrario $\mathcal{H}$ y un núcleo

  $$K : \Omega \ni t \mapsto K(t) \in \mathcal{H},$$

  con $\Omega \subseteq \mathbb{R}$ (o $\mathbb{C}$), y considerar entonces, para cada $x \in \mathcal{H}$, la función

  $$f_x(t) = \langle x, K(t) \rangle_{\mathcal{H}}, \ t \in \Omega.$$

  Ver, por ejemplo, Refs. [40, 50, 61, 123]. Así obtenemos un espacio de Hilbert con núcleo reproductor (RKHS en sus siglas inglesas) $\mathcal{H}_K$ puesto que el funcional evaluación $E_t : f \mapsto f(t)$ es acotado para cada $t \in \Omega$. Por tanto, para cada $t \in \Omega$, el teorema de representación de Riesz asegura la existencia de un único $k_t \in \mathcal{H}_K$ tal que $f(t) = \langle f, k_t \rangle$ para todo $f \in \mathcal{H}_K$. El núcleo reproductor del espacio $\mathcal{H}_K$ viene dado por

  $$k(t, s) := \langle k_s, k_t \rangle = k_s(t), \qquad (t, s) \in \Omega \times \Omega.$$

  Los espacios RKHS tienen la propiedad importante de que la convergencia en norma implica convergencia puntual en Ω, que será uniforme en aquellos subconjuntos de Ω en donde la función $t \mapsto \|k_t\|$ esté acotada.

  Si existe una base ortonormal en $\mathcal{H}_K$ de la forma $\{k(\cdot, t_n)\}_{n\in\mathbb{Z}}$, donde la sucesión $\{t_n\}_{n\in\mathbb{Z}} \subset \Omega$, la siguiente fórmula de muestreo se cumple para todo $f \in \mathcal{H}_K$:

  $$f(t) = \sum_{n\in\mathbb{Z}} f(t_n) \frac{k(t, t_n)}{k(t_n, t_n)}, \qquad t \in \Omega.$$

El espacio de Paley-Wiener PW_π es un RKHS y su núcleo reproductor está dado por $k_\pi(t,s) = \frac{\sin\pi(t-s)}{\pi(t-s)}$, $t,s \in \mathbb{R}$. En particular, para la sucesión $\{t_n = n\}_{n\in\mathbb{Z}}$ se tiene que $\big\{\frac{\sin\pi(t-n)}{\pi(t-n)}\big\}_{n\in\mathbb{Z}}$ es una base ortonormal para PW_π, y la fórmula de muestreo anterior es precisamente la de Shannon.

- De acuerdo con el teorema de Shannon el espacio de Paley-Wiener PW_π es un subespacio invariante por traslación de $L^2(\mathbb{R})$ generado por la función sinc, i.e., $\operatorname{sinc} t := \sin\pi t/\pi t$, $t \in \mathbb{R}\backslash\{0\}$ y $\operatorname{sinc} 0 = 1$. Puede ser descrito por tanto como

$$PW_\pi \equiv \Big\{ \sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(t-n),\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

Otra generalización consiste en reemplazar la función sinc por otra función generadora $\varphi \in L^2(\mathbb{R})$ que presente mejores propiedades computacionales (ver, por ejemplo, Refs. [108, 109]. En otras palabras, tomar en consideración subespacios de $L^2(\mathbb{R})$ de la forma

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n)\,,\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}$$

que no es otra cosa que

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, T^n\varphi(\cdot)\,,\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}$$

donde T es el operador shift $f(t) \mapsto f(t-1)$. Considerando en vez de T un operador unitario U en un espacio de Hilbert abstracto $\mathcal{H}$, se obtienen los subespacios U-invariantes. El estudio de la teoría de muestreo en estos subespacios será el objetivo del Capítulo 4. Más referencias acerca de esta última extensión se pueden encontrar a lo largo del manuscrito.

Este manuscrito estudia la última de estas dos posibles extensiones.

El primer capítulo comienza con una introducción histórica de las principales ramas de la matemática que se abordan en esta memoria, la teoría de muestreo de Shannon y la teoría de frames. También incluimos, como motivación, la extensión de la fórmula de Shannon a espacios invariantes por traslación, resultado obtenido, por primera vez, por G. Walter en [113].

El segundo capítulo está dedicado al estudio de subespacios invariantes por traslación de $L^2(\mathbb{R}^d)$ con un conjunto de múltiples generadores. Vale la pena mencionar que las muestras no son precisamente valores de la señales en un conjunto discreto, estas son obtenidas mediante la acción sobre la señal de un sistema de convolución. Si los generadores son funciones con soporte compacto la complejidad computacional es baja y se evitan los errores de truncamiento; este caso será analizado. Es natural también considerar sucesiones de muestras perturbando los puntos en donde se obtienen las mismas; en este marco encontramos condiciones que hacen posible la reconstrucción. Los desarrollos aquí obtenidos son inútiles desde un punto de vista práctico ya que las funciones de reconstrucción dependen de la sucesión de errores que es, obviamente, desconocida. No obstante, un algoritmo frame es implementado para soslayar este problema.

El Capítulo 3 va un poco más allá; esta vez suponemos que las señales pertenecen a un subespacio de $L^p_\nu(\mathbb{R}^d)$, donde ν es una función peso. Una función f pertenece a $L^p_\nu(\mathbb{R}^d)$ si νf pertence a $L^p(\mathbb{R}^d)$. Esta función peso controla el decaimiento o crecimiento de las señales. También obtenemos aquí fórmulas de reconstrucción usando un método similar al del Capítulo 2. En este caso consideramos generadores que sean funciones localmente en $L^\infty_\nu(\mathbb{R}^d)$ y globalmente en $L^1_\nu(\mathbb{R}^d)$. Además el espacio auxiliar deberá tener estructura de álgebra de Wiener; esto requerirá supuestos adicionales en las funciones peso. Obtenemos fórmulas de muestreo regular asociadas a dos tipos de sistemas lineales: los obtenidos mediante convolución con ciertas funciones prefijadas y los que la respuesta impulsional es una delta de Dirac trasladada.

El Capítulo 4 justifica por si mismo el título de la tesis. Es bien conocido que el operador de traslación $T : f(t) \mapsto f(t-1)$ es unitario en $L^2(\mathbb{R})$. En los capítulos anteriores tratamos con espacios de la forma $\overline{\operatorname{span}}_{L^2(\mathbb{R})}\big\{\varphi(t-n),\ n \in \mathbb{Z}\big\}$, donde la función generadora φ pertenece a $L^2(\mathbb{R})$. Una extensión natural es considerar un operador unitario $U : \mathcal{H} \to \mathcal{H}$, donde $\mathcal{H}$ es un espacio de Hilbert separable y desarrollar una teoría de muestreo generalizada en subespacios de la forma

$$\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}\,,$$

donde a es un elemento fijo en $\mathcal{H}$. Con el fin de generalizar los sistemas de convolución y, principalmente, obtener resultados de perturbación, suponemos que el operador U está incluido en un *grupo continuo de operadores unitarios* $\{U^t\}_{t\in\mathbb{R}}$. Obtenemos resultados interesantes en este marco abstracto, usando técnicas de teoría de frames, teoría espectral y sucesiones estacionarias, entre otras. En nuestra opinión, este capítulo es uno de los logros más importantes y originales de la memoria.

Brief description of the manuscript

Roughly speaking sampling theory deals with determining whether we can or can not recover a continuous function from some discrete set of its values. The most important result and main pillar of this theory is the well-known Shannon's sampling theorem wich states that:

If a signal $f(t)$ contains no frequencies higher than $1/2$ cycles per second, it is completely determined by giving its ordinates at a sequence of points spaced one second apart, and can be reconstructed from these ordinates, via the formula

$$f(t) = \sum_{k\in\mathbb{Z}} f(k)\frac{\sin \pi(t-k)}{\pi(t-k)}\,, \quad t \in \mathbb{R}\,.$$

In other words, formula above is valid for band-limited functions(to the interval $[-\pi, \pi]$), i.e. functions for which the Fourier transform vanishes outside certain interval ($[-\pi, \pi]$ in this case). This crucial result in spite of its impact has several problems that researchers(mathematicians, physicists and engineers) have tried to solve. That is why the study of shift-invariant spaces have gained great importance for the scientist community working on signal processing.

An important tool that we will recurrently use is the well known Fourier transform for functions in $L^2(\mathbb{R})$. It is defined in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}wt}\,dt$$

or

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-2\pi\mathrm{i}\xi t}\,dt$$

and then extended, by a density argument, to the whole $L^2(\mathbb{R})$. Their respective inverse formulae are

$$f(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(t)\mathrm{e}^{\mathrm{i}wt}\,dt$$

or

$$f(\xi) = \int_{-\infty}^{\infty} \widehat{f}(t) e^{2\pi i \xi t}\, dt$$

Along this memoir we will use both of them interchangeably. The first one is measuring the angular frequency in radians per second while in the second ξ represents the frequency in hertz, or cycles per second. If we choose, for instance, the first one, we define the Paley-Wiener space in the following way:

$$PW_\pi = \big\{ f \in L^2(\mathbb{R}) \; : \; \operatorname{supp} \widehat{f} \subseteq [-\pi, \pi] \big\}.$$

In other words

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(w) e^{iwt}\, dw = \Big\langle \widehat{f}, \frac{e^{-iwt}}{\sqrt{2\pi}} \Big\rangle_{L^2[-\pi,\pi]}, \quad t \in \mathbb{R}.$$

Analogously, we can define the space $PW_{\sigma\pi}$ with $\sigma > 0$. Using the second definition of Fourier transform the involved condition would be $\operatorname{supp} \widehat{f} \subseteq [-1/2, 1/2]$.

The most common extensions of Paley-Wiener spaces are the following:

- The first one consists in substituting the Hilbert space $L^2[-\pi, \pi]$ and the Fourier kernel in the expression above by an arbitrary Hilbert space $\mathcal{H}$ and a kernel

  $$K : \Omega \ni t \mapsto K(t) \in \mathcal{H},$$

  with $\Omega \subseteq \mathbb{R}$ (or $\mathbb{C}$), and thus consider, for each $x \in \mathcal{H}$, the function

  $$f_x(t) = \langle x, K(t) \rangle_{\mathcal{H}}, \quad t \in \Omega.$$

  See, for instance, Refs. [40, 50, 61, 123]. In this extension we obtain a Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}_K$ since the evaluation functional $E_t : f \mapsto f(t)$ is bounded for all $t \in \Omega$. Therefore, for each $t \in \Omega$, Riesz representation theorem assures the existence of an unique $k_t \in \mathcal{H}_K$ such that $f(t) = \langle f, k_t \rangle$ for all $f \in \mathcal{H}_K$. The reproducing kernel of the space $\mathcal{H}_K$ is given by

  $$k(t,s) := \langle k_s, k_t \rangle = k_s(t), \qquad (t,s) \in \Omega \times \Omega.$$

  The RKHS has the important property that convergence in norm implies pointwise convergence in Ω, which will be uniform on those subsets of Ω where the function $t \mapsto \|k_t\|$ is bounded.

  If there exists an orthonormal basis for $\mathcal{H}_K$ of the form $\{k(\cdot, t_n)\}_{n\in\mathbb{Z}}$, where the sequence $\{t_n\}_{n\in\mathbb{Z}} \subset \Omega$, then the following sampling formula holds for every $f \in \mathcal{H}_K$:

  $$f(t) = \sum_{n\in\mathbb{Z}} f(t_n) \frac{k(t, t_n)}{k(t_n, t_n)}, \qquad t \in \Omega.$$

The Paley-Wiener space PW_π is a RKHS and its reproducing kernel is given by $k_\pi(t,s) = \frac{\sin\pi(t-s)}{\pi(t-s)}$, $t,s \in \mathbb{R}$. In particular, taken $\{t_n = n\}_{n\in\mathbb{Z}}$ it is known that $\{\frac{\sin\pi(t-n)}{\pi(t-n)}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for PW_π, and the above sampling formula reduces to Shannon's one.

- According to Shannon's sampling theorem the Paley-Wiener space PW_π is a shift-invariant subspace of $L^2(\mathbb{R})$ generated by the sinc function, i.e., the function defined as $\operatorname{sinc} t := \sin\pi t/\pi t$, $t \in \mathbb{R}\setminus\{0\}$ and $\operatorname{sinc} 0 = 1$. It can be described as
$$PW_\pi \equiv \Big\{ \sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(t-n),\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$
Other generalization consists of replacing the sinc function by another generating function $\varphi \in L^2(\mathbb{R})$ having better convergence properties (see, for instance, Refs. [108, 109]). In other words, take into account subspaces of $L^2(\mathbb{R})$ of the form
$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n)\,,\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}$$
which is nothing but
$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, T^n\varphi(\cdot)\,,\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}$$
where T is the shift operator. The replacement of T by an arbitrary unitary operator U on an abstract Hilbert space $\mathcal{H}$ gives the U-invariant subspaces. The study of a sampling theory in these spaces is the subject of Chapter 4. More references concerning this last extension are profusely given along the manuscript.

This manuscript concerns with the last one of these two possible extensions.

The first chapter begins with an historical introduction of the main mathematical branches this memoir deal with, Shannon sampling theory and frame theory. We also include as a motivation the extension of Shannon's formula to shift-invariant spaces, a work done by G. Walter in [113].

The second chapter is devoted to study $L^2(\mathbb{R}^d)$ shift-invariant spaces with a set of multiple stable generators, in which we obtain generalized sampling formulas. It is worth to mention that samples are not precisely values of the signal at some discrete set, they are obtained by the action of a convolution system on the signal. If the generators are functions with compact support the computational complexity is lower and truncations errors are avoided, this case is also analyzed. It is also natural to consider error sequences perturbing the samples; in this setting, we found conditions to make possible the reconstruction. The sampling expansions here obtained are useless from a practical point of view because the reconstruction functions depend on the error sequence which is obviously unknown. Nevertheless, a frame type algorithm is implemented to overcome this problem.

Chapter 3 goes a little further. This time we suppose that signals belong to a subspace of $L^p_\nu(\mathbb{R}^d)$, where ν is a weight function. A function f belongs to $L^p_\nu(\mathbb{R}^d)$ if νf belongs to $L^p(\mathbb{R}^d)$. This weight function controls the decay or growth of the signals. We also obtained here reconstruction formulas using a similar approach to the one in Chapter 2. We have to consider generators which are functions locally in $L^\infty_\nu(\mathbb{R}^d)$ and globally in $L^1_\nu(\mathbb{R}^d)$. Furthermore the auxiliary sampling space should have a Wiener algebra structure, this will require further assumptions on the weight functions. We derive regular sampling formulas involving two types of linear systems, the ones obtained by convolution with certain fixed functions and the ones in which the impulse response is a translated Dirac delta.

Chapter 4 justifies by itself the title of the thesis. It is well known that the shift operator $T : f(t) \mapsto f(t-1)$ is unitary in $L^2(\mathbb{R})$. We dealt in the previous chapters with spaces of the form $\overline{\operatorname{span}}_{L^2(\mathbb{R})}\big\{\varphi(t-n),\ n \in \mathbb{Z}\big\}$, where the generator function φ belongs to $L^2(\mathbb{R})$. A natural extension is to consider an unitary operator $U : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space and develop a generalized sampling theory in subspaces of the form

$$\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}\,,$$

where a is a fixed element in $\mathcal{H}$. In order to generalize convolution systems and mainly to obtain some perturbation results, we assume that the operator U is included in a continuous *group of unitary operators* $\{U^t\}_{t\in\mathbb{R}}$. We obtain interesting results in this abstract setting, using techniques from frame theory, spectral theory, stationary sequences, among other branches of mathematics. In our opinion this chapter is one of the most important and original achievements of the memoir.

1

Introduction to Sampling Theory

1.1 A little bit of history

Let us suppose that we have a function f defined on some domain D, and has a series representation there of the form

$$f(t) = \sum_{k \in \mathbb{Z}} f(t_k) S_k(t), \quad t \in D\,, \tag{1.1}$$

where $\{t_k\}_{k \in \mathbb{Z}}$ is a discrete collection of points in D, and $\{S_k\}_{k \in \mathbb{Z}}$ is some set of suitable expansion functions. An expansion like (1.1) is called sampling series and the first thing that comes to mind is how the function can be represented in terms of its values at just a discrete set of its domain. Series of this kind and their generalizations are the main interest of sampling theory.

Sampling theory as we know it today is about sixty-five years old, but its foundations relies on the work of several renowned mathematicians, such as Poisson, Borel, Hadamard, de la Vallée Poussin, and E. T. Whittaker. Actually, there are studies which make us think that the very first sampling result can be deduced from Cauchy's work, but for some authors the evidence of this is poor and could not be firmly substantiated. Nevertheless, the reader can check Refs. [62, 74], both authors coincides that classical sampling theorem may come from papers by Cauchy, they also agree in the fact that the strength of the evidence is debatable.

The fundamental result of sampling theory states that if a signal $f(t)$ [1] contains no frequencies higher than $W/2$ cycles per second, it is completely determined by giving its ordinates at a sequence of points spaced $1/W$ seconds apart, say $t_k = k/W$, $k \in \mathbb{Z}$, and can be reconstructed from these ordinates, via the formula

$$f(t) = \sum_{k\in\mathbb{Z}} f\Big(\frac{k}{W}\Big)\frac{\sin \pi(Wt-k)}{\pi(Wt-k)}\,, \quad t \in \mathbb{R}\,. \tag{1.2}$$

Here, as we have mention before, we can see how all the information of the function is contained in the sample values that are taken, this time, at equidistantly spaced instants. We also noticed that in this setting, the reconstruction functions $\{S_k\}_{k\in\mathbb{Z}}$ are given by

$$S_k(t) = \frac{\sin \pi(Wt-k)}{\pi(Wt-k)} = \operatorname{sinc}(Wt-k),$$

where the cardinal sine function sinc is defined as

$$\operatorname{sinc} v := \begin{cases} \frac{\sin \pi v}{\pi v}, & v \neq 0; \\ 1, & v = 0. \end{cases}$$

The cardinal series (1.2) is the key of the sampling theory's birth and that is why we expose here how this series was obtained by many mathematicians working on different branches of mathematics.

At the end of the nineteenth century Borel in [18] was dealing with the problem of how the coefficients $\{a_n\}_{n\in\mathbb{N}}$ of a function $f(z) = \sum a_n z^n$ determine its singularities. One way to attack this problem is to construct an auxiliary function ψ determined (in part) by the conditions $\psi(n) = a_n$, at that time Borel chose

$$\psi(z) = \frac{\sin \pi z}{\pi}\sum_{n\in\mathbb{N}}\frac{a_n}{z-n},$$

with $\sum |a_n| < \infty$ for convergence. This has the appearance of the cardinal series, but with a closer look we will detect a few differences. A couple years earlier, Borel in [17] had been studying the general Lagrange-type formula

$$f(z) = \sum_n \frac{c_n \phi(z)}{\phi'(a_n)(z-a_n)},$$

the amazing fact here is that Borel explicitly mentioned that under certain conditions, if we know the functions at the integer points we know the entire function. A few years later Hadamard [56] made a much more extensive study of the same problem, quoting Borel's work, but he also missed the precise cardinal series.

[1] A signal is nothing but a function $f(t)$, both terms will be used throughout the manuscript.

An interpolation scheme due to de la Vallée Poussin [112] is often cited as being an early form of the sampling theorem, he considered the finite interpolation formula

$$\frac{\sin mt}{m}\sum_{a}^{b}(-1)^n\frac{f(n\pi/m)}{t-n\pi/m},$$

where f is defined on $[a,b]$, and the summation is understood to be over those n for which $n\pi/m \in [a,b)$. The limit $m \to \infty$ is now taken, and de la Valle Poussin's main result is that the formula converges to $f(t)$ at any point t in a neighborhood of which f is continuous and of bounded variation. The work of de la Vallée Poussin was applied and extended by many mathematicians: Steffensen [94], Theis [105], Ferrar [39], the last one reported that Steffensen seems to have been the first to relate cardinal series to other interpolation series, in this case Newton's divided difference formula.

The cardinal series can be obtained formally by considering the Lagrange interpolation formula in the form

$$H_m(z)\Big\{\frac{f(0)}{z}+\sum_{n=1}^{m}\Big[\frac{f(n)}{H'_m(n)(z-n)}+\frac{f(-n)}{H'_m(-nn)(z+n)}\Big]\Big\},$$

where

$$H_m(z)=z\prod_{n=1}^{m}\Big(1-\frac{z^2}{n^2}\Big),$$

which interpolates $f(z)$ at $z=-m,\dots,0,\dots,m$. Since

$$\frac{\sin\pi z}{\pi z}=z\prod_{n=1}^{\infty}\Big(1-\frac{z^2}{n^2}\Big),$$

the cardinal series is obtained by letting $m \to \infty$, we can find the details of this approach in the work of T. A. Brown [19], Ferrar [38] and also J. M. Whittaker [116]. On the other hand, one can consider a special case of Cauchy's partial fractions expansion for a suitably restricted meromorphic function F with poles at the points p_n, namely

$$F(z)=\sum_{n}\text{res}\left.\frac{F(w)}{(z-w)}\right|_{w=p_n};$$

one applies this to $F(z) = f(z)/\sin\pi z$, where f is entire, and the cardinal series results, this was developed by Ferrar (1925).

The sampling theorem involving the formula (1.2), which we will discuss in the next section, is mainly attributed to E. T. Whittaker [115] and further develop by his second son J. M. Whittaker, but actually, who was the first person to discovered this seminal result is a question far from being clarified. Indeed, Ferrar claimed that another mathematician, F. J. W. Whipple, had discovered it five years before E. T. Whittaker, but did not publish his findings.

It is also worth to mention the work of the Japanese mathematician K. Ogura [82]. In the paper [21], Butzer, et al. assert that the firts clear statement of the classical sampling theorem was made by Ogura; they also pointed out that the hypotheses and the formulation were both correct. The proof, which is simple and rigorous, Ogura just quote that it can be easily obtained using calculus of residues, in [21] the authors also ascertain this fact.

The Whittakers' work was purely mathematical and we can deduce that neither one of them had any application in mind. They did not mention the engineering words, signal or cycle or bounded frequencies, but used their mathematical counterparts. This is one of the many examples we have as a motivation for doing mathematics, application is not always necessary, we just make math and time will endorse the right value to our work.

E. T. Whittaker's result was later retaken and introduced in information theory and communication engineering by C. E. Shannon in 1940, though it did not appear in the literature until after World War II in 1949. In his two famous papers [92] and [93] which granted him several awards, Shannon acknowledge the work of E. T. Whittaker.

In the late fifties, it became known in the western world that Shannon's result had been discovered earlier in 1933 by a russian engineer, V. Kotel'nikov [73], who applied it in communication engineering earlier than Shannon, and it was known by his name in the russian and eastern european literature.

Concerning the cardinal series everything here exposed is contained in the superb article [59] and the amazing story surrounding Shannon's theorem was taken from [123]. For more anecdotes and details the reader can check references therein. From now and on, we shall call the sampling theorem, Whittaker-Shannon-Kotel'nikov sampling theorem (WSK theorem) as A. I. Zayed does in his great book [123], there, the author present several facts which justified this sharing of the credit.

As a recent advances in Shannon theory we can cite Refs. [20, 22]. In [20] the authors shows the equivalence of six well known results: WSK theorem, Poisson's summation formula, general Parseval formula, the reproducing kernel formula, the Paley-Wiener theorem of Fourier analysis and the Valiron-Tschakaloff sampling formula. Meanwhile, paper [22] is concerned with Shannon sampling reconstruction formulae of derivatives of bandlimited signals as well as of derivatives of their Hilbert transform, and their application to Boas-type formulae for higher order derivatives.

If we perform a search of the word "frame" in this manuscript we will obtain over two hundred instances. This is because we have attacked the proposed problems with many of the tools which frame theory can provide. The theory of frames for a Hilbert spaces plays a fundamental role in signal processing, image processing, data compression, sampling theory and could be used even for abstract mathematical purposes. Among the classical books related to frame theory we can cite Refs. [23, 25, 26, 49, 57, 58, 122].

To locate the origins of frames we have to cite the landmark work by Duffin and Schaeffer [33], where they dealt with some problems in nonharmonic Fourier series, a branch of mathematics concerned with the completeness and expansion properties of sets of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^p[-\pi,\pi]$. The foundations of the theory of nonharmonic Fourier series lie in the works by a couple of mathematicians very related with this memoir, R. Paley and N. Wiener.

Duffin and Schaeffer in [33] defined as a frame, any infinite sequence of nonzero vectors $\{\phi_n\}_{n\in\mathbb{N}}$ on the Hilbert space $\mathcal{H}$ such that for an arbitrary vector $v \in \mathcal{H}$,

$$A\|v\|^2 \leqslant \sum_{n\in\mathbb{N}} |\langle v, \phi_n\rangle|^2 \leqslant B\|v\|^2 \,,$$

where A and B are positive constants independent of v.

Inexplicably, frames were living inside the theory of nonharmonic Fourier series for years. However, they were brought back to life in 1986 by Daubechies, Grossman and Meyer with the work [30], at the beginning of the Wavelet era. In this great paper the authors emphasize the power of the “overcomplete” property of frames, the loss of the uniqueness of the coefficients in the expansions $x = \sum_n c_n\, x_n$ is indeed a very good thing.

As a final comment here we emphasize that in the whole manuscript the samples are not taken from the signal itself but from some new functions obtained by the action on the signal of some linear operators; concretely the samples will be of the form

$$\big\{\mathcal{L}f(t_n) = (f * h)(t_n)\big\}_{n\in\mathbb{Z}}$$

where h is a fixed function. Following engineering jargon we are taking our samples from the filter $\mathcal{L}$ with impulse response h; the average function h reflects the characteristics of the acquisition device of the samples. This concept of generalized sampling, also known as average sampling was first introduced by A. Papoulis in [83].

1.2 By way of motivation: introducing our technique

The classical Whittaker-Shannon-Kotel’nikov sampling theorem (WSK theorem) states that any function f band-limited to $[-1/2, 1/2]$, that is,

$$f(t) = \int_{-1/2}^{1/2} \widehat{f}(w) \mathrm{e}^{2\pi \mathrm{i} t w} dw \,, \quad t \in \mathbb{R}$$

can be reconstructed from the sequence of samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, \frac{\sin\, \pi(t-n)}{\pi(t-n)} \,, \quad t \in \mathbb{R}\,.$$

Thus, the Paley-Wiener space $PW_{1/2}$ of band-limited functions to $[-1/2, 1/2]$ (that is, $\operatorname{supp} \widehat{f} \subseteq [-1/2, 1/2]$) is generated by the integer shifts of the cardinal sine function, $\operatorname{sinc}(t) := \sin \pi t/\pi t$. A simple proof of this result is given by using the Fourier duality technique which uses that the Fourier transform

$$\begin{array}{rccc} \mathcal{F}: & PW_{1/2} & \longrightarrow & L^2[-1/2, 1/2] \\ & f & \longmapsto & \widehat{f} \end{array}$$

is an unitary operator from $PW_{1/2}$ onto $L^2[-1/2, 1/2]$. Thus, the Fourier series of $\widehat{f}$ in $L^2[-1/2, 1/2]$ is

$$\widehat{f} = \sum_{n=-\infty}^{\infty} \langle \widehat{f}, \mathrm{e}^{-2\pi i n w} \rangle \mathrm{e}^{-2\pi i n w} = \sum_{n=-\infty}^{\infty} f(n)\, \mathrm{e}^{-2\pi i n w} .$$

By applying the inverse Fourier transform $\mathcal{F}^{-1}$, we get

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(n) \mathcal{F}^{-1}\big[\mathrm{e}^{-2\pi i n w} \chi_{[-\pi,\pi]}(w)\big](t) \\ &= \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \quad \text{in} \quad L^2(\mathbb{R}) . \end{aligned}$$

The pointwise convergence comes from the fact that $PW_{1/2}$ is a reproducing kernel Hilbert space (written shortly as RKHS) where convergence in norm implies pointwise convergence (which in this case is uniform on $\mathbb{R}$); this comes out from the inequality:

$$|f(t)| \leqslant \|f\| \quad \text{for each } t \in \mathbb{R} \text{ and } f \in PW_{1/2} .$$

For the RKHS's theory and applications, see, for instance, Ref. [90].

The WSK theorem has its d-dimensional counterpart. Any function f band-limited to the d-dimensional cube $[-1/2, 1/2]^d$, i.e.,

$$f(t) = \int_{[-1/2,1/2]^d} \widehat{f}(x) \mathrm{e}^{2\pi i x^\top t} dx$$

for each $t \in \mathbb{R}^d$ (here we are using the vector notation $x^\top t := x_1 t_1 + \cdots + x_d t_d$ identifying elements in $\mathbb{R}^d$ with column vectors), may be reconstructed from the sequence of samples $\{f(\alpha)\}_{\alpha \in \mathbb{Z}^d}$ as

$$f(t) = \sum_{\alpha \in \mathbb{Z}^d} f(\alpha) \frac{\sin \pi(t_1 - \alpha_1)}{\pi(t_1 - \alpha_1)} \cdots \frac{\sin \pi(t_d - \alpha_d)}{\pi(t_d - \alpha_d)} , \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d .$$

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in Refs. [108, 109]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay at infinity which makes computation in the signal domain very inefficient. Besides, in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval. Moreover, many applied problems impose different a priori constraints on the type of signals. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces; signals are assumed to belong to some shift-invariant space of the form:

$$V_\varphi^2 := \overline{\operatorname{span}}_{L^2}\{\varphi(t-\alpha) : \alpha \in \mathbb{Z}^d\},$$

where the function φ in $L^2(\mathbb{R}^d)$ is called the generator of V_φ^2. See, for instance, Refs. [8, 6, 11, 13, 24, 109, 113, 118, 119, 124] and the references therein.

In this new context, the analogous of the WSK sampling theorem in a shift-invariant space V_φ^2 was first time proved by Walter in [113]:

1.3 Walter's sampling theorem in shift-invariant spaces

Let $\varphi \in L^2(\mathbb{R})$ be a stable generator for the shift-invariant space V_φ^2 which means that the sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ^2. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n=1}^\infty$ has a unique biorthogonal (dual) Riesz basis $\{y_n\}_{n=1}^\infty$, i.e., $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n=1}^\infty \langle x, y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n=1}^\infty \langle x, x_n\rangle_{\mathcal{H}}\, y_n\,,$$

hold for every $x \in \mathcal{H}$ (see [25, 122] for more details and proofs). Recall that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for V_φ^2 (see, for instance, [25, p. 143]) if and only if there exist two positive constants $0 < A \leqslant B$ such that

$$A \leqslant \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2 \leqslant B\,, \quad \text{a.e. } w \in [0,1]\,.$$

Thus we have that

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(\cdot - n) : \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})\,.$$

We assume that the functions in the shift-invariant space V_φ^2 are continuous on $\mathbb{R}$. Equivalently, that the generator φ is continuous on $\mathbb{R}$ and the function $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$ (see [98]). Thus, any $f \in V_\varphi^2$ is defined on $\mathbb{R}$ as the pointwise sum $f(t) = \sum_{n\in\mathbb{Z}} a_n\varphi(t-n)$ for each $t \in \mathbb{R}$.

On the other hand, the space V_φ^2 is the image of $L^2[0,1]$ by means of the isomorphism

$$\begin{array}{cccc} \mathcal{T}_\varphi : & L^2[0,1] & \longrightarrow & V_\varphi^2 \\ & \{\mathrm{e}^{-2\pi i n x}\}_{n\in\mathbb{Z}} & \longmapsto & \{\varphi(t-n)\}_{n\in\mathbb{Z}}\,, \end{array}$$

which maps the orthonormal basis $\{\mathrm{e}^{-2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2[0,1]$ onto the Riesz basis $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ for V_φ^2. For any $F \in L^2[0,1]$ we have

$$\begin{aligned} \mathcal{T}_\varphi F(t) &= \sum_{n\in\mathbb{Z}} \langle F, \mathrm{e}^{-2\pi i n x}\rangle\, \varphi(t-n) \\ &= \langle F, \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)}\mathrm{e}^{-2\pi i n x}\rangle \\ &= \langle F, K_t\rangle_{L^2[0,1]}\,, \quad t \in \mathbb{R}\,, \end{aligned}$$

where, for each $t \in \mathbb{R}$, the function $K_t \in L^2[0,1]$ is given by

$$K_t(x) = \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)}\mathrm{e}^{-2\pi i n x} = \overline{\sum_{n\in\mathbb{Z}} \varphi(t+n)\mathrm{e}^{-2\pi i n x}} = \overline{Z\varphi(t,x)}\,.$$

Here, $Z\varphi(t,x) := \sum_{n\in\mathbb{Z}} \varphi(t+n)\mathrm{e}^{-2\pi i n x}$ denotes the Zak transform of the function φ. See [25, 53, 65] for properties and uses of the Zak transform.

As a consequence, the samples $\{f(a+m)\}_{m\in\mathbb{Z}}$ of $f \in V_\varphi^2$, where $a \in [0,1)$ is fixed, can be expressed as

$$f(a+m) = \langle F, K_{a+m}\rangle = \langle F, \mathrm{e}^{-2\pi i m x} K_a\rangle\,, \quad m \in \mathbb{Z} \ \text{ where } F = \mathcal{T}_\varphi^{-1} f\,.$$

Then, the stable recovery of $f \in V_\varphi^2$ from its samples $\{f(a+m)\}_{m\in\mathbb{Z}}$, reduces to the study of the sequence $\left\{\mathrm{e}^{-2\pi i m x} K_a(x)\right\}_{m\in\mathbb{Z}}$ in $L^2[0,1]$. The following result is easy to prove, having in mind that the multiplication operator

$$\begin{array}{cccc} m_F : & L^2[0,1] & \longrightarrow & L^2[0,1] \\ & f & \longmapsto & Ff\,, \end{array}$$

is well-defined if and only if $F \in L^\infty[0,1]$; in this case, it is bounded and $\|m_F\| = \|F\|_\infty$ (see, for instance, Ref. [121] for a proof).

Theorem 1.1. *The sequence of functions* $\left\{\mathrm{e}^{-2\pi i m x} K_a(x)\right\}_{m\in\mathbb{Z}}$ *is a Riesz basis for* $L^2[0,1]$ *if and only if the inequalities* $0 < \|K_a\|_0 \leqslant \|K_a\|_\infty < \infty$ *hold, where* $\|K_a\|_0 := \operatorname{ess\,inf}_{x\in[0,1]} |K_a(x)|$ *and* $\|K_a\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]} |K_a(x)|$. *Moreover, its biorthogonal Riesz basis is* $\left\{\mathrm{e}^{-2\pi i m x}/\overline{K_a(x)}\right\}_{m\in\mathbb{Z}}$.

Note that the above basis is an orthonormal one if and only if $|K_a(x)| = 1$ a.e. in $[0,1]$.

Let a be a real number in $[0,1)$ such that $0 < \|K_a\|_0 \leqslant \|K_a\|_\infty < \infty$; next we prove Walter's sampling theorem for V_φ^2 in [113]. Given $f \in V_\varphi^2$, we expand the function $F = \mathcal{T}_\varphi^{-1} f \in L^2[0,1]$ with respect to the Riesz basis $\big\{\mathrm{e}^{-2\pi \mathrm{i} n x}/\overline{K_a(x)}\big\}_{n\in\mathbb{Z}}$. Thus we get

$$F = \sum_{n\in\mathbb{Z}} \langle F, K_{a+n}\rangle \frac{\mathrm{e}^{-2\pi \mathrm{i} n x}}{\overline{K_a(x)}} = \sum_{n\in\mathbb{Z}} f(a+n) \frac{\mathrm{e}^{-2\pi \mathrm{i} n x}}{\overline{K_a(x)}} \quad \text{in} \quad L^2[0,1]\,.$$

Applying the operator $\mathcal{T}_\varphi$ to the above expansion we obtain

$$\begin{aligned} f &= \sum_{n\in\mathbb{Z}} f(a+n) \mathcal{T}_\varphi(\mathrm{e}^{-2\pi \mathrm{i} n x}/\overline{K_a(x)}) \\ &= \sum_{n\in\mathbb{Z}} f(a+n) S_a(\cdot - n) \quad \text{in} \quad L^2(\mathbb{R})\,, \end{aligned}$$

where we have used the shifting property

$$\mathcal{T}_\varphi(\mathrm{e}^{-2\pi \mathrm{i} n x} F)(t) = (\mathcal{T}_\varphi F)(t-n), \qquad t \in \mathbb{R},\ n \in \mathbb{Z}$$

for the function $S_a := \mathcal{T}_\varphi(1/\overline{K_a}) \in V_\varphi^2$. As in the Paley-Wiener case, the shift-invariant space V_φ^2 is a reproducing kernel Hilbert space. Indeed, for each $t \in \mathbb{R}$, the evaluation functional at t is bounded:

$$\begin{aligned} |f(t)| &\leqslant \|F\| \|K_t\| \leqslant \|\mathcal{T}_\varphi^{-1}\| \|K_t\| \|f\| \\ &= \|\mathcal{T}_\varphi^{-1}\| \Big(\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2 \Big)^{1/2} \|f\|\,, \quad f \in V_\varphi^2\,. \end{aligned}$$

Therefore, the L^2-convergence implies pointwise convergence which here is uniform on $\mathbb{R}$. The convergence is also absolute due to the unconditional convergence of a Riesz expansion. Thus, for each $f \in V_\varphi^2$ we get the sampling formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(a+n)\, S_a(t-n)\,, \quad t \in \mathbb{R}\,.$$

This mathematical technique, which mimics the Fourier duality technique for Paley-Wiener spaces [60], has been successfully used in deriving sampling formulas in other sampling settings [41, 43, 46, 47, 63, 69, 71]. Here, it will be used for obtaining generalized sampling formulas in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators and also in more general settings.

2

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

2.1 Statement of the general problem

Assume that our functions (signals) belong to some shift-invariant space of the form:

$$V_\Phi^2 := \overline{\operatorname{span}}_{L^2(\mathbb{R}^d)}\big\{\varphi_k(t-\alpha) \,:\, k = 1, 2, \ldots, r \text{ and } \alpha \in \mathbb{Z}^d\big\}\,,$$

where the functions in $\Phi := \{\varphi_1, \ldots, \varphi_r\}$ in $L^2(\mathbb{R}^d)$ are called a set of generators for V_Φ^2. Assuming that the sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, k=1,2\ldots,r}$ is a Riesz basis for V_Φ^2, the shift-invariant space V_Φ^2 can be described as

$$V_\Phi^2 = \Big\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha) \,:\, d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2\ldots, r\Big\}\,. \tag{2.1}$$

The general theory of shift-invariant spaces and their applications can be seen, for instance, in Refs. [15, 16, 87]. These spaces and the scaling functions $\Phi = \{\varphi_1, \ldots, \varphi_r\}$ appear in the multiwavelet setting. Multiwavelets lead to multiresolution analyses and fast algorithms just as scalar wavelets, but they have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal (see, for instance, Ref. [70]). Classical sampling in multiwavelet subspaces has been studied in Refs. [91, 99].

On the other hand, in many common situations the available data are samples of some filtered versions $f * \mathsf{h}_j$ of the signal f itself, where the average function h_j reflects the characteristics of the acquisition device. This leads to generalized sampling (also called average sampling) in V_Φ^2 (see, among others, Refs. [3, 10, 41, 46, 43, 68, 80, 83, 96, 97, 99]).

Suppose that s convolution systems (linear time-invariant systems or filters in engineering jargon) $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, are defined on the shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$. Assume also that the sequence of samples

$$\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$$

for f in V_Φ^2 is available, where the samples are taken at the sub-lattice $M\mathbb{Z}^d$ of $\mathbb{Z}^d$, where M denotes a matrix of integer entries with positive determinant. If we sample any function $f \in V_\Phi^2$ on $M\mathbb{Z}^d$, we are using the sampling rate $1/r(\det M)$ and, roughly speaking, we will need, for the recovery of $f \in V_\Phi^2$, the sequence of generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ coming from $s \geqslant r(\det M)$ convolution systems $\mathcal{L}_j$.

Assume that the sequences of generalized samples satisfy the following stability condition: There exist two positive constants $0 < A \leqslant B$ such that

$$A\|f\|^2 \leqslant \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j f(M\alpha)|^2 \leqslant B\|f\|^2 \quad \text{for all } \ f \in V_\Phi^2\,.$$

In [10] the set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_s\}$ is said to be an M-stable filtering sampler for V_Φ^2. The goal here is to obtain sampling formulas in V_Φ^2 having the form

$$f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,, \tag{2.2}$$

such that the sequence of reconstruction functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is a frame for the shift-invariant space V_Φ^2. This will be done in the light of the frame theory for separable Hilbert spaces, by using a similar mathematical technique as in the above chapter.

Recall that a sequence $\{x_n\}$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|x\|^2 \leqslant \sum_n |\langle x, x_n\rangle|^2 \leqslant B\|x\|^2 \ \text{ for all } x \in \mathcal{H}\,.$$

Given a frame $\{x_n\}$ for $\mathcal{H}$ the representation property of any vector $x \in \mathcal{H}$ as a series $x = \sum_n c_n x_n$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients $\{c_n\}$, depending linearly and continuously on x, are obtained by using the dual frames $\{y_n\}$ of $\{x_n\}$, i.e., the sequence $\{y_n\}$ is another frame for $\mathcal{H}$ such that, for each $x \in \mathcal{H}$, the expansions $x = \sum_n \langle x, y_n\rangle x_n = \sum_n \langle x, x_n\rangle y_n$ hold. For more details on the frame theory see Appendix A which collects the main results of the superb monograph [25].

2.2 Preliminaries on $L^2(\mathbb{R}^d)$ shift-invariant subspaces

Let $\Phi := \{\varphi_1, \varphi_2, \dots, \varphi_r\}$ be a set of functions, where $\varphi_k \in L^2(\mathbb{R}^d)$ $k = 1, 2, \dots, r$, such that the sequence $\big\{\varphi_k(t-\alpha)\big\}_{\alpha\in\mathbb{Z}^d;\, k=1,2\dots,r}$ is a Riesz basis for the shift-invariant space

$$V_\Phi^2 := \Big\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha) \,:\, d_k \in \ell^2(\mathbb{Z}^d),\, k = 1, 2\dots, r \Big\} \subset L^2(\mathbb{R}^d)\,.$$

There exists a necessary and sufficient condition involving the Gramian matrix-function

$$G_\Phi(w) := \sum_{\alpha\in\mathbb{Z}^d} \widehat{\Phi}(w+\alpha)\overline{\widehat{\Phi}(w+\alpha)}^\top, \text{ where } \widehat{\Phi} := (\widehat{\varphi}_1, \widehat{\varphi}_2, \dots, \widehat{\varphi}_r)^\top,$$

which assures that the sequence $\{\varphi_k(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, k=1,2\dots,r}$ is a Riesz basis for V_Φ^2; namely (see, for instance, [10]): There exist two positive constants c and C such that

$$c\,\mathbb{I}_r \leqslant G_\Phi(w) \leqslant C\,\mathbb{I}_r \quad \text{a.e. } w \in [0,1)^d\,. \tag{2.3}$$

We assume throughout the paper that the functions in the shift-invariant space V_Φ^2 are continuous on $\mathbb{R}^d$. As in the case of one generator, this is equivalent to the generators Φ being continuous on $\mathbb{R}^d$ with $\sum_{\alpha\in\mathbb{Z}^d} |\Phi(t-\alpha)|^2$ uniformly bounded on $\mathbb{R}^d$, the proof of this equivalence can be found in [98], and the generalization to L^p-Banach spaces in [44]. Thus, any $f \in V_\Phi^2$ is defined on $\mathbb{R}^d$ as the pointwise sum

$$f(t) = \sum_{k=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} d_k(\alpha)\, \varphi_k(t-\alpha)\,, \quad t \in \mathbb{R}^d\,. \tag{2.4}$$

Besides, the space V_Φ^2 is a RKHS since the evaluation functionals, $E_t f := f(t)$ are bounded on V_Φ^2 for each $t \in \mathbb{R}^d$. Indeed, for each fixed $t \in \mathbb{R}^d$ we have

$$\begin{aligned}
|f(t)|^2 &= \Big| \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha) \Big|^2 \\
&\leqslant \Big(\sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} |d_k(\alpha)|^2 \Big) \Big(\sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} |\varphi_k(t-\alpha)|^2 \Big) \\
&= \Big(\sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} |d_k(\alpha)|^2 \Big) \Big(\sum_{\alpha\in\mathbb{Z}^d} |\Phi(t-\alpha)|^2 \Big) \\
&\leqslant \frac{\|f\|^2}{c} \sum_{\alpha\in\mathbb{Z}^d} |\Phi(t-\alpha)|^2, \quad f \in V_\Phi^2\,,
\end{aligned}$$

where we have used Cauchy-Schwarz's inequality in (2.4), and the inequality satisfied for any lower Riesz bound c of the Riesz basis $\{\varphi_k(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, k=1,2\dots,r}$ for V_Φ^2, that is,

$$c \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} |d_k(\alpha)|^2 \leqslant \|f\|^2 .$$

Thus, the convergence in V_Φ^2 in the $L^2(\mathbb{R}^d)$-sense implies pointwise convergence which is uniform on $\mathbb{R}^d$ having in mind the boundedness of $\sum_{\alpha\in\mathbb{Z}^d} |\Phi(t-\alpha)|^2$ in $\mathbb{R}^d$.

The product space

$$L_r^2[0,1)^d := \left\{\mathbf{F} = (F_1, F_2, \dots, F_r)^\top : F_k \in L^2[0,1)^d ,\ k = 1,2,\dots,r\right\}$$

with its usual inner product

$$\langle \mathbf{F}, \mathbf{H}\rangle_{L_r^2[0,1)^d} := \sum_{k=1}^{r} \langle F_k, H_k\rangle_{L^2[0,1)^d} = \int_{[0,1)^d} \mathbf{H}^*(w)\mathbf{F}(w)dw$$

becomes a Hilbert space. Similarly, we introduce the product Banach space $L_r^\infty[0,1)^d$ which will be used later.

The system $\left\{\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top w}\mathbf{e}_k\right\}_{\alpha\in\mathbb{Z}^d;\, k=1,2,\dots,r}$, where $\mathbf{e}_k$ denotes the vector of $\mathbb{R}^r$ with all the components null except the k-th component which is equal to one, is an orthonormal basis for $L_r^2[0,1)^d$.

The shift-invariant space V_Φ^2 is the image of $L_r^2[0,1)^d$ by means of the isomorphism

$$\begin{array}{llcl} \mathcal{T}_\Phi : & L_r^2[0,1)^d & \longrightarrow & V_\Phi^2 \\ & \{\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top w}\mathbf{e}_k\}_{\alpha\in\mathbb{Z}^d;\, k=1,2,\dots,r} & \longmapsto & \{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, k=1,2,\dots,r} , \end{array}$$

which maps the orthonormal basis $\left\{\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top w}\mathbf{e}_k\right\}_{\alpha\in\mathbb{Z}^d;\, k=1,2,\dots,r}$ for $L_r^2[0,1)^d$ onto the Riesz basis $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, k=1,2,\dots,r}$ for V_Φ^2. For each $\mathbf{F} = (F_1,\dots,F_r)^\top \in L_r^2[0,1)^d$ we have

$$\mathcal{T}_\Phi\mathbf{F}(t) := \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} \left\langle F_k, \mathrm{e}^{-2\pi\mathrm{i}\alpha^\top\cdot}\right\rangle_{L^2[0,1)^d} \varphi_k(t-\alpha) , \quad t\in\mathbb{R}^d . \tag{2.5}$$

It is routine to check that the isomorphism $\mathcal{T}_\Phi$ can also be expressed by

$$f(t) = \mathcal{T}_\Phi\mathbf{F}(t) = \langle \mathbf{F}, \mathbf{K}_t\rangle_{L_r^2[0,1)^d} , \quad t\in\mathbb{R}^d ,$$

where the kernel transform $\mathbb{R}^d \ni t \mapsto \mathbf{K}_t \in L_r^2[0,1)^d$ is defined as $\mathbf{K}_t(x) := \overline{\mathbf{Z}\Phi}(t,x)$, and $\mathbf{Z}\Phi$ denotes the Zak transform of Φ, i.e.,

$$(\mathbf{Z}\Phi)(t,w) := \sum_{\alpha\in\mathbb{Z}^d} \Phi(t+\alpha)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top w} .$$

Note that $(\mathbf{Z}\Phi) = (Z\varphi_1, \ldots, Z\varphi_r)^\top$ where $\mathbf{Z}$ denotes the usual Zak transform. See [25, 53, 65] for properties and uses of the Zak transform.

The following shifting property of $\mathcal{T}_\Phi$ will be used later: For $\mathbf{F} \in L^2_r[0,1)^d$ and $\alpha \in \mathbb{Z}^d$ we have

$$\mathcal{T}_\Phi\big[\mathbf{F}(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top\cdot}\big](t) = \mathcal{T}_\Phi\mathbf{F}(t-\alpha)\,, \quad t \in \mathbb{R}^d\,. \tag{2.6}$$

Indeed, using (2.5)

$$\begin{aligned}
\mathcal{T}_\Phi\big[\mathbf{F}(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top\cdot}\big](t) &= \sum_{\beta\in\mathbb{Z}^d}\sum_{k=1}^{r}\big\langle F_k(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top\cdot}, \mathrm{e}^{-2\pi\mathrm{i}\beta^\top\cdot}\big\rangle_{L^2[0,1)^d}\varphi_k(t-\beta)\\
&= \sum_{\beta\in\mathbb{Z}^d}\sum_{k=1}^{r}\big\langle F_k(\cdot), \mathrm{e}^{-2\pi\mathrm{i}(\beta^\top-\alpha^\top)\cdot}\big\rangle_{L^2[0,1)^d}\varphi_k(t-\beta)\\
&= \sum_{\beta\in\mathbb{Z}^d}\sum_{k=1}^{r}\big\langle F_k(\cdot), \mathrm{e}^{-2\pi\mathrm{i}\beta^\top\cdot}\big\rangle_{L^2[0,1)^d}\varphi_k(t-\alpha-\beta)\\
&= \mathcal{T}_\Phi\mathbf{F}(t-\alpha)\,.
\end{aligned}$$

2.2.1 The convolution systems $\mathcal{L}_j$ on V^2_Φ

We consider s convolution systems $\mathcal{L}_j f = f * \mathsf{h}_j$, $j = 1, 2, \ldots, s$, defined for $f \in V^2_\Phi$ where each impulse response h_j belongs to one of the following three types:

(a) The impulse response h_j is a linear combination of partial derivatives of shifted delta functionals, i.e.,

$$\big(\mathcal{L}_j f\big)(t) := \sum_{|\beta|\leqslant N_j} c_{j,\beta} D^\beta f(t + d_{j,\beta})\,, \quad t \in \mathbb{R}^d\,.$$

If there is a system of this type, we also assume that $\sum_{\alpha\in\mathbb{Z}^d} |D^\beta\varphi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$ for $|\beta| \leqslant N_j$.

(b) The impulse response h_j of $\mathcal{L}_j$ belongs to $L^2(\mathbb{R}^d)$. Thus, for any $f \in V^2_\varphi$ we have

$$\big(\mathcal{L}_j f\big)(t) := [f * \mathsf{h}_j](t) = \int_{\mathbb{R}^d} f(x)\mathsf{h}_j(t-x)dx, \quad t \in \mathbb{R}^d.$$

(c) The function $\widehat{\mathsf{h}}_j \in L^\infty(\mathbb{R}^d)$ whenever $H_{\varphi_k}(x) := \sum_{\alpha\in\mathbb{Z}^d} |\widehat{\varphi}_k(x+\alpha)| \in L^2[0,1)^d$ for all $k = 1, 2, \ldots, r$.

Lemma 2.1. *Let $\mathcal{L}$ be a convolution system of the type (b) or (c). Then for each fixed $t \in \mathbb{R}^d$ the sequence $\big\{\big(\mathcal{L}\varphi_k\big)(t+\alpha)\big\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^2(\mathbb{Z}^d)$ for each $k = 1, \ldots, r$.*

Proof. First assume that $\mathsf{h} \in L^2(\mathbb{R}^d)$; then we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}\varphi_k(t+\alpha)|^2 &= \Big\| \sum_{\alpha\in\mathbb{Z}^d} \mathcal{L}\varphi_k(t+\alpha)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}\Big\|^2_{L^2[0,1)^d} \\ &= \big\|Z\mathcal{L}\varphi_k(t,x)\big\|^2_{L^2[0,1)^d} \\ &= \Big\| \sum_{\alpha\in\mathbb{Z}^d} \big(\widehat{\mathcal{L}\varphi_k}\big)(x+\alpha)\mathrm{e}^{2\pi\mathrm{i}(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d},\end{aligned}$$

where, in the last equality, we have used a version of the Poisson summation formula [45, Lemma 2.1]. Notice that $\widehat{\varphi}_k, \widehat{\mathsf{h}} \in L^2(\mathbb{R}^d)$ implies, by Cauchy-Schwarz's inequality, that $\widehat{\varphi}_k\widehat{\mathsf{h}} = \widehat{\mathcal{L}\varphi_k} \in L^1(\mathbb{R}^d)$.

Now,

$$\begin{aligned}&\Big\| \sum_{\alpha\in\mathbb{Z}^d} \big(\widehat{\mathcal{L}\varphi_k}\big)(x+\alpha)\mathrm{e}^{2\pi\mathrm{i}(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d} \\ &= \Big\| \sum_{\alpha\in\mathbb{Z}^d} \widehat{\varphi}_k(x+\alpha)\widehat{\mathsf{h}}(x+\alpha)\mathrm{e}^{2\pi\mathrm{i}(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d} \\ &\leqslant \bigg\|\Big(\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\varphi}_k(x+\alpha)|^2\Big)^{1/2}\Big(\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathsf{h}}(x+\alpha)|^2\Big)^{1/2}\bigg\|^2_{L^2[0,1)^d} \\ &\leqslant C^{1/2}\|\mathsf{h}\|^2_{L^2[0,1)^d},\end{aligned}$$

where we have used (2.3) and the fact that $\|\mathsf{h}\|^2_{L^2(\mathbb{R}^d)} = \|\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathsf{h}}(x+\alpha)|^2\|_{L^1[0,1)^d}$. Finally, assume that $H_{\varphi_k} \in L^2[0,1)^d$; since $\widehat{\varphi}_k \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we obtain that

$$\widehat{\mathcal{L}\varphi_k} = \widehat{\varphi}_k\,\widehat{\mathsf{h}} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

Since $\sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathcal{L}\varphi_k}(x+\alpha)| \leqslant \|\widehat{\mathsf{h}}\|_{L^\infty(\mathbb{R}^d)}H_{\varphi_k}(x)$, using again [45, Lemma 2.1] we get

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}\varphi_k(t+\alpha)|^2 &= \Big\| \sum_{\alpha\in\mathbb{Z}^d} \big(\widehat{\mathcal{L}\varphi_k}\big)(x+\alpha)\mathrm{e}^{2\pi\mathrm{i}(x+\alpha)^\top t}\Big\|^2_{L^2[0,1)^d} \\ &\leqslant \Big\| \sum_{\alpha\in\mathbb{Z}^d} |\widehat{\mathcal{L}\varphi_k}(x+\alpha)|\Big\|^2_{L^2[0,1)^d} \\ &\leqslant \|\widehat{\mathsf{h}}\|^2_{L^\infty(\mathbb{R}^d)}\|H_{\varphi_k}\|^2_{L^2[0,1)^d}.\end{aligned}$$

□

Notice that above result becomes trivial for systems of type (a).

Lemma 2.2. *Let $\mathcal{L}$ be a convolution system of the type (a), (b) or (c). Then, for each $f \in V^2_\Phi$ we have*

$$\big(\mathcal{L}f\big)(t) = \big\langle \mathbf{F}, \big(\overline{\mathbf{Z}\mathcal{L}\Phi}\big)(t,\cdot)\big\rangle_{L^2_r[0,1)^d}, \quad t \in \mathbb{R}^d,$$

where $\mathbf{F} = \mathcal{T}_\Phi^{-1}f$.

Proof. Assume that $\mathcal{L}$ is a convolution system of type (a). Under our hypothesis on $\mathcal{L}$, for $m = 0, 1, 2 \dots, N$ we have that

$$f^{(m)}(t) = \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\rangle\varphi_k^{(m)}(t-\alpha)\,.$$

Having in mind we have assumed that $\sum_{\alpha\in\mathbb{Z}^d}|\Phi^{(m)}(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$, we obtain that

$$\begin{aligned}
(\mathcal{L}f)(t) &= \sum_{m=0}^{N} c_m f^{(m)}(t+d_m)\\
&= \sum_{m=0}^{N} c_m \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\Big\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\Big\rangle\varphi_k^{(m)}(t+d_m-\alpha)\\
&= \sum_{k=1}^{r}\Big\langle F_k, \sum_{m=0}^{N}\overline{c_m}\sum_{\alpha\in\mathbb{Z}^d}\overline{\varphi}_k^{(m)}(t+d_m-\alpha)\mathrm{e}^{-2\pi i\alpha^\top\cdot}\Big\rangle_{L^2[0,1)^d}\\
&= \sum_{k=1}^{r}\Big\langle F_k, \sum_{\alpha\in\mathbb{Z}^d}\overline{\mathcal{L}\varphi_k}(t-\alpha)\mathrm{e}^{-2\pi i\alpha^\top\cdot}\Big\rangle_{L^2[0,1)^d}\\
&= \sum_{k=1}^{r}\Big\langle F_k, (\overline{Z\mathcal{L}\varphi_k})(t,\cdot)\Big\rangle_{L^2[0,1)^d}\,.
\end{aligned}$$

Assume now that $\mathcal{L}$ is a convolution system of the type (b) or (c). For each $t\in\mathbb{R}^d$, considering the function $\psi(x) := \overline{\mathsf{h}(-x)}$, $x\in\mathbb{R}^d$, we have

$$\begin{aligned}
(\mathcal{L}f)(t) &= \big\langle f, \psi(\cdot - t)\big\rangle_{L^2(\mathbb{R}^d)}\\
&= \Big\langle \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\big\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\big\rangle\varphi_k(\cdot-\alpha), \psi(\cdot-t)\Big\rangle_{L^2(\mathbb{R}^d)}\\
&= \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\big\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\big\rangle_{L^2[0,1)^d}\Big\langle\varphi_k, \psi(\cdot-t+\alpha)\Big\rangle_{L^2(\mathbb{R}^d)}\\
&= \sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\big\langle F_k, \mathrm{e}^{-2\pi i\alpha^\top\cdot}\big\rangle_{L^2[0,1)^d}\mathcal{L}\varphi_k(t-\alpha)\,.
\end{aligned}$$

Since the sequence $\{(\mathcal{L}\varphi_k)(t+\alpha)\}_{\alpha\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$, Parseval's equality gives

$$\begin{aligned}
(\mathcal{L}f)(t) &= \sum_{k=1}^{r}\big\langle F_k, \sum_{\alpha\in\mathbb{Z}^d}\overline{\mathcal{L}\varphi_k}(t-\alpha)\,\mathrm{e}^{-2\pi i\alpha^\top\cdot}\big\rangle_{L^2[0,1)^d}\\
&= \big\langle \mathbf{F}, (\overline{\mathbf{Z}\mathcal{L}\Phi})(t,\cdot)\big\rangle_{L^2_r(0,1)}\,, \quad t\in\mathbb{R}^d\,,
\end{aligned}$$

which ends the proof. □

2.2.2 Sampling at a lattice of $\mathbb{Z}^d$: An expression for the samples

Given a nonsingular matrix M with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e.,

$$\Lambda_M := \{M\alpha : \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d .$$

Without loss of generality we can assume that $\det M > 0$; otherwise we can consider $M' = ME$ where E is some $d \times d$ integer matrix satisfying $\det E = -1$. Trivially, $\Lambda_M = \Lambda'_M$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of M and M^{-1} respectively. The following useful generalized orthogonal relationship holds (see [111]):

$$\sum_{p\in\mathcal{N}(M^\top)} \mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^{-T}p} = \begin{cases}\det M, & \alpha \in \Lambda_M \\ 0 & \alpha \in \mathbb{Z}^d \backslash \Lambda_M\end{cases} \tag{2.7}$$

where

$$\mathcal{N}(M^\top) := \mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\} \tag{2.8}$$

The set $\mathcal{N}(M^\top)$ has $\det M$ elements (see [111] or [117]). One of these elements is zero, say $i_1 = 0$; we denote the rest of elements by $i_2, \dots, i_{\det M}$ ordered in any form; from now on,

$$\mathcal{N}(M^\top) = \{i_1 = 0, i_2, \dots, i_{\det M}\} \subset \mathbb{Z}^d .$$

Notice that the sets, defined as $Q_l := M^{-\top} i_l + M^{-\top}[0,1)^d$, $l = 1, 2, \dots, \det M$, satisfy (see [117, p. 110]):

$$Q_l \cap Q_{l'} = \varnothing \quad \text{if} \quad l \neq l' \qquad \text{and} \qquad \mathrm{Vol}\Big(\bigcup_{l=1}^{\det M} Q_l\Big) = 1 .$$

Thus, for any function F integrable in $[0,1)^d$ and $\mathbb{Z}^d$-periodic we have

$$\int_{[0,1)^d} F(x)dx = \sum_{l=1}^{\det M} \int_{Q_l} F(x)dx$$

Now assume that we sample the filtered versions $\mathcal{L}_j f$ of $f \in V_\Phi^2$, $j = 1, 2, \dots, s$, at a lattice Λ_M. Having in mind Lemma 2.2, for $j = 1, 2, \dots, s$ and $\alpha \in \mathbb{Z}^d$ we obtain that

$$\begin{aligned}\big(\mathcal{L}_j f\big)(M\alpha) &= \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(M\alpha, \cdot)\rangle \\ &= \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(0,\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\rangle_{L_r^2[0,1)^d} ,\end{aligned} \tag{2.9}$$

where $\mathbf{F} = \mathcal{T}_\Phi^{-1} f \in L_r^2[0,1)^d$. Denote

$$\mathbf{g}_j(x) := \mathbf{Z}\mathcal{L}_j\Phi(0, x) , \quad j = 1, 2, \dots, s ; \tag{2.10}$$

in other words,

$$\mathbf{g}_j^\top(x) := \big(g_{j,1}(x), g_{j,2}(x), \dots, g_{j,r}(x)\big)\,,$$

where $g_{j,k}(x) = Z\mathcal{L}_j\varphi_k(0,x)$ for $1 \leqslant j \leqslant s$ and $1 \leqslant k \leqslant r$.

As a consequence of expression (2.9) for generalized samples, a challenge problem is to study the completeness, Bessel, frame, or Riesz basis properties of the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ in $L^2_r[0,1)^d$.

To this end we introduce the $s \times r(\det M)$ matrix of functions

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x+M^{-\top}i_2) & \cdots & \mathbf{g}_1^\top(x+M^{-\top}i_{\det M}) \\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x+M^{-\top}i_2) & \cdots & \mathbf{g}_2^\top(x+M^{-\top}i_{\det M}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x+M^{-\top}i_2) & \cdots & \mathbf{g}_s^\top(x+M^{-\top}i_{\det M}) \end{bmatrix}, \tag{2.11}$$

and its related constants

$$A_{\mathbb{G}} := \operatorname*{ess\,inf}_{x\in[0,1)^d} \lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)],$$

$$B_{\mathbb{G}} := \operatorname*{ess\,sup}_{x\in[0,1)^d} \lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)]\,,$$

where $\mathbb{G}^*(x)$ denotes the transpose conjugate of the matrix $\mathbb{G}(x)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(x)\mathbb{G}(x)$. Observe that $0 \leqslant A_{\mathbb{G}} \leqslant B_{\mathbb{G}} \leqslant \infty$. Note that in the definition of the matrix $\mathbb{G}(x)$ we are considering the $\mathbb{Z}^d$-periodic extension of the involved functions $\mathbf{g}_j,\ j = 1,2,\dots,s$.

Regardless the functions $\mathbf{g}_j$ in $L^2_r[0,1)^d$, $j = 1,2,\dots,s$, are given by (2.10), the following result holds:

Lemma 2.3. *Let $\mathbf{g}_j$ be in $L^2_r[0,1)^d$ for $j = 1,2,\dots,s$ and let $\mathbb{G}(x)$ be its associated matrix as in* (3.9). *Then,*

(a) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a complete system for $L^2_r[0,1)^d$ if and only if the rank of the matrix $\mathbb{G}(x)$ is $r(\det M)$ a.e. in $[0,1)^d$.

(b) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a Bessel sequence for $L^2_r[0,1)^d$ if and only if $\mathbf{g}_j \in L^\infty_r[0,1)^d$ (or equivalently $B_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $B_{\mathbb{G}}/(\det M)$.

(c) The sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a frame for $L^2_r[0,1)^d$ if and only if $0 < A_{\mathbb{G}} \leqslant B_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $A_{\mathbb{G}}/(\det M)$ and $B_{\mathbb{G}}/(\det M)$.

(d) The sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ *is a Riesz basis for* $L^2_r[0,1)^d$ *if and only if it is a frame and* $s=r(\det M)$.

Proof. For any $\mathbf{F}\in L^2_r[0,1)^d$ we have

$$\begin{aligned}
\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d} &= \int_{[0,1)^d}\sum_{k=1}^{r} F_k(x)g_{j,k}(x)\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}\,dx\\
&= \sum_{k=1}^{r}\sum_{l=1}^{\det M}\int_{Q_l} F_k(x)g_{j,k}(x)\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}\,dx\\
&= \sum_{k=1}^{r}\int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M} F_k(x+M^{-\top}i_l)g_{j,k}(x+M^{-\top}i_l)\;\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}\,dx\\
&= \int_{M^{-\top}[0,1)^d}\sum_{k=1}^{r}\sum_{l=1}^{\det M} F_k(x+M^{-\top}i_l)g_{j,k}(x+M^{-\top}i_l)\;\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}\,dx\\
&= \int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M} \mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{F}(x+M^{-\top}i_l)\;\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}\,dx\,,
\end{aligned} \tag{2.12}$$

where we have considered the $\mathbb{Z}^d$-periodic extension of $\mathbf{F}$. By using that the sequence $\{\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ is an orthogonal basis for $L^2\big(M^{-\top}[0,1)^d\big)$ we obtain

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2 =$$
$$\frac{1}{\det M}\sum_{j=1}^{s}\Big\|\sum_{l=1}^{\det M}\mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{F}(x+M^{-\top}i_l)\Big\|^2_{L^2_r(M^{-\top}[0,1)^d)}.$$

Denoting

$$\mathbb{F}(x):=\big[\mathbf{F}^\top(x),\mathbf{F}^\top(x+M^{-\top}i_2),\cdots,\mathbf{F}^\top(x+M^{-\top}i_{\det M})\big]^\top,$$

the equality above reads

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\Big|^2 = \frac{1}{\det M}\big\|\mathbb{G}(x)\mathbb{F}(x)\big\|^2_{L^2_s(M^{-\top}[0,1)^d)}\,. \tag{2.13}$$

On the other hand, using that the function $\mathbf{g}_j$ is $\mathbb{Z}^d$-periodic, we obtain that the set

$$\big\{\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_1),\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_2),\dots,\mathbf{g}_j(x+M^{-\top}i_l+M^{-\top}i_{\det M})\big\}$$

has the same elements as

$$\big\{\mathbf{g}_j(x+M^{-\top}i_1),\mathbf{g}_j(x+M^{-\top}i_2),\dots,\mathbf{g}_j(x+M^{-\top}i_{\det M})\big\}\,.$$

Thus the matrix $\mathbb{G}(x + M^{-\top} i_l)$ has the same columns of $\mathbb{G}(x)$, possibly in a different order. Hence, $\operatorname{rank}\mathbb{G}(x) = r(\det M)$ a.e. in $[0,1)^d$ if and only if $\operatorname{rank}\mathbb{G}(x) = r(\det M)$ a.e. in $M^{-\top}[0,1)^d$. Moreover,

$$
\begin{aligned}
A_{\mathbb{G}} &= \operatorname*{ess\,inf}_{x\in M^{-\top}[0,1)^d} \lambda_{\min}[\mathbb{G}^*(x)\mathbb{G}(x)],\\
B_{\mathbb{G}} &= \operatorname*{ess\,sup}_{x\in M^{-\top}[0,1)^d} \lambda_{\max}[\mathbb{G}^*(x)\mathbb{G}(x)]\,.
\end{aligned}
\tag{2.14}
$$

To prove (a), assume that there exists a set $\Omega \subseteq M^{-\top}[0,1)^d$ with positive measure such that $\operatorname{rank}\mathbb{G}(x) < r(\det M)$ for ech $x \in \Omega$. Then, there exists a measurable function $v(x)$, $x \in \Omega$, such that $\mathbb{G}(x)v(x) = 0$ and $\|v(x)\|_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)} = 1$ in Ω. This function can be constructed as in [67, Lemma 2.4]. Define $\mathbf{F} \in L^2_r[0,1)^d$ such that $\mathbb{F}(x) = v(x)$ if $x \in \Omega$, and $\mathbb{F}(x) = 0$ if $x \in M^{-\top}[0,1)^d \backslash \Omega$. Hence, from (2.13) we obtain that the system is not complete. Conversely, if the system is not complete, by using (2.13) we obtain a $\mathbb{F}(x)$ different from 0 in a set with positive measure such that $\mathbb{G}(x)\mathbb{F}(x) = 0$. Thus $\operatorname{rank}\mathbb{G}(x) < r(\det M)$ on a set with positive measure.
To prove (b) notice that

$$
\begin{aligned}
&\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\right|^2 = \frac{1}{\det M}\left\|\mathbb{G}(x)\mathbb{F}(x)\right\|^2_{L^2_s(M^{-\top}[0,1)^d)}\\
&= \frac{1}{\det M}\int_{M^{-\top}[0,1)^d} \mathbb{F}^*(x)\mathbb{G}^*(x)\mathbb{G}(x)\mathbb{F}(x)dx\,.
\end{aligned}
\tag{2.15}
$$

If $B_{\mathbb{G}} < \infty$ then, for each $\mathbb{F}$, we have

$$
\begin{aligned}
\frac{1}{\det M}\int_{M^{-\top}[0,1)^d} \mathbb{F}^*(x)\mathbb{G}^*(x)\mathbb{G}(x)\mathbb{F}(x)dx &\leqslant \frac{B_{\mathbb{G}}}{\det M}\|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)}\\
&= \frac{B_{\mathbb{G}}}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\,,
\end{aligned}
\tag{2.16}
$$

from which the sequence $\left\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\right\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is a Bessel sequence and its optimal Bessel bound is less than or equal to $B_{\mathbb{G}}/(\det M)$.
Let $K < B_{\mathbf{G}}$; there exists a set $\Omega_K \subset M^{-\top}[0,1)^d$ with positive measure such that $\lambda_{\max_{x\in\Omega_K}}[\mathbb{G}^*(x)\mathbb{G}(x)] \geqslant K$. Let $\mathbf{F} \in L^2_r[0,1)^d$ such that its associated vector function $\mathbb{F}$ is 0 if $x \in M^{-\top}[0,1)^d \backslash \Omega_K$ and $\mathbb{F}$ is an eigenvector of norm 1 associated with the largest eigenvalue of $\mathbb{G}^*(x)\mathbb{G}(x)$ if $x \in \Omega_K$. Using (2.15), we obtain

$$
\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\right|^2 \geqslant \frac{K}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\,.
$$

Therefore if $B_{\mathbf{G}} = \infty$ the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is not a Bessel sequence, and the optimal Bessel bound is $B_{\mathbb{G}}/(\det M)$.
To prove (c) assume first that $0 < A_{\mathbb{G}} \leqslant B_{\mathbb{G}} < \infty$. By using part (b), the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a Bessel sequence in $L^2_r[0,1)^d$.

Moreover, using (2.15) and the Rayleigh-Ritz theorem (see [64, p. 176]), for each $\mathbf{F} \in L^2_r[0,1)^d$ we obtain

$$\begin{aligned}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\right|^2 &\geqslant \frac{A_{\mathbb{G}}}{\det M}\|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)}\\ &= \frac{A_{\mathbb{G}}}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\end{aligned} \tag{2.17}$$

Hence the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a frame with optimal lower bound larger that or equal to $A_{\mathbb{G}}/(\det M)$.

Conversely if $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a frame for $L^2_r[0,1)^d$ we know by part (b) that $B_{\mathbb{G}} < \infty$. In order to prove that $A_{\mathbb{G}} > 0$, consider any constant $K > A_{\mathbb{G}}$. Then there exists a set $\Omega_K \subset M^{-\top}[0,1)^d$ with positive measure such that $\lambda_{\min_{x\in\Omega_K}}[\mathbb{G}^*(x)\mathbb{G}(x)] \leqslant K$. Let $\mathbf{F} \in L^2_r[0,1)^d$ such that its associated $\mathbb{F}(x)$ is 0 if $x \in M^{-\top}[0,1)^d\backslash\Omega_K$ and $\mathbb{F}(x)$ is an eigenvector of norm 1 associated with the smallest eigenvalue of $\mathbb{G}^*(x)\mathbb{G}(x)$ if $x \in \Omega_K$. Since $\mathbb{F}$ is bounded, we have that $\mathbb{G}(x)\mathbb{F}(x) \in L^2_s(M^{-\top}[0,1)^d)$. From (2.15) we get

$$\begin{aligned}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\left|\langle \mathbf{F}(x), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2_r[0,1)^d}\right|^2 &\leqslant \frac{K}{\det M}\|\mathbb{F}\|^2_{L^2_{r(\det M)}(M^{-\top}[0,1)^d)}\\ &= \frac{K}{\det M}\|\mathbf{F}\|^2_{L^2_r[0,1)^d}\,.\end{aligned} \tag{2.18}$$

Denoting by A the optimal lower frame bound of $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$, we have obtained that $K/(\det M) \geqslant A$ for each $K > A_{\mathbb{G}}$; thus $A_{\mathbb{G}}/(\det M) \geqslant A$ and consequently, $A_{\mathbb{G}} > 0$. Moreover, under the hypotheses of part (c) we deduce that $A_{\mathbb{G}}/(\det M)$ and $B_{\mathbb{G}}/(\det M)$ are the optimal frame bounds.

The proof of (d) is based on Theorem A.7: A frame is a Riesz basis if and only if it has a biorthogonal sequence. Assume that the sequence

$$\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$$

is a Riesz basis for $L^2_r[0,1)^d$ being the sequence $\{\mathbf{h}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ its biorthogonal sequence. Using (2.12) we get

$$\int_{M^{-\top}[0,1)^d}\sum_{l=1}^{\det M}\mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{h}_{j',0}(x+M^{-\top}i_l)\ \mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}\,dx$$

$$= \langle \mathbf{h}_{j',0}(\cdot), \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot} \rangle = \delta_{j,j'}\delta_{\alpha,0}\,.$$

Therefore,

$$\sum_{l=1}^{\det M} \mathbf{g}_j^\top(x+M^{-\top}i_l)\mathbf{h}_{j',0}(x+M^{-\top}i_l)\ \mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x} = (\det M)\delta_{j,j'}\,,$$

a.e. in $M^{-\top}[0,1)^d$.Thus the matrix $\mathbb{G}(x)$ has a right inverse a.e. in $M^{-\top}[0,1)^d$ and, in particular, $s \leqslant r(\det M)$. On the other hand, $A_{\mathbb{G}} > 0$ implies that $\det[\mathbb{G}^*(x)\mathbb{G}(x)] > 0$, a.e. in $M^{-\top}[0,1)^d$, and there exists the matrix

$$[\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x) \qquad \text{a.e. in } M^{-\top}[0,1)^d.$$

This matrix is a left inverse of the matrix $\mathbb{G}(x)$ which implies $s \geqslant r(\det M)$. Thus, we obtain that $r(\det M) = s$.

Conversely, assume that $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is a frame for $L_r^2[0,1)^d$ and $r(\det M) = s$. In this case $\mathbb{G}(x)$ is a square matrix and $\det[\mathbb{G}(x)^*(x)\mathbb{G}(x)(x)] > 0$ a.e. in $M^{-\top}[0,1)^d$ implies that $\det\mathbb{G}(x) \neq 0$ a.e. in $M^{-\top}[0,1)^d$. Having in mind the structure of $\mathbb{G}(x)$ its inverse must be the $r(\det M)\times s$ matrix

$$\mathbb{G}^{-1}(x) = \begin{bmatrix} \mathbf{c}_1(x) & \ldots & \mathbf{c}_s(x) \\ \mathbf{c}_1(x+M^{-\top}i_2) & \ldots & \mathbf{c}_s(x+M^{-\top}i_2) \\ \vdots & \ddots & \vdots \\ \mathbf{c}_1(x+M^{-\top}i_{\det M}) & \ldots & \mathbf{c}_s(x+M^{-\top}i_{\det M}) \end{bmatrix},$$

where, for each $j=1,2,\ldots,s$, the function $\mathbf{c}_j \in L_r^2[0,1)^d$.
It is easy to verify that the sequence $\big\{(\det M)\mathbf{c}_j(x)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is a biorthogonal sequence of $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ and therefore, it is a Riesz basis for $L_r^2[0,1)^d$. □

2.3 Generalized regular sampling in V_Φ^2

In this section we prove that expression (2.9) allows us to obtain $\mathbf{F} = \mathcal{T}_\Phi^{-1}f$ from the generalized samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$; as a consequence, applying the isomorphism $\mathcal{T}_\Phi$ we recover the function f in V_Φ^2.

Assume that the functions $\mathbf{g}_j$ given in (2.10) belong to $\in L_r^\infty[0,1)^d$ for $j=1,2,\ldots,s$; thus, $\mathbf{g}_j^\top(x)\mathbf{F}(x) \in L^2[0,1)^d$. Having in mind (2.7) and the expression

(2.9) for the generalized samples, we have that

$$
\begin{aligned}
&(\det M)\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x}\\
&=\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(\alpha)\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top x}\sum_{p\in\mathcal{N}(M^\top)}\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^{-\top}p}\\
&=\sum_{p\in\mathcal{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(\alpha)\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top (x+M^{-\top}p)}\\
&=\sum_{p\in\mathcal{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\langle \mathbf{F},\overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top \cdot}\rangle_{L^2_r[0,1)^d}\,\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top (x+M^{-\top}p)}\\
&=\sum_{p\in\mathcal{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}\Big(\int_{[0,1)^d}\sum_{k=1}^{r}F_k(y)g_{j,k}(y)\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top y}dy\Big)\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top (x+M^{-\top}p)}\\
&=\sum_{p\in\mathcal{N}(M^\top)}\sum_{k=1}^{r}F_k(x+M^{-\top}p)g_{j,k}(x+M^{-\top}p)\\
&=\sum_{p\in\mathcal{N}(M^\top)}\mathbf{g}_j^\top(x+M^{-\top}p)\,\mathbf{F}(x+M^{-\top}p)\,.
\end{aligned}
$$

Defining

$$
\mathbb{F}(x):=\big[\mathbf{F}^\top(x),\mathbf{F}^\top(x+M^{-\top}i_2),\dots,\mathbf{F}^\top(x+M^{-\top}i_{\det M})\big]^\top,
$$

the above equality allows us to writte, in matrix form, that $\mathbb{G}(x)\,\mathbb{F}(x)$ equals to

$$
(\det M)\Big[\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_1 f\big)(M\alpha)\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x},\dots,\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_s f\big)(M\alpha)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\Big]^\top.
$$

In order to recover the function $\mathbf{F}=\mathcal{T}_\Phi^{-1}f$, assume the existence of an $r\times s$ matrix $\mathbf{a}(x):=[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]$, with entries in $L^\infty[0,1)^d$, such that

$$
\big[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)\big]\,\mathbb{G}(x)=[\mathbb{I}_r,\mathbb{O}_{(\det M-1)r\times r}]\quad \text{a.e. in } [0,1)^d\,.
$$

If we left multiply $\mathbb{G}(x)\mathbb{F}(x)$ by $\mathbf{a}(x)$, we get

$$
\mathbf{F}(x)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)\,\mathbf{a}_j(x)\mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x}\quad \text{in } L^2_r[0,1)^d\,. \tag{2.19}
$$

Finally, the isomorphism $\mathcal{T}_\Phi$ gives

$$
f(t)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f\big)(M\alpha)(\mathcal{T}_\Phi\mathbf{a}_j)(t-M\alpha)\,,\quad t\in\mathbb{R}^d\,,
$$

where we have used the shifting property (2.6) and that the space V_Φ^2 is a RKHS. Much more can be said about the above sampling result. In fact, the following theorem holds:

Theorem 2.1. *Assume that the functions* $\mathbf{g}_j$ *given in* (2.10) *belong to* $L_r^\infty[0,1)^d$ *for each* $j=1,2,\dots,s$. *Let* $\mathbb{G}(x)$ *be the associated matrix defined in* $[0,1)^d$ *as in* (2.11), *and its related constant* $A_\mathbb{G}$. *The following statements are equivalents:*

(a) $A_\mathbb{G} > 0$.

(b) There exists an $r \times s$ *matrix* $\mathbf{a}(x) := \big[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)\big]$ *with columns* $\mathbf{a}_j \in L_r^\infty[0,1)^d$, *and satisfying*

$$\big[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)\big]\mathbb{G}(x) = \big[\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}\big] \quad \text{a.e. in } [0,1)^d\,. \tag{2.20}$$

(c) There exists a frame for V_Φ^2 *having the form* $\{S_{j,\mathbf{a}}(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ *such that for any* $f \in V_\Phi^2$

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)\, S_{j,\mathbf{a}}(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d)\,. \tag{2.21}$$

(d) There exists a frame $\{S_{j,\alpha}(\cdot)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ *for* V_Φ^2 *such that for any* $f \in V_\Phi^2$

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)\, S_{j,\alpha} \quad \text{in } L^2(\mathbb{R}^d)\,. \tag{2.22}$$

Proof. First we prove that (a) implies (b). As the determinant of the positive semidefinite matrix $\mathbb{G}^*(x)\mathbb{G}(x)$ is equal to the product of its eigenvalues, condition (a) implies that $\operatorname{ess\,inf}_{x\in\mathbb{R}^d} \det[\mathbb{G}^*(x)\mathbb{G}(x)] > 0$. Hence, the Moore-Penrose pseudo inverse matrix is given by $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$, a.e. in $[0,1)^d$, and it satisfies $\mathbb{G}^\dagger(x)\mathbb{G}(x) = \mathbb{I}_{r(\det M)}$. The first r rows of $\mathbb{G}^\dagger(x)$ form an $r\times s$ matrix $\big[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)\big]$ which satisfies (2.20). Moreover, the functions $\mathbf{a}_j(x)$, $j=1,\dots,s$, are essentially bounded since the condition $\operatorname{ess\,inf}_{x\in[0,1)^d} \det[\mathbb{G}^*(x)\mathbb{G}(x)] > 0$ holds.

Next, we prove that (b) implies (c). For $j=1,2,\dots,s$, let $\mathbf{a}_j(x)$ be a function in $L_r^\infty[0,1)^d$, and satisfying $\big[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)\big]\mathbb{G}(x) = \big[\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}\big]$. In (2.19) we have proved that, for each $\mathbf{F} = \mathcal{T}_\Phi^{-1}(f) \in L_r^2[0,1)^d$, we have the expansion

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \big(\mathcal{L}_j f\big)(M\alpha)\, \mathbf{a}_j(x) \mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x} \quad \text{in } L_r^2[0,1)^d\,,$$

from which

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \big(\mathcal{L}_j f\big)(M\alpha)\, S_{j,\mathbf{a}}(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d)\,,$$

where $S_{j,\mathbf{a}} := \mathcal{T}_\Phi \mathbf{a}_j$ for $j=1,2,\dots,s$. Since we have assumed that $\mathbf{g}_j \in L_r^\infty[0,1)^d$ for each $j=1,2,\dots,s$, the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a Bessel sequence in $L^2_r[0,1)^d$ by using part (b) in Lemma 2.3. The same argument proves that the sequence $\big\{(\det M)\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is also a Bessel sequence in $L^2_r[0,1)^d$. These two Bessel sequences satisfy for each $\mathbf{F}\in L^2_r[0,1)^d$

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\langle \mathbf{F}, \overline{\mathbf{g}_j}\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle \mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L^2_r[0,1)^d\,.$$

Hence, they are a pair of dual frames for $L^2_r[0,1)^d$ (see Proposition A.4). Since $\mathcal{T}_\Phi$ is an isomorphism, the sequence $\big\{S_{j,\mathbf{a}}(t-M\alpha)\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a frame for V^2_Φ; hence (b) implies (c). Statement (c) implies (d) trivially.

Assume condition (d), applying the isomorphism $\mathcal{T}_\Phi^{-1}$ to the expansion (2.22) we get

$$\mathbf{F}(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\langle \mathbf{F}, \overline{\mathbf{g}_j}\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle \mathcal{T}_\Phi^{-1}(S_{j,\alpha})(x) \quad \text{in } L^2_r[0,1)^d\,, \tag{2.23}$$

where $\big\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a frame for $L^2_r[0,1)^d$. By using Lemma 2.3, the sequence $\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a Bessel sequence; expansion (2.23) implies that is also a frame (see A.4). Hence, by using again Lemma 2.3, condition (a) holds. □

In the case that the functions $\mathbf{g}_j$, $j=1,2,\dots,s$, are continuous on $\mathbb{R}^d$ (for instance, if the sequences of generalized samples $\big\{\mathcal{L}_j\varphi_k(\alpha)\big\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d)$ for $1\leqslant j\leqslant s$ and $1\leqslant k\leqslant r$), the following corollary holds:

Corollary 2.1. *Assume that the functions $\mathbf{g}_j$, $j=1,2,\dots,s$, in* (2.10) *are continuous on $\mathbb{R}^d$. Then, the following assertions are equivalents:*

(a) rank $\mathbb{G}(x) = r(\det M)$ for all $x\in\mathbb{R}^d$.

(b) There exists a frame $\{S_{j,\mathbf{a}}(\cdot - rn)\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ for V^2_Φ satisfying the sampling formula (2.21).

Proof. Whenever the functions $\mathbf{g}_j$, $j=1,2,\dots,s$, are continuous on $\mathbb{R}^d$, condition $A_\mathbb{G} > 0$ is equivalent to that $\det\big[\mathbb{G}^*(x)\mathbb{G}(x)\big]\neq 0$ for all $x\in\mathbb{R}^d$. Indeed, if $\det\mathbb{G}^*(x)\mathbb{G}(x) > 0$ then the r first rows of the matrix

$$\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x),$$

give an $r\times s$ matrix $\mathbf{a}(x) = [\mathbf{a}_1(x),\mathbf{a}_2(x),\dots,\mathbf{a}_s(x)]$ satisfying statement (b) in Theorem 2.1, and therefore $A_\mathbb{G} > 0$.

The reciprocal follows from the fact that $\det\big[\mathbb{G}^*(x)\mathbb{G}(x)\big]\geqslant A_\mathbb{G}^{r(\det M)}$ for all $x\in\mathbb{R}^d$. Since $\det\big[\mathbb{G}^*(x)\mathbb{G}(x)\big]\neq 0$ is equivalent to rank $\mathbb{G}(x)=r(\det M)$ for all $x\in\mathbb{R}^d$, the result is a consequence of Theorem 2.1. □

The reconstruction functions $S_{j,\mathbf{a}}$, $j = 1, 2, \ldots, s$, are determined from the Fourier coefficients of the components of

$$\mathbf{a}_j(x) := [a_{1,j}(x), a_{2,j}(x), \ldots, a_{r,j}]^\top, \quad j = 1, 2, \ldots, s.$$

More specifically, if $\widehat{a}_{k,j}(\alpha) := \int_{[0,1)^d} a_{k,j}(x) \mathrm{e}^{2\pi \mathrm{i}\alpha^\top x} dx$ we get (see (2.5))

$$S_{j,\mathbf{a}}(t) = \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} \widehat{a}_{k,j}(\alpha)\varphi_k(t-\alpha), \quad t \in \mathbb{R}^d. \tag{2.24}$$

The Fourier transform in (2.24) gives $\widehat{S}_{j,\mathbf{a}}(x) = \sum_{k=1}^r a_{k,j}(x)\widehat{\varphi}_k(x)$.

Assume that the $r \times s$ matrix $\mathbf{a}(x) = \big[\mathbf{a}_1(x), \mathbf{a}_2(x), \ldots, \mathbf{a}_s(x)\big]$ satisfies (2.20). We consider the periodic extension of $a_{k,j}$, i.e., $a_{k,j}(x+\alpha) = a_{k,j}(x)$, $\alpha \in \mathbb{Z}^d$. For all $x \in [0,1)^d$, the $r(\det M) \times s$ matrix

$$\mathbb{A}^\top(x) := \begin{bmatrix} \mathbf{a}_1(x) & \mathbf{a}_2(x) & \cdots & \mathbf{a}_s(x) \\ \mathbf{a}_1(x + M^{-\top} i_2) & \mathbf{a}_2(x + M^{-\top} i_2) & \cdots & \mathbf{a}_s(x + M^{-\top} i_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_1(x + M^{-\top} i_{\det M}) & \mathbf{a}_2(x + M^{-\top} i_{\det M}) & \cdots & \mathbf{a}_s(x + M^{-\top} i_{\det M}) \end{bmatrix}$$

is a left inverse matrix of $\mathbb{G}(x)$, i.e., $\mathbb{A}^\top(x)\mathbb{G}(x) = \mathbb{I}_{r(\det M)}$.

Provided that condition (2.20) is satisfied, it can be easily checked that all matrices $\mathbf{a}(x)$ with entries in $L^\infty[0,1)^d$, and satisfying (2.20) correspond to the first r rows of the matrices of the form

$$\mathbb{A}^\top(x) = \mathbb{G}^\dagger(x) + \mathbb{U}(x)\big[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\big], \tag{2.25}$$

where $\mathbb{U}(x)$ is any $r(\det M) \times s$ matrix with entries in $L^\infty[0,1)^d$, and $\mathbb{G}^\dagger$ denotes the Moore-Penrose pseudo inverse $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$.

Notice that if $s = r(\det M)$ there exists a unique matrix $\mathbf{a}(x)$, given by the first r rows of $\mathbb{G}^{-1}(x)$; if $s > r(\det M)$ there are infinitely many solutions according to (2.25).

Moreover, the sequence $\big\{(\det M)\mathbf{a}_j^\dagger(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$, associated with the $r \times s$ matrix $[\mathbf{a}_1^\dagger(x), \mathbf{a}_2^\dagger(x), \ldots, \mathbf{a}_s^\dagger(x)]$ obtained from the r first rows of $\mathbb{G}^\dagger(x)$, gives precisely the canonical dual frame of the frame

$$\big\{\overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}.$$

Indeed, the frame operator $\mathcal{S}$ associated to $\big\{\overline{\mathbf{g}_j(\cdot)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\big\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is given by

$$\mathcal{S}\mathbf{F}(x) = \frac{1}{\det M}\big[\overline{\mathbf{g}_1(x)}, \overline{\mathbf{g}_2(x)}, \ldots, \overline{\mathbf{g}_s(x)}\big]\mathbb{G}(x)\mathbb{F}(x), \quad \mathbf{F} \in L_r^2[0,1)^d,$$

from which one gets

$$\mathcal{S}\big[(\det M)\mathbf{a}_j^\dagger(\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\big](x) = \overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\,,\ j=1,2,\ldots,s \text{ and } \alpha\in\mathbb{Z}^d\,.$$

Something more can be said in the case where $s = r(\det M)$:

Theorem 2.2. *Assume that the functions $\mathbf{g}_j$, $j = 1,2,\ldots,s$, given in* (2.10) *belong to $L_r^\infty[0,1)^d$ and $s = r(\det M)$. The following statements are equivalent:*

(a) $A_{\mathbb{G}} > 0$

(b) There exists a Riesz basis $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\ldots,s}$ for V_Φ^2 such that for any $f\in V_\Phi^2$, the expansion

$$f = (\det M)\sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\big(\mathcal{L}_j f\big)(M\alpha)\,S_{j,\alpha}\,, \tag{2.26}$$

holds in $L^2(\mathbb{R}^d)$.

In case the equivalent conditions are satisfied, necessarily $S_{j,\alpha}(t) = S_{j,\mathbf{a}}(t-M\alpha)$, $t\in\mathbb{R}^d$, where $S_{j,\mathbf{a}} = \mathcal{T}_\Phi(\mathbf{a}_j)$, $j=1,2,\ldots,s$, and the $r\times s$ matrix $\mathbf{a} := [\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_s]$ is formed with the r first rows of the inverse matrix $\mathbb{G}^{-1}$. The sampling functions $S_{j,\mathbf{a}}$, $j=1,2,\ldots,s$, satisfy the interpolation property $(\mathcal{L}_{j'}S_{j,\mathbf{a}})(M\alpha) = \delta_{j,j'}\delta_{\alpha,0}$, where $j,j' = 1,2,\ldots,s$ and $\alpha\in\mathbb{Z}^d$.

Proof. Assume that $A_{\mathbb{G}} > 0$; since $\mathbb{G}(x)$ is a square matrix, this implies that $\operatorname{ess\,inf}_{x\in\mathbb{R}^d}|\det\mathbb{G}(x)| > 0$. Therefore, the r first rows of $\mathbb{G}^{-1}(x)$ gives a solution of the equation

$$[\mathbf{a}_1(x),\ldots,\mathbf{a}_s(x)]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}]$$

with $\mathbf{a}_j\in L_r^\infty[0,1)^d$ for $j=1,2,\ldots,s$.

According to Theorem 2.1, the sequence

$$\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\ldots,s} := \{S_{j,\mathbf{a}}(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\ldots,s}\,,$$

where $S_{j,\mathbf{a}} = \mathcal{T}_\Phi(\mathbf{a}_j)$, satisfies the sampling formula (2.26). Moreover, the sequence

$$\big\{(\det M)\mathbf{a}_j(x)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\ldots,s} = \big\{\mathcal{T}_\Phi^{-1}S_{j,\mathbf{a}}(\cdot-M\alpha)\big\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\ldots,s}$$

is a frame for $L_r^2[0,1)^d$. Since $r(\det M) = s$, according to Lemma 2.3 it is a Riesz basis for $L_r^2[0,1)^d$. Hence, the sequence $\{S_{j,\mathbf{a}}(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\ldots,s}$ is a Riesz basis for V_Φ^2 and condition (b) is proved.

Conversely, assume now that $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\ldots,s}$ is a Riesz basis for V_Φ^2 satisfying (2.26). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition $(\mathcal{L}_{j'}S_{j,\alpha})(M\alpha') = \delta_{j,j'}\delta_{\alpha,\alpha'}$ holds for $j, j' = 1, 2, \ldots, s$ and $\alpha, \alpha' \in \mathbb{Z}^d$. Since $\mathcal{T}_\Phi^{-1}$ is an isomorphism, $\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is a Riesz basis for $L_r^2[0,1)^d$. Expanding the function $\overline{\mathbf{g}_{j'}(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha'^\top M^\top x}$ with respect to the dual basis of $\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$, denoted by $\{G_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$, we obtain

$$\begin{aligned}\overline{\mathbf{g}_{j'}(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha'^\top M^\top x} &= \sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\langle\overline{\mathbf{g}_{j'}(\cdot)}\mathrm{e}^{-2\pi\mathrm{i}\alpha'^\top M^\top\cdot}, \mathcal{T}_\Phi^{-1}S_{j,\alpha}\rangle_{L^2[0,1)^d}G_{j,\alpha}(x)\\ &= \sum_{\alpha\in\mathbb{Z}^d}\overline{\mathcal{L}_{j'}S_{j,\alpha}}(M\alpha')G_{j,\alpha}(x) = G_{j',\alpha'}(x)\,.\end{aligned}$$

Therefore, the sequence $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is the dual basis of the Riesz basis $\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$. In particular it is a Riesz basis for $L_r^2[0,1)^d$, which implies, according to Lemma 2.3, that $A_{\mathbf{G}} > 0$; this proves (a). Moreover, the sequence $\{\mathcal{T}_\Phi^{-1}S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is necessarily the unique dual basis of the Riesz basis $\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$. Therefore, this proves the uniqueness of the Riesz basis $\{S_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ for V_Φ^2 satisfying (2.26). □

2.3.1 Reconstruction functions with prescribed properties

A generalized sampling formula in the shift-invariant space V_Φ^2 as

$$f(t) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_jf)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)\,, \quad t\in\mathbb{R}^d\,, \tag{2.27}$$

can be read as a filter bank. Indeed, introducing the expression for the sampling functions

$$S_{j,\mathbf{a}}(t) = \sum_{\beta\in\mathbb{Z}^d}\sum_{k=1}^{r}\widehat{a}_{k,j}(\beta)\varphi_k(t-\beta)\,, \qquad t\in\mathbb{R}^d,$$

the change $\gamma := \beta + M\alpha$ in the summation's index gives

$$f(t) = (\det M)\sum_{k=1}^{r}\sum_{\gamma\in\mathbb{Z}^d}\left\{\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_jf)(M\alpha)\widehat{a}_{k,j}(\gamma-M\alpha)\right\}\varphi_k(t-\gamma)\,, \quad t\in\mathbb{R}^d\,.$$

Thus, the relevant data for the recovery of the signal $f\in V_\Phi^2$,

$$d_k(\gamma) := \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_jf)(M\alpha)\widehat{a}_{k,j}(\gamma-M\alpha)\,, \quad \gamma\in\mathbb{Z}^d\,, \quad 1\leqslant k\leqslant r\,,$$

is obtained by means of r filter banks whose impulse responses involve the Fourier coefficients of the entries of the $r\times s$ matrix $\mathbf{a} := \big[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s\big]$ in (2.20), and the input is given by the sampling data.

Notice that reconstruction functions $S_{j,\mathbf{a}}$ with compact support in the above sampling formula implies low computational complexities and avoids truncation errors. This occurs whenever the generators φ_k have compact support and the sum in (2.24) is finite. These sums are finite if and only if the entries of the $r \times s$ matrix $\mathbf{a}$ are trigonometric polynomials. In this case, all the filter banks involved in the reconstruction process are FIR (finite impulse response) filters.

Before to give a necessary and sufficient condition assuring compactly supported reconstruction functions $S_{j,\mathbf{a}}$ in formula (2.27), we introduce first some complex notation, more convenient for this study. We denote $\mathbf{z}^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_d^{\alpha_d}$ for $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$, and the d-torus by

$$\mathbb{T}^d := \{\mathbf{z} \in \mathbb{C}^d : |z_1| = |z_2| = \dots = |z_d| = 1\}.$$

For $1 \leqslant j \leqslant s$ and $1 \leqslant k \leqslant r$ we define

$$\mathsf{g}_{j,k}(\mathbf{z}) := \sum_{\mu \in \mathbb{Z}^d} \mathcal{L}_j \varphi_k(\mu) \mathbf{z}^{-\mu}, \quad \mathsf{g}_j^\top(\mathbf{z}) := \big(\mathsf{g}_{j,1}(\mathbf{z}), \mathsf{g}_{j,2}(\mathbf{z}), \dots, \mathsf{g}_{j,r}(\mathbf{z})\big),$$

and the $s \times r(\det M)$ matrix

$$\mathsf{G}(\mathbf{z}) := \Big[\mathsf{g}_j^\top(z_1 \mathrm{e}^{2\pi \mathrm{i} m_1^\top i_l}, \dots, z_d \mathrm{e}^{2\pi \mathrm{i} m_d^\top i_l})\Big]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r;\ l=1,2,\dots,\det M}} \tag{2.28}$$

where $m_1, \dots, m_d$ denote the columns of the matrix M^{-1}. Recall that $i_1, i_2, \dots, i_{\det M}$ in $\mathbb{Z}^d$ are the elements of $\mathcal{N}(M^\top)$ defined in (2.8). Note also that for the values $x = (x_1, \dots, x_d) \in [0,1)^d$ and $\mathbf{z} = (\mathrm{e}^{2\pi \mathrm{i} x_1}, \dots, \mathrm{e}^{2\pi \mathrm{i} x_d}) \in \mathbb{T}^d$ we have $\mathbb{G}(x) = \mathsf{G}(\mathbf{z})$.

Provided that the functions $\mathbf{g}_j$ are continuous on $\mathbb{R}^d$, Corollary 2.1 can be reformulated as follows: There exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z}) = \big[\mathsf{a}_1(\mathbf{z}), \dots, \mathsf{a}_s(\mathbf{z})\big]$ with entries essentially bounded in the torus $\mathbb{T}^d$ and satisfying

$$\mathsf{a}(\mathbf{z})\mathsf{G}(\mathbf{z}) = [\mathbb{I}_r, \mathbb{O}_{(\det M - 1)r \times r}] \quad \text{for all } \mathbf{z} \in \mathbb{T}^d \tag{2.29}$$

if and only if

$$\operatorname{rank} \mathsf{G}(\mathbf{z}) = r(\det M) \quad \text{for all } \mathbf{z} \in \mathbb{T}^d. \tag{2.30}$$

Denoting the columns of the matrix $\mathsf{a}(\mathbf{z})$ as $\mathsf{a}_j^\top(\mathbf{z}) = \big(\mathsf{a}_{1,j}(\mathbf{z}), \dots, \mathsf{a}_{r,j}(\mathbf{z})\big)$, $j = 1, 2, \dots, s$, the corresponding reconstruction functions $S_{j,\mathsf{a}}$ in sampling formula (2.27) are

$$S_{j,\mathsf{a}}(t) = \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^{r} \widehat{\mathsf{a}}_{k,j}(\alpha) \varphi(t - \alpha), \quad t \in \mathbb{R}^d, \tag{2.31}$$

where $\widehat{\mathsf{a}}_{k,j}(\alpha)$, $\alpha \in \mathbb{Z}^d$, are the Laurent coefficients of the functions $\mathsf{a}_{k,j}(\mathbf{z})$, that is,

$$\mathsf{a}_{k,j}(\mathbf{z}) = \sum_{\alpha \in \mathbb{Z}^d} \widehat{\mathsf{a}}_{k,j}(\alpha) \mathbf{z}^{-\alpha}. \tag{2.32}$$

Note that, in order to obtain compactly supported reconstruction functions $S_{j,\mathsf{a}}$ in (2.27), we need an $r \times s$ matrix $\mathsf{a}(\mathbf{z})$ whose entries are Laurent polynomials, i.e., the sum in (2.32) is finite. The following result, which proof can be found in [46] under minor changes, holds:

Theorem 2.3. *Assume that the generators φ_k and the functions $\mathcal{L}_j\varphi_k$, $1 \leqslant k \leqslant r$ and $1 \leqslant j \leqslant s$, have compact support. Then, there exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z})$ whose entries are Laurent polynomials and satisfying* (2.29) *if and only if*

$$\textit{rank}\ \mathsf{G}(\mathbf{z}) = r(\det M) \ \ \textit{for all} \ \ \mathbf{z} \in (\mathbb{C}\backslash\{0\})^d .$$

The reconstruction functions $S_{j,\mathsf{a}}$, $j = 1, 2, \dots, s$, obtained from such matrix $\mathsf{a}(\mathbf{z})$ through Eq. (2.31) *have compact support.*

From one of these $r \times s$ matrices, say $\widetilde{\mathsf{a}}(\mathbf{z}) = [\widetilde{\mathsf{a}}_1(\mathbf{z}), \dots, \widetilde{\mathsf{a}}_s(\mathbf{z})]$, we can get all of them. Indeed, it is easy to check that they are given by the r first rows of the $r(\det M) \times s$ matrices of the form

$$\mathsf{A}(\mathbf{z}) = \widetilde{\mathsf{A}}(\mathbf{z}) + \mathsf{U}(\mathbf{z})\big[\mathbb{I}_s - \mathsf{G}(\mathbf{z})\widetilde{\mathsf{A}}(\mathbf{z})\big] , \tag{2.33}$$

where

$$\widetilde{\mathsf{A}}(\mathbf{z}) := \Big[\widetilde{\mathsf{a}}_j(z_1 e^{2\pi i m_1^\top i_l}, \dots, z_d e^{2\pi i m_d^\top i_l})\Big]_{\substack{k=1,2,\dots,r;\ l=1,2,\dots,\det M \\ j=1,2,\dots,s}} ,$$

and $\mathsf{U}(\mathbf{z})$ is any $r(\det M) \times s$ matrix with Laurent polynomial entries. Remember that $m_1, \dots, m_d$ denote the columns of the matrix M^{-1}, and $i_1, \dots, i_{\det M}$ the elements of $\mathcal{N}(M^\top)$ defined in (2.8).

Next we study the existence of reconstruction functions $S_{j,\mathsf{a}}$, $j = 1, 2, \dots, s$, in (2.27) having exponential decay; it means that there exist constants $C > 0$ and $q \in (0, 1)$ such that $|S_{j,\mathsf{a}}(t)| \leqslant Cq^{|t|}$ for each $t \in \mathbb{R}^d$. In so doing, we introduce the algebra $\mathcal{H}(\mathbb{T}^d)$ of all holomorphic functions in a neighborhood of the d-torus $\mathbb{T}^d$. Note that the elements in $\mathcal{H}(\mathbb{T}^d)$ are characterized as admitting a Laurent series where the sequence of coefficients decays exponentially fast [66].

The following theorem, which proof can be found in [46] under minor changes, holds:

Theorem 2.4. *Assume that the generators φ_k and the functions $\mathcal{L}_j\varphi_k$, $j = 1, 2, \dots, s$ and $k = 1, 2, \dots, r$, have exponential decay. Then, there exists an $r \times s$ matrix $\mathsf{a}(\mathbf{z}) = [\mathsf{a}_1(\mathbf{z}), \dots, \mathsf{a}_s(\mathbf{z})]$ with entries in $\mathcal{H}(\mathbb{T}^d)$ and satisfying* (2.29) *if and only if*

$$\textit{rank}\ \mathsf{G}(\mathbf{z}) = r(\det M) \quad \textit{for all}\ \mathbf{z} \in \mathbb{T}^d .$$

In this case, all of such matrices $\mathsf{a}(\mathbf{z})$ are given as the first r rows of a $r(\det M) \times s$ matrix $\mathsf{A}(\mathbf{z})$ of the form

$$\mathsf{A}(\mathbf{z}) = \mathsf{G}^\dagger(\mathbf{z}) + \mathsf{U}(\mathbf{z})\big[\mathbb{I}_s - \mathsf{G}(\mathbf{z})\mathsf{G}^\dagger(\mathbf{z})\big] , \tag{2.34}$$

where $\mathsf{U}(\mathbf{z})$ *denotes any* $r(\det M) \times s$ *matrix with entries in the algebra* $\mathcal{H}(\mathbb{T}^d)$ *and* $\mathsf{G}^\dagger(\mathbf{z}) := \big[\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})\big]^{-1}\mathsf{G}^*(\mathbf{z})$. *The corresponding reconstruction functions* $S_{j,\mathsf{a}}$, $j = 1, 2, \dots, s$, *given by* (2.31) *have exponential decay.*

2.3.2 Some illustrative examples

We include here some examples illustrating Theorem 2.3, a particular case of Theorem 2.1, by taking B-splines as generators; they certainly are important for practical purposes [108].

The case $d = 1$, $r = 1$, $M = 2$ and $s = 3$

Let $N_3(t) := \chi_{[0,1)} * \chi_{[0,1)} * \chi_{[0,1)}(t)$ be the quadratic B-spline, where $\chi_{[0,1)}$ denotes the characteristic function of the interval $[0, 1)$, and let $\mathcal{L}_j$, $j = 1, 2, 3$, be the systems:

$$\mathcal{L}_1 f(t) = f(t); \quad \mathcal{L}_2 f(t) = f(t + \frac{2}{3}) \quad \text{and} \quad \mathcal{L}_3 f(t) = f(t + \frac{4}{3})\,.$$

Since the functions $\mathcal{L}_j N_3$, $j = 1, 2, 3$, have compact support, then the entries of the 3×2 matrix $\mathsf{G}(z)$ in (2.28) are Laurent polynomials and we can try to search a vector $\mathbf{a}(z) := [\mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z)]$ satisfying (2.29) with Laurent polynomials entries also. This implies reconstruction functions $S_{j,\mathbf{a}}$, $j = 1, 2, 3$, with compact support. Proceeding as in [41] we obtain that any function $f \in V^2_{N_3}$ can be recovered through the sampling formula:

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{3} \mathcal{L}_j f(2n)\, S_{j,\mathbf{a}}(t - 2n)\,, \quad t \in \mathbb{R}\,,$$

where the reconstruction functions, according to (2.31), are given by

$$\begin{aligned} S_{1,\mathbf{a}}(t) &= \frac{1}{16}\big[N_3(t+3) - 3N_3(t+2) - 3N_3(t+1) + N_3(t)\big], \\ S_{2,\mathbf{a}}(t) &= \frac{1}{16}\big[27N_3(t+1) - 9N_3(t)\big], \\ S_{3,\mathbf{a}}(t) &= \frac{1}{16}\big[-9N_3(t+1) + 27N_3(t)\big]\,, \quad t \in \mathbb{R}\,. \end{aligned}$$

The case $d = 1$, $r = 2$, $M = 1$ and $s = 3$

Consider the Hermite cubic splines defined as

$$\varphi_1(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1, 0] \\ (1-t)^2(1+2t), & t \in [0, 1] \\ 0, & |t| > 1 \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} (t+1)^2 t, & t \in [-1, 0] \\ (1-t)^2 t, & t \in [0, 1] \\ 0, & |t| > 1 \end{cases}\,.$$

They are stable generators for the space $V^2_{\varphi_1,\varphi_2}$ (see Ref. [29]). Consider the sampling period $M = 1$ and the systems $\mathcal{L}_j$, $j = 1, 2, 3$, defined by

$$\mathcal{L}_1 f(t) := \int_t^{t+1/3} f(u)du\,, \quad \mathcal{L}_2 f(t) := \mathcal{L}_1 f\big(t+\frac{1}{3}\big), \quad \mathcal{L}_3 f(t) := \mathcal{L}_1 f\big(t+\frac{2}{3}\big)\,.$$

Since the functions $\mathcal{L}_j\varphi_k$, $j = 1, 2, 3$ and $k = 1, 2$, have compact support, then the entries of the 3×2 matrix $\mathsf{G}(z)$ in (2.28) are Laurent polynomials and we can try to search an 2×3 matrix $\mathbf{a}(z) := \big[\mathbf{a}_1(z), \mathbf{a}_2(z), \mathbf{a}_3(z)\big]$ satisfying (2.29) with Laurent polynomials entries also. This leads to reconstruction functions $S_{j,\mathbf{a}}$, $j = 1, 2, 3$, with compact support. Proceeding as in [43] we obtain in $V^2_{\varphi_1,\varphi_2}$ the following sampling formula:

$$f(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{3} \mathcal{L}_j f(n)\, S_{j,\mathbf{a}}(t-n)\,, \quad t \in \mathbb{R}\,,$$

where the sampling functions, according to (2.31), are

$$\begin{aligned}
S_{1,\mathbf{a}}(t) &:= \frac{85}{44}\varphi_1(t) + \frac{1}{11}\varphi_1(t-1) + \frac{85}{4}\varphi_2(t) - \varphi_2(t-1)\,,\\
S_{2,\mathbf{a}}(t) &:= \frac{-23}{44}\varphi_1(t) - \frac{23}{44}\varphi_1(t-1) - \frac{23}{4}\varphi_2(t) + \frac{23}{4}\varphi_2(t-1)\,,\\
S_{3,\mathbf{a}}(t) &:= \frac{1}{11}\varphi_1(t) + \frac{85}{44}\varphi_1(t-1) + \varphi_2(t) - \frac{85}{4}\varphi_2(t-1)\,, \quad t \in \mathbb{R}\,.
\end{aligned}$$

2.3.3 L^2-approximation properties

Consider an $r \times s$ matrix $\mathbf{a}(x) := \big[\mathbf{a}_1(x), \mathbf{a}_2(x), \dots, \mathbf{a}_s(x)\big]$ with entries $a_{k,j} \in L^\infty[0,1)^d$, $1 \leqslant k \leqslant r$, $1 \leqslant j \leqslant s$, and satisfying (2.20). Let $S_{j,\mathbf{a}}$ be the associated reconstruction functions, $j = 1, 2, \dots, s$, given in Theorem 2.1. The aim of this section is to show that if the set of generators Φ satisfies the Strang-Fix conditions of order ℓ, then the scaled version of the sampling operator

$$\Gamma_{\mathbf{a}} f(t) := \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t-M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

gives L^2- approximation order ℓ for any smooth function f (in a Sobolev space). In so doing, we take advantage of the good approximation properties of the scaled space $\sigma_{1/h}V_\Phi^2$, where for $h > 0$ we are using the notation: $\sigma_h f(t) := f(ht)$, $t \in \mathbb{R}^d$.

The set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ is said to satisfy the Strang-Fix conditions of order ℓ if there exist r finitely supported sequences $b_k : \mathbb{Z}^d \to \mathbb{C}$ such that the function $\varphi(t) = \sum_{k=1}^r \sum_{\alpha\in\mathbb{Z}^d} b_k(\alpha)\varphi_k(t-\alpha)$ satisfies the Strang-Fix conditions of order ℓ, i.e.,

$$\widehat{\varphi}(0) \neq 0, \quad D^\beta\widehat{\varphi}(\alpha) = 0, \quad |\beta| < \ell, \quad \alpha \in \mathbb{Z}^d\backslash\{0\}\,. \tag{2.35}$$

We denote by

$$W_2^\ell(\mathbb{R}^d) := \big\{ f \text{ measurable} \,:\, \|D^\gamma f\|_2 < \infty\,,\, |\gamma| \leqslant \ell \big\}$$

the usual Sobolev space, and by $|f|_{\ell,2} := \sum_{|\beta|=\ell} \|D^\beta f\|_2$ the corresponding seminorm of a function $f \in W_2^\ell(\mathbb{R}^d)$. When $2\ell > d$ we identify $f \in W_2^\ell(\mathbb{R}^d)$ with its continuous choice (see [1]).

It is well-known that if Φ satisfies the Strang-Fix conditions of order ℓ, and the generators φ_k satisfy a suitable decay condition, the space V_Φ^2 provides L^2-approximation order ℓ for any function f regular enough. For instance, Lei et al. proved in [75, Theorem 5.2] the following result: If a set $\Phi = \{\varphi_k\}_{k=1}^r$ of stable generators satisfies the Strang-Fix conditions of order ℓ, and the decay condition $\varphi_k(t) = O\big([1+|t|]^{-d-\ell-\epsilon}\big)$ for each $k = 1, 2, \ldots, r$ and some $\epsilon > 0$, then, for any $f \in W_2^\ell(\mathbb{R}^d)$, there exists a function $f_h \in \sigma_{1/h} V_\Phi^2$ such that

$$\|f - f_h\|_2 \leqslant C\, |f|_{\ell,2}\, h^\ell\,, \tag{2.36}$$

where the constant C does not depend on h and f.

In this section we assume that all the systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, are of type (b), i.e., $\mathcal{L}_j f = f * \mathsf{h}_j$, belonging the impulse response h_j to the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$. Recall that a Lebesgue measurable function $\mathsf{h} : \mathbb{R}^d \longrightarrow \mathbb{C}$ belongs to the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$ if

$$|\mathsf{h}|_2 := \bigg(\int_{[0,1)^d} \Big(\sum_{\alpha \in \mathbb{Z}^d} |\mathsf{h}(t-\alpha)| \Big)^2 dt \bigg)^{1/2} < \infty\,.$$

Notice that $\mathcal{L}^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$ and $\mathsf{h} \in \mathcal{L}^2(\mathbb{R}^d)$, the following inequality holds: $\big\|\{\mathsf{h} * f(\alpha)\}_{\alpha\in\mathbb{Z}^d}\big\|_2 \leqslant |\mathsf{h}|_2\, \|f\|_2$ (see [66, Theorem 3.1]); thus the sequence of generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ belongs to $\ell^2(\mathbb{Z}^d)$ for any $f \in L^2(\mathbb{R}^d)$.

First we note that the operator $\Gamma_{\mathbf{a}} : \big(L^2(\mathbb{R}^d), \|\cdot\|_2\big) \longrightarrow \big(V_\Phi^2, \|\cdot\|_2\big)$ given by

$$(\Gamma_{\mathbf{a}} f)(t) := (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

is a well-defined bounded operator onto V_Φ^2. Besides, $\Gamma_{\mathbf{d}} f = f$ for all $f \in V_\Phi^2$.

Under appropriate hypotheses we prove that the scaled operator $\Gamma_{\mathbf{a}}^h := \sigma_{1/h} \Gamma_{\mathbf{a}} \sigma_h$ approximates, in the L^2-norm sense, any function f in the Sobolev space $W_2^\ell(\mathbb{R}^d)$ as $h \to 0^+$. Specifically we have:

Theorem 2.5. *Assume $2\ell > d$ and that all the systems $\mathcal{L}_j$ satisfy $\mathcal{L}_j f = f * \mathsf{h}_j$ with $\mathsf{h}_j \in \mathcal{L}^2(\mathbb{R}^d)$, $j = 1, \ldots, s$. Then,*

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \leqslant (1 + \|\Gamma_{\mathbf{a}}\|) \inf_{g \in \sigma_{1/h} V_\Phi^2} \|f - g\|_2, \quad f \in W_2^\ell(\mathbb{R}^d),$$

where $\|\Gamma_{\mathbf{a}}\|$ denotes the norm of the sampling operator $\Gamma_{\mathbf{a}}$. If the set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ satisfies the Strang-Fix conditions of order ℓ and, for each $k = 1, 2, \ldots, r$, the decay condition $\varphi_k(t) = O\big([1+|t|]^{-d-\ell-\epsilon}\big)$ for some $\epsilon > 0$, then

$$\|f - \Gamma_{\mathbf{a}}^h f\|_p \leqslant C\,|f|_{\ell,2}\,h^\ell\ , \quad \textit{for all } f \in W_2^\ell(\mathbb{R}^d),$$

where the constant C does not depend on h and f.

Proof. Using that $\Gamma_{\mathbf{a}}^h g = g$ for each $g \in \sigma_{1/h}V_\Phi^2$ then, for each $f \in L^2(\mathbb{R}^d)$ and $g \in \sigma_{1/h}V_\Phi^2$, Lebesgue's Lemma [31, p. 30] gives

$$\begin{aligned}\|f - \Gamma_{\mathbf{a}}^h f\|_2 &\leqslant \|f-g\|_2 + \|\Gamma_{\mathbf{a}}^h g - \Gamma_{\mathbf{a}}^h f\|_2 \\ &\leqslant (1+\|\Gamma_{\mathbf{a}}\|) \inf_{g\in\sigma_{1/h}V_\Phi^2} \|f-g\|_2\,,\end{aligned}$$

where we have used that $\|\Gamma_{\mathbf{a}}^h\| = \|\Gamma_{\mathbf{a}}\|$ for $h > 0$. Now, for each $f \in W_2^\ell(\mathbb{R}^d)$ and $h > 0$, there exists a function $f_h \in \sigma_{1/h}V_\Phi^2$ such that (2.36) holds, from which we obtain the desired result. □

More results on approximation by means of generalized sampling formulas can be found in Refs. [44, 48].

2.4 Irregular sampling in V_Φ^2: time-jitter error

Given an error sequence $\varepsilon := \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ in $\mathbb{R}^d$, this section aims to study when it is possible to recover any function $f \in V_\Phi^2$ from the sequence of perturbed samples $\{(\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$. Keeping in mind expression (2.9) for the systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, for $f = \mathcal{T}_\Phi \mathbf{F} \in V_\Phi^2$ we have

$$(\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha}) = \big\langle \mathbf{F}, (\overline{\mathbf{Z}\mathcal{L}_j\Phi})(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\big\rangle_{L_r^2[0,1)^d}\,, \quad \alpha \in \mathbb{Z}^d\,, \quad (2.37)$$

where we have used that

$$(Z\mathcal{L}_j\Phi)(M\beta + \varepsilon_{j,\beta}, x) = (Z\mathcal{L}_j\Phi)(\varepsilon_{j,\beta}, x)\mathrm{e}^{2\pi\mathrm{i}\beta^\top M^\top x}\,, \qquad \beta \in \mathbb{Z}^d\,.$$

Equation (2.37) leads us to study the frame property of the perturbed sequence

$$\{(\overline{Z\mathcal{L}_j\Phi})(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top \cdot}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s} \qquad (2.38)$$

in $L_r^2[0,1)^d$. On the other hand, we know that, whenever $0 < A_{\mathbf{G}} \leqslant B_{\mathbf{G}} < \infty$, the sequence (2.38) is a frame for $L_r^2[0,1)^d$ with optimal frame bounds $A_{\mathbf{G}}/(\det M)$ and $B_{\mathbf{G}}/(\det M)$. In the case of $s = r(\det M)$, the above sequence is a Riesz basis for $L_r^2[0,1)^d$.

2.4.1 The perturbed sequence

One possibility is to use frame perturbation theory in order to find the suitable error sequences for which the sequence (2.38) is a frame for $L^2_r[0,1)^d$. Given an error sequence $\varepsilon := \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s} \subset \mathbb{R}^d$ we define on $\ell^2_r(\mathbb{Z}^d)$ the operator $D_\varepsilon = \big[D_{\varepsilon,1},\dots,D_{\varepsilon,s}\big]$, where

$$D_{\varepsilon,j}\, c := \Big\{ \sum_{k=1}^{r} \sum_{\beta\in\mathbb{Z}^d} \big[\mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha-\beta)\big] c_{k\beta} \Big\}_{\alpha\in\mathbb{Z}^d}$$

for each $c = \big(\{c_{1\beta}\}_{\beta\in\mathbb{Z}^d},\dots,\{c_{r\beta}\}_{\beta\in\mathbb{Z}^d}\big) \in \ell^2_r(\mathbb{Z}^d)$. The operator norm is defined as usual

$$\|D_\varepsilon\| := \sup_{c\in\ell^2_r(\mathbb{Z}^d)\setminus\{0\}} \frac{\|D_\varepsilon\, c\|_{\ell^2_s(\mathbb{Z}^d)}}{\|c\|_{\ell^2_r(\mathbb{Z}^d)}}\,,$$

where $\|D_\varepsilon\, c\|^2_{\ell^2_s(\mathbb{Z}^d)} := \sum_{j=1}^{s} \|D_{\varepsilon,j}\, c\|^2_{\ell^2(\mathbb{Z}^d)}$ for each $c \in \ell^2(\mathbb{Z}^d)$.

Theorem 2.6. *Assume that $g_j \in L^\infty_r[0,1)^d$ for $j = 1,2,\dots,s$ with $A_{\mathbf{G}} > 0$. If the error sequence $\varepsilon := \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ satisfies the inequality $\|D_\varepsilon\|^2 < A_{\mathbf{G}}/(\det M)$, then there exists a frame $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ for V^2_Φ such that, for any $f \in V^2_\Phi$*

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \big(\mathcal{L}_j f\big)(M\alpha+\varepsilon_{j,\alpha}) S^\varepsilon_{j,\alpha}(t)\,, \quad t \in \mathbb{R}^d\,, \tag{2.39}$$

where the convergence of the series is in the $L^2(\mathbb{R}^d)$-sense, absolute and uniform on $\mathbb{R}^d$. Moreover, when $s = r(\det M)$ the sequence $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a Riesz basis for V^2_Φ, and the interpolation property $(\mathcal{L}_l S^\varepsilon_{j,\alpha})(M\beta+\varepsilon_{j,\beta}) = \delta_{j,l}\delta_{\alpha,\beta}$ holds.

Proof. The sequence $\{(\overline{Z\mathcal{L}_j\Phi})(0,\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ is a frame (a Riesz basis if $r(\det M)$=s) for $L^2_r[0,1)^d$ with frame (Riesz) bounds $A_{\mathbf{G}}/(\det M)$ and $B_{\mathbf{G}}/(\det M)$. For any

$$F(x) = \Big(\sum_{\gamma\in\mathbb{Z}^d} c_{1\gamma}\mathrm{e}^{-2\pi\mathrm{i}\gamma^\top x},\dots, \sum_{\gamma\in\mathbb{Z}^d} c_{r\gamma}\mathrm{e}^{-2\pi\mathrm{i}\gamma^\top x}\Big) \text{ in } L^2[0,1)^d$$

we have

$$\begin{aligned}
&\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\Big\langle\overline{(Z\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}-\overline{(Z\mathcal{L}_j\Phi)}(0,\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot},\,F(\cdot)\Big\rangle_{L_r^2[0,1)^d}\Big|^2\\
&=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\Big\langle\sum_{\beta\in\mathbb{Z}^d}[\overline{\mathcal{L}_j\Phi}(\beta+\varepsilon_{j,\alpha})-\overline{\mathcal{L}_j\Phi}(\beta)]\mathrm{e}^{-2\pi\mathrm{i}(M\alpha-\beta)^\top\cdot},\,F(\cdot)\Big\rangle_{L^2[0,1)^d}\Big|^2\\
&=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^d}[\overline{\mathcal{L}_j\varphi_k}(\beta+\varepsilon_{j,\alpha})-\overline{\mathcal{L}_j\varphi_k}(\beta)]\,\overline{c}_{k\,M\alpha-\beta}\Big|^2\\
&=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\Big|\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^d}[\mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha})-\mathcal{L}_j\varphi_k(M\alpha-\beta)]c_{k\beta}\Big|^2\\
&=\sum_{j=1}^{s}\|D_{\varepsilon,j}c\|^2_{\ell^2(\mathbb{Z}^d)}=\|D_\varepsilon c\|^2_{\ell_s^2(\mathbb{Z}^d)}\leqslant\|D_\varepsilon\|^2\|c\|^2_{\ell_r^2(\mathbb{Z}^d)}=\|D_\varepsilon\|^2\|F\|^2_{L_r^2[0,1)^d}\,.
\end{aligned}$$

By using Lemma A.8, the sequence $\{\overline{(Z\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$ is a frame for $L_r^2[0,1)^d$ (a Riesz basis if $r(\det M)=s$). Let $\{h_{j,\alpha}^\varepsilon\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F\in L_r^2[0,1)^d$ we have

$$\begin{aligned}
\mathbf{F}&=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big\langle\mathbf{F}(\cdot),\overline{(Z\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\big\rangle_{L_r^2[0,1)^d}\,h_{j,\alpha}^\varepsilon\\
&=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_jf\big)(M\alpha+\varepsilon_{j,\alpha})h_{j,\alpha}^\varepsilon\quad\text{ in }\;L_r^2[0,1)^d\,.
\end{aligned}$$

Applying the isomorphism $\mathcal{T}_\Phi$, one gets (2.39) in $L^2(\mathbb{R}^d)$ where $S_{j,\alpha}^\varepsilon=\mathcal{T}_\Phi h_{j,\alpha}^\varepsilon$. Since $\mathcal{T}_\Phi$ is an isomorphism between $L_r^2[0,1)^d$ and V_Φ^2, the sequence $\{S_{j,\alpha}^\varepsilon\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$ is a frame for V_Φ^2 (a Riesz basis if $r(\det M)=s$).

Pointwise convergence in the sampling series is absolute due to the unconditional character of a frame. The uniform convergence on $\mathbb{R}^d$ is a consequence of the reproducing property in V_Φ^2. The interpolatory property in the case $r(\det M)=s$ follows from the uniqueness of the coefficients with respect to a Riesz basis. □

Following the techniques in [42] (see also Refs. [34, 104]), whenever the generator φ and the impulse responses of the systems $\mathcal{L}_j$, $j=1,2,\dots,s$, are compactly supported one could obtain a bound for $\|D_\varepsilon\|$ in terms of $\delta:=\sup_{j,\alpha}\|\varepsilon_{j,\alpha}\|_\infty$. Finally, it is worth to mention the recent related Refs. [78, 118].

The next result yields a uniform bound of the norm $\|D_\varepsilon\|$ regardless the sequence ε such that $\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d}$ is in $[\alpha_j,\beta_j]^d\subset[-r,r]^d$, for each $j=1,2,\dots,s$.

Theorem 2.7. *For any sequence ε such that $\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d} \subset [\alpha_j,\beta_j]^d \subset [-r,r]^d$ for each $j=1,2,\ldots,s$ the following inequality holds*

$$\|D_\varepsilon\|^2 \leqslant \sum_{j=1}^{s} \Lambda_j \Gamma_j\,, \tag{2.40}$$

where, for each $j=1,2,\ldots,s$, the constants Λ_j and Γ_j are given by

$$\Lambda_j := \sup_{\substack{\beta\in\mathcal{N}(M)\\ \{d_k\}\subset[\alpha_j,\beta_j]^d}} \sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(M\alpha+\beta+d_k) - \mathcal{L}_j\varphi_k(M\alpha+\beta)|\,,$$

$$\Gamma_j := \sup_{d\in[\alpha_j,\beta_j]^d} \sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(\alpha+d) - \mathcal{L}_j\varphi_k(\alpha)|\,.$$

$\mathcal{N}(M) = \{i_1 = 0, i_2, \ldots, i_{\det M}\}$ *defined as in section 2.2.2.*

Proof. Suppose that $\sum_{j=1}^{s} \Lambda_j\Gamma_j < \infty$; otherwise the result obviously holds. Denoting

$$d_{\alpha,\beta}^{(jk)} := \mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha-\beta)\,,$$

for a fixed β, let β' such that $M\beta'-\beta \in \mathcal{N}(M)$; without lose of generality, set $M\beta'-\beta = i_p$. Thus,

$$\begin{aligned}
\sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} |d_{\alpha,\beta}^{(jk)}| &= \sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha-\beta)| \\
&= \sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(M\alpha+i_p-M\beta'+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha+i_p-M\beta')| \\
&= \sum_{k=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(M\alpha+i_p+\varepsilon_{j,\alpha+\beta'}) - \mathcal{L}_j\varphi_k(M\alpha+i_p)| \leqslant \Lambda_j\,.
\end{aligned}$$

For a fixed α,

$$\begin{aligned}
\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^d} |d_{\alpha,\beta}^{(jk)}| &= \sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha-\beta)| \\
&= \sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(\beta+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(\beta)| \leqslant \Gamma_j\,.
\end{aligned}$$

For any $c = \big(\{c_{1\alpha}\}_{\alpha\in\mathbb{Z}^d}, \dots, \{c_{r\alpha}\}_{\alpha\in\mathbb{Z}^d}\big) \in \ell_r^2(\mathbb{Z}^d)$ we have

$$
\begin{aligned}
\|D_\varepsilon c\|^2_{\ell_s^2(\mathbb{Z})} &= \sum_{j=1}^s \|D_{\varepsilon j} c\|^2_{\ell^2(\mathbb{Z}^d)} \\
&= \sum_{j=1}^s \sum_{\alpha\in\mathbb{Z}^d} \Big| \sum_{k=1}^r \sum_{\beta\in\mathbb{Z}^d} \big[\mathcal{L}_j\varphi_k(M\alpha+\beta+d_k) - \mathcal{L}_j\varphi_k(M\alpha+\beta)\big] c_{k\beta}\Big|^2 \\
&= \sum_{j=1}^s \sum_{\alpha\in\mathbb{Z}^d} \Big| \sum_{k=1}^r \sum_{\beta\in\mathbb{Z}^d} d_{\alpha,\beta}^{(jk)} c_{k\beta}\Big|^2 \leqslant \sum_{j=1}^s \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^r \sum_{k'=1}^r \sum_{\beta,\beta'\in\mathbb{Z}^d} |d_{\alpha,\beta}^{(jk)} c_{k\beta} \overline{d}_{\alpha,\beta'}^{(jk')} \overline{c}_{k'\beta'}| \\
&\leqslant \sum_{j=1}^s \sum_{k=1}^r \sum_{k'=1}^r \sum_{\beta,\beta'\in\mathbb{Z}^d} |c_{k\beta}||c_{k'\beta'}| \sum_{\alpha\in\mathbb{Z}^d} |d_{\alpha,\beta}^{(jk)}||d_{\alpha,\beta'}^{(jk')}| \\
&\leqslant \sum_{j=1}^s \sum_{k=1}^r \sum_{k'=1}^r \sum_{\beta,\beta'\in\mathbb{Z}^d} \frac{|c_{k\beta}|^2 + |c_{k'\beta'}|^2}{2} \sum_{\alpha\in\mathbb{Z}^d} |d_{\alpha,\beta}^{(jk)}||d_{\alpha,\beta'}^{(jk')}| \\
&= \sum_{j=1}^s \sum_{k=1}^r \sum_{\beta\in\mathbb{Z}^d} |c_{k\beta}|^2 \sum_{\alpha\in\mathbb{Z}^d} |d_{\alpha,\beta}^{(jk)}| \sum_{k'=1}^r \sum_{\beta'\in\mathbb{Z}^d} |d_{\alpha,\beta'}^{(jk')}| \\
&\leqslant \sum_{j=1}^s \sum_{k=1}^r \sum_{\beta\in\mathbb{Z}^d} |c_{k\beta}|^2 \sum_{\alpha\in\mathbb{Z}^d} |d_{\alpha,\beta}^{(jk)}| \varGamma_j \\
&\leqslant \sum_{j=1}^s \sum_{k=1}^r \sum_{\beta\in\mathbb{Z}^d} |c_{k\beta}|^2 \varLambda_j \varGamma_j \leqslant \Big(\sum_{j=1}^s \varLambda_j\varGamma_j\Big) \sum_{k=1}^r \sum_{\beta\in\mathbb{Z}^d} |c_{k\beta}|^2 \\
&\leqslant \Big(\sum_{j=1}^s \varLambda_j\varGamma_j\Big) \|c\|^2_{\ell_r^2(\mathbb{Z}^d)} ,
\end{aligned}
$$

which concludes the proof. □

It is worth to mention that, in some important examples, the value of $\sum_{j=1}^s \varLambda_j\varGamma_j$ in (2.42) can be explicitely computed in terms of $\delta := \sup_{j,\alpha} |\varepsilon_{j,\alpha}|$. We include two of them in the one dimension case taken from Ref. [42]; for details te reader can check also references therein.

Recovering functions in $V_{N_4}^2$ where N_4 denotes the cubic B-spline

For each fixed $m \in \mathbb{N}$, the B-spline N_m is defined as $N_m := N_1 * N_1 * \ldots * N_1$ (m times) where N_1 denotes the characteristic function of the interval $(0,1)$. It is known [25] that $\{N_m(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R})$. The corresponding shift-invariant space $V_{N_m}^2$ is the space of splines of degree $m-1$ in $L^2(\mathbb{R})$ with nodes at the integers.

By means of samples from f and f'

For $r = s = 2$, consider the systems $(\mathcal{L}_1 f)(t) := f(t + a)$ and $(\mathcal{L}_2 f)(t) := f'(t + a)$. For $a = 0$ we have

$$\begin{aligned}\mathbb{G}(w) &= \begin{pmatrix} ZN_4(0, w) & ZN_4(0, w + 1/2) \\ ZN_4'(0, w) & ZN_4'(0, w + 1/2) \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} z + 4z + z^3 & -z + 4z^2 - z^3 \\ 3z - 3z^3 & -3z + 3z^3 \end{pmatrix}\end{aligned}$$

where $z = \mathrm{e}^{-2\pi \mathrm{i} w}$. Since $\det \mathbb{G} = 2(z^5 - z^3)/3$ vanishes at $w = 0$, it follows that $\alpha_{\mathbb{G}} = 0$. Hence, Theorem 2.6 does not apply. However, taking $a = 1/2$ we obtain

$$\begin{aligned} g_1(w) &= (ZN_4)(1/2, w) = \frac{1}{48} + \frac{23}{48}\mathrm{e}^{-2\pi \mathrm{i} w} + \frac{23}{48}\mathrm{e}^{-4\pi \mathrm{i} w} + \frac{1}{48}\mathrm{e}^{-6\pi \mathrm{i} w} \\ g_2(w) &= (ZN_4')(1/2, w) = \frac{1}{8} + \frac{5}{8}\mathrm{e}^{-2\pi \mathrm{i} w} - \frac{5}{8}\mathrm{e}^{-4\pi \mathrm{i} w} - \frac{1}{8}\mathrm{e}^{-6\pi \mathrm{i} w} \end{aligned}$$

The eigenvalues of the matrix $\mathbb{G}^*(w)\mathbb{G}(w)$ are

$$1 + \frac{157}{288}\sin^2 2\pi w \pm \frac{7}{288}\sqrt{576 \sin^2 2\pi w + 265 \sin^4 2\pi w}$$

The minimum on $(0, 1/2)$ of the smallest eigenvalue is attained at $w = \frac{1}{2\pi}\arctan\sqrt{\frac{392}{403}}$ and takes the value $\alpha_{\mathbb{G}} = \frac{216}{265}$. Besides, the maximum on $(0, 1/2)$ of the largest eigenvalue is $\beta_{\mathbb{G}} = 9/4$ attained at $w = 1/4$.

For $d \in [0, 1/2]$, we have

$$\begin{aligned}\sum_{k=0}^{3} |N_4(k + 1/2 - d) - N_4(k)| &= \sum_{k=0}^{3} |N_4(k + 1/2 + d) - N_4(k)| \\ &= \frac{3}{2}d - \frac{2}{3}d^3\end{aligned}$$

For $d \in (0, 1/3)$ the inequality $N_4'(5/2) > N_4'(5/2 + d)$ holds. Thus, for $d \in [0, 1/3)$ we get

$$\sum_{k=0}^{3} |N_4'(k + 1/2 - d) - N_4'(k)| = \sum_{k=0}^{3} |N_4'(k + 1/2 + d) - N_4'(k)| = 2d$$

Therefore, whenever $[\alpha_1, \beta_1] = [\alpha_2, \beta_2] = [-\delta, \delta]$, with $0 < \delta < 1/3$, we obtain that $\Gamma_1 = (3/2)\delta - (2/3)\delta^3$ and $\Gamma_2 = 2\delta$.

Now, having in mind the symmetry of N_4 and the inequalities

$$N_4(1/2 + \delta) - N_4(1/2) > N_4(1/2) - N_4(1/2 - \delta)$$

and

$$N_4(5/2) - N_4(5/2 + d) > N_4(5/2 - d) - N_4(5/2)\,,$$

we get

$$
\begin{aligned}
&\sup_{d\in[-\delta,\delta]} |N_4(3/2+\delta) - N_4(3/2)| + \sup_{d\in[-\delta,\delta]} |N_4(7/2+\delta) - N_4(7/2)| \\
&= \sup_{d\in[-\delta,\delta]} |N_4(1/2+\delta) - N_4(1/2)| + \sup_{d\in[-\delta,\delta]} |N_4(5/2+\delta) - N_4(5/2)| \\
&= \frac{3\delta}{4} + \frac{\delta^2}{2} - \frac{\delta^3}{3}
\end{aligned}
$$

Analogously, using the symmetry of N_4', the inequality

$$N_4'(1/2+\delta) - N_4'(1/2) > N_4'(1/2) - N_4'(1/2-\delta)\,,$$

and that

$$\sup_{d\in[-\delta,\delta]} |N_4'(5/2+d) - N_4(5/2)| = N_4'(5/2-\delta) - N_4(5/2)\,,$$

we get

$$
\begin{aligned}
&\sup_{d\in[-\delta,\delta]} |N_4'(3/2+\delta) - N_4'(3/2)| + \sup_{d\in[-\delta,\delta]} |N_4'(7/2+\delta) - N_4'(7/2)| \\
&= \sup_{d\in[-\delta,\delta]} |N_4'(1/2+\delta) - N_4'(1/2)| + \sup_{d\in[-\delta,\delta]} |N_4'(5/2+\delta) - N_4'(5/2)| \\
&= \delta + 2\delta^2
\end{aligned}
$$

Hence, $\Lambda_1 = (3/4)\,\delta + (1/2)\,\delta^2 - (1/3)\,\delta^3$ and $\Lambda_2 = \delta + 2\delta^2$. Thus, for any sequence $\varepsilon = \{\varepsilon_{j,n}\}_{n\in\mathbb{Z};\, j=1,2} \subset [\delta,\delta]$, where $\delta < 1/3$, we have that

$$\|D_\varepsilon\|^2 \leqslant \Lambda_1\Gamma_1 + \Lambda_2\Gamma_2 = \frac{25\,\delta^2}{8} + \frac{19\,\delta^3}{4} - \delta^4 - \frac{\delta^5}{3} + \frac{2\,\delta^6}{9}\,.$$

Thus, from Theorem 2.6, whenever $\sup_{j,n}|\varepsilon_{j,n}| < C \approx 0.3022$, where C is the root of

$$25\delta^2/8 + 19\delta^3/4 - \delta^4 - \delta^5/3 + 2\delta^6/9 - 108/265 = 0$$

in $(0,1/3)$, there exists a Riesz basis $\{S^\varepsilon_{j,n}\}_{n\in\mathbb{Z};\, j=1,2}$ for $V^2_{N_4}$ such that the expansion

$$f(t) = \sum_{n\in\mathbb{Z}} \big[f(2n+1/2+\varepsilon_{1,n})\, S^\varepsilon_{1,n}(t) + f'(2n+1/2+\varepsilon_{2,n})\, S^\varepsilon_{2,n}(t) \big]\,, \quad t\in\mathbb{R}\,,$$

holds.

By means of average sampling

For each $f \in V^2_{N_4}$ consider the system defined as $(\mathcal{L}_1 f)(t) := \int_{t-1/2}^{t+1/2} f(x)dx$. For $d \in [0,1/2]$ we have

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} |(\mathcal{L}_1N_4)(k-d) - (\mathcal{L}_1N_4)(k)| = \sum_{k\in\mathbb{Z}} |(\mathcal{L}_1N_4)(k+d) - (\mathcal{L}_1N_4)(k)| \\
&= \frac{23d}{24} + \frac{5d^2}{8} - \frac{d^3}{6} - \frac{d^4}{4}
\end{aligned}
$$

where we have used the symmetry of $\mathcal{L}_1 N_4$ with respect the line $t = 2$. Thus, for $[\alpha_1, \beta_1] = [-\delta, \delta]$, where $\delta \leqslant 1/2$, we obtain

$$\Gamma_1 = 23\delta/24 + 5\delta^2/8 - \delta^3/6 - \delta^4/4\,.$$

Besides,

$$\begin{aligned}\Lambda_1 &= 2 \sup_{d\in[-\delta,\delta]} \big|(\mathcal{L}_1 N_4)(d) - (\mathcal{L}_1 N_4)(0)\big| + 2 \sup_{d\in[-\delta,\delta]} \big|(\mathcal{L}_1 N_4)(1+d) - (\mathcal{L}_1 N_4)(1)\big| \\ &+ \sup_{d\in[-\delta,\delta]} \big|(\mathcal{L}_1 N_4)(2) - (\mathcal{L}_1 N_4)(2+d)\big| = \frac{5\delta^2}{4} - \frac{\delta^3}{6} - \frac{\delta^4}{2} + \frac{23\delta}{24}\end{aligned}$$

Hence, for any sequence $\{\varepsilon_n\}_{n\in\mathbb{Z}} \subset [-\delta, \delta]$, $\delta \leqslant 1/2$, we have that

$$\|D_\varepsilon\|^2 \leqslant \Lambda_1 \Gamma_1 = \frac{529\delta^2}{576} + \frac{115\delta^3}{64} + \frac{133\delta^4}{288} - \frac{33\delta^5}{32} - \frac{43\delta^6}{72} + \frac{\delta^7}{8} + \frac{\delta^8}{8}$$

Moreover

$$\begin{aligned}\alpha_{\mathbb{G}} &= \inf_{w\in(0,1)} |g_1(w)|^2 \\ &= \inf_{w\in(0,1)} \Big|\frac{1 + 76\mathrm{e}^{-2\pi \mathrm{i} w} + 230\mathrm{e}^{-4\pi \mathrm{i} w} + 76e^{-6\pi i n w} + \mathrm{e}^{-8\pi i n w}}{384}\Big|^2 = \frac{25}{576}\end{aligned}$$

Hence, from Theorem 4.8, whenever $\sup_n |\varepsilon_n| < C \approx 0.185$, where C is the root of $529\delta^2/576 + 115\delta^3/64 + 133\delta^4/288 - 33\delta^5/32 - 43\delta^6/72 + \delta^7/8 + \delta^8/8 - 25/576 = 0$ in $(0, 1/2)$, there exists a Riesz basis $\{S_n^\varepsilon\}_{n\in\mathbb{Z}}$ for $V_{N_4}^2$ such that the expansion

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathcal{L}_1 f)(n + \varepsilon_n)\, S_n^\varepsilon(t)\,, \qquad t \in \mathbb{R}\,,$$

holds for each $f \in V_{N_4}^2$.

2.4.2 Another approach

In order to obtain reconstruction formulas like (2.39) we can focus the problem with a different sight, this time taking into account the differential operator acting on the image of the generators $\{\varphi_k\}_{k=1,2,\dots r}$ by the filters $\{\mathcal{L}_j\}_{j=1,2,\dots s}$, we are obviously assuming that $\mathcal{L}_j \varphi_k$ are continuously differentiable for $j = 1, 2, \dots s$ and $k = 1, 2, \dots r$.

For a fixed $\Psi \in L^2(\mathbb{R}^d)$ consider the function

$$C_\Psi(t) = \sum_{\beta\in\mathbb{Z}^d} \|\Psi(t + \beta)\|^2.$$

It is not difficult to check that C_Ψ is always $\mathbb{Z}^d$-periodic and belongs to $L^1[0, 1)^d$. We will require strong conditions on the ℓ^2-norm of the error sequence as well as on the supremum norm of the rs functions $C_{\nabla \mathcal{L}_j \varphi_k}$.

Theorem 2.8. *Assume that* $g_j \in L_r^\infty[0,1)^d$ *for* $j = 1, 2, \dots, s$ *with* $A_{\mathbf{G}} > 0$. *If the error sequence* $\varepsilon := \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ *satisfies* $\|\{\varepsilon_{j,\alpha}\}\|_{\ell^2(\mathbb{Z}^d)} < 1$ *for all* $j = 1, 2, \dots s$ *and* $C_{\nabla\mathcal{L}_j\varphi_k} \in L^\infty(\mathbb{R}^d)$ *for all* $j = 1, 2, \dots s$ *and* $k = 1, 2, \dots r$ *with*

$$\mathbf{u} := \sum_{j=1}^{s}\sum_{k=1}^{r} \|C_{\nabla\mathcal{L}_j\varphi_k}\|_\infty < A_{\mathbf{G}}/(\det M)\,,$$

then there exists a frame $\{S_{j,\alpha}^\varepsilon\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ *for* V_Φ^2 *such that, for any* $f \in V_\Phi^2$

$$f(t) = \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \big(\mathcal{L}_j f\big)(M\alpha + \varepsilon_{j,\alpha}) S_{j,\alpha}^\varepsilon(t)\,, \quad t \in \mathbb{R}^d\,, \tag{2.41}$$

where the convergence of the series is in the $L^2(\mathbb{R}^d)$*-sense, absolute and uniform on* $\mathbb{R}^d$. *Moreover, when* $s = r(\det M)$ *the sequence* $\{S_{j,\alpha}^\varepsilon\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ *is a Riesz basis for* V_Φ^2, *and the interpolation property* $(\mathcal{L}_l S_{j,\alpha}^\varepsilon)(M\beta + \varepsilon_{j,\beta}) = \delta_{j,l}\delta_{\alpha,\beta}$ *holds.*

Proof. By the fundamental theorem of calculus,

$$\mathcal{L}_j\varphi_k(M\alpha - \beta + \varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha - \beta) = \int_0^1 \nabla\mathcal{L}_j\varphi_k(t\varepsilon_{j\alpha} + M\alpha - \beta)\cdot\varepsilon_{j\alpha}dt$$

then

$$\begin{aligned}
&\Big|\mathcal{L}_j\varphi_k(M\alpha - \beta + \varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha - \beta)\Big|^2 \\
&= \Big|\int_0^1 \nabla\mathcal{L}_j\varphi_k(t\varepsilon_{j\alpha} + M\alpha - \beta)\cdot\varepsilon_{j\alpha}dt\Big|^2 \\
&\leqslant \int_0^1 \|\nabla\mathcal{L}_j\varphi_k(t\varepsilon_{j\alpha} + M\alpha - \beta)\|^2 dt \int_0^1 \|\varepsilon_{j\alpha}\|^2 dt \\
&= \|\varepsilon_{j\alpha}\|^2 \int_0^1 \|\nabla\mathcal{L}_j\varphi_k(t\varepsilon_{j\alpha} + M\alpha - \beta)\|^2 dt
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{\beta\in\mathbb{Z}^d} |\mathcal{L}_j\varphi_k(M\alpha - \beta + \varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha - \beta)|^2 \\
&\leqslant \|\varepsilon_{j\alpha}\|^2 \int_0^1 \sum_{\beta\in\mathbb{Z}^d} \|\nabla\mathcal{L}_j\varphi_k(t\varepsilon_{j\alpha} + M\alpha - \beta)\|^2 dt \\
&\leqslant \|\varepsilon_{j\alpha}\|^2 \|C_{\nabla\mathcal{L}_j\varphi_k}\|_\infty\,.
\end{aligned}$$

Now,

$$
\begin{aligned}
&\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big|\big\langle\overline{(Z\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}-\overline{(Z\mathcal{L}_j\Phi)}(0,\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\,,\,F(\cdot)\big\rangle_{L^2_r[0,1)^d}\big|^2\\
&=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big|\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^d}[\mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha})-\mathcal{L}_j\varphi_k(M\alpha-\beta)]c_{k\beta}\big|^2\\
&\leqslant\|c\|^2_{\ell^2_r(\mathbb{Z}^d)}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^d}|\mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha})-\mathcal{L}_j\varphi_k(M\alpha-\beta)|^2\\
&\leqslant\|c\|^2_{\ell^2_r(\mathbb{Z}^d)}\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\|\varepsilon_{j\alpha}\|^2\|C_{\nabla\mathcal{L}_j\varphi_k}\|_\infty\\
&\leqslant\|c\|^2_{\ell^2_r(\mathbb{Z}^d)}\sum_{k=1}^{r}\sum_{j=1}^{s}\|C_{\nabla\mathcal{L}_j\varphi_k}\|_\infty\sum_{\alpha\in\mathbb{Z}^d}\|\varepsilon_{j\alpha}\|^2\\
&\leqslant\mathbf{u}\|c\|^2_{\ell^2_r(\mathbb{Z}^d)}=\mathbf{u}\|F\|^2_{L^2_r[0,1)^d}\,.
\end{aligned}
$$

which concludes the proof as the one in Theorem 2.6. □

We have obtained formula (2.39) with two differents hypothesis, but from a practical point of view it is useless, since the frame $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$, which depends on the error sequence $\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\;j=1,2,\dots,s}$, is impossible to determine. As a consequence, to recover any function $f\in V^2_\Phi$ from the samples

$$\{\mathcal{L}_jf(M\alpha+\varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$$

we should use the frame algorithm (see [36]). In order to approximate the sequence $\{a_{k\alpha}\}_{\alpha\in\mathbb{Z}^d;\,k=1,\dots,r}\in\ell^2(\mathbb{Z}^d)$ associated to $f=\sum_{k=1}^r\sum_{\alpha\in\mathbb{Z}^d}a_{k\alpha}\,\varphi_k(t-\alpha)\in V^2_\Phi$, the frame algorithm can be implemented in the $\ell^2(\mathbb{Z}^d)$ setting as in Ref. [42].

2.4.3 The frame algorithm

Now we are going to implement a frame algorithm in the $\ell^2_r(\mathbb{Z}^d)$ setting. To this end, consider the canonical isometry $\mathcal{U}:\ell^2_r(\mathbb{Z}^d)\to L^2_r[0,1)^d$

$$\mathcal{U}\,c:=\Big(\sum_{\beta\in\mathbb{Z}^d}c_{1\beta}\mathrm{e}^{-2\pi\mathrm{i}\beta^\top M^\top\cdot},\dots,\sum_{\beta\in\mathbb{Z}^d}c_{r\beta}\mathrm{e}^{-2\pi\mathrm{i}\beta^\top M^\top\cdot}\Big)^\top,$$

where $c=(\{c_{1\beta}\}_{\beta\in\mathbb{Z}^d},\dots,\{c_{r\beta}\}_{\beta\in\mathbb{Z}^d})^\top\in\ell^2_r(\mathbb{Z}^d)$.
For $f(t)=\sum_{k=1}^r\sum_{\alpha\in\mathbb{Z}^d}c_{k\alpha}\varphi_k(t-\alpha)\in V^2_\Phi$, denote by $\mathbb{F}$ the sequence

$$\mathbb{F}:=\mathcal{U}^{-1}\mathbf{F}=\mathcal{U}^{-1}\mathcal{T}_\Phi^{-1}f=(\{c_{1\alpha}\}_{\alpha\in\mathbb{Z}^d},\dots,\{c_{r\alpha}\}_{\alpha\in\mathbb{Z}^d})^\top\in\ell^2_r(\mathbb{Z}^d)\,.$$

The samples $\{(\mathcal{L}_j f)(M\alpha+\varepsilon_{j,\alpha})\}$ can be written as

$$\big(\mathcal{L}_j f\big)(M\alpha+\varepsilon_{j,\alpha}) = \big\langle \mathbf{F}(\cdot), \overline{(Z\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\big\rangle_{L_r^2[0,1)^d} = \big\langle \mathbb{F}, \mathbb{L}_{j,\alpha}\big\rangle_{\ell_r^2(\mathbb{Z}^d)}$$

where, for $j=1,2,\dots,s$ and $\alpha\in\mathbb{Z}^d$,

$$\begin{aligned}
\mathbb{L}_{j,\alpha} &:= \mathcal{U}^{-1}\big(\overline{(Z\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\big)\\
&= \mathcal{U}^{-1}\Big[\sum_{\beta\in\mathbb{Z}^d}\mathrm{e}^{2\pi\mathrm{i}(\beta^\top-\alpha^\top M^\top)\cdot}\big(\overline{(\mathcal{L}_j\varphi_1)}(\varepsilon_{j,\alpha}+\beta),\dots,\overline{(\mathcal{L}_j\varphi_r)}(\varepsilon_{j,\alpha}+\beta)\big)^\top\Big]\\
&= \mathcal{U}^{-1}\Big[\sum_{\beta\in\mathbb{Z}^d}\mathrm{e}^{2\pi\mathrm{i}\beta^\top\cdot}\big(\overline{(\mathcal{L}_j\varphi_1)}(\varepsilon_{j,\alpha}+\beta+M\alpha),\dots,\overline{(\mathcal{L}_j\varphi_r)}(\varepsilon_{j,\alpha}+\beta+M\alpha)\big)^\top\Big]\\
&= \mathcal{U}^{-1}\Big[\sum_{\beta\in\mathbb{Z}^d}\mathrm{e}^{-2\pi\mathrm{i}\beta^\top\cdot}\big(\overline{(\mathcal{L}_j\varphi_1)}(\varepsilon_{j,\alpha}-\beta+M\alpha),\dots,\overline{(\mathcal{L}_j\varphi_r)}(\varepsilon_{j,\alpha}-\beta+M\alpha)\big)^\top\Big]\\
&= \Big(\{\overline{(\mathcal{L}_j\varphi_1)}(\varepsilon_{j,\alpha}-\beta+M\alpha)\}_{\beta\in\mathbb{Z}^d},\dots,\{\overline{(\mathcal{L}_j\varphi_r)}(\varepsilon_{j,\alpha}-\beta+M\alpha)\}_{\beta\in\mathbb{Z}^d}\Big)^\top
\end{aligned}$$

The sequence $\{\mathbb{L}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$ is a frame for $\ell_r^2(\mathbb{Z}^d)$. Indeed, assume that

$$\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d}\subset[\alpha_j,\beta_j]^d\subset[-r,r]^d$$

for each $j=1,2,\dots,s$, and that $\sum_{j=1}^s \Lambda_j\Gamma_j < A_{\mathbf{G}}/\det M$. According to the proof of Theorem 2.6 and Lemma A.8, the sequence $\{\overline{(Z\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$ is a frame for $L_r^2[0,1)^d$ with bounds

$$\begin{aligned}
A &:= \frac{A_{\mathbf{G}}}{\det M}\left(1-\sqrt{\frac{\det M}{A_{\mathbf{G}}}\sum_{j=1}^s\Lambda_j\Gamma_j}\right)^2,\\
B &:= \frac{B_{\mathbf{G}}}{\det M}\left(1+\sqrt{\frac{\det M}{B_{\mathbf{G}}}\sum_{j=1}^s\Lambda_j\Gamma_j}\right)^2
\end{aligned}\tag{2.42}$$

Since $\mathcal{U}^{-1}$ is an isometry, the sequence $\{\mathbb{L}_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$ is a frame for $\ell_r^2(\mathbb{Z}^d)$ with the same bounds.

Hence, the recovering of the function $f=\mathcal{T}_\Phi\,\mathcal{U}\,\mathbb{F}\in V_\Phi^2$ from the samples $\{(\mathcal{L}_j f)(M\alpha+\varepsilon_{j,\alpha})\}$ is reduced to recover $\mathbb{F}$ from the sequence

$$\big\{\langle\mathbb{F},\mathbb{L}_{j,\alpha}\rangle_{\ell_r^2(\mathbb{Z}^d)}\big\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}.$$

In so doing, the classical frame algorithm reads:

Consider

$$\mathbb{F}_0 = \mathcal{A}\,\mathbb{F} := \frac{2}{A+B}\sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\langle \mathbb{F}, \mathbb{L}_{j,\alpha}\rangle_{\ell^2(\mathbb{Z}^d)}\mathbb{L}_{j,\alpha}$$
$$= \frac{2}{A+B}\sum_{\alpha\in\mathbb{Z}^d}\sum_{j=1}^{s}\big(\mathcal{L}_j f\big)(M\alpha+\varepsilon_{j,\alpha})\mathbb{L}_{j,\alpha}$$

and define recursively $\mathbb{F}_{k+1} = \mathbb{F}_k + \mathcal{A}(\mathbb{F}-\mathbb{F}_k)\,,\quad k\in\mathbb{N}$. Then, the sequence $\{f_k\}_{k\in\mathbb{N}}$ in V_Φ^2 given by

$$f_k(t) = \sum_{m=1}^{r}\sum_{\alpha\in\mathbb{Z}^d} a_{m\alpha}^{(k)}\,\varphi_m(t-\alpha)\,,\quad t\in\mathbb{R}^d\,,$$

where

$$\mathbb{F}_k = \big(\{a_{1\alpha}^{(k)}\}_{\alpha\in\mathbb{Z}^d},\ldots,\{a_{r\alpha}^{(k)}\}_{\alpha\in\mathbb{Z}^d}\big)\,,$$

satisfies

$$\|f-f_k\|_{L^2(\mathbb{R}^d)} \leqslant \|\mathcal{T}_\Phi\|\;\|\mathbb{F}-\mathbb{F}_k\|_{\ell_r^2(\mathbb{Z}^d)} \leqslant \|\mathcal{T}_\Phi\|\,\gamma^{k+1}\|\mathbb{F}\|_{\ell_r^2(\mathbb{Z}^d)}$$
$$\leqslant \|\mathcal{T}_\Phi\|\,\|\mathcal{T}_\Phi^{-1}\|\,\gamma^{k+1}\|f\|_{L^2(\mathbb{R}^d)} = \sqrt{\frac{\|\Phi\|_\infty}{\|\Phi\|_0}}\,\gamma^{k+1}\|f\|_{L^2(\mathbb{R}^d)}\,,$$

where $\gamma := (B-A)/(B+A)$, and we have used that $\|\mathcal{T}_\Phi^{-1}\|^{-2} = \|\Phi\|_0$ and $\|\mathcal{T}_\Phi\|^2 = \|\Phi\|_\infty$, see Theorem A.5. In order to improve this algorithm, specially when the ratio B/A is large, we can use the methods of acceleration of the frame algorithm proposed by Gröchenig in [52].

Finally it is worth to mention that the general irregular sampling case has been treated, for instance, in Refs. [6, 8, 9, 36, 96, 100, 101, 102, 103, 119].

3

Uniform average sampling in frame generated weighted shift-invariant spaces

3.1 Statement of the problem

We have already mentioned the possible drawbacks behind Shannon's sampling theory (Section 1.2), that is why we have focused our attention in sampling on spaces like

$$V_\Phi^2 = \Big\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha) \,:\, d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2 \ldots, r \Big\}\,.$$

where the sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, k=1,2\ldots,r}$ is a Riesz basis for V_Φ^2.

In the preceding chapter we have obatained a formula of the form

$$f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,, \tag{3.1}$$

where the sequence of reconstruction functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ is a frame for the shift-invariant space V_Φ^2. Besides the samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,s}$ are some filtered versions of the signal itself, in other words, for each $j = i, 2, \ldots, s$, $\mathcal{L}_j$ is a linear operator acting on the function f, these operators reflect features of the acquisition device which provides the samples.

Besides, to model decay or growth of real signals one can assume that they belong to a $L^p_\nu(\mathbb{R}^d)$ space with weight function ν. Notice that a function f belongs to $L^p_\nu(\mathbb{R}^d)$ if νf belongs to $L^p(\mathbb{R}^d)$. If the weight function ν grows rapidly as $|t| \to \infty$, then the functions in $L^p_\nu(\mathbb{R}^d)$ decay roughly at a corresponding rate. Conversely, if the weight function ν decays rapidly, then the functions in $L^p_\nu(\mathbb{R}^d)$ may grow as $|t| \to \infty$ (see, for instance, [6, 54, 118]).

In this chapter we deal with generalized (average) regular sampling in a weighted shift-invariant space $V^p_\nu(\varPhi)$ in $L^p_\nu(\mathbb{R}^d)$, formally defined as

$$V^p_\nu(\varPhi) := \Big\{ \sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(t-\alpha) \ : \ \{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^p_\nu(\mathbb{Z}^d),\ j=1,2,\dots,r \Big\} .$$

That is, we derive sampling formulas like (2.2) valid in $V^p_\nu(\varPhi)$. The set of generators $\varPhi := \{\phi_j\}_{j=1}^r$ is contained in the Wiener amalgam space $W(L^1_\nu)$, i.e., the generators are functions locally in $L^\infty(\mathbb{R}^d)$ and globally in $L^1_\nu(\mathbb{R}^d)$.

The sequence $\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ is assumed to be a p-frame for $V^p_\nu(\varPhi)$; thus $V^p_\nu(\varPhi)$ is a closed subspace in $L^p_\nu(\mathbb{R}^d)$. See Section 3.2 below for the precise results.

In order to obtain our appropriate sampling functions S_l, $l = 1,2,\dots,s$, we use Wiener's Lemma for the weighted Wiener algebra $\mathcal{A}_\nu$; thus, we need a submultiplicative weight ν satisfying also the so called GRS-condition (see [54, 55]). Typical subexponential or Sobolev weights satisfy our requirements. Our main sampling result (see Theorem 3.3 in Section 3.3) will be first proved in $\operatorname{span}\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ (see Lemma 3.5 in Section 3.3) and then proved in $V^p_\nu(\varPhi)$ by means of a density argument. Finally, the recovery of any function $f \in V^p_\nu(\varPhi)$ from a sequence of its samples is also treated in Section 3.3; in this case we assume in addition that the set of generators $\varPhi = \{\phi_j\}_{j=1}^r$ has L^p_ν-stable shifts.

Firstly, we collect the needed preliminaries in Section 3.2.

3.2 Weighted shift-invariant spaces $V^p_\nu(\varPhi)$ $(1 \leqslant p \leqslant \infty)$

3.2.1 Some needed preliminaries for weighted spaces

Let ν be a weight function which in general means a non-negative function on $\mathbb{R}^d$. Given a sequence $c := \{c(\alpha)\}_{\alpha\in\mathbb{Z}^d}$, for $1 \leqslant p < \infty$ the weighted $\ell^p_\nu(\mathbb{Z}^d)$ space is defined by the norm

$$\|c\|_{\ell^p_\nu} := \Big(\sum_{\alpha\in\mathbb{Z}^d} |c(\alpha)|^p \nu(\alpha)^p \Big)^{1/p} ,$$

and for $p = \infty$, we have

$$\|c\|_{\ell^\infty_\nu} := \sup_{\alpha\in\mathbb{Z}^d} |c(\alpha)| \nu(\alpha) .$$

A function f belongs to $L_\nu^p(\mathbb{R}^d)$ if νf belongs to $L^p(\mathbb{R}^d)$. The norm is defined by $\|f\|_{L_\nu^p(\mathbb{R}^d)} = \|\nu f\|_{L^p(\mathbb{R}^d)}$. Equipped with these norms, the spaces $\ell_\nu^p(\mathbb{Z}^d)$ and $L_\nu^p(\mathbb{R}^d)$ are Banach spaces; when $\nu \equiv 1$, we obtain the usual ℓ^p and L^p spaces.

Given a set of functions $\Phi := \{\phi_j\}_{j=1}^r$, the weighted multiply generated shift-invariant space $V_\nu^p(\Phi)$ is formally defined as

$$V_\nu^p(\Phi) := \Big\{ \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\, \phi_j(t-\alpha) \;:\; \{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell_\nu^p(\mathbb{Z}^d),\ j=1,2,\ldots,r \Big\}.$$

In order to give a complete sense to these spaces as (closed) subspaces of $L_\nu^p(\mathbb{R}^d)$, the convergence properties of the series $\sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\,\phi_j(t-\alpha)$ should be studied. Thus, suitable hypotheses on the generators Φ must be imposed (see Section 3.2.2 infra).

Throughout the paper the weight function ν is always assumed to be continuous, symmetric, i.e., $\nu(x) = \nu(-x)$, positive and submultiplicative, i.e.,

$$0 < \nu(x+y) \leqslant \nu(x)\nu(y), \quad \text{for all } x, y \in \mathbb{R}^d .$$

It is straightforward to deduce that $\nu(x) \geqslant 1$ for all $x \in \mathbb{R}^d$. Some typical examples of weight functions are the subexponential weight $\nu(x) = \mathrm{e}^{\alpha|x|^\beta}$ with $\alpha \geqslant 0$, $\beta \in [0,1]$, and the Sobolev weight $\nu(x) = (1+|x|)^\alpha$, with $\alpha \geqslant 0$.

For $1 \leqslant p < \infty$ we consider the Wiener amalgam spaces

$$W(L_\nu^p) := \Big\{ f \text{ measurable} \;:\; \|f\|_{W(L_\nu^p)}^p := \sum_{\alpha\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d} \big\{ |f(x+\alpha)|^p \nu(\alpha)^p \big\} < \infty \Big\},$$

and for $p = \infty$

$$W(L_\nu^\infty) := \Big\{ f \text{ measurable} \;:\; \|f\|_{W(L_\nu^\infty)} := \sup_{\alpha\in\mathbb{Z}^d} \big\{ \operatorname*{ess\,sup}_{x\in[0,1]^d} \{|f(x+\alpha)|\nu(\alpha)\} \big\} < \infty \Big\}.$$

Endowed with above norms, these spaces become Banach spaces. Furthermore, they are also translation-invariant spaces.

The subspace of continuous functions in $W(L_\nu^p)$, denoted as $W_0(L_\nu^p)$, is a closed subspace of $W(L_\nu^p)$ and thus also a Banach space. The inclusion

$$W_0(L_\nu^p) \subset W_0(L_\nu^q)\,, \quad \text{where } 1 \leqslant p \leqslant q \leqslant \infty$$

holds (see [6]). From Ref. [85] we also have the following inclusions

$$W(L_\nu^p) \subset W(L_\nu^q) \subset L_\nu^q\,, \quad \text{where } 1 \leqslant p \leqslant q \leqslant \infty\,.$$

Given a function ϕ and a sequence a, the semi-discrete convolution product is formally defined by

$$\phi *' a := \sum_{\alpha\in\mathbb{Z}^d} a(\alpha)\phi(\cdot - \alpha)\,.$$

A weight function ν is called moderate with respect to the submultiplicative weight ω, or simply ω-moderate, if it is continuous, symmetric, and positive and satisfies $\nu(x+y) \leqslant C\omega(x)\nu(y)$ for all $x, y \in \mathbb{R}^d$. As a submultiplicative weight ν is, in particular, ν-moderate (with constant $C = 1$) we have the following inequalities (see, for instance, [6, 85]):

Lemma 3.1. *(a) If $f \in L^p_\nu$, $g \in L^1_\nu$ and $1 \leqslant p \leqslant \infty$, then $\|f * g\|_{L^p_\nu} \leqslant \|f\|_{L^p_\nu}\|g\|_{L^1_\nu}$.*

(b) If $f \in L^p_\nu$, $g \in W(L^1_\nu)$ and $1 \leqslant p \leqslant \infty$, then $\|f * g\|_{W(L^p_\nu)} \leqslant C\|f\|_{L^p_\nu}\|g\|_{W(L^1_\nu)}$, for some positive constant C.

(c) If $a \in \ell^p_\nu$, $b \in \ell^1_\nu$ and $1 \leqslant p \leqslant \infty$, then $\|a * b\|_{\ell^p_\nu} \leqslant \|a\|_{\ell^p_\nu}\|b\|_{\ell^1_\nu}$.

(d) If $f \in W(L^p_\nu)$, $c \in \ell^1_\nu$ and $1 \leqslant p \leqslant \infty$, then $\|f *' c\|_{W(L^p_\nu)} \leqslant \|c\|_{\ell^1_\nu}\|f\|_{W(L^p_\nu)}$.

(e) If $f \in W(L^1_\nu)$, $c \in \ell^p_\nu$ and $1 \leqslant p \leqslant \infty$, then $\|f *' c\|_{W(L^p_\nu)} \leqslant \|c\|_{\ell^p_\nu}\|f\|_{W(L^1_\nu)}$.

Lemma 3.2. *If $f \in L^p_\nu$ and $g \in W(L^1_\nu)$, then the sequence d defined by*

$$d := \Big\{ \int_{\mathbb{R}^d} f(x)\overline{g(x-\alpha)}dx \Big\}_{\alpha \in \mathbb{Z}^d}$$

belongs to ℓ^p_ν, and we have $\|d\|_{\ell^p_\nu} \leqslant \|f\|_{L^p_\nu}\|g\|_{W(L^1_\nu)}$.

Notice that from Lemma 3.2, for $f \in L^p_\nu$ and $h \in W(L^1_\nu)$, it is easy to deduce the inequality $\|\{(f * h)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} \leqslant C\|f\|_{L^p_\nu}\|h\|_{W(L^1_\nu)}$ for some positive constant C.

For the sake of completeness we include the following result borrowed from [6, Theorem 3.1]):

Lemma 3.3. *Assume that $\{\phi_j\}_{j=1}^r \subset W_0(L^1_\nu)$ and $1 \leqslant p \leqslant \infty$. Then the following inclusion holds:*

$$V^p_\nu(\Phi) \subset W_0(L^p_\nu).$$

Proof. Let $f = \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\phi_j(\cdot - \alpha) \in V^p_\nu(\Phi)$, then Lemma 3.1($e$) implies that

$$\|f\|_{W(L^p_\nu)} \leqslant \sum_{j=1}^r \|a_j\|_{\ell^p_\nu}\|\phi_j\|_{W(L^1_\nu)}. \tag{3.2}$$

On the other hand, it is easy to prove that there exists a positive constant C such that

$$\|f\|_{L^\infty_\nu} \leqslant C\|f\|_{W(L^\infty_\nu)} \leqslant C\|f\|_{W(L^p_\nu)}. \tag{3.3}$$

To verify the continuity of f in the case $1 \leqslant p < \infty$, for each $N \in \mathbb{N}$ consider $f_N = \sum_{j=1}^r \sum_{|\alpha|\leqslant N} a_j(\alpha)\phi_j(\cdot - \alpha)$, where $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_d|$; then (3.2) and (3.3) imply that

$$\begin{aligned} \|f - f_N\|_{L^\infty_\nu} &\leqslant C\|f - f_N\|_{W(L^p_\nu)} \\ &\leqslant C \max_{j=1,2,\ldots,r} \Big\{\|\phi_j\|_{W(L^1_\nu)}\Big\} \sum_{j=1}^r \Big(\sum_{|\alpha|>N} |a_j(\alpha)|^p \nu(\alpha)^p \Big)^{1/p}. \end{aligned}$$

Therefore, the sequence of continuous functions νf_N converges uniformly to the continuous function νf. Since ν is positive and continuous, f must be continuous as well.

For the case $p = \infty$, we choose sequences $\phi_j^{(n)}$ of continuous functions with compact support such that $\|\phi_j - \phi_j^{(n)}\|_{W(L^1_\nu)} \to 0$ as $n \to \infty$ for all $j = 1, 2, \ldots, r$. Set

$$f_n(x) = \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} a_j(\alpha)\phi_j^{(n)}(x - \alpha)\,.$$

Since the sum is locally finite, each f_n is continuous. Using again (3.2) and (3.3) we estimate

$$\begin{aligned}\|f - f_n\|_{L^\infty_\nu} &\leqslant C\|f - f_n\|_{W(L^\infty_\nu)} \\ &\leqslant C \max_{j=1,2,\ldots,r} \Big\{\|c_j\|_{\ell^\infty_\nu}\Big\}\Big(\sum_{j=1}^{r} \|\phi_j - \phi_j^{(n)}\|_{W(L^1_\nu)}\Big)\,.\end{aligned}$$

It follows that the sequence νf_n converges uniformly to νf, and again, since ν is positive and continuous, f is continuous. □

3.2.2 On the generators of $V_\nu^p(\Phi)$ $(1 \leqslant p \leqslant \infty)$

Let $\Phi = \{\phi_j\}_{j=1}^r$ be the set of generators for $V_\nu^p(\Phi)$; in most of the papers in the mathematical literature it is assumed that the sequence $\{\phi_j(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\ldots,r}$ is a Riesz basis for $V_\nu^p(\Phi)$. Here we assume a more general condition: the sequence $\{\phi_j(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\ldots,r}$ is a p-frame for $V_\nu^p(\Phi)$. Following [7] or [85], we introduce the concept of p-frame:

Definition 3.1. *A collection $\{\phi_j(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\ldots,r}$ is said to be a p-frame for $V_\nu^p(\Phi)$ if there exists a positive constant C (depending on Φ, p and ν) such that*

$$C^{-1}\|f\|_{L^p_\nu} \leqslant \sum_{j=1}^{r} \Big\|\Big\{\int_{\mathbb{R}^d} f(x)\overline{\phi_j(x - \alpha)}dx\Big\}_{\alpha \in \mathbb{Z}^d}\Big\|_{\ell^p_\nu} \leqslant C\|f\|_{L^p_\nu}, \qquad f \in V_\nu^p(\Phi). \tag{3.4}$$

Next result, obtained from a theorem in [85] (see also [118]), gives a characterization of the space $V_\nu^p(\Phi)$ as a closed subspace of L^p_ν; it generalizes the result given in [7] for the non-weighted case:

Theorem 3.1. *Let $\Phi = \{\phi_j\}_{j=1}^r \subset W(L^1_\nu)$ and $1 \leqslant p \leqslant \infty$. Then the following statements are equivalent:*

i) $V_\nu^p(\Phi)$ is a closed subspace in L^p_ν.

ii) $\{\phi_j(\cdot - \alpha)\}_{\alpha \in \mathbb{Z}^d;\, j=1,2,\ldots,r}$ is a p-frame for $V_\nu^p(\Phi)$.

iii) There exists a positive constant C such that

$$C^{-1}[\widehat{\Phi},\widehat{\Phi}](\xi) \leqslant [\widehat{\Phi},\widehat{\Phi}](\xi)\overline{[\widehat{\Phi},\widehat{\Phi}](\xi)^T} \leqslant C[\widehat{\Phi},\widehat{\Phi}](\xi), \qquad \xi \in [-\pi,\pi]^d.$$

iv) There exist positive constants C_1 and C_2 (depending on Φ and w) such that

$$C_1\|f\|_{L^p_\nu} \leqslant \inf_{f=\sum_{i=1}^r \phi_i *' c_i} \sum_{j=1}^r \left\|\{c_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}\right\|_{\ell^p_\nu} \leqslant C_2\|f\|_{L^p_\nu}, \qquad f \in V^p_\nu(\Phi).$$

v) There exists a set of functions $\Psi := \{\psi_j\}_{j=1}^r \subset W(L^1_\nu)$, such that

$$\begin{aligned} f &= \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} \langle f, \psi_j(\cdot-\alpha)\rangle \phi_j(\cdot-\alpha) \\ &= \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} \langle f, \phi_j(\cdot-\alpha)\rangle \psi_j(\cdot-\alpha), \qquad f \in V^p_\nu(\Phi). \end{aligned}$$

In the above theorem the matrix of functions $[\widehat{\Phi},\widehat{\Phi}](\xi)$ is defined by

$$[\widehat{\Phi},\widehat{\Phi}](\xi) = \Big[\sum_{\alpha\in\mathbb{Z}^d} \widehat{\phi}_i(\xi+2\pi\alpha)\overline{\widehat{\phi}_j(\xi+2\pi\alpha)}\Big]_{1\leqslant i,j\leqslant r},$$

and we are assuming that $\widehat{\phi}_i(\xi)\overline{\widehat{\phi}_j(\xi)}$ is integrable for any $1 \leqslant i,j \leqslant r$.

Some comments about the above result are in order:

1. If the sequence $\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,r}$ is a p_0-frame for $V^{p_0}_\nu(\Phi)$, then it is a p-frame for $V^p_\nu(\Phi)$ for any $1 \leqslant p \leqslant \infty$. This fact is proved in Corollary 3.13 in [85], and in Corollary 1 in [7] for the non-weighted case.
2. Theorem 2.4 in [6] assures us that if $\Phi \subset W(L^1_\nu)$ then the space $V^p_\nu(\Phi)$ is a subspace (not necessarily closed) of L^p_ν and $W(L^p_\nu)$ for any $1 \leqslant p \leqslant \infty$. Hence we have $V^p_\nu(\Phi) \subset \overline{\operatorname{span}}_{L^p_\nu}\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,r}$. On the other hand, if one of the statements in the previous theorem is satisfied we have the other inclusion; in other words

$$V^p_\nu(\Phi) = \overline{\operatorname{span}}_{L^p_\nu}\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,r}\,.$$

3. Finally it is worth to mention that for $f \in V^p_\nu(\Phi)$ we do not have uniqueness for the coefficients $\{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^p_\nu(\mathbb{Z}^d)$ in the expansion

$$f = \sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\,\phi_j(\cdot-\alpha) \quad \text{in } L^p_\nu(\mathbb{R}^d)\,.$$

3.3 Uniform average sampling in $V_\nu^p(\Phi)$ $(1 \leqslant p \leqslant \infty)$

3.3.1 The convolution systems $\mathcal{L}_l$ $(1 \leqslant l \leqslant s)$

Throughout this chapter we consider again s convolution systems $\mathcal{L}_l$, $1 \leqslant l \leqslant s$, of the following type: the impulse response h_l of the system $\mathcal{L}_l$ belongs to $W(L^1_\nu)$, i.e.,

$$\big(\mathcal{L}_l f\big)(t) := [f * \mathsf{h}_l](t) = \int_{\mathbb{R}^d} f(x)\mathsf{h}_l(t-x)dx\,, \quad t \in \mathbb{R}^d\,.$$

Whenever $f \in V_\nu^p(\Phi)$ the above convolution $f * \mathsf{h}_l$ is well-defined as a function in L^p_ν: see Lemma 3.1(a). Besides, provided that $\phi_j \in W(L^1_\nu)$, $j = 1, 2, \ldots, r$, the sequence $\{\mathcal{L}_l\phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^1_\nu(\mathbb{Z}^d)$; this is a consequence of the inclusion $W(L^1_\nu) \subset L^1_\nu$ and Lemma 3.2.

For the submultiplicative weight ν, let $\mathcal{A}_\nu$ be the weighted Wiener algebra of the functions

$$f(x) = \sum_{\alpha\in\mathbb{Z}^d} a(\alpha)\, \mathrm{e}^{2\pi \mathrm{i}\alpha^\top x}\,,$$

with $a := \{a(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^1_\nu(\mathbb{Z}^d)$; here we are using the notation $\alpha^\top x := \sum_{k=1}^d \alpha_k x_k$ for $\alpha \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$. This space $\mathcal{A}_\nu$, normed by $\|f\|_{\mathcal{A}_\nu} := \|a\|_{\ell^1_\nu}$ and with pointwise multiplication becomes a commutative Banach algebra.

A weight function ν satisfies the so called GRS-condition (Gelfand-Raikov-Shilov) if for each $\alpha \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \nu(n\alpha)^{1/n} = 1\,.$$

Then the Wiener's Lemma holds:

Theorem 3.2. *Let ν be a weight satisfying the GRS-condition. If $f \in \mathcal{A}_\nu$ and $f(x) \neq 0$ for every $x \in \mathbb{R}^d$, the function $1/f$ is also in $\mathcal{A}_\nu$.*

See the proof, for instance, in [54, 55]).

Thus, for $l = 1, 2, \ldots, s$ and $j = 1, 2, \ldots r$, the Fourier transform of the sequence $\{\mathcal{L}_l\phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ belongs to the Wiener algebra $\mathcal{A}_\nu$, and it will play an important role in the sequel. We denote it by

$$g_{l,j}(x) := \sum_{\alpha\in\mathbb{Z}^d} \big(\mathcal{L}_l\phi_j\big)(\alpha)\, \mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}\,, \quad x \in \mathbb{R}^d\,,$$

and

$$\mathbf{g}_l^\top(x) := \big(g_{l,1}(x), g_{l,2}(x), \ldots, g_{l,r}(x)\big)\,, \quad 1 \leqslant l \leqslant s\,. \tag{3.5}$$

In order to recover any function $f \in V_\nu^p(\Phi)$ from its generalized samples at a lattice $M\mathbb{Z}^d$, i.e., from the sequence of samples $\big\{(\mathcal{L}_l f)(M\alpha)\big\}_{\alpha\in\mathbb{Z}^d;\, l=1,2,\ldots,s}$, a suitable expression for the samples will be useful. As a consequence of Lemma 3.2, the sequence $\{(\mathcal{L}_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d;\, l=1,2,\dots,s}$ belongs to ℓ^p_ν; in order that the sequence of samples $\{(\mathcal{L}_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, l=1,2,\dots,s}$ belongs also to ℓ^p_ν, we will need to assume the following compatibility condition:

Definition 3.2. *Given a submultiplicative weight ν and a lattice $M\mathbb{Z}^d$, we say that ν is M-compatible if the ratio $\nu(\alpha)/\nu(M\alpha)$ remains bounded as $|\alpha|$ goes to infinity.*

The compatibility condition in Definition 3.2 is not always true; for a subexponential weight there exists a nonsingular matrix M with integer entries for which the condition fails. For instance, consider the matrix

$$M = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \qquad \text{and } (\beta,-2\beta)^\top \in \mathbb{Z}^2 \text{ with } \beta\in\mathbb{Z}\text{; we have } M(\beta,-2\beta)^\top = (\beta,0)^\top\,.$$

For the weight $\nu(x) = \mathrm{e}^{|x|}$, the ratio $\nu\big((\beta,-2\beta)\big)/\nu\big((\beta,0)\big) = \mathrm{e}^{(\sqrt{5}-1)|\beta|}$ remains unbounded as $|\beta|\to\infty$.

However, one can prove that any Sobolev weight is compatible with respect to any lattice $M\mathbb{Z}^d$. Also, subexponential weights are compatible with respect to any diagonal lattice. From now on, the submultiplicative weight ν will be considered M-compatible.

3.3.2 An expression for the samples

Recall that ν is a submultiplicative weight so that $\mathcal{A}_\nu$ is a Banach algebra. Consider the map

$$\begin{array}{rccl} \mathcal{T}_\Phi: & \mathcal{A}_\nu\times\ldots\times\mathcal{A}_\nu & \longrightarrow & L^p_\nu(\mathbb{R}^d) \\ & \mathbf{F}^\top := (f_1,\ldots,f_r) & \longmapsto & \sum_{j=1}^r \phi_j *' a_j\,, \end{array} \tag{3.6}$$

where

$$f_j(x) = \sum_{\alpha\in\mathbb{Z}^d} a_j(\alpha)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x} \in \mathcal{A}_\nu\,,\ j=1,2,\ldots,r\,.$$

It is easy to deduce the existence of a positive constant C such that $\|f\|_{L^p_\nu} \leqslant C\|f\|_{W(L^p_\nu)}$.

Thus,

$$\begin{aligned} \Big\|\sum_{j=1}^r \phi_j *' a_j\Big\|_{L^p_\nu} &\leqslant C\sum_{j=1}^r \|\phi_j *' a_j\|_{W(L^p_\nu)} \\ &\leqslant C \max_{j=1,2,\ldots,r}\big\{\|\phi_j\|_{W(L^p_\nu)}\big\}\sum_{j=1}^r \|a_j\|_{\ell^1_\nu}\,, \end{aligned}$$

where we have used Lemma 3.1(d). Now, with the inclusion $W(L^1_\nu)\subset W(L^p_\nu)$ we get that $\mathcal{T}_\Phi$ is a well-defined bounded operator by considering in $\mathcal{A}_\nu\times\ldots\times\mathcal{A}_\nu$ the norm $\|\mathbf{F}\| := \sum_{j=1}^r \|a_j\|_{\ell^1_\nu}$.

For $f \in \operatorname{span}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ let $\mathbf{a} := \{(a_1(\alpha), \dots, a_r(\alpha))\}$ be the finite sequence such that $f = \sum_{j=1}^r \phi_j *' a_j$ and the corresponding trigonometric polynomial

$$\begin{aligned}\mathbf{F}^\top(x) :&= \Big(\sum_\alpha a_1(\alpha)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}, \dots, \sum_\alpha a_r(\alpha)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}\Big)\\ &= \sum_\alpha \mathbf{a}(\alpha)\,\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}\,,\end{aligned}$$

so that $\mathcal{T}_\Phi\mathbf{F} = f$. For any $l = 1, 2, \dots, s$ and $\alpha \in \mathbb{Z}^d$, we have

$$\begin{aligned}(\mathcal{L}_l f)(M\alpha) &= \sum_{\beta\in\mathbb{Z}^d}\sum_{j=1}^r a_j(\beta)(\mathcal{L}_l\phi_j)(M\alpha - \beta)\\ &= \langle \mathbf{F}, \overline{\mathbf{g}}_l\,\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\rangle_{L^2[0,1)^d}\\ &= \int_{[0,1)^d} \mathbf{F}^\top(x)\mathbf{g}_l(x)\,\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}dx\,.\end{aligned} \tag{3.7}$$

As the sequence $\{\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ is an orthogonal basis for $L^2(M^{-\top}[0,1)^d)$, we can exploit this fact in computing the above integral as follows

$$\begin{aligned}(\mathcal{L}_l f)(M\alpha) &= \sum_{k=1}^{\det M}\int_{Q_k} \mathbf{F}^\top(x)\mathbf{g}_l(x)\,\mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}dx\\ &= \int_{M^{-\top}[0,1)^d}\sum_{k=1}^{\det M} \mathbf{F}^\top(x + M^{-\top}i_k)\mathbf{g}_l(x + M^{-\top}i_k)\ \mathrm{e}^{2\pi\mathrm{i}\alpha^\top M^\top x}dx\,.\end{aligned} \tag{3.8}$$

This leads us to introduce the $s \times (\det M)r$ matrix of functions $\mathbb{G}(x)$, $x \in [0,1)^d$, which, involving the functions in (3.5), is given by

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x + M^{-\top}i_2) & \cdots & \mathbf{g}_1^\top(x + M^{-\top}i_{\det M})\\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x + M^{-\top}i_2) & \cdots & \mathbf{g}_2^\top(x + M^{-\top}i_{\det M})\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x + M^{-\top}i_2) & \cdots & \mathbf{g}_s^\top(x + M^{-\top}i_{\det M})\end{bmatrix} \tag{3.9}$$

$$= \Big[\mathbf{g}_l^\top\big(x + M^{-\top}i_k\big)\Big]_{\substack{l=1,2,\dots,s\\ k=1,2,\dots,\det M}}\,.$$

As we will see in next section, the reconstruction functions S_l, $l = 1, 2, \dots, s$, appearing in formula (3.1) rely on the existence of left inverse matrices of $\mathbb{G}(x)$ having entries in the weighted Wiener algebra $\mathcal{A}_\nu$.

Lemma 3.4. *There exists an* $r \times s$ *matrix* $\mathbf{d}(x) := \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \ldots, \mathbf{d}_s(x)\big)$ *with entries* $d_{j,l} \in \mathcal{A}_\nu$, $j = 1, 2, \ldots, r$, $l = 1, 2, \ldots, s$ *and satisfying*

$$\mathbf{d}(x)\mathbb{G}(x) = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix} = \big[\mathbb{I}_r, \mathbb{O}_{r\times(\det M-1)r}\big], \quad x \in [0,1)^d, \tag{3.10}$$

if and only if rank $\mathbb{G}(x) = (\det M)r$ *for all* $x \in \mathbb{R}^d$.

Proof. Notice that rank $\mathbb{G}(x) = (\det M)r$ if and only if $\det(\mathbb{G}^*(x)\mathbb{G}(x)) \neq 0$ where $\mathbb{G}^*(x)$ denotes the conjugate transpose of $\mathbb{G}(x)$. If rank $\mathbb{G}(x) = (\det M)r$ then the first r rows of the Moore-Penrose pseudo inverse of $\mathbb{G}(x)$,

$$\mathbb{G}^\dagger(x) := \big(\mathbb{G}^*(x)\mathbb{G}(x)\big)^{-1}\mathbb{G}^*(x),$$

satisfy (3.10); moreover, according to Wiener's Lemma (see [55]) the entries of $\mathbb{G}^\dagger$ belong to $\mathcal{A}_\nu$.

Conversely, assume that the $r \times s$ matrix $\mathbf{d}(x) = \big(\mathbf{d}_1(x), \mathbf{d}_2(x), \ldots, \mathbf{d}_s(x)\big)$ satisfies (3.10). We consider the periodic extension of $d_{j,l}$, i.e., $d_{j,l}(x+\alpha) = d_{j,l}(x)$, $\alpha \in \mathbb{Z}^d$. For all $x \in [0,1)^d$, the matrix

$$\mathbb{D}(x) := \begin{bmatrix} \mathbf{d}_1(x) & \mathbf{d}_2(x) & \cdots & \mathbf{d}_s(x) \\ \mathbf{d}_1(x+M^{-\top}i_2) & \mathbf{d}_2(x+M^{-\top}i_2) & \cdots & \mathbf{d}_s(x+M^{-\top}i_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{d}_1(x+M^{-\top}i_{\det M}) & \mathbf{d}_2(x+M^{-\top}i_{\det M}) & \cdots & \mathbf{d}_s(x+M^{-\top}i_{\det M}) \end{bmatrix} \tag{3.11}$$

is a left inverse matrix of $\mathbb{G}(x)$. Therefore, necessarily rank $\mathbb{G}(x) = (\det M)r$, for all $x \in [0,1)^d$. □

Provided that the condition (3.10) in Lemma 3.4 is satisfied, it can be easily checked that all matrices $\mathbf{d}(x)$ with entries in $\mathcal{A}_\nu$, and satisfying (3.10) correspond to the first r rows of the matrices of the form

$$\mathbb{D}(x) = \mathbb{G}^\dagger(x) + \mathbb{U}(x)\big[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\big], \tag{3.12}$$

where $\mathbb{U}(x)$ is any $(\det M)r \times s$ matrix with entries in $\mathcal{A}_\nu$ and $\mathbb{G}^\dagger(x)$ denotes the Moore-Penrose pseudo-inverse of $\mathbb{G}(x)$. Notice that if $s = (\det M)r$ there exists a unique matrix $\mathbf{d}(x)$, given by the first r rows of $\mathbb{G}^{-1}(x)$; if $s > (\det M)r$ there are many solutions according to (3.12).

As it was pointed out in the beginning of this chapter, in proving our sampling result for $V_\nu^p(\Phi)$, $1 \leqslant p \leqslant \infty$, we are going to prove it first for the linear span of

$\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,r}$. In so doing, assume that the set of generators $\Phi = \{\phi_j\}_{j=1}^r$ satisfies, for $j = 1, 2, \cdots, r$, that $\phi_j \in W_0(L^1_\nu)$, this condition ensures that functions in $V_\nu^p(\Phi)$ are continuous (see Lemma 3.3). Consider also s convolution systems $\mathcal{L}_l$, $l = 1, 2, \cdots, s$, with $\mathsf{h}_l \in W(L^1_\nu)$. Under these circumstances we have:

Lemma 3.5. *Let* $\mathbf{d}(x) = \big(d_1(x), d_2(x), \ldots, d_s(x)\big)$ *be an* $r \times s$ *matrix with entries* $d_{j,l} \in \mathcal{A}_\nu$, $j = 1, 2, \ldots, r$, $l = 1, 2, \ldots, s$, *and satisfying condition* (3.10). *Then, for any* $f \in \operatorname{span}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,r}$ *the following sampling expansion holds:*

$$f = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha) \quad \text{in } L_\nu^p(\mathbb{R}^d)\,, \tag{3.13}$$

where the reconstruction function $S_{l,\mathbf{d}}$ *is given by*

$$S_{l,\mathbf{d}}(t) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} \sum_{j=1}^{r} \widehat{d}_{j,l}(\alpha)\phi_j(t-\alpha)\,, \quad t \in \mathbb{R}^d\,, \tag{3.14}$$

with $\widehat{d}_{j,l}(\alpha) := \int_{[0,1)^d} d_{j,l}(x)\, \mathrm{e}^{2\pi \mathrm{i}\alpha^\top x}\, dx$, $\alpha \in \mathbb{Z}^d$, *the Fourier coefficients of the functions* $d_{l,j} \in \mathcal{A}_\nu$, $j = 1, 2, \ldots, r$ *and* $l = 1, 2, \ldots, s$.

Proof. For $f \in \operatorname{span}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\ldots,r}$ let $\mathbf{a} = \{(a_1(\alpha), \ldots, a_r(\alpha))\}$ be the finite sequence such that $f = \sum_{j=1}^r \phi_j *' a_j$ and

$$\begin{aligned}\mathbf{F}^\top(x) : &= \Big(\sum_\alpha a_1(\alpha)\, \mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}, \ldots, \sum_\alpha a_r(\alpha)\, \mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}\Big)\\ &= \sum_\alpha \mathbf{a}(\alpha)\, \mathrm{e}^{-2\pi\mathrm{i}\alpha^\top x}\end{aligned}$$

the corresponding trigonometric polynomial such that $\mathcal{T}_\Phi \mathbf{F} = f$ (see (3.6)).

Having in mind expression (3.8), the sequence of samples $\{(\mathcal{L}_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d}$ forms the Fourier coefficients of the continuous function

$$\sum_{k=1}^{\det M} \mathbf{F}^\top(x + M^{-\top} i_k)\mathbf{g}_l(x + M^{-\top} i_k)$$

with respect to the orthogonal basis $\{\mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ for $L^2(M^{-\top}[0,1)^d)$.

Since $\{\mathcal{L}_l\phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^1_\nu(\mathbb{Z}^d)$ we have that $\{\mathcal{L}_l f(M\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^1_\nu(\mathbb{Z}^d)$; remind that $(\mathcal{L}_l f)(M\alpha)$ is a finite sum $\sum_\beta \sum_{j=1}^r a_j(\beta)(\mathcal{L}_l\phi_j)(M\alpha - \beta)$ and ν is M-compatible. Therefore, for $l = 1, 2 \ldots, s$, we have

$$\sum_{k=1}^{\det M} \mathbf{F}^\top(x + M^{-\top} i_k)\mathbf{g}_l(x + M^{-\top} i_k) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha)\, \mathrm{e}^{-2\pi\mathrm{i}\alpha^\top M^\top x}\,,$$

where $x \in M^{-\top}[0,1)^d$. By periodicity, the above equality also holds for all $x \in [0,1)^d$. Hence we can write

$$\mathbb{G}(x)\mathbb{F}(x) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} \mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x} \Big((\mathcal{L}_1 f)(M\alpha), \dots, (\mathcal{L}_s f)(M\alpha) \Big)^\top$$

where $\mathbb{G}(x)$ is the $s \times (\det M)r$ matrix, defined in (3.9) and

$$\mathbb{F}(x) := \left(\mathbf{F}^\top(x), \mathbf{F}^\top(x + M^{-\top} i_2), \cdots, \mathbf{F}^\top(x + M^{-\top} i_{\det M})\right)^\top .$$

Multiplying on the left by the matrix $\mathbf{d}(x)$ we obtain $\mathbf{F}(x)$ by means of the generalized samples

$$\mathbf{F}(x) = (\det M) \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) \mathbf{d}_l(x)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x}, \quad x \in [0,1)^d . \tag{3.15}$$

Since $\{(\mathcal{L}_l f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d}$ belongs to $\ell^1_\nu(\mathbb{Z}^d)$ and $d_{l,j} \in \mathcal{A}_\nu$, the series in (3.15) also converges in the norm of $\mathcal{A}_\nu \times \ldots \times \mathcal{A}_\nu$. Indeed, for $N \in \mathbb{N}$,

$$\begin{aligned} \Big\| \sum_{|\alpha|>N} (\mathcal{L}_l f)(M\alpha) \mathbf{d}_l(x)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x} \Big\| &\leqslant \|\mathbf{d}_l\| \Big\| \sum_{|\alpha|>N} (\mathcal{L}_l f)(M\alpha)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top x} \Big\|_{\mathcal{A}_\nu} \\ &= \|\mathbf{d}_l\| \sum_{|\alpha|>N} |(\mathcal{L}_l f)(M\alpha)| \nu(\alpha) . \end{aligned}$$

Applying $\mathcal{T}_\Phi$ to both sides of the equality (3.15), and using that

$$\left[\mathcal{T}_\Phi \mathbf{d}_l(\cdot)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha^\top M^\top \cdot}\right](t) = \left[\mathcal{T}_\Phi \mathbf{d}_l\right](t - M\alpha), \quad \alpha \in \mathbb{Z}^d ,$$

we deduce that

$$f = \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha) \quad \text{in } L^p_\nu(\mathbb{R}^d) ,$$

where $S_{l,\mathbf{d}} = (\det M)\mathcal{T}_\Phi \mathbf{d}_l$, for $l = 1, 2, \dots, s$.

The reconstruction functions $S_{l,\mathbf{d}}$, $l = 1, 2, \dots, s$, are determined from the Fourier coefficients of $d_{j,l}$, $\widehat{d}_{j,l}(\alpha) := \int_{[0,1)^d} d_{j,l}(x)\, \mathrm{e}^{2\pi \mathrm{i}\alpha^\top x} dx$. More specifically,

$$S_{l,\mathbf{d}}(t) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} \sum_{j=1}^{r} \widehat{d}_{j,l}(\alpha) \phi_j(t - \alpha) , \quad t \in \mathbb{R}^d . \tag{3.16}$$

The sequence $\widehat{d}_{j,l} \in \ell^1_\nu(\mathbb{Z}^d)$ because the function $d_{j,l}(x) = \sum_{\alpha\in\mathbb{Z}^d} \widehat{d}_{j,l}(\alpha)\, \mathrm{e}^{-2\pi \mathrm{i}\alpha^\top x}$ belongs to $\mathcal{A}_\nu$. As a consequence, the function $S_{l,\mathbf{d}} \in V^1_\nu(\Phi)$. □

Some comments about Lemma 3.5 are in order:

1. We are assuming that rank $\mathbb{G}(x) = (\det M)r$ for all $x \in \mathbb{R}^d$ and, consequently $s \geqslant r(\det M)$.

2. The Fourier transform of $S_{l,\mathbf{d}}$ can be determined from the functions $d_{j,l}$. Indeed, from (3.14), we obtain that

$$\widehat{S}_{l,\mathbf{d}}(w) = (\det M) \sum_{j=1}^{r} d_{j,l}(w)\widehat{\phi}_j(w)\,, \quad w \in \mathbb{R}^d\,.$$

3. In the case $s = (\det M)r$, there is a unique $r \times s$ matrix $\mathbf{d}(x)$ satisfying (3.10), which is that one formed with the first r rows of the matrix $\mathbb{G}^{-1}(x) = \mathbb{D}(x)$ in the notation of (3.11). Then, using (3.8), we obtain that the reconstruction functions $S_{l,\mathbf{d}}$ satisfy in this case an interpolatory property. Namely:

$$\begin{aligned}
&(\mathcal{L}_{l'} S_{l,\mathbf{d}})(M\alpha)\\
&= (\det M) \int_{M^{-\top}[0,1)^d} \sum_{k=1}^{\det M} \mathbf{d}_l(x + M^{-\top} i_k)\mathbf{g}_{l'}(x + M^{-\top} i_k)\, \mathrm{e}^{2\pi \mathrm{i} \alpha^\top M^\top x} dx\\
&= \delta_{l',l}\ (\det M) \int_{M^{-\top}[0,1)^d} \mathrm{e}^{2\pi \mathrm{i} \alpha^\top M^\top x} dx = \begin{cases} 1 & \text{if } l = l' \text{ and } \alpha = 0\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

3.3.3 The average sampling result in $V_\nu^p(\Phi)$ $(1 \leqslant p \leqslant \infty)$

Assume that $\Phi \subset W_0(L_\nu^1)$ and that we have s systems $\mathcal{L}_l$ with $\mathsf{h}_l \in W(L_\nu^1)$ such that there exists an $r \times s$ matrix $\mathbf{d}(x) = \big(d_1(x), d_2(x), \dots, d_s(x)\big)$ with entries $d_{j,l} \in \mathcal{A}_\nu$, $j = 1, 2, \dots, r$, $l = 1, 2, \dots, s$ satisfying condition (3.10). Thus, a density argument allows us to prove that sampling formula (3.13) in Lemma 3.5 is also valid for the whole space $V_\nu^p(\Phi)$. In fact, the following theorem holds:

Theorem 3.3. *Under the above assumptions, for any $f \in V_\nu^p(\Phi)$, $1 \leqslant p \leqslant \infty$, the sampling formula*

$$f = \sum_{l=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha)\,, \tag{3.17}$$

holds in the L_ν^p-sense. The series in (3.17) *also converges absolutely and uniformly to f on $\mathbb{R}^d$.*

Proof. We define on $V_\nu^p(\Phi)$ the sampling operator

$$\begin{array}{rccl}
\Gamma_{\mathbf{d}}: & V_\nu^p(\Phi) & \longrightarrow & V_\nu^p(\Phi)\\
& f & \longmapsto & \Gamma_{\mathbf{d}} f := \sum_{l=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) S_{l,\mathbf{d}}(\cdot - M\alpha)\,.
\end{array}$$

It is a well-defined and bounded operator; indeed, having in mind (3.16) we have

$$
\begin{aligned}
(\Gamma_{\mathbf{d}} f)(t) &= (\det M) \sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) \sum_{\beta\in\mathbb{Z}^d} \sum_{j=1}^{r} \widehat{d}_{j,l}(\beta)\phi_j(t-M\alpha-\beta) \\
&= (\det M) \sum_{j=1}^{r} \sum_{\delta\in\mathbb{Z}^d} \Big(\sum_{l=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha)\widehat{d}_{j,l}(\delta-M\alpha)\Big)\phi_j(t-\delta) \\
&= (\det M) \sum_{j=1}^{r} \sum_{l=1}^{s} \Big(a_{jl} *' \phi_j\Big)(t)\,,
\end{aligned}
$$

where $a_{jl}(\delta) := \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha)\widehat{d}_{j,l}(\delta-M\alpha)$. Notice that,

$$
\begin{aligned}
|a_{jl}(\delta)|\nu(\delta) &= \Big| \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha)\, \widehat{d}_{j,l}(\delta-M\alpha)\Big|\nu(\delta) \\
&\leqslant \sum_{\alpha\in\mathbb{Z}^d} \Big|(\mathcal{L}_l f)(M\alpha)\, \widehat{d}_{j,l}(\delta-M\alpha)\Big|\nu(\delta) \\
&\leqslant \sum_{\alpha\in\mathbb{Z}^d} \Big|(\mathcal{L}_l f)(\alpha)\, \widehat{d}_{j,l}(\delta-\alpha)\Big|\nu(\delta) \\
&\leqslant \sum_{\alpha\in\mathbb{Z}^d} \Big|(\mathcal{L}_l f)(\alpha)\, \widehat{d}_{j,l}(\delta-\alpha)\Big|\nu(\alpha)\nu(\delta-\alpha) \\
&= \Big(\{|(\mathcal{L}_l f)(\alpha)|\nu(\alpha)\} * \{|\widehat{d}_{j,l}(\alpha)|\nu(\alpha)\}\Big)(\delta)\,.
\end{aligned}
$$

Thus, Lemma 3.1(c) gives

$$
\begin{aligned}
\|a_{jl}\|_{\ell^p_\nu} &\leqslant \Big\|\{(\mathcal{L}_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\Big\|_{\ell^p_\nu} \Big\|\{\widehat{d}_{j,l}(\alpha)\}_{\alpha\in\mathbb{Z}^d}\Big\|_{\ell^1_\nu} \\
&\leqslant \|f\|_{L^p_\nu} \|h_l\|_{W(L^1_\nu)} \Big\|\{\widehat{d}_{j,l}(\alpha)\}_{\alpha\in\mathbb{Z}^d}\Big\|_{\ell^1_\nu}\,.
\end{aligned}
\tag{3.18}
$$

In the last step we have used Lemma 3.2. Now, taking into account Lemma 3.1(e), and the fact that the continuous inclusion $W(L^p_\nu) \subset L^p_\nu$ provides a positive constant C such that $\|f\|_{L^p_\nu} \leqslant C\,\|f\|_{W(L^p_\nu)}$, we obtain

$$
\begin{aligned}
\|\Gamma_{\mathbf{d}} f\|_{L^p_\nu} &\leqslant (\det M) \sum_{j=1}^{r} \sum_{l=1}^{s} \Big\|a_{jl} *' \phi_j\Big\|_{L^p_\nu} \\
&\leqslant C\,(\det M) \sum_{j=1}^{r} \sum_{l=1}^{s} \Big\|a_{jl} *' \phi_j\Big\|_{W(L^p_\nu)} \\
&\leqslant C\,(\det M) \sum_{j=1}^{r} \sum_{l=1}^{s} \|a_{jl}\|_{\ell^p_\nu} \|\phi_j\|_{W(L^1_\nu)}\,.
\end{aligned}
\tag{3.19}
$$

Combining (3.19) and (3.18) we deduce the boundedness of the operator $\Gamma_{\mathbf{d}}$.

Now, given $f \in V_\nu^p(\Phi)$, there exists a sequence of functions $\{f_N\}$ contained in $\operatorname{span}\{\phi_j(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,r}$ such that $\|f_N - f\|_{L_\nu^p} \to 0$ as $N \to \infty$. By using Lemma 3.5 we have,

$$\begin{aligned} 0 \leqslant \|f - \Gamma_{\mathbf{d}} f\|_{L_\nu^p} &= \|f - f_N + \Gamma_{\mathbf{d}} f_N - \Gamma_{\mathbf{d}} f\|_{L_\nu^p} \\ &\leqslant \big(1 + \|\Gamma_{\mathbf{d}}\|\big)\|f_N - f\|_{L_\nu^p} \to 0\,,\ N \to \infty\,, \end{aligned}$$

which implies that $\Gamma_{\mathbf{d}} f = f$ in $L_\nu^p(\mathbb{R}^d)$, i.e., the validity of expansion (3.13) in $V_\nu^p(\Phi)$.

The series $\sum_{l=1}^s \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)$ converges, absolutely and uniformly on $\mathbb{R}^d$, to the continuous function f. Indeed,

$$\begin{aligned} \sum_{|\alpha|>N} \big|(\mathcal{L}_l f)(M\alpha) S_{l,\mathbf{d}}(t - M\alpha)\big| &\leqslant \sup_{|\alpha|>N} \big|(\mathcal{L}_l f)(M\alpha)\big| \sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} \big|S_{l,\mathbf{d}}(t - M\alpha)\big| \\ &\leqslant \sup_{|\alpha|>N} \big|(\mathcal{L}_l f)(M\alpha)\big|\nu(M\alpha) \sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} \big|S_{l,\mathbf{d}}(t - M\alpha)\big| \\ &\leqslant \sup_{|\alpha|>N} \big|(\mathcal{L}_l f)(M\alpha)\big|\nu(M\alpha)\, \|S_{l,\mathbf{d}}\|_{W(L_\nu^1)} \longrightarrow 0 \quad \text{as } N \to \infty\,, \end{aligned}$$

uniformly on $\mathbb{R}^d$. In the last inequality we have used that $S_{l,\mathbf{d}} \in V_\nu^1(\Phi) \subset W_0(L_\nu^1)$, $l = 1, 2, \dots, s$ (see Lemma 3.3), and

$$\begin{aligned} \|S_{l,\mathbf{d}}\|_{W(L_\nu^1)} &= \sum_{\alpha\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{t\in[0,1)^d} \big|S_{l,\mathbf{d}}(t + \alpha)\big|\nu(\alpha) \\ &\geqslant \sup_{t\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} \big|S_{l,\mathbf{d}}(t - M\alpha)\big|\,. \end{aligned}$$

□

3.3.4 Dirac sampling case

This subsection is devoted to study another type of linear systems: their impulse response is a translated Dirac delta, i.e., $\big(\mathcal{L}_l f\big)(t) := f(t + c_l),\quad t \in \mathbb{R}^d$, where c_l is a fixed vector in $\mathbb{R}^d$. Provided that $\phi_j \in W(L_\nu^1)$, $j = 1, 2, \dots, r$, the sequence $\{\mathcal{L}_l \phi_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}$ also belongs to $\ell_\nu^1(\mathbb{Z}^d)$. Indeed, for a fixed $c_l \in \mathbb{R}^d$, with $c_l = d_l + x_l,\ x_l \in [0,1)^d$ and $d_l \in \mathbb{Z}^d$, we have

$$\begin{aligned} \sum_{\alpha\in\mathbb{Z}^d} |\phi(\alpha + c_l)|\nu(\alpha) &= \sum_{\beta\in\mathbb{Z}^d} |\phi(\beta + x_l)|\nu(\beta - d_l) \\ &\leqslant \sum_{\beta\in\mathbb{Z}^d} |\phi(\beta + x_l)|\nu(\beta)\nu(d_l) \\ &\leqslant \nu(d_l) \sum_{\beta\in\mathbb{Z}^d} |\phi(\beta + x_l)|\nu(\beta) \\ &\leqslant \nu(d_l) \sum_{\beta\in\mathbb{Z}^d} \operatorname*{ess\,sup}_{x\in[0,1]^d} |\phi(\beta + x)|\nu(\beta) = \nu(d_l)\|\phi\|_{W(L_\nu^1)}\,. \end{aligned}$$

Thus, for these new systems the functions defined in (3.5) make sense.

In order to extend Theorem 3.3 for the case $1 \leqslant p < \infty$ we need to assume stronger hypotheses on the set of generators $\Phi = \{\phi_j\}_{j=1}^r$ in $W_0(L^1_\nu)$. Next, we state the L^p_ν-stable shifts concept as established in [75] for the non-weighted case. Note that the space $W_0(L^1_\nu)$ is included in the corresponding $\mathcal{L}^\infty_\nu(\mathbb{R}^d)$ space, defined in [75] as

$$\mathcal{L}^\infty_\nu(\mathbb{R}^d) := \Big\{ f \text{ measurable} \ : \ \|f\|_{\mathcal{L}^\infty_\nu} := \operatorname*{ess\,sup}_{x\in[0,1)^d} \sum_{\alpha\in\mathbb{Z}^d} |f(x+\alpha)|\nu(x+\alpha) < \infty \Big\} .$$

Definition 3.3. *For $1 \leqslant p < \infty$, a finite subset $\Phi = \{\phi_j\}_{j=1}^r$ of $W_0(L^1_\nu)$ is said to have L^p_ν-stable shifts if there exist positive constants $0 < A \leqslant B$ (depending on p and Φ) such that*

$$A \sum_{j=1}^r \|a_j\|_{\ell^p_\nu} \leqslant \Big\| \sum_{j=1}^r \phi_j *' a_j \Big\|_{L^p_\nu} \leqslant B \sum_{j=1}^r \|a_j\|_{\ell^p_\nu} , \tag{3.20}$$

for any sequence $a_j \in \ell^p_\nu(\mathbb{Z}^d)$, $j = 1, 2, \dots, r$, when $1 \leqslant p < \infty$.

Given $f \in V^p_\nu(\Phi)$, i.e., $f(t) = \sum_{j=1}^r \sum_{\beta\in\mathbb{Z}^d} a_j(\beta)\, \phi_j(t-\beta)$ with $\{a_j(\beta)\}_{\beta\in\mathbb{Z}^d} \in \ell^p_\nu$ for $j = 1, 2, \dots, r$, we have

$$\begin{aligned}
(\mathcal{L}_l f)(\alpha) = f(\alpha + c_l) &= \sum_{j=1}^r \sum_{\beta\in\mathbb{Z}^d} a_j(\beta)\, \phi_j(\alpha + c_l - \beta) \\
&= \sum_{j=1}^r \Big(\{a_j(\beta)\}_{\beta\in\mathbb{Z}^d} * \{\phi_j(\beta + c_l)\}_{\beta\in\mathbb{Z}^d} \Big)(\alpha) .
\end{aligned}$$

Having in mind the first inequality in (3.18), in proving Theorem 3.3 we just need an inequality like

$$\big\| \{(\mathcal{L}_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d} \big\|_{\ell^p_\nu} \leqslant K \, \|f\|_{L^p_\nu} .$$

Since $\{\phi_j(\beta + c_l)\}_{\beta\in\mathbb{Z}^d} \in \ell^1_\nu$, from Lemma 3.1$(c)$ there exists a positive constant K_1 such that

$$\|\{(\mathcal{L}_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} \leqslant K_1 \sum_{j=1}^r \|\{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} .$$

Finally, from the left inequality in (3.20) we get

$$\|\{(\mathcal{L}_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^p_\nu} \leqslant K_2 \|f\|_{L^p_\nu} ,$$

where K_2 is a positive constant. Thus, Theorem 3.3 can be extended to Dirac's systems, whenever $1 \leqslant p < \infty$. Due to the inequality $\|\{(\mathcal{L}_l f)(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_{\ell^\infty_\nu} \leqslant \|f\|_{L^\infty_\nu}$, the case $p = \infty$ becomes trivial.

Finally, it is worth to mention that Theorem 3.3 remains true for linear combinations of average and Dirac's systems.

4

Sampling theory in U-invariant spaces

4.1 By way of motivation

The aim in this chapter is to derive a generalized sampling theory for U-invariant subspaces of a separable Hilbert space $\mathcal{H}$, where $U : \mathcal{H} \to \mathcal{H}$ denotes an unitary operator. The motivation for this work can be found in the previous chapters. To be more precise in the generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$; there $\mathcal{H} := L^2(\mathbb{R})$ and U is the shift operator $T : f(t) \mapsto f(t-1)$ in $L^2(\mathbb{R})$. Namely, assume that our functions (signals) belong to some (principal) shift-invariant subspace

$$V_\varphi^2 := \overline{\operatorname{span}}_{L^2(\mathbb{R})}\big\{\varphi(t-n),\ n \in \mathbb{Z}\big\}\,,$$

where the generator function φ belongs to $L^2(\mathbb{R})$ and the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$. Recall that a *Riesz sequence* in $\mathcal{H}$ is a Riesz basis for its closed span. Thus, the shift-invariant space V_φ^2 can be described as

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, \varphi(t-n) \,:\, \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}\,.$$

Mathematically, the generalized sampling problem consists of the stable recovery of any $f \in V_\varphi^2$ from the above sequence of samples, i.e., to obtain sampling formulas in

V_φ^2 having the form

$$f(t) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \big(\mathcal{L}_j f\big)(rm)\, S_j(t-rm)\,, \quad t \in \mathbb{R}\,,$$

such that the sequence of reconstruction functions $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for the shift-invariant space V_φ^2. In this case the sequence of samples

$$\{(\mathcal{L}_j f)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

has been obtained by means of s convolution systems $\mathcal{L}_j f := f * \mathsf{h}_j$, $j = 1, 2, \dots, s$, which are defined on V_φ^2.

As it was said in Chapter 2, sampling in shift-invariant spaces of $L^2(\mathbb{R})$ (or $L^2(\mathbb{R}^d)$), with one or multiple generators, has been profusely treated in the mathematical literature. See, for instance, Refs. [8, 6, 27, 28, 32, 37, 41, 45, 68, 83, 102, 106, 107, 110, 113] and references therein.

In the present chapter we provide a generalization of the above problem in the following sense. Let U be an unitary operator in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by

$$\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}\,.$$

In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}\,.$$

In order to generalize convolution systems and mainly to obtain some perturbation results in this new setting, we assume that the operator U is included in a continuous *group of unitary operators* $\{U^t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ as $U := U^1$. Recall that $\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying (see Ref. [4, vol. 2; p. 29]; see also Refs. [14, 89, 114]):

(1) $U^t\, U^{t'} = U^{t+t'}$,

(2) $U^0 = I_{\mathcal{H}}$,

(3) $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

Note that $(U^t)^{-1} = U^{-t}$, and since $(U^t)^* = (U^t)^{-1}$, we have $(U^t)^* = U^{-t}$. For more details concerning continuous group of operators and the results we will use in what follows we refer to Appendix B and references therein.

Thus, for $b \in \mathcal{H}$ we consider the linear operator $\mathcal{H} \ni x \longmapsto \mathcal{L}_b x \in C(\mathbb{R})$ such that

$$\big(\mathcal{L}_b x\big)(t) := \langle x, U^t b\rangle_{\mathcal{H}} \qquad \text{for every } t \in \mathbb{R}\,.$$

Given U-systems $\mathcal{L}_j$, $j = 1, 2, \dots, s$, corresponding to s elements $b_j \in \mathcal{H}$, i.e., $\mathcal{L}_j \equiv \mathcal{L}_{b_j}$ for each $j = 1, 2, \dots, s$, the generalized regular sampling problem in $\mathcal{A}_a$ consists of the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of the samples

$$\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s} \quad \text{where} \quad r \in \mathbb{N}\,, r \geqslant 1\,.$$

This U-sampling problem has been treated, for the first time, in some recent papers [79, 86]. Sampling in shift-invariant subspaces or in modulation-invariant subspaces of $L^2(\mathbb{R})$ becomes a particular case of U-sampling associated, respectively, with the translation operator $T_a : f(t) \mapsto f(t-a)$ or with the modulation operator $M_b : f(t) \mapsto \mathrm{e}^{\mathrm{i}bt} f(t)$ in $L^2(\mathbb{R})$.

The operators $\mathcal{L}_b$ can be seen as a generalization of the convolution systems in $L^2(\mathbb{R})$. Note that, for the shift operator $U : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$, we have

$$\langle f, U^t b\rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(u)\, \overline{b(u-t)} du = (f * h)(t)\,, \quad t \in \mathbb{R}\,,$$

where $h(u) := \overline{b(-u)}$, $u \in \mathbb{R}$.

Here we propose a completely different approach which allows to analyze in depth the U-sampling problem. In Section 4.3 we prove the existence of frames in $\mathcal{A}_a$, having the form $\big\{U^{rm} c_j\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$, where $c_j \in \mathcal{A}_a$ for $j = 1, 2, \dots, s$, such that for each $x \in \mathcal{A}_a$ the sampling expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, U^{rm} c_j \quad \text{in } \mathcal{H} \tag{4.1}$$

holds. To this end, as in the shift-invariant case (see, for instance, Refs. [41, 45]), we use that the above sampling formula is intimately related with some special dual frames in $L^2(0,1)$ (see Section 4.2 below) via the isomorphism

$$\mathcal{T}_{U,a} : L^2(0,1) \longrightarrow \mathcal{A}_a$$

mapping the orthonormal basis $\{\mathrm{e}^{2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$. In [86] regular sampling expansions like (4.1) are obtained by using a completely different technique; basically, they use the cross-covariance function $\langle U^n a, b_j\rangle_{\mathcal{H}}$ between the sequences $\{U^n a\}_{n\in\mathbb{Z}}$ and $\{U^n b_j\}_{n\in\mathbb{Z}}$, $j = 1, 2, \dots, s$. We developed a version of this work adapted to our framework in Subsection 4.3.1 below.

Strictly speaking, we do not need the formalism of the continuous group of unitary operators to derive the sampling results in Section 4.3 since we only use the discrete group $\{U^n\}_{n\in\mathbb{Z}}$ which is completely determined by U. However, for the study, in Section 4.4, of the time-jitter error in sampling formulas as in (4.1), the continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ becomes essential. In this case we dispose of a perturbed sequence of samples

$$\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

with errors $\epsilon_{mj} \in \mathbb{R}$, for the recovery of $x \in \mathcal{A}_a$. We prove that, for small enough errors ϵ_{mj}, the stable recovery of any $x \in \mathcal{A}_a$ is still possible. Finally, in Section 4.5 we deal with the case of multiple stable generators. We only sketch the procedure since it is essentially identical to the one generator case.

4.2 On sampling in U-invariant subspaces

For a fixed $a \in \mathcal{H}$, assume that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$. Thus, the U-invariant subspace $\mathcal{A}_a := \overline{\operatorname{span}}\{U^n a,\ n \in \mathbb{Z}\}$ can be expressed as

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\} .$$

For simplicity and ease of notation we are considering the one generator setting; as we have said before the same sampling results for the general case can be obtained by analogy, and it will be drawn in Section 4.5.

Since the inner product $\langle U^n a, U^m a\rangle_{\mathcal{H}}$ depends only on the difference $n - m \in \mathbb{Z}$, the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is an *stationary sequence*. Moreover, the *auto-covariance* R_a of the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta) , \qquad k \in \mathbb{Z} ,$$

in terms of a positive Borel measure μ_a on $(-\pi, \pi)$ called the *spectral measure* of the sequence (see [72]). This is obtained from the integral representation of the unitary operator U on $\mathcal{H}$ (see, for instance, Refs. [4, 120]). The spectral measure μ_a can be decomposed into an absolute continuous and a singular part as

$$d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta) .$$

A necessary and sufficient condition in order for the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ to be a Riesz sequence for $\mathcal{H}$ is given in next theorem in terms of the decomposition of the spectral measure μ_a:

Theorem 4.1. *Let $\{U^n a\}_{n\in\mathbb{Z}}$ be a sequence obtained from an unitary operator in a separable Hilbert space $\mathcal{H}$ with spectral measure $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$, and let $\mathcal{A}_a$ be the closed subspace spanned by $\{U^n a\}_{n\in\mathbb{Z}}$. Then the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and the function ϕ_a (called the spectral density of the stationary sequence) satisfies*

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \leqslant \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty .$$

Proof. Theorem 4.1 is just the one generator case ($L = 1$) of Theorem 4.11 below. □

Proposition 4.1. *Let* $\{U^n a\}_{n\in\mathbb{Z}}$ *be a Riesz basis for* $\mathcal{A}_a$ *with* spectral density $\phi_a(\theta)$, *then the dual Riesz basis is given by the sequence* $\{U^n b\}_{n\in\mathbb{Z}}$, *where the coefficients of* b *with respect to* $\{U^n a\}_{n\in\mathbb{Z}}$ *are the Fourier coefficients of the function* $1/\phi_a \in L^2(-\pi,\pi)$. *Moreover the spectral density of the Riesz basis* $\{U^n b\}_{n\in\mathbb{Z}}$ *is precisely* $\phi_b(\theta) = \frac{1}{\phi_a(\theta)}$.

Proof. Consider the expansion $b = \sum_{k\in\mathbb{Z}} b_k\, U^k a \in \mathcal{A}_a$, the biorthogonality between the sequences $\{U^n a\}_{n\in\mathbb{Z}}$ and $\{U^n b\}_{n\in\mathbb{Z}}$ means

$$\begin{aligned}\delta_{m,0} = \langle U^m a, b\rangle_{\mathcal{H}} &= \langle U^m a, \sum_{k\in\mathbb{Z}} b_k\, U^k a\rangle_{\mathcal{H}} = \sum_{k\in\mathbb{Z}} \overline{b}_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(m-k)\theta}\phi_a(\theta)d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(\sum_{k\in\mathbb{Z}} \overline{b}_k \mathrm{e}^{-ik\theta}\big)\phi_a(\theta)\mathrm{e}^{im\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\theta)\phi_a(\theta)\mathrm{e}^{-im\theta}\, d\theta\,,\end{aligned}$$

where $B(\theta) := \sum_{k\in\mathbb{Z}} b_k\, \mathrm{e}^{ik\theta}$; in other words, we have $B(\theta)\phi_a(\theta) \equiv 1$ in $L^2(-\pi,\pi)$, as a consequence, the terms of the sequence $\{b_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ are the Fourier coefficients of the function $1/\phi_a(\theta) \in L^2(-\pi,\pi)$. Moreover,

$$\begin{aligned}\langle U^n b, b\rangle_{\mathcal{H}} &= \langle \sum_{k\in\mathbb{Z}} b_k\, U^{n+k} a, \sum_{l\in\mathbb{Z}} b_l\, U^l a\rangle_{\mathcal{H}} \\ &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} b_k \overline{b}_l \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i(n+k-l)\theta}\phi_a(\theta)d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{in\theta}\big(\sum_{k\in\mathbb{Z}} b_k \mathrm{e}^{ik\theta}\big)\big(\sum_{l\in\mathbb{Z}} \overline{b}_l \mathrm{e}^{-ik\theta}\big)\phi_a(\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{in\theta}\big|B(\theta)\big|^2\phi_a(\theta)\, d\theta\,.\end{aligned}$$

Therefore, $\phi_b(\theta) = \big|B(\theta)\big|^2\phi_a(\theta) = 1/\phi_a(\theta), \theta \in (-\pi,\pi)$; that is, for $n \in \mathbb{Z}$ we obtain

$$\langle U^n b, b\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{in\theta}\frac{d\theta}{\phi_a(\theta)}\,.$$

□

Finally, for the shift operator $T : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$, Theorem 4.1 allows to recover the classical necessary and sufficient condition for the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$, where $\varphi \in L^2(\mathbb{R})$, to be a Riesz basis for the corresponding shift-invariant subspace V_φ^2 in $L^2(\mathbb{R})$. Indeed, consider the Fourier transform as $\widehat{\varphi}(\theta) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(t)\, \mathrm{e}^{-it\theta}\, d\theta$ in

$L^1(\mathbb{R}) \cap L^2(\mathbb{R})$; using the Parseval's equality one easily gets

$$\begin{aligned}
\langle T^k\varphi, \varphi\rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} \varphi(u-k)\,\overline{\varphi(u)}\,du = \int_{-\infty}^{\infty} \widehat{\varphi(u-k)}(\theta)\overline{\widehat{\varphi}(\theta)}\,d\theta \\
&= \int_{-\infty}^{\infty} |\widehat{\varphi}(\theta)|^2\,\mathrm{e}^{-ik\theta}\,d\theta = \int_{-\pi}^{\pi} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2\,\mathrm{e}^{-ik\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-ik\theta}\,2\pi \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(-\theta+2\pi n)|^2\,d\theta\,,
\end{aligned}$$

that is,

$$\phi_\varphi(\theta) = 2\pi \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(-\theta+2\pi n)|^2\,, \quad \theta\in(-\pi,\pi)\,.$$

Thus, Theorem 4.1 yields the aforementioned classical condition (see, for instance, [25, p.143]):

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 \leqslant \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 < \infty\,.$$

The following isomorphism between $L^2(0,1)$ and $\mathcal{A}_a$ will be crucial along the chapter:

The isomorphism $\mathcal{T}_{U,a}$

We define the isomorphism $\mathcal{T}_{U,a}$ which maps the orthonormal basis $\{\mathrm{e}^{2\pi\mathrm{i}nw}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$, that is,

$$\begin{array}{rccc}
\mathcal{T}_{U,a}: & L^2(0,1) & \longrightarrow & \mathcal{A}_a \\
& F=\displaystyle\sum_{n\in\mathbb{Z}} \alpha_n\,\mathrm{e}^{2\pi\mathrm{i}nw} & \longmapsto & x=\displaystyle\sum_{n\in\mathbb{Z}} \alpha_n\,U^n a\,.
\end{array}$$

The following U-shift property holds: For any $F\in L^2(0,1)$ and $N\in\mathbb{Z}$, we have

$$\mathcal{T}_{U,a}\Big(F\,\mathrm{e}^{2\pi\mathrm{i}Nw}\Big) = U^N\big(\mathcal{T}_{U,a}F\big)\,. \tag{4.2}$$

The U-systems

For any fixed $b\in\mathcal{H}$ we define the U-system $\mathcal{L}_b$ as the linear operator between $\mathcal{H}$ and the set $C(\mathbb{R})$ of the continuous functions on $\mathbb{R}$ given by

$$\mathcal{H}\ni x\longmapsto \mathcal{L}_b x\in C(\mathbb{R}) \ \text{ such that } \ \mathcal{L}_b x(t) := \langle x, U^t b\rangle_{\mathcal{H}}\,, \quad t\in\mathbb{R}\,.$$

For any $x \in \mathcal{A}_a$ and $t \in \mathbb{R}$, by using the Plancherel equality for the orthonormal basis $\{\mathrm{e}^{2\pi i n w}\}_{n\in\mathbb{Z}}$ in $L^2(0,1)$, we have

$$\begin{aligned}\mathcal{L}_b x(t) &= \langle x, U^t b\rangle_{\mathcal{H}} = \Big\langle \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a, U^t b\Big\rangle_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \alpha_n\, \overline{\langle U^t b, U^n a\rangle_{\mathcal{H}}} \\ &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^t b, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i n w}\Big\rangle_{L^2(0,1)} = \big\langle F, K_t\big\rangle_{L^2(0,1)}\,,\end{aligned} \tag{4.3}$$

where $\mathcal{T}_{U,a}F = x$, and the function

$$K_t(w) := \sum_{n\in\mathbb{Z}} \langle U^t b, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i n w} = \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a(t-n)}\, \mathrm{e}^{2\pi i n w}$$

belongs to $L^2(0,1)$ since the sequence $\big\{\langle U^t b, U^n a\rangle_{\mathcal{H}}\big\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ for each $t \in \mathbb{R}$; note that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$.

An expression for the generalized samples

Suppose that s vectors $b_j \in \mathcal{H}$, $j = 1,2,\ldots,s$, are given and consider their associated U-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$, $j = 1,2,\ldots,s$. Our aim is the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of samples $\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s}$ where $r \geqslant 1$. To this end, first we obtain a suitable expression for the samples. For $x \in \mathcal{A}_a$ let $F \in L^2(0,1)$ such that $\mathcal{T}_{U,a}F = x$; by using (4.3), for $j = 1,2,\ldots s$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned}\mathcal{L}_j x(rm) &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^{rm} b_j, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i n w}\Big\rangle_{L^2(0,1)} \\ &= \Big\langle F, \sum_{k\in\mathbb{Z}} \langle U^k b_j, a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i (rm-k) w}\Big\rangle_{L^2(0,1)} \\ &= \Big\langle F, \Big[\sum_{k\in\mathbb{Z}} \overline{\langle a, U^k b_j\rangle_{\mathcal{H}}}\, \mathrm{e}^{-2\pi i k w}\Big]\, \mathrm{e}^{2\pi i r m w}\Big\rangle_{L^2(0,1)}\,,\end{aligned}$$

where the change in the summation's index $k := rm - n$ has been done. Hence,

$$\mathcal{L}_j x(rm) = \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\rangle_{L^2(0,1)} \quad \text{for } m \in \mathbb{Z} \text{ and } j = 1,2,\ldots,s\,, \tag{4.4}$$

where the function

$$g_j(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k)\, \mathrm{e}^{2\pi i k w} \tag{4.5}$$

belongs to $L^2(0,1)$ for each $j = 1,2,\ldots,s$.

As a consequence of (4.4), the stable recovery of any $x \in \mathcal{A}_a$ depends on whether the sequence

$$\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s}$$

forms a frame for $L^2(0,1)$. First we need to introduce some notations. Namely, consider the $s \times r$ matrix of functions in $L^2(0,1)$

$$\mathbb{G}(w) := \begin{bmatrix} g_1(w) & g_1(w+\frac{1}{r}) & \cdots & g_1(w+\frac{r-1}{r}) \\ g_2(w) & g_2(w+\frac{1}{r}) & \cdots & g_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & \ddots & \vdots \\ g_s(w) & g_s(w+\frac{1}{r}) & \cdots & g_s(w+\frac{r-1}{r}) \end{bmatrix} \tag{4.6}$$

$$= \left[g_j\Big(w + \frac{k-1}{r}\Big)\right]_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,r}}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

where $\mathbb{G}^*(w)$ denotes the transpose conjugate of the matrix $\mathbb{G}(w)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(w)\mathbb{G}(w)$. Observe that $0 \leqslant \alpha_{\mathbb{G}} \leqslant \beta_{\mathbb{G}} \leqslant \infty$. Notice that in the definition of the matrix $\mathbb{G}(w)$ we are considering 1-periodic extensions of the involved functions $g_j,\ j = 1,2,\ldots,s$.

A complete characterization of the sequence $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s}$ in the space $L^2(0,1)$ is obtained from Lemma 2.3, as a particular case; here the dimension $d = 1$, the number of generators is $r = 1$ and the sampling lattice M is an scalar $r \in \mathbb{N}$ (see also Refs. [41, 45]):

Lemma 4.1. *For the functions $g_j \in L^2(0,1)$, $j = 1,2,\ldots,s$, consider the associated matrix $\mathbb{G}(w)$ given in* (4.6). *Then, the following results hold:*

(a) The sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a complete system for $L^2(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is r a.e. in $(0,1/r)$.

(b) The sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a Bessel sequence for $L^2(0,1)$ if and only if $g_j \in L^\infty(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.

(c) The sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \leqslant \beta_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.

(d) The sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a Riesz basis for $L^2(0,1)$ if and only if is a frame and $s = r$.

A comment about Lemma 4.1 in terms of the average sampling terminology introduced by Aldroubi et al. in [10] is in order. According to [10] we say that

1. The set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an *r-determining U-sampler* for $\mathcal{A}_a$ if the only vector $x \in \mathcal{A}_a$, satisfying $\mathcal{L}_j x(rm) = 0$ for all $j = 1, 2, \dots, s$ and $m \in \mathbb{Z}$, is $x = 0$.
2. The set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an *r-stable U-sampler* for $\mathcal{A}_a$ if there exist positive constants A and B such that
$$A\|x\|^2 \leqslant \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} |\mathcal{L}_j x(rm)|^2 \leqslant B\|x\|^2 \quad \text{for all } x \in \mathcal{A}_a.$$

Hence, parts (a) and (c) of Lemma 4.1 can be read, by using (4.4), as follows:

i. The set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an r-determining U-sampler for $\mathcal{A}_a$ if and only if $\operatorname{rank} \mathbb{G}(w) = r$ a.e. in $(0, 1)$ (and hence, necessarily, $s \geqslant r$).

ii. The set $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an r-stable U-sampler for $\mathcal{A}_a$ if and only if
$$0 < \alpha_{\mathbb{G}} \leqslant \beta_{\mathbb{G}} < \infty\,.$$

An r-determining U-sampler for $\mathcal{A}_a$ can distinguish between two distinct elements in $\mathcal{A}_a$, but the recovery, if any, is not necessarily stable. If the system $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an r-stable U-sampler for $\mathcal{A}_a$, then any $x \in \mathcal{A}_a$ can be recovered, in a stable way, from the sequence of generalized samples $\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$, where necessarily the inequality $s \geqslant r$ holds. Roughly speaking, the operator which maps
$$\mathcal{A}_a \ni x \longmapsto \big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s} \in \ell^2_s(\mathbb{Z}) := \underbrace{\ell^2(\mathbb{Z}) \times \cdots \times \ell^2(\mathbb{Z})}_{(s \text{ times})}$$
has a bounded inverse.

Having in mind (4.4), from the sequence of samples $\big\{\mathcal{L}_j x(rm)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ we recover $F \in L^2(0, 1)$, and by means of the isomorphism $\mathcal{T}_{U,a}$, the vector $x = \mathcal{T}_{U,a} F \in \mathcal{A}_a$. This will be the main goal in the next section:

4.3 Generalized regular sampling in $\mathcal{A}_a$

Along with the characterization of the sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ as a frame in $L^2(0, 1)$, in [41] a family of dual frames are also given: Choose functions h_j in $L^\infty(0, 1)$, $j = 1, 2, \dots, s$, such that
$$\big[h_1(w), h_2(w), \dots, h_s(w)\big]\mathbb{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0, 1)\,. \tag{4.7}$$

It was proven in [41] that the sequence $\{rh_j(w)\,\mathrm{e}^{2\pi i r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a dual frame of the sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i r n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ in $L^2(0,1)$. In other words, taking into account (4.4), we have for any $F \in L^2(0,1)$ the expansion

$$F = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, rh_j(w)\,\mathrm{e}^{2\pi i r m w} \quad \text{in } L^2(0,1)\,. \tag{4.8}$$

This a particular case of Eq. (2.19).

Concerning to the existence of the functions h_j, $j = 1,2,\dots,s$, consider the first row of the $r \times s$ Moore-Penrose pseudo inverse $\mathbb{G}^\dagger(w)$ of $\mathbb{G}(w)$ given by

$$\mathbb{G}^\dagger(w) := \big[\mathbb{G}^*(w)\,\mathbb{G}(w)\big]^{-1}\,\mathbb{G}^*(w)\,.$$

Its entries are essentially bounded in $(0,1)$ since the functions g_j, $j = 1,2,\dots,s$, and $\det^{-1}\big[\mathbb{G}^*(w)\,\mathbb{G}(w)\big]$ are essentially bounded in $(0,1)$, and (4.7) trivially holds. All the possible solutions of (4.7) are given by the first row of the $r \times s$ matrices given by

$$\mathbb{H}_{\mathbb{U}}(w) := \mathbb{G}^\dagger(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^\dagger(w)\big]\,, \tag{4.9}$$

where $\mathbb{U}(w)$ denotes any $r \times s$ matrix with entries in $L^\infty(0,1)$, and $\mathbb{I}_s$ is the identity matrix of order s.

Applying the isomorphism $\mathcal{T}_{U,a}$ in (4.8), for $x = \mathcal{T}_{U,a}F \in \mathcal{A}_a$ we obtain the sampling expansion:

$$\begin{aligned} x &= \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\,\mathcal{T}_{U,a}\big[rh_j(\cdot)\,\mathrm{e}^{2\pi i r m\,\cdot}\big] \\ &= \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\,U^{rm}\big[\mathcal{T}_{U,a}(rh_j)\big] \\ &= \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\,U^{rm} c_{j,h} \quad \text{in } \mathcal{H}\,, \end{aligned} \tag{4.10}$$

where $c_{j,h} := \mathcal{T}_{U,a}(rh_j) \in \mathcal{A}_a$, $j = 1,2,\dots,s$, and we have used the U-shift property (4.2). Besides, the sequence $\big\{U^{rm}c_{j,h}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$. In fact, the following result holds:

Theorem 4.2. *Let b_j be in $\mathcal{H}$ and let $\mathcal{L}_j$ be its associated U-system for $j = 1,2,\dots,s$. Assume that the function g_j, $j = 1,2,\dots,s$, given in* (4.5) *belongs to $L^\infty(0,1)$; or equivalently, $\beta_{\mathbb{G}} < \infty$ for the associated $s \times r$ matrix $\mathbb{G}(w)$. The following statements are equivalent:*

(a) $\alpha_{\mathbb{G}} > 0$

(b) *There exists a vector* $\big[h_1(w), h_2(w), \dots, h_s(w)\big]$ *with entries in* $L^\infty(0,1)$ *satisfying*

$$\big[h_1(w), h_2(w), \dots, h_s(w)\big]\mathbb{G}(w) = [1, 0, \dots, 0] \quad \textit{a.e. in } (0,1)\,.$$

(c) *There exist* $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, *such that the sequence* $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ *is a frame for* $\mathcal{A}_a$, *and for any* $x \in \mathcal{A}_a$ *the expansion*

$$x = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\, U^{rk}c_j \quad \textit{in } \mathcal{H}\,, \tag{4.11}$$

holds.

(d) *There exists a frame* $\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ *for* $\mathcal{A}_a$ *such that, for each* $x \in \mathcal{A}_a$ *the expansion*

$$x = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\, C_{j,k} \quad \textit{in } \mathcal{H}\,,$$

holds.

Proof. We have already proved that (a) implies (b) and that (b) implies (c). Obviously, (c) implies (d). As a consequence, we only need to prove that (d) implies (a). Applying the isomorphism $\mathcal{T}_{U,a}^{-1}$ to the expansion in (d), and taking into account (4.4) we obtain

$$\begin{aligned} F = \mathcal{T}_{U,a}^{-1}x &= \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\, \mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big) \\ &= \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw}\big\rangle_{L^2(0,1)}\, \mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big) \quad \text{in } L^2(0,1)\,, \end{aligned}$$

where the sequence $\big\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $L^2(0,1)$. The sequence $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ is a Bessel sequence in $L^2(0,1)$ since $\beta_{\mathbb{G}} < \infty$, and satisfying the above expansion in $L^2(0,1)$. According to Proposition A.4 the sequences $\big\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ and $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rkw}\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ form a pair of dual frames in $L^2(0,1)$; in particular, by using Lemma 4.1 we obtain that $\alpha_{\mathbb{G}} > 0$ which concludes the proof. □

In case the functions g_j, $j = 1, 2, \dots, s$ are continuous on $\mathbb{R}$, condition (a) in Theorem 4.2 can be expressed in terms of the rank of the matrix $\mathbb{G}(w)$; notice that this occurs, for example, whenever the sequences $\{\mathcal{L}_j a(k)\}_{k\in\mathbb{Z}}$, $j = 1, 2, \dots, s$, belong to $\ell^1(\mathbb{Z})$.

Corollary 4.1. *Assume that the* 1*-periodic extension of the functions* g_j, $j = 1, 2, \dots, s$, *given in* (4.5) *are continuous on* $\mathbb{R}$. *Then, the following conditions are equivalent:*

(i) $\operatorname{rank} \mathbb{G}(w) = r$ *for all* $w \in \mathbb{R}$.

(ii) There exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, *such that the sequence* $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ *is a frame for* $\mathcal{A}_a$, *and the sampling formula* (4.11) *holds for each* $x \in \mathcal{A}_a$.

Proof. Whenever the functions g_j, $j = 1, 2, \ldots, s$, are continuous on $\mathbb{R}$, the condition $\alpha_{\mathbb{G}} > 0$ is equivalent to $\det \big[\mathbb{G}^*(w)\mathbb{G}(w)\big] \neq 0$ for all $w \in \mathbb{R}$.

Indeed, if $\det \mathbb{G}^*(w)\mathbb{G}(w) > 0$ then the first row of the matrix

$$\mathbb{G}^{\dagger}(w) := [\mathbb{G}^*(w)\mathbb{G}(w)]^{-1}\mathbb{G}^*(w)\,,$$

gives a vector $[h_1, h_2, \ldots, h_s]$ satisfying the statement (b) in Theorem 4.2 and, as a consequence, $\alpha_{\mathbb{G}} > 0$.

The reciprocal follows from the fact that $\det \big[\mathbb{G}^*(w)\mathbb{G}(w)\big] \geqslant \alpha_{\mathbb{G}}^r$ for all $w \in \mathbb{R}$.

Since, $\det \big[\mathbb{G}^*(w)\mathbb{G}(w)\big] \neq 0$ is equivalent to rank $\mathbb{G}(w) = r$ for all $w \in \mathbb{R}$, the result is a consequence of Theorem 4.2. □

Whenever the sampling period r equals the number of U-systems s we are in the presence of Riesz bases, and there exists a unique sampling expansion in Theorem 4.2:

Corollary 4.2. *Let* b_j *be in* $\mathcal{H}$ *for* $j = 1, 2, \ldots, r$, *i.e.,* $r = s$ *in Theorem 4.2. Let* $\mathcal{L}_j$ *be its associated* U*-system for* $j = 1, 2, \ldots, r$. *Assume that the function* g_j, $j = 1, 2, \ldots, r$, *given in* (4.5) *belongs to* $L^\infty(0, 1)$*; or equivalently,* $\beta_{\mathbb{G}} < \infty$ *for the associated* $r \times r$ *matrix* $\mathbb{G}(w)$. *The following statements are equivalent:*

(1) $\alpha_{\mathbb{G}} > 0$.

(2) There exists a Riesz basis $\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ *such that for any* $x \in \mathcal{A}_a$ *the expansion*

$$x = \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\, C_{j,k} \quad \text{in } \mathcal{H} \tag{4.12}$$

holds.

In case the equivalent conditions are satisfied, necessarily there exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, r$, *such that* $C_{j,k} = U^{rk} c_j$ *for* $k \in \mathbb{Z}$ *and* $j = 1, 2, \ldots, r$. *Moreover, the interpolation property* $\mathcal{L}_{j'} c_j(rk) = \delta_{j,j'}\, \delta_{k,0}$, *where* $k \in \mathbb{Z}$ *and* $j, j' = 1, 2, \ldots, r$, *holds.*

Proof. Assume that $\alpha_{\mathbb{G}} > 0$; since $\mathbb{G}(w)$ is a square matrix, this implies that

$$\operatorname*{ess\,inf}_{w\in\mathbb{R}} |\det \mathbb{G}(w)| > 0\,.$$

Therefore, the first row of $\mathbb{G}^{-1}(w)$ gives the unique solution $[h_1(w), h_2(w), \dots, h_r(w)]$ of (4.7) with $h_j \in L^\infty(0,1)$ for $j = 1, 2, \dots, r$.
According to Theorem 4.2, the sequence

$$\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r} := \{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}\,,$$

where $c_j = \mathcal{T}_{U,a}(rh_j)$, satisfies the sampling formula (4.12). Moreover, the sequence

$$\{rh_j(w)\,\mathrm{e}^{2\pi \mathrm{i} rkw}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r} = \{\mathcal{T}_{U,a}^{-1}\big(U^{rk}c_j\big)\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$$

is a frame for $L^2(0,1)$. Since $r = s$, according to Lemma 4.1, it is a Riesz basis. Hence, $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a Riesz basis for $\mathcal{A}_a$ and (2) is proved.

Conversely, assume now that $\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a Riesz basis for $\mathcal{A}_a$ satisfying (4.12). From the uniqueness of the coefficients in a Riesz basis, we get that the interpolatory condition $(\mathcal{L}_{j'}C_{j,k})(rk') = \delta_{j,j'}\delta_{k,k'}$ holds for $j, j' = 1, 2, \dots, r$ and $k, k' \in \mathbb{Z}$. Since $\mathcal{T}_{U,a}^{-1}$ is an isomorphism, the sequence $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a Riesz basis for $L^2(0,1)$. Expanding the function $\overline{g_{j'}(w)}\,\mathrm{e}^{-2\pi \mathrm{i} rk'w}$ with respect to the dual basis of $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$, denoted by $\{D_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$, and having in mind (4.4) we obtain

$$\begin{aligned}\overline{g_{j'}(w)}\,\mathrm{e}^{2\pi \mathrm{i} rk'w} &= \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} \big\langle \overline{g_{j'}(\cdot)}\,\mathrm{e}^{2\pi \mathrm{i} rk'\cdot}, \mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\big\rangle_{L^2(0,1)} D_{j,k}(w) \\ &= \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} \overline{\mathcal{L}_{j'}\, C_{j,k}(rk')}\, D_{j,k}(w) = D_{j',k'}(w)\,.\end{aligned}$$

Therefore, the sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rkw}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is the dual basis of the Riesz basis $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$. In particular, it is a Riesz basis for $L^2(0,1)$, which implies, according to Lemma 4.1, that $\alpha_{\mathbb{G}} > 0$, i.e., condition (1). Moreover, the sequence $\{\mathcal{T}_{U,a}^{-1}\big(C_{j,k}\big)\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is necessarily the unique dual basis of the Riesz basis $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rkw}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$. Therefore, this proves the uniqueness of the Riesz basis $\{C_{j,k}\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ for $\mathcal{A}_a$ satisfying (4.12). □

4.3.1 Another approach involving the left shift and decimation operators

In a Hilbert space $\mathcal{H}$, the U-invariant subspaces are intimately related to stationary sequences, in this section we develop another approach to the sampling problem in $\mathcal{A}_a$.

Recall that a sequence $\mathbf{s} = \{s_k\}_{k\in\mathbb{Z}}$ is said to be stationary if the inner product $\langle s_m, s_n\rangle_{\mathcal{H}}$ depends only on the difference $m - n$, for every $m, n \in \mathbb{Z}$. Besides, two stationary sequences $\mathbf{s} = \{s_k\}_{k\in\mathbb{Z}}$ and $\mathbf{w} = \{w_k\}_{k\in\mathbb{Z}}$ are said to be *stationary correlated* if

$$\langle s_{m+k}, w_{n+k}\rangle_{\mathcal{H}} = \langle s_m, w_n\rangle_{\mathcal{H}} \quad \text{for all } m, n, k \in \mathbb{Z},$$

and $R_{\mathbf{s},\mathbf{w}}(k) := \langle s_k, w_0\rangle_{\mathcal{H}}$, for every $k \in \mathbb{Z}$ defines the corresponding *cross-covariance* function. The following result is a well-known characterization of stationary sequences (see [72]):

Lemma 4.2. *To every stationary sequence $\mathbf{s} = \{s_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ there exists a unique unitary operator $U : \mathcal{H} \to \mathcal{H}$ and $s \in \mathcal{H}$ such that $s_n = U^n s$ for all $n \in \mathbb{Z}$. Conversely every pair (U, s) of a unitary operator U and an $s \in \mathcal{H}$ defines by $s_n = U^n s$, $n \in \mathbb{Z}$, a stationary sequence $\mathbf{s} = \{s_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$.*
Moreover, two stationary sequence $\mathbf{s}$ and $\mathbf{w}$ are stationary correlated if and only if they are generated by the same unitary operator U, i.e., $s_n = U^n s$ and $w_n = U^n w$ for some $s, w \in \mathcal{H}$.

Again, the cross-covariance $R_{\mathbf{s},\mathbf{w}}$ functions admits a spectral representation which is related to the integral representation of the unitary operator U (see [72]). For every two stationary correlated sequences $\mathbf{s} = \{s_n\}_{n\in\mathbb{Z}}$, $\mathbf{w} = \{w_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ the cross-covariance function admits a spectral representation

$$R_{\mathbf{s},\mathbf{w}}(k) = \langle s_k, w_0\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta}\, d\mu_{\mathbf{s},\mathbf{w}}(\theta)\,, \quad k \in \mathbb{Z}\,, \tag{4.13}$$

in the form of an integral with respect to a (complex) *spectral measure* $\mu_{\mathbf{s},\mathbf{w}}$.

Studying the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ in $\mathcal{A}_a$

For $b_j \in \mathcal{A}_a$, $j = 1, 2, \ldots s$, consider the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$. For every $j = 1, 2, \ldots s$, the spectral measure $\mu_{\mathbf{a},\mathbf{b_j}}$ in the integral representation of the cross-covariance function of the sequences $\mathbf{a} := \{U^k a\}_{k\in\mathbb{Z}}$ and $\mathbf{b_j} := \{U^k b_j\}_{k\in\mathbb{Z}}$ has no singular part since the sequence $\{U^k a\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$. Indeed, according to Theorem 4.1, the spectral measure associated with the auto-covariance function of the sequence $\{U^k a\}_{k\in\mathbb{Z}}$ has no singular part; then by using the Cauchy-Schwarz type inequality in [14, p. 125] we get the result.

In the sequel we will use the abridged notation $b_{k,j} := U^{rk}b_j$; our goal in this section is to study the sequence $\{b_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ in $\mathcal{A}_a$ in terms of an $s \times r$ matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})$ introduced below. For the sake of completeness we include some needed calculations borrowed from Ref. [86].

First of all, we have

$$\langle U^k a, b_{n,j}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-rn)\theta}\phi_{\mathbf{a},\mathbf{b}_j}(\mathrm{e}^{\mathrm{i}\theta})d\theta\,, \tag{4.14}$$

where $\phi_{\mathbf{a},\mathbf{b}_j}$ stands for the *cross spectral density* of the stationary correlated sequences $\mathbf{a} := \{U^k a\}_{k\in\mathbb{Z}}$ and $\mathbf{b}_j := \{U^k b_j\}_{k\in\mathbb{Z}}$. Define

$$\boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(\phi_{\mathbf{a},\mathbf{b}_1}(\mathrm{e}^{\mathrm{i}\theta}), \phi_{\mathbf{a},\mathbf{b}_2}(\mathrm{e}^{\mathrm{i}\theta}), \ldots, \phi_{\mathbf{a},\mathbf{b}_s}(\mathrm{e}^{\mathrm{i}\theta})\big)^\top .$$

In what follows we will use the *left-shift operator* S defined as

$$\begin{array}{rccc} S: & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}k\theta} & \longmapsto & \sum_{k\in\mathbb{Z}} a_{k+1} \mathrm{e}^{\mathrm{i}k\theta}\,, \end{array}$$

or equivalently, by $(Sf)(\mathrm{e}^{\mathrm{i}\theta}) = f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}\theta}$, where $\mathbb{T} := \{\mathrm{e}^{\mathrm{i}\theta} : \theta \in [-\pi, \pi)\}$ denotes the unidimensional torus.

Also, we will consider the *decimation operator* D_r, r a positive integer, defined as

$$\begin{array}{rccc} D_r: & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}k\theta} & \longmapsto & \sum_{k\in\mathbb{Z}} a_{rk} \mathrm{e}^{\mathrm{i}k\theta}\,, \end{array}$$

which can equivalently be written as

$$(D_r f)(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{r}\sum_{k=0}^{r-1} f(\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{r}})\,.$$

For each $l = 0, 1, \ldots, r-1$, set the $s \times 1$ matrix of functions on the torus $\mathbb{T}$

$$\boldsymbol{\Psi}^l_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(D_r S^{-l} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}\big)(\mathrm{e}^{\mathrm{i}\theta})\,,$$

and define the $s \times r$ matrix of functions on the torus $\mathbb{T}$

$$\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := \Big(\boldsymbol{\Psi}^0_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\ \boldsymbol{\Psi}^1_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\ \ldots\ldots\ \boldsymbol{\Psi}^{r-1}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\Big)\,. \tag{4.15}$$

It is worth to mention that the matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}$ was explicitly computed in [86] for the translation and modulation cases in $L^2(\mathbb{R})$ (see Section 4.3.2 below).

Next, for any $x \in \mathcal{A}_a$, we obtain an expression for the inner products

$$\alpha_{n,j} := \langle x, U^{rn} b_j\rangle\,, \quad n \in \mathbb{Z} \text{ and } j = 1, 2, \ldots, s\,.$$

Indeed, writing $x = \sum_{k\in\mathbb{Z}} x_k U^k a$ where $\{x_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have:

$$\begin{aligned} \alpha_{n,j} &= \langle x, U^{rn} b_j\rangle = \sum_{k\in\mathbb{Z}} x_k \langle U^k a, U^{rn} b_j\rangle \\ &= \sum_{k\in\mathbb{Z}} x_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-rn)\theta} \phi_{\mathbf{a},\mathbf{b}_j}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} x_k \mathrm{e}^{\mathrm{i}k\theta} \phi_{\mathbf{a},\mathbf{b}_j}(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}rn\theta} d\theta\,, \end{aligned}$$

that is,

$$\boldsymbol{\alpha}_n := \big(\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,s}\big)^\top = \frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) X(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}rn\theta} d\theta\,, \tag{4.16}$$

where $X(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k\in\mathbb{Z}} x_k \mathrm{e}^{\mathrm{i}k\theta}$.
Now, for $l = 0, 1, \dots, r-1$, define the sequence $x^{(l)} := \{x_k^{(l)} := x_{kr+l}\}_{k\in\mathbb{Z}}$. Thus, we can write

$$X(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{l=0}^{r-1} \sum_{k\in\mathbb{Z}} x_{kr+l} \mathrm{e}^{\mathrm{i}(kr+l)\theta} = \sum_{l=0}^{r-1} X^{(l)}(\mathrm{e}^{\mathrm{i}r\theta}) \mathrm{e}^{\mathrm{i}l\theta}\,, \tag{4.17}$$

where $X^{(l)}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k\in\mathbb{Z}} x_k^{(l)} \mathrm{e}^{\mathrm{i}k\theta}$.
Using Eq. (4.17) in Eq. (4.16), we obtain

$$\boldsymbol{\alpha}_n = \sum_{l=0}^{r-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) X^{(l)}(\mathrm{e}^{\mathrm{i}r\theta}) \mathrm{e}^{\mathrm{i}l\theta} \mathrm{e}^{-\mathrm{i}rn\theta} d\theta\,.$$

After some easy calculations we get

$$\begin{aligned}
\boldsymbol{\alpha}_n &= \sum_{l=0}^{r-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} S^{-l} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) X^{(l)}(\mathrm{e}^{\mathrm{i}r\theta}) \mathrm{e}^{-\mathrm{i}rn\theta} d\theta \\
&= \sum_{l=0}^{r-1} \frac{1}{2\pi} \int_{-r\pi}^{r\pi} \frac{S^{-l} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\frac{\theta}{r}})}{r} X^{(l)}(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}n\theta} d\theta \\
&= \sum_{l=0}^{r-1} \sum_{k=0}^{r-1} \int_{2\pi k}^{2\pi(k+1)} \frac{S^{-l} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\frac{\theta}{r}})}{2\pi r} X^{(l)}(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}n\theta} d\theta \\
&= \int_{0}^{2\pi} \sum_{l=0}^{r-1} \sum_{k=0}^{r-1} \frac{S^{-l} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\frac{\theta+2\pi k}{r}})}{2\pi r} X^{(l)}(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}n\theta} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{l=0}^{r-1} (D_r S^{-l} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}})(\mathrm{e}^{\mathrm{i}\theta}) X^{(l)}(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}n\theta} d\theta\,.
\end{aligned} \tag{4.18}$$

Defining $\mathbf{C}(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k\, \mathrm{e}^{\mathrm{i}k\theta}$, Eq. (4.18) implies that

$$\mathbf{C}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{l=0}^{r-1} \big(D_r S^{-l} \boldsymbol{\Phi}_{\mathbf{a},\mathbf{b}}\big)(\mathrm{e}^{\mathrm{i}\theta}) X^{(l)}(\mathrm{e}^{\mathrm{i}\theta})\,,$$

which can be written in matrix form as,

$$\begin{aligned}
\mathbf{C}(\mathrm{e}^{\mathrm{i}\theta}) &= \Big(\sum_{k\in\mathbb{Z}} \alpha_{k,1}\, \mathrm{e}^{\mathrm{i}k\theta}, \sum_{k\in\mathbb{Z}} \alpha_{k,2}\, \mathrm{e}^{\mathrm{i}k\theta}, \dots, \sum_{k\in\mathbb{Z}} \alpha_{k,s}\, \mathrm{e}^{\mathrm{i}k\theta} \Big)^{\top} \\
&= \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) \big(X^{(0)}(\mathrm{e}^{\mathrm{i}\theta}), X^{(1)}(\mathrm{e}^{\mathrm{i}\theta}), \dots, X^{(r-1)}(\mathrm{e}^{\mathrm{i}\theta}) \big)^{\top} \\
&= \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) \widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}) = (L_{\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}} \widetilde{\mathbf{X}})(\mathrm{e}^{\mathrm{i}\theta})
\end{aligned} \tag{4.19}$$

where $L_{\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}} : L_r^2(\mathbb{T}) \longrightarrow L_s^2(\mathbb{T})$ denotes the multiplication operator by $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}$ and

$$\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(X^{(0)}(\mathrm{e}^{\mathrm{i}\theta}), X^{(1)}(\mathrm{e}^{\mathrm{i}\theta}), \dots, X^{(r-1)}(\mathrm{e}^{\mathrm{i}\theta}) \big)^{\top}\,. \tag{4.20}$$

We denote by $L^2_r(\mathbb{T})$ (respectively $L^2_s(\mathbb{T})$) the product Hilbert space

$$\underbrace{L^2(\mathbb{T}) \times \cdots \times L^2(\mathbb{T})}_{r \text{ times}} \quad \text{(respectively } s \text{ times)}\,.$$

Thus,

$$\begin{aligned} \|\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}), \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\rangle_{\mathbb{C}^r}\, d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle \boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}), \widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\rangle_{\mathbb{C}^r}\, d\theta\,. \end{aligned} \tag{4.21}$$

The above calculations let us prove the following result:

Theorem 4.3. *Let $b_j \in \mathcal{A}_a$ for $j = 1, 2, \ldots, s$ and let $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}$ be the associated matrix given in* (4.15)*. Then, the following results hold:*

(a) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a complete system in $\mathcal{A}_a$ if and only if the rank of the matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)$ is r a.e. ζ in $\mathbb{T}$.

(b) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Bessel sequence for $\mathcal{A}_a$ if and only if there exists a constant $B < \infty$ such that

$$\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta) \leqslant B\,\mathbb{I}_r \qquad \text{a.e. } \zeta \text{ in } \mathbb{T}. \tag{4.22}$$

(c) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$ if and only if there exist constants $0 < A \leqslant B < \infty$ such that

$$A\,\mathbb{I}_r \leqslant \boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta) \leqslant B\,\mathbb{I}_r \qquad \text{a.e. } \zeta \text{ in } \mathbb{T}. \tag{4.23}$$

(d) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Riesz basis for $\mathcal{A}_a$ if and only if it is a frame and $s = r$.

Proof. To prove (a), assume that there exists a set $\Omega \subseteq \mathbb{T}$ with positive measure such that $\operatorname{rank}\left[\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\right] < r$ for each $\zeta \in \Omega$. Then, there exists a measurable function $v(\zeta)$, $\zeta \in \Omega$, such that $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)v(\zeta) = 0$ and $\|v(\zeta)\|_{L^2_r(\mathbb{T})} = 1$ in Ω. This function can be constructed as in [67, Lemma 2.4]. Define $\widetilde{\mathbf{V}} \in L^2_r(\mathbb{T})$ such that $\widetilde{\mathbf{V}}(\zeta) = v(\zeta)$ if $\zeta \in \Omega$, and $\widetilde{\mathbf{V}}(\zeta) = 0$ if $\zeta \in \mathbb{T}\backslash\Omega$. Hence, from (4.19) we obtain that the system is not complete.

Conversely, if the system is not complete, by using (4.19) we obtain a $\widetilde{\mathbf{V}}(\zeta)$ different from 0 in a set with positive measure such that $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\widetilde{\mathbf{V}}(\zeta) = 0$. Thus $\operatorname{rank} \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta) < r$ on a set with positive measure.

To prove (b), we keep in mind that $\{U^k a\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$, the mapping $T:\ell^2(\mathbb{Z})\to\mathcal{A}_a$, given by $T\{x_k\}_{k\in\mathbb{Z}}=x=\sum_{k\in\mathbb{Z}}x_kU^ka$ is bijective and there exist two constants $0<m_a\leqslant M_a<\infty$ such that

$$m_a\|\{x_k\}\|^2_{\ell^2}\leqslant\|T\{x_k\}\|^2_{\mathcal{H}}\leqslant M_a\|\{x_k\}\|^2_{\ell^2}\,. \tag{4.24}$$

Assume first that (4.22) is satisfied. It follows from (4.19) and (4.21) that

$$\|\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}\leqslant B\|\widetilde{\mathbf{X}}\|^2_{L^2_r(\mathbb{T})}\,. \tag{4.25}$$

By construction $\|\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}=\sum_{j=1}^s\sum_{k\in\mathbb{Z}}|\langle x,b_{k,j}\rangle|^2$ and $\|\widetilde{\mathbf{X}}\|^2_{L^2_r(\mathbb{T})}=\|\{x_k\}_{k\in\mathbb{Z}}\|^2_{\ell^2}$. Using (4.24), it follows from (4.25) that

$$\sum_{j=1}^s\sum_{k\in\mathbb{Z}}|\langle x,b_{k,j}\rangle|^2\leqslant\frac{B}{m_a}\|x\|^2_{\mathcal{H}}$$

Conversely, assume that $\{b_{kj}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a Bessel sequence for $\mathcal{A}_a$, then there exists $0<B'<\infty$ such that

$$\sum_{j=1}^s\sum_{k\in\mathbb{Z}}|\langle x,b_{k,j}\rangle|^2\leqslant B'\|x\|^2_{\mathcal{H}}\,.$$

Using (4.24), this implies

$$\|\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}\leqslant B'\,M_a\|\widetilde{\mathbf{X}}\|^2_{L^2_r(\mathbb{T})}$$

for all $\widetilde{\mathbf{X}}\in L^2_r(\mathbb{T})$. Inserting the right hand side of (4.21) for $\|\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}$, it is straightforward to see that (4.22) holds with $B=B'\,M_a$.

The proof of (c) is completed proceeding as in (b).

To prove (d) consider the mapping

$$\begin{aligned} S:\mathcal{A}_a &\longrightarrow \ell^2_s(\mathbb{Z})\\ x&\longmapsto\{\langle x,b_{k,j}\rangle\}_{k\in\mathbb{Z};\,j=1,2,\dots s}\,.\end{aligned}$$

According to (4.19), the mapping S is isometric equivalent to $L_{\Psi_{a,\mathbf{b}}}$, and assuming that $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a frame, it is a Riesz basis if and only if S is surjective.
First, if $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a Riesz basis, then it is a frame and S is surjective.

Applying (a) yields that $L_{\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}}$ is bijective, and therefore $L^*_{\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}}=L_{\Psi^*_{\mathbf{a},\mathbf{b}}}$ is bijective. Hence, $\operatorname{rank}[\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)]$ is s for almost every ζ in $\mathbb{T}$ so

$$r=\operatorname{rank}[\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)]=\operatorname{rank}[\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)]=s,$$

and finally $s=r$.

Conversely, if $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a frame and $s=r$, (a) implies that $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)$ is invertible for almost every ζ in $\mathbb{T}$, which implies that $L_{\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}}$ is surjective, then S is surjective and $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a Riesz basis. $\square$

The following lemma will allow us to restate Theorem 4.3:

Lemma 4.3. *Let* $\mathbf{G}(\zeta)$ *be an* $s \times r$ *matrix with entries in* $L^2(\mathbb{T})$*, and consider the constants*

$$A_{\mathbf{G}} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \ \lambda_{\min}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]\,,$$
$$B_{\mathbf{G}} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \ \lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]\,,$$

where $\lambda_{\min}$ *(respectively* $\lambda_{\max}$*) denotes the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix* $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$*. Then,*

(a) The matrix $\mathbf{G}(\zeta)$ *has essentially bounded entries on* $\mathbb{T}$ *if and only if* $B_{\mathbf{G}} < \infty$*.*

(b) There exist constants $0 < A \leqslant B < \infty$ *such that*

$$A\,\mathbb{I}_r \leqslant \mathbf{G}^*(\zeta)\mathbf{G}(\zeta) \leqslant B\,\mathbb{I}_r\,, \quad \textit{a.e. } \zeta\in\mathbb{T}$$

if and only if

$$0 < A_{\mathbf{G}} \leqslant B_{\mathbf{G}} < \infty\,.$$

Proof. The first part of lemma follows from that $\lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big] = \|\mathbf{G}(\zeta)\|_2^2$, and

$$\max_{i,\,j} |a_{ij}| \leqslant \|\mathbf{A}\|_2 \leqslant \sqrt{mn}\ \max_{i,\,j} |a_{ij}| \qquad \text{for any matrix } \mathbf{A} = \big[a_{ij}\big]_{\substack{i=1,2\ldots,m\\ j=1,2\ldots,n}}\,,$$

where $\|\mathbf{A}\|_2$ denotes the spectral norm of the matrix $\mathbf{A}$ (see, for instance, Ref. [64])

Now we prove the second part of the lemma. Since $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta) \leqslant B\,\mathbb{I}_r$ means that $\langle Bx - \mathbf{G}^*(\zeta)\mathbf{G}(\zeta)x, x\rangle \geqslant 0$ for all $x \in \mathbb{C}^r$, in particular, taking an eigenvector x associated to the largest eigenvalue $\lambda_{\max}$ of $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$ such that $\|x\| = 1$, one has that $B \geqslant \lambda_{\max}(\mathbf{G}^*(\zeta)\mathbf{G}(\zeta))$. Hence, $B \geqslant \operatorname{ess\,sup}_{\zeta\in\mathbb{T}} \lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]$. In a similar way, $A\,\mathbb{I}_r \leqslant \mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$ implies that $A \leqslant \operatorname{ess\,inf}_{\zeta\in\mathbb{T}} \lambda_{\min}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]$.
Conversely, Rayleigh-Ritz theorem [64, p. 176] yields that

$$\lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big] = \max_{x\in\mathbb{C}^r} \frac{x^*\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)}{x^*x} = \max_{x\in\mathbb{C}^r} \frac{\langle \mathbf{G}^*(\zeta)\mathbf{G}(\zeta)x, x\rangle}{\langle x, x\rangle}$$

Thus, $\operatorname{ess\,sup}_{\zeta\in\mathbb{T}} \lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big] = B_{\mathbf{G}}$ implies that

$$\max_{x\in\mathbb{C}^r} \frac{\langle \mathbf{G}^*(\zeta)\mathbf{G}(\zeta)x, x\rangle}{\langle x, x\rangle} \leqslant B_{\mathbf{G}}\,, \qquad \text{a.e. } \zeta\in\mathbb{T}\,.$$

In other words, $B_{\mathbf{G}}\mathbb{I}_r \geqslant \mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$; analogously, $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta) \geqslant A_{\mathbf{G}}\mathbb{I}_r$.
It is easy to deduce from the proof that $A_{\mathbf{G}}$ and $B_{\mathbf{G}}$ are the optimal constants $A > 0$ and $B < \infty$ satisfying the inequalities $A\,\mathbb{I}_r \leqslant \mathbf{G}^*(\zeta)\mathbf{G}(\zeta) \leqslant B\,\mathbb{I}_r$, a.e. $\zeta\in\mathbb{T}$. □

As a consequence of Lemma 4.3, statements (b) and (c) in Theorem 4.3 can be restated in terms of the constants

$$A_{\Psi} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \ \lambda_{\min}\big[\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\big]\,;$$
$$B_{\Psi} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \ \lambda_{\max}\big[\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\big] \tag{4.26}$$

as:

Theorem 4.4. *Let $b_j \in \mathcal{A}_a$ for $j = 1, 2, \ldots, s$, and let $\Psi_{\mathbf{a},\mathbf{b}}$ be the associated matrix given in* (4.15) *and its related constants* (4.26)*. Then, the following results hold:*

(i) The sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Bessel sequence for $\mathcal{A}_a$ if and only if the constant $B_{\Psi} < \infty$.

(ii) The sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$ if and only if the constants A_{Ψ} and B_{Ψ} satisfy $0 < A_{\Psi} \leqslant B_{\Psi} < \infty$. In this case, A_{Ψ} and B_{Ψ} are the optimal frame bounds for $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$.

The frame expansion

Define the $r \times s$ matrix of functions on the torus $\mathbb{T}$

$$\boldsymbol{\Gamma}(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k\, \mathrm{e}^{\mathrm{i}k\theta} = \big[\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\big]^{-1}\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) \tag{4.27}$$

It is worth to mention that the following procedure also works with any left-inverse of the matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})$; see Eq. (4.31) below.

Firstly, the following expansion involving the inner products $\alpha_{n,j} = \langle x, U^{rn}b_j\rangle$ of $x \in \mathcal{A}_a$ holds:

Lemma 4.4. *Assume that the matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)$ has essentially bounded entries on $\mathbb{T}$. For any $x = \sum_{k\in\mathbb{Z}} x_k U^k a \in \mathcal{A}_a$ we have*

$$\widetilde{\mathbf{x}}_{\mathbf{n}} = \sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k\, \boldsymbol{\alpha}_{n-k}\,,$$

where $\widetilde{\mathbf{x}}_{\mathbf{n}}$ denotes the n-th Fourier coefficient of the function $\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})$ defined in (4.20)*, and the sequence $\{\boldsymbol{\alpha}_n\}_{n\in\mathbb{Z}}$ is given in* (4.16)*.*

Proof. Indeed,

$$\begin{aligned}
\widetilde{\mathbf{x}}_{\mathbf{n}} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k \mathrm{e}^{\mathrm{i}k\theta}\Big)\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}d\theta \\
&= \sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{i\theta})\mathrm{e}^{-\mathrm{i}(n-k)\theta}d\theta = \sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k\, \boldsymbol{\alpha}_{n-k}\,.
\end{aligned}$$

□

At this point, we are ready to prove the following expansion result:

Theorem 4.5. *Let $b_j \in \mathcal{A}_a$ for $j = 1, 2, \ldots, s$, and assume that the associated matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}$ given in (4.15) has essentially bounded entries on $\mathbb{T}$, i.e., $B_{\boldsymbol{\Psi}} < \infty$. The following statements are equivalent:*

(i) The constant $A_{\boldsymbol{\Psi}} > 0$.

(ii) There exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, such that the sequence $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$, yielding, for any $x \in \mathcal{A}_a$, the expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{rk} b_j \rangle U^{rk} c_j \quad \text{in } \mathcal{H}\,.$$

In case the equivalent conditions hold, $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ and $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ form a pair of dual frames in $\mathcal{A}_a$.

Proof. First we prove that (i) implies (ii). Observe that $x = \sum_{k\in\mathbb{Z}} x_k\, U^k a$ can be written as $\sum_{n\in\mathbb{Z}} \widetilde{\mathbf{x}}_{\mathbf{n}}^\top \widetilde{\mathbf{a}}_{\mathbf{n}}$ where $\widetilde{\mathbf{a}}_{\mathbf{n}} = \big(U^{nr} a, U^{nr+1} a, \cdots, U^{nr+r-1} a\big)^\top$. Next,

$$\begin{aligned} x &= \sum_{n\in\mathbb{Z}} \widetilde{\mathbf{x}}_{\mathbf{n}}^\top \widetilde{\mathbf{a}}_{\mathbf{n}} = \sum_{n\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k \boldsymbol{\alpha}_{n-k}\Big)^\top \widetilde{\mathbf{a}}_{\mathbf{n}} = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_{n-k}^\top \boldsymbol{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n}} \\ &= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top \boldsymbol{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n+k}} = \sum_{n\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top \Big(\sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n+k}} \Big) \end{aligned} \tag{4.28}$$

For $l \in \mathbb{Z}$ and $j = 1, 2, \ldots, s$ define $c_{l,j} := U^{rl} c_j$, where

$$\big(c_1, c_2, \ldots, c_s\big)^\top = \sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k^\top\, \widetilde{\mathbf{a}}_{\mathbf{k}}\,,$$

and $b_{l,j} := U^{rl} b_j$. Then Eq. (4.28) implies

$$\begin{aligned} x &= \sum_{n\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top \Big(\sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n+k}} \Big) = \sum_{n\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top U^{nr} \Big(\sum_{k\in\mathbb{Z}} \boldsymbol{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{k}} \Big) \\ &= \sum_{l=1}^{s} \sum_{n\in\mathbb{Z}} \langle x, b_{n,l} \rangle\, c_{n,l} \quad \text{in } \mathcal{H}\,. \end{aligned} \tag{4.29}$$

In order to be allowed to use Proposition A.4, we have to prove that the above constructed sequence $\{c_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a Bessel sequence for $\mathcal{A}_a$. To this end, we compute the corresponding $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}$ matrix for $\mathbf{c} := \{c_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$. Setting $[\boldsymbol{\Gamma}_k^\top]_{ij} = a_{ij}^k$, we obtain

$$
\begin{aligned}
\langle U^k a, c_{n,j}\rangle &= \sum_{l\in\mathbb{Z}}\sum_{i=1}^{r}\langle U^k a, U^{nr}\big(a_{ji}^l\, U^{lr+ir+i-1}a\big)\rangle \\
&= \sum_{l\in\mathbb{Z}}\sum_{i=1}^{r}\overline{a}_{ji}^l\,\langle U^{k-nr-lr-i+1}a, a\rangle \\
&= \sum_{l\in\mathbb{Z}}\sum_{i=1}^{r}\overline{a}_{ji}^l\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-nr-lr-i+1)\theta}\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta\,.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\begin{pmatrix}\langle U^k a, c_{n,1}\rangle\\ \langle U^k a, c_{n,2}\rangle\\ \vdots\\ \langle U^k a, c_{n,s}\rangle\end{pmatrix} &= \sum_{l\in\mathbb{Z}}\overline{\boldsymbol{\Gamma}}_l^\top
\begin{pmatrix}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(k-nr-lr)\theta}\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta\\
\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(k-nr-lr-1)\theta}\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta\\
\vdots\\
\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(k-nr-lr-r+1)\theta}\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta
\end{pmatrix}\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{l\in\mathbb{Z}}\overline{\boldsymbol{\Gamma}}_l^\top\,\mathrm{e}^{-\mathrm{i}lr\theta}
\begin{pmatrix}\mathrm{e}^{\mathrm{i}(k-nr)\theta}\\ \mathrm{e}^{\mathrm{i}(k-nr-1)\theta}\\ \vdots\\ \mathrm{e}^{\mathrm{i}(k-nr-r+1)\theta}\end{pmatrix}\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(k-nr)\theta}\boldsymbol{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})\widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta})\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta\,,
\end{aligned}
$$

where $\widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(1, \mathrm{e}^{-\mathrm{i}\theta},\dots,\mathrm{e}^{-\mathrm{i}(r-1)\theta}\big)^\top$. Hence, we have deduced that

$$\boldsymbol{\Phi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta}) = \boldsymbol{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})\widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta})\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})\,.$$

Therefore, for $l = 0, 1, \dots, r-1$, we have

$$\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}^l(\mathrm{e}^{\mathrm{i}\theta}) := D_r S^{-l}\big[\boldsymbol{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})\widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta})\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})\big]\,,$$

and consequently, the $s\times r$ matrix

$$\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta}) := \Big(\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}^0(\mathrm{e}^{\mathrm{i}\theta}), \boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}^1(\mathrm{e}^{\mathrm{i}\theta}),\dots,\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}^{r-1}(\mathrm{e}^{\mathrm{i}\theta})\Big)$$

can be written as

$$\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta}) = D_r\big[\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})\boldsymbol{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})\widetilde{\mathbf{E}}(\mathrm{e}^{\mathrm{i}\theta})\big]\,, \tag{4.30}$$

where

$$\widetilde{\mathbf{E}}(\mathrm{e}^{\mathrm{i}\theta}) := \begin{pmatrix}
1 & \mathrm{e}^{\mathrm{i}\theta} & \cdots & \mathrm{e}^{\mathrm{i}(r-1)\theta}\\
\mathrm{e}^{-\mathrm{i}\theta} & 1 & \cdots & \mathrm{e}^{\mathrm{i}(r-2)\theta}\\
\vdots & \vdots & \ddots & \vdots\\
\mathrm{e}^{-\mathrm{i}(r-1)\theta} & \mathrm{e}^{-\mathrm{i}(r-2)\theta} & \cdots & 1
\end{pmatrix}.$$

As a consequence of Theorem 4.4, the proof ends if we prove that the matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta})$ has essentially bounded entries: Clearly, the decimation operator D_r sends bounded functions into bounded functions; Theorem 4.1 implies that $\phi_{\mathbf{a}}$ is bounded so, taking into account (4.30) it remains to check that the matrix $\boldsymbol{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})$ has essentially bounded entries.
Now, since

$$\boldsymbol{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta}) = \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\big[\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\big]^{-1},$$

the lower bound condition (c) in Theorem 4.3 and Lemma 4.3 imply that the matrix $\big[\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\big]^{-1}$ has bounded entries, and therefore the matrix $\boldsymbol{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})$ has bounded entries. We have shown that $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta})$ has bounded entries, then Theorem 4.4, part (a) and Lemma 4.3 guarantee that the sequence $\{c_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Bessel sequence; then, the sequences $\{b_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ and $\{c_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ form a pair of dual frames in $\mathcal{A}_a$ (see A.4).

Finally, condition (ii) implies condition (i). According to Proposition A.4, the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$ since it is a Bessel sequence and the expansion in (ii) holds. By using Theorem 4.4 we obtain that $A_{\boldsymbol{\Psi}} > 0$. □

It is worth to observe that the analysis done in Theorem 4.2 provides a whole family of dual frames for the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$. In fact, everything works if we replace $\boldsymbol{\Gamma}(\mathrm{e}^{\mathrm{i}\theta})$ in (4.27) by any matrix of the form,

$$\boldsymbol{\Gamma}_{\mathbb{U}}(\mathrm{e}^{\mathrm{i}\theta}) := \boldsymbol{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) + \mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})\big[\mathbb{I}_s - \boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\boldsymbol{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\big], \tag{4.31}$$

where $\mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})$ is any $r \times s$ matrix with entries in $L^\infty(\mathbb{T})$, and $\boldsymbol{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}$ denotes the Moore-Penrose pseudo inverse $\boldsymbol{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := [\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})]^{-1}\boldsymbol{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})$. Note that we need essentially bounded entries in the matrix $\boldsymbol{\Gamma}_{\mathbb{U}}(\mathrm{e}^{\mathrm{i}\theta})$ since the multiplication operator $M_F : f \mapsto Ff$ in $L^2(\mathbb{T})$ is well-defined (and consequently bounded) if and only if $F \in L^\infty(\mathbb{T})$.

Notice that if $s = r$, we have $\boldsymbol{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}} = \boldsymbol{\Psi}^{-1}_{\mathbf{a},\mathbf{b}}$ which implies a unique $\boldsymbol{\Gamma}_{\mathbb{U}}$, and we are in presence of a pair of dual Riesz bases. In fact, the following result holds:

Corollary 4.3. *Let $b_j \in \mathcal{A}_a$ for $j = 1, 2, \dots, r$, i.e., $r = s$ in Theorem 4.2. Assume that the square matrix $\boldsymbol{\Psi}_{\mathbf{a},\mathbf{b}}$ given in* (4.15) *has entries essentially bounded on $\mathbb{T}$, i.e., $B_{\boldsymbol{\Psi}} < \infty$. The following statements are equivalent:*

(a) The constant $A_{\boldsymbol{\Psi}} > 0$.

(b) There exists a Riesz basis $\{C_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ such that for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}}\langle x, U^{rk}b_j\rangle C_{k,j} \quad \text{in } \mathcal{H}$$

holds.

In case the equivalent conditions are satisfied, necessarily there exist $c_j \in \mathcal{A}_a$*,* $j = 1, 2, \ldots, r$*, such that* $C_{k,j} = U^{rk} c_j$ *for* $k \in \mathbb{Z}$ *and* $j = 1, 2, \ldots, r$*. Moreover, the sequences* $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ *and* $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ *are dual Riesz bases in* $\mathcal{A}_a$*, and the interpolation property* $\langle c_j, U^{rk} b_{j'} \rangle = \delta_{j,j'}\, \delta_{k,0}$*, where* $k \in \mathbb{Z}$ *and* $j, j' = 1, 2, \ldots, r$*, holds.*

Proof. To prove $(a) \Rightarrow (b)$ we use Theorem 4.2; whenever $0 < A_{\boldsymbol{\Psi}} \leqslant B_{\boldsymbol{\Psi}} < \infty$ there exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, such that the sequence $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$ and, for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{rk} b_j \rangle U^{rk} c_j \quad \text{in } \mathcal{H}\,,$$

holds. Actually, from Theorem 4.3 we get that $r = s$ implies that $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Riesz basis, and consequently, $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is indeed its dual Riesz basis.

The converse follows easily from the fact that if $\{C_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Riesz basis, then (b) implies that $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is its dual Riesz basis; hence, Theorem 4.3 provides $A_{\boldsymbol{\Psi}} > 0$. The interpolation property comes out from the biorthogonal condition of a pair of dual Riesz bases. □

Closing this section it is worth to mention that the results stated and proved in this subsection mathematically enrich some of the remarkable results concerning regular sampling contained in the interesting Ref. [86]. Here we have assumed only one generator $a \in \mathcal{H}$ and that $b_j \in \mathcal{A}_a$ for all $j = 1, 2, \ldots, s$. If $b_j \notin \mathcal{A}_a$ for some j, see the additional remarks see the next section. The case of several generators $a_l \in \mathcal{H}$, $l = 1, 2, \ldots, L$, can be essentially treated in the same way.

4.3.2 Some comments on the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$

Concerning Theorem 4.2, more can be said about the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$, where the vectors $b_j \in \mathcal{H}$ define the U-systems $\mathcal{L}_j \equiv \mathcal{L}_{b_j}$, $j = 1, 2, \ldots, s$. Having in mind (4.4) and the isomorphism $\mathcal{T}_{U,a}$, we obtain that

$$\frac{\alpha_{\mathbb{G}}}{r} \|\mathcal{T}_{U,a}\|^{-2} \|x\|^2 \leqslant \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} |\langle x, U^{rk} b_j \rangle|^2 \leqslant \frac{\beta_{\mathbb{G}}}{r} \|\mathcal{T}_{U,a}^{-1}\|^2 \|x\|^2 \quad \text{for all } x \in \mathcal{A}_a\,. \tag{4.32}$$

- In case that $b_j \in \mathcal{A}_a$ for each $j = 1, 2, \ldots, s$, we derive that $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$, and it is dual to the frame $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ in $\mathcal{A}_a$. Thus, the sampling expansion (4.11) is nothing but a frame expansion in $\mathcal{A}_a$.

- In case that some $b_j \notin \mathcal{A}_a$, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is not contained in $\mathcal{A}_a$. However, inequalities (4.32) hold. Therefore, the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a pseudo-dual frame for the frame $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ in $\mathcal{A}_a$ (see [76, 77]). Denoting by $P_{\mathcal{A}_a}$ the orthogonal projection onto $\mathcal{A}_a$, we derive from (4.32) that the sequence $\{P_{\mathcal{A}_a}(U^{rk}b_j)\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a dual frame of $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ in $\mathcal{A}_a$.
- The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ will be a Riesz basis or a pseudo-Riesz basis for $\mathcal{A}_a$ whenever $r = s$.

Translation and Modulation cases

We can notice that in the subsection 4.3.1 the information which the operator U provides is contained in the spectral density of the representation (4.14). Now, we are going to deduce these densities for the, by far, most famous examples of sampling operators.

Consider the translation operator, T defined as $T : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$, then for $f, g \in L^2(\mathbb{R})$ and we have

$$\begin{aligned}
\langle T^k f, g\rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} f(u-k)\,\overline{g(u)}\,du = \Big(f(-\cdot) * \overline{g(\cdot)}\Big)(k)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Big(\widehat{f(-\cdot) * \overline{g(\cdot)}}\Big)(w)\,\mathrm{e}^{\mathrm{i}kw}\,dw\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{2\pi}\,\widehat{f}(-w)\,\overline{\widehat{f}(-w)}\,\mathrm{e}^{\mathrm{i}kw}\,dw\\
&= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kw}\,\widehat{f}(-w)\,\overline{\widehat{g}(-w)}\,dw\\
&= \sum_{n\in\mathbb{Z}}\int_{2\pi n}^{2\pi n+2\pi} \mathrm{e}^{\mathrm{i}kw}\,\widehat{f}(-w)\,\overline{\widehat{g}(-w)}\,dw\\
&= \sum_{n\in\mathbb{Z}}\int_{0}^{2\pi} \mathrm{e}^{\mathrm{i}kw}\,\widehat{f}(-w+2\pi n)\,\overline{\widehat{g}(-w+2\pi n)}\,dw\\
&= \int_{0}^{2\pi} \mathrm{e}^{\mathrm{i}kw}\sum_{n\in\mathbb{Z}}\widehat{f}(-w+2\pi n)\,\overline{\widehat{g}(-w+2\pi n)}\,dw\,.
\end{aligned}$$

Then the cross spectral density associated to the sequences $\{T^k f\}_{k\in\mathbb{Z}}$ and $\{T^k g\}_{k\in\mathbb{Z}}$ is

$$\phi_{f,g}(w) = 2\pi\sum_{n\in\mathbb{Z}}\widehat{f}(-w+2\pi n)\,\overline{\widehat{g}(-w+2\pi n)}\,.$$

For the modulation operator, M defined as $M : f(u) \mapsto \mathrm{e}^{\mathrm{i}u} f(u)$ in $L^2(\mathbb{R})$ is simpler; indeed, for $f, g \in L^2(\mathbb{R})$ and we have

$$\begin{aligned}
\langle M^k f, g\rangle_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ku} f(u)\,\overline{g(u)}\,du \\
&= \sum_{n\in\mathbb{Z}} \int_{2\pi n}^{2\pi n+2\pi} \mathrm{e}^{\mathrm{i}ku} f(u)\,\overline{g(u)}\,du \\
&= \sum_{n\in\mathbb{Z}} \int_{0}^{2\pi} \mathrm{e}^{\mathrm{i}ku} f(u+2\pi n)\,\overline{g(u+2\pi n)}\,du \\
&= \int_{0}^{2\pi} \mathrm{e}^{\mathrm{i}ku} \sum_{n\in\mathbb{Z}} f(u+2\pi n)\,\overline{g(u+2\pi n)}\,du \\
&= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}ku} \sum_{n\in\mathbb{Z}} f(u+2\pi n)\,\overline{g(u+2\pi n)}\,du\,.
\end{aligned}$$

In this case the cross spectral density is

$$\phi_{f,g}(w) = 2\pi \sum_{n\in\mathbb{Z}} f(u+2\pi n)\,\overline{g(u+2\pi n)}\,.$$

As a final remark we consider worth to mention that in the main motivation of this approach [86], the authors explicitly compute the matrix given in (4.15) for the translation and modulation operators. The entry in the m-th row and l-th column of the matrix $\boldsymbol{\Psi}$ for the translation case, adapted to our setting, is

$$[\boldsymbol{\Psi}(\mathrm{e}^{\mathrm{i}\theta})]_{m,l} = \frac{1}{r}\sum_{k=0}^{r-1} \mathrm{e}^{\mathrm{i}l(\frac{\theta+2k\pi}{r})} \sum_{n\in\mathbb{Z}} \widehat{a}\Big(2\pi n - \frac{2k\pi}{r} - \frac{\theta}{r}\Big)\overline{\widehat{b}_{m+1}\Big(2\pi n - \frac{2k\pi}{r} - \frac{\theta}{r}\Big)}$$

and for the modulation case

$$[\boldsymbol{\Psi}(\mathrm{e}^{\mathrm{i}\theta})]_{m,l} = \frac{1}{r}\sum_{k=0}^{r-1} \mathrm{e}^{\mathrm{i}l(\frac{\theta+2k\pi}{r})} \sum_{n\in\mathbb{Z}} a\Big(2\pi n + \frac{2k\pi}{r} + \frac{\theta}{r}\Big)\overline{b_{m+1}\Big(2\pi n + \frac{2k\pi}{r} + \frac{\theta}{r}\Big)}$$

in both cases $m = 0, 1, \dots, s-1$ and $l = 0, 1, \dots, r-1$, and the functions $\{b_j\}_{j=1,2\dots,s}$ belong to $L^2(\mathbb{R})$ as well as the generator a.

4.3.3 Sampling formulas with prescribed properties

The sampling formula (4.11) can be thought as a filter-bank. Indeed, assume that for $j = 1, 2, \dots, s$ we have

$$c_{j,h} = \mathcal{T}_{U,a}(rh_j) = r\sum_{n\in\mathbb{Z}} \widehat{h}_j(n)\, U^n a$$

where

$$\widehat{h}_j(n) = \int_0^1 h_j(w)\, \mathrm{e}^{-2\pi i n w} dw\,,\ \ n \in \mathbb{Z}\,.$$

Substituting in (4.11), after the change of summation index $m := rk + n$ we obtain

$$x = \sum_{m\in\mathbb{Z}} \Big\{ \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} r\mathcal{L}_j x(rk)\, \widehat{h}_j(m-rk) \Big\}\, U^m a\,,$$

that is, the relevant data is the output of a filter-bank:

$$\alpha_m := \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} r\mathcal{L}_j x(rk)\, \widehat{h}_j(m-rk)\,, \quad m \in \mathbb{Z}\,.$$

where the input is the given samples and the impulse responses depends on the sampling vectors $c_{j,h}$, $j = 1, 2, \dots, s$.

In the oversampling setting, i.e., $s > r$, according to (4.9) there exist infinitely many sampling vectors $c_{j,h}$, $j = 1, 2, \dots, s$, for which the sampling formula (4.11) holds. A natural question is whether we can choose the sampling vectors $c_{j,h}$, $j = 1, 2, \dots, s$, with prescribed properties.

For instance, a challenging problem is to ask under what conditions we are in the presence of a FIR (finite impulse response) filter-bank; i.e,

$$c_{j,h} = r \sum_{\text{finite}} \widehat{h}_j(n)\, U^n a\,, \quad j = 1, 2, \dots, s\,,$$

or equivalently, when the functions h_j, $j = 1, \dots, s$, are 2π-periodic trigonometric polynomials. Instead, we deal with Laurent polynomials by using the variable $z = \mathrm{e}^{2\pi i w}$, that is,

$$\mathsf{g}_j(z) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k)\, z^k\,, \quad j = 1, 2, \dots, s\,.$$

We introduce the $s \times r$ matrix

$$\mathsf{G}(z) := \begin{bmatrix} \mathsf{g}_1(z) & \mathsf{g}_1(zW) & \cdots & \mathsf{g}_1(zW^{r-1}) \\ \mathsf{g}_2(z) & \mathsf{g}_2(zW) & \cdots & \mathsf{g}_2(zW^{r-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{g}_s(z) & \mathsf{g}_s(zW) & \cdots & \mathsf{g}_s(zW^{r-1}) \end{bmatrix}$$

$$= \Big[\mathsf{g}_j\big(zW^k\big)\Big]_{\substack{j=1,2,\dots,s \\ k=0,1,\dots,r-1}}\,,$$

where $W := \mathrm{e}^{2\pi i/r}$. In case the functions $\mathsf{g}_j(z)$, $j = 1, 2, \dots, s$, are Laurent polynomials, the matrix $\mathsf{G}(z)$ has Laurent polynomials entries. Besides, the relationship $\mathbb{G}(w) = \mathsf{G}(\mathrm{e}^{2\pi i w})$, $w \in (0, 1)$, holds.

So that, we are interested in finding Laurent polynomials $\mathsf{h}_j(z)$, $j = 1, 2 \ldots, s$, satisfying

$$\big[\mathsf{h}_1(z), \mathsf{h}_2(z), \ldots, \mathsf{h}_s(z)\big]\, \mathsf{G}(z) = [1, 0, \ldots, 0]\,.$$

Thus, the trigonometric polynomials $h_j(w) := \mathsf{h}_j(\mathrm{e}^{2\pi \mathrm{i} w})$, $j = 1, 2, \ldots, s$, satisfy (4.7), and the corresponding reconstruction vectors $c_{j,h} = \mathcal{T}_{U,a}(r h_j)$, $j = 1, 2, \ldots, s$, can be expanded in $\mathcal{A}_a$ with just a finite number of terms. Namely,

$$c_{j,h} = r \sum_{\text{finite}} \widehat{h}_j(n)\, U^n a\,,$$

where

$$\mathsf{h}_j(z) = \sum_{\text{finite}} \widehat{h}_j(n)\, z^n\,, \;\; j = 1, 2, \ldots, s\,.$$

The following result holds:

Theorem 4.6. *Assume that the sequences $\big\{\mathcal{L}_j a(k)\big\}_{k\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$, contain only a finite number of nonzero terms. Then, there exists a vector*

$$\mathsf{h}(z) := \big[\mathsf{h}_1(z), \mathsf{h}_2(z), \ldots, \mathsf{h}_s(z)\big]$$

whose entries are Laurent polynomials, and satisfying

$$\mathsf{h}(z)\, \mathsf{G}(z) = [1, 0, \ldots, 0]$$

if and only if

$$\operatorname{rank}\, \mathsf{G}(z) = r \quad \textit{for all} \;\; z \in \mathbb{C}\backslash\{0\}\,.$$

Proof. This result is a consequence of the next lemma which proof can be found in [125, Theorems 5.1 and 5.6]:

Lemma 4.5. *Let $\mathsf{G}(z)$ be an $s \times r$ matrix whose entries are Laurent polynomials. Then, there exists an $r \times s$ matrix $\mathsf{H}(z)$ whose entries are also Laurent polynomials satisfying $\mathsf{H}(z)\mathsf{G}(z) = \mathbb{I}_r$ if and only if*

$$\operatorname{rank} \mathsf{G}(z) = r \quad \textit{for all}\; z \in \mathbb{C}\backslash\{0\}\,.$$

$\square$

Analogously we can consider the case where the coefficients of the reconstruction vectors $c_{j,h} = r \sum_{n\in\mathbb{Z}} \widehat{h}_j(n)\, U^n a$, $j = 1, 2, \ldots, s$, have exponential decay, i.e., there exist $C > 0$ and $q \in (0, 1)$ such that

$$|\widehat{h}_j(n)| \leqslant C q^{|n|}\,, \quad n \in \mathbb{Z}\,,\; j = 1, 2, \ldots, s\,.$$

Assuming that the sequences $\big\{\mathcal{L}_j a(k)\big\}_{k\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay then, we can find reconstruction vectors $c_{j,h}$ such that the sequences $\{\widehat{h}_j(n)\}_{n\in\mathbb{Z}}$, $j = 1, 2, \ldots, s$, have exponential decay if and only if $\operatorname{rank}\, \mathsf{G}(z) = r$ for all $z \in \mathbb{C}$ such that $|z| = 1$. For the details, see [46] and references therein.

4.3.4 Asymmetric sampling

This subsection is based on the recent work [69]; there the authors deal with an asymmetric multi-channel sampling problem. They also use a Fourier type duality and here we extend their results to the U-sampling framework; the computations perfectly fit to our more general setting.

An expression for the generalized asymmetric samples

Suppose that s vectors $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, are given and consider their associated U-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$, $j = 1, 2, \dots, s$. Our aim is the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of asymmetric samples

$$\big\{\mathcal{L}_j x(\sigma_j + r_j m)\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$$

where $\sigma_j \in \mathbb{R}$ and $r_j \in \mathbb{N}$. To this end, first we obtain a suitable expression for the above samples. Notice here that the samples are no longer taken on integer numbers, then we should consider the operator U to be included in a continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$.

For $x \in \mathcal{A}_a$ let $F \in L^2(0,1)$ such that $\mathcal{T}_{U,a}F = x$; by using (4.3), for $j = 1, 2, \dots s$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned}
\mathcal{L}_j x(\sigma_j + r_j m) &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^{\sigma_j + r_j m} b_j, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} n w} \Big\rangle_{L^2(0,1)} \\
&= \Big\langle F, \sum_{k\in\mathbb{Z}} \langle U^{\sigma_j + k} b_j, a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i}(r_j m - k) w} \Big\rangle_{L^2(0,1)} \\
&= \Big\langle F, \Big[\sum_{k\in\mathbb{Z}} \overline{\langle a, U^{\sigma_j + k} b_j\rangle_{\mathcal{H}}}\, \mathrm{e}^{-2\pi \mathrm{i} k w} \Big] \mathrm{e}^{2\pi \mathrm{i} r_j m w} \Big\rangle_{L^2(0,1)},
\end{aligned}$$

where the change in the summation's index $k := r_j m - n$ has been done. Hence,

$$\mathcal{L}_j x(\sigma_j + r_j m) = \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r_j m w} \big\rangle_{L^2(0,1)} \quad \text{for } m \in \mathbb{Z} \text{ and } j = 1, 2, \dots, s, \tag{4.33}$$

where the function

$$g_j(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(\sigma_j + k)\, \mathrm{e}^{2\pi \mathrm{i} k w} \tag{4.34}$$

belongs to $L^2(0,1)$ for each $j = 1, 2, \dots, s$.

As a consequence of (4.33), the stable recovery of any $x \in \mathcal{A}_a$ depends on whether the sequence $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r_j m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ forms a frame for $L^2(0,1)$.

Following [69] we have

$$\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r_j m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s} = \big\{\overline{g_{j,n_j}(w)}\, \mathrm{e}^{2\pi \mathrm{i} r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s;\, n_j=1,2,\dots,r/r_j}$$

where $r := \operatorname{lcm}\{r_j : j = 1, 2, \ldots, s\}$ and $g_{j,n_j}(w) := g_j(w)\, \mathrm{e}^{2\pi \mathrm{i} r_j (n_j-1)w}$.

Let $\mathcal{D}$ be the unitary operator

$$\begin{array}{rccc} \mathcal{D}: & L^2(0,1) & \longrightarrow & L^2_r(0,1/r) \\ & F & \longmapsto & \mathcal{D}F\,, \end{array}$$

where

$$\mathcal{D}F(w) := \Big(F(w), F(w+\frac{1}{r}), \ldots, F(w+\frac{r-1}{r})\Big)^\top, \quad w \in (0,1/r).$$

We also consider the $\big(\sum_{j=1}^{s} \frac{r}{r_j}\big) \times r$ matrix on $(0,1/r)$

$$\mathbb{G}(w) := \Big[\mathcal{D}g_{1,1}(w) \quad \ldots \quad \mathcal{D}g_{1,\frac{r}{r_1}}(w) \quad \ldots \quad \mathcal{D}g_{s,1}(w) \quad \ldots \quad \mathcal{D}g_{s,\frac{r}{r_s}}(w)\Big]^\top \tag{4.35}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,,$$

where $\mathbb{G}^*(w)$ denotes the transpose conjugate of the matrix $\mathbb{G}(w)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(w)\mathbb{G}(w)$. Observe that $0 \leqslant \alpha_{\mathbb{G}} \leqslant \beta_{\mathbb{G}} \leqslant \infty$. Notice that in the definition of the matrix $\mathbb{G}(w)$ we are considering 1-periodic extensions of the involved functions $g_j,\ j = 1, 2, \ldots, s$.

Proceeding as in Section 4.2 we state a complete characterization of the sequence

$$\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r_j m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s} \quad \text{in } L^2(0,1)\,.$$

The result is obtained from Lemma 2.3, as a particular case; here the dimension $d = 1$, the number of generators is $r = 1$ and the sampling lattice M is now a collection of scalars $\{r_j\}_{j=1,2,\ldots s} \subset \mathbb{N}$:

Lemma 4.6. *For the functions $g_j \in L^2(0,1)$, $j = 1, 2, \ldots, s$, consider the associated matrix $\mathbb{G}(w)$ given in* (4.6). *Then, the following results hold:*

(a) The sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r_j n w}\}_{n\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a complete system for $L^2(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is r a.e. in $(0,1/r)$.

(b) The sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r_j n w}\}_{n\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a Bessel sequence for $L^2(0,1)$ if and only if $g_j \in L^\infty(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.

(c) The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} r_j n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ *is a frame for* $L^2(0,1)$ *if and only if* $0<\alpha_{\mathbb{G}}\leqslant \beta_{\mathbb{G}}<\infty$. *In this case, the optimal frame bounds are* $\alpha_{\mathbb{G}}/r$ *and* $\beta_{\mathbb{G}}/r$.

(d) The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} r_j n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ *is a Riesz basis for* $L^2(0,1)$ *if and only if is a frame and* $\sum_{j=1}^{s}\frac{1}{r_j}=1$.

Asymmetric regular sampling in $\mathcal{A}_a$

Once we have characterized the sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} r_j n w}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ as a frame in $L^2(0,1)$ we can mimic the technique of Section 4.3, that is: Choose in $L^\infty(0,1)$ functions h_{j,n_j} with, $j=1,2,\dots,s$ and $1\leqslant n_j\leqslant \frac{r}{r_j}$, such that

$$\big[h_{1,1}(w),\dots,h_{1,\frac{r}{r_1}}(w),\dots,h_{s,1}(w)\dots,h_{s,\frac{r}{r_s}}(w)\big]\mathbb{G}(w)=[1,0,\dots,0]\,. \tag{4.36}$$

a.e. in $(0,1)$. Again we have that the sequence

$$\big\{r h_{j,n_j}(w)\,\mathrm{e}^{2\pi \mathrm{i} r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s;\, n_j=1,2,\dots,r/r_j}$$

is a dual frame of the sequence

$$\big\{\overline{g_{j,n_j}(w)}\,\mathrm{e}^{2\pi \mathrm{i} r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s;\, n_j=1,2,\dots,r/r_j}\,.$$

In other words, taking into account (4.33), we have for any $F\in L^2(0,1)$ the expansion

$$F=\sum_{j=1}^{s}\sum_{l_j=1}^{\frac{r}{r_j}}\sum_{k\in\mathbb{Z}}\mathcal{L}_j x\big(\sigma_j+rk+r_j(l_j-1)\big)\, r h_{j,l_j}(w)\,\mathrm{e}^{2\pi \mathrm{i} r k w}\quad \text{in } L^2(0,1)\,. \tag{4.37}$$

We have used,

$$\mathcal{L}_j x(\sigma_j+r_j m)=\big\langle F,\overline{g_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} r_j m w}\big\rangle_{L^2(0,1)}=\big\langle F,\overline{g_{j,l_j}(w)}\,\mathrm{e}^{2\pi \mathrm{i} r k w}\big\rangle_{L^2(0,1)}\,,$$

with $m=k\,\frac{r}{r_j}+l_j-1,\ j=1,2,\dots,s$ and $1\leqslant l_j\leqslant\frac{r}{r_j}$.

This time the existence of the functions h_{j,n_j}, $j=1,2,\dots,s$; $1\leqslant n_j\leqslant\frac{r}{r_j}$, depends on the first row of the $r\times\big(\sum_{j=1}^{s}\frac{r}{r_j}\big)$ Moore-Penrose pseudo-inverse $\mathbb{G}^\dagger(w)$ of $\mathbb{G}(w)$ given in this case by

$$\mathbb{G}^\dagger(w):=\big[\mathbb{G}^*(w)\,\mathbb{G}(w)\big]^{-1}\,\mathbb{G}^*(w)\,.$$

Its entries are essentially bounded in $(0,1)$ since the functions g_{j,n_j}, $j=1,2,\dots,s$; $1\leqslant n_j\leqslant\frac{r}{r_j}$, and $\det^{-1}\big[\mathbb{G}^*(w)\,\mathbb{G}(w)\big]$ are essentially bounded in $(0,1)$, and (4.36) trivially holds. All the possible solutions of (4.36) are given by the first row of the $r\times\big(\sum_{j=1}^{s}\frac{r}{r_j}\big)$ matrices given by

$$\mathbb{H}_{\mathbb{U}}(w):=\mathbb{G}^\dagger(w)+\mathbb{U}(w)\big[\mathbb{I}-\mathbb{G}(w)\mathbb{G}^\dagger(w)\big]\,, \tag{4.38}$$

where $\mathbb{U}(w)$ denotes any $r \times s$ matrix with entries in $L^\infty(0,1)$, and $\mathbb{I}$ is the identity matrix of order $\sum_{j=1}^{s} \frac{r}{r_j}$.

Applying the isomorphism $\mathcal{T}_{U,a}$ in (4.37), for $x = \mathcal{T}_{U,a}F \in \mathcal{A}_a$ we obtain the sampling expansion:

$$\begin{aligned} x &= \sum_{j=1}^{s} \sum_{l_j=1}^{\frac{r}{r_j}} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x\big(\sigma_j + rk + r_j(l_j-1)\big)\, \mathcal{T}_{U,a}\big[r h_{j,l_j}(\cdot)\, \mathrm{e}^{2\pi \mathrm{i} rk\,\cdot}\big] \\ &= \sum_{j=1}^{s} \sum_{l_j=1}^{\frac{r}{r_j}} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x\big(\sigma_j + rk + r_j(l_j-1)\big)\, U^{rk}\big[\mathcal{T}_{U,a}(r h_{j,l_j})\big] \qquad (4.39) \\ &= \sum_{j=1}^{s} \sum_{l_j=1}^{\frac{r}{r_j}} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x\big(\sigma_j + rk + r_j(l_j-1)\big)\, U^{rk} c_{j,l_j}\,, \end{aligned}$$

where $c_{j,l_j} := \mathcal{T}_{U,a}(r h_{j,l_j}) \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, and we have used the U-shift property (4.2). Besides, the sequence $\{U^{rk} c_{j,l_j}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s;\, l_j=1,2,\dots,\frac{r}{r_j}}$ is a frame for $\mathcal{A}_a$. In fact, the following result holds:

Theorem 4.7. *Let b_j be in $\mathcal{H}$ and let $\mathcal{L}_j$ be its associated U-system for $j = 1, 2, \dots, s$. Assume that the function g_j, $j = 1, 2, \dots, s$, given in* (4.34) *belongs to $L^\infty(0,1)$; or equivalently, $\beta_{\mathbb{G}} < \infty$ for the associated $\big(\sum_{j=1}^{s} \frac{r}{r_j}\big) \times r$ matrix $\mathbb{G}(w)$. The following statements are equivalent:*

(a) $\alpha_{\mathbb{G}} > 0$.

(b) There exists a vector $\big[h_{1,1}(w), \dots, h_{1,\frac{r}{r_1}}(w), \dots, h_{s,1}(w) \dots, h_{s,\frac{r}{r_s}}(w)\big]$ with entries in $L^\infty(0,1)$ satisfying

$$\big[h_{1,1}(w), \dots, h_{1,\frac{r}{r_1}}(w), \dots, h_{s,1}(w) \dots, h_{s,\frac{r}{r_s}}(w)\big]\mathbb{G}(w) = [1, 0, \dots, 0]$$

a.e. in $(0,1)$.

(c) There exist $c_{j,l_j} \in \mathcal{A}_a$, $j = 1, 2, \dots, s$; $l_j = 1, 2, \dots, \frac{r}{r_j}$, such that the sequence $\{U^{rk} c_{j,l_j}\}_{k\in\mathbb{Z};\, j=1,2,\dots s;\, l_j=1,2,\dots,\frac{r}{r_j}}$ is a frame for $\mathcal{A}_a$, and for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s} \sum_{l_j=1}^{\frac{r}{r_j}} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x\big(\sigma_j + rk + r_j(l_j-1)\big)\, U^{rk} c_{j,l_j} \qquad (4.40)$$

holds.

(d) *There exists a frame* $\{C_{j,l_j,k}\}_{k\in\mathbb{Z};\, j=1,2,\ldots s;\, l_j=1,2,\ldots,\frac{r}{r_j}}$ *for* $\mathcal{A}_a$ *such that, for each* $x \in \mathcal{A}_a$ *the expansion*

$$x = \sum_{j=1}^{s} \sum_{l_j=1}^{\frac{r}{r_j}} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x\big(\sigma_j + rk + r_j(l_j-1)\big)\; C_{j,l_j,k} \quad \text{in } \mathcal{H}$$

holds.

Proof. The proof is analogous to that in Theorem 4.2. □

4.4 Time-jitter error: irregular sampling in $\mathcal{A}_a$

A close look to Section 4.3 shows that all the regular sampling results have been proved without the formalism of a continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$: we have only used the integer powers $\{U^n\}_{n\in\mathbb{Z}}$ which are completely determined from the unitary operator U. However, if we are concerned with the jitter-error in a sampling formula as (4.11), the group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ becomes essential. Let T be the infinitesimal generator of this continuous group with domain D_T (see Appendix B). Here, we dispose of a perturbed sequence of samples

$$\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}\,,$$

with errors $\epsilon_{mj} \in \mathbb{R}$, for the recovery of $x \in \mathcal{A}_a$. By using (4.4) and (4.3) we obtain:

$$\mathcal{L}_j x(rm) = \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi i rmw} \big\rangle_{L^2(0,1)}$$

and

$$\mathcal{L}_j x(rm + \epsilon_{mj}) = \big\langle F, \overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi i rmw} \big\rangle_{L^2(0,1)}\,,$$

where the functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k)\, \mathrm{e}^{2\pi i kw} \quad \text{and} \quad g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \mathcal{L}_j a(k+\epsilon_{mj})\, \mathrm{e}^{2\pi i kw}\,,$$

belong to $L^2(0,1)$. Let $\mathbb{G}(w)$ be the $s \times r$ matrix given in (4.6), associated with the functions g_j, $j = 1, 2, \ldots, s$. In the case that $0 < \alpha_{\mathbb{G}} \leqslant \beta_{\mathbb{G}} < \infty$, the sequence

$$\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s}$$

is a frame for $L^2(0,1)$ with optimal frame bounds $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$. Thus, as in [42], we can see the sequence

$$\big\{\overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi i rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s} \quad \text{in } L^2(0,1)$$

as a perturbation of the above frame.

The time-jitter error sampling expansion

Given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$, assume that the operator

$$\begin{array}{rccl} D_{\boldsymbol{\epsilon}}: & \ell^2(\mathbb{Z}) & \longrightarrow & \ell^2_s(\mathbb{Z}) \\ & c=\{c_l\}_{l\in\mathbb{Z}} & \longmapsto & D_{\boldsymbol{\epsilon}}\, c := \big(D_{\boldsymbol{\epsilon},1}\, c, \dots, D_{\boldsymbol{\epsilon},s}\, c\big)\,, \end{array}$$

is well-defined, where, for $j=1,2,\dots,s$,

$$D_{\boldsymbol{\epsilon},j}\, c := \Big\{ \sum_{k\in\mathbb{Z}} \big[\mathcal{L}_j a(rm-k+\epsilon_{mj}) - \mathcal{L}_j a(rm-k)\big] c_k \Big\}_{m\in\mathbb{Z}}\,. \tag{4.41}$$

The operator norm (it could be infinity) is defined as usual

$$\|D_{\boldsymbol{\epsilon}}\| := \sup_{c\in\ell^2(\mathbb{Z})\setminus\{0\}} \frac{\|D_{\boldsymbol{\epsilon}}\, c\|_{\ell^2_s(\mathbb{Z})}}{\|c\|_{\ell^2(\mathbb{Z})}}\,,$$

where $\|D_\epsilon\, c\|^2_{\ell^2_s(\mathbb{Z})} := \sum_{j=1}^s \|D_{\epsilon,j}\, c\|^2_{\ell^2(\mathbb{Z})}$ for each $c\in\ell^2(\mathbb{Z})$.

Theorem 4.8. *Assume that for the functions g_j, $j=1,2,\dots,s$, given in* (4.5) *we have $0<\alpha_{\mathbb{G}}\leqslant\beta_{\mathbb{G}}<\infty$. Let $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$ be an error sequence satisfying the inequality $\|D_{\boldsymbol{\epsilon}}\|^2<\alpha_{\mathbb{G}}/r$. Then, there exists a frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x\in\mathcal{A}_a$, the sampling expansion*

$$x=\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm+\epsilon_{mj})\, C^{\boldsymbol{\epsilon}}_{j,m} \quad \text{in } \mathcal{H}\,, \tag{4.42}$$

holds. Moreover, when $r=s$ the sequence $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz basis for $\mathcal{A}_a$, and the interpolation property $(\mathcal{L}_l\, C^{\boldsymbol{\epsilon}}_{j,n})(rm+\epsilon_{mj})=\delta_{j,l}\,\delta_{n,m}$ holds.

Proof. We already know that the sequence $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame (a Riesz basis if $r=s$) for $L^2(0,1)$ with optimal frame (Riesz) bounds $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.

For any

$$F(w)=\sum_{l\in\mathbb{Z}} a_l\, \mathrm{e}^{2\pi i l w} \quad \text{in } L^2(0,1)$$

we have

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\left|\left\langle \overline{g_{m,j}(\cdot)}\,\mathrm{e}^{2\pi \mathrm{i} rm\cdot}-\overline{g_j(\cdot)}\,\mathrm{e}^{2\pi \mathrm{i} rm\cdot},F(\cdot)\right\rangle_{L^2(0,1)}\right|^2\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\left|\left\langle \sum_{k\in\mathbb{Z}}\left(\overline{\mathcal{L}_j a(k+\epsilon_{mj})}-\overline{\mathcal{L}_j a(k)}\right)\mathrm{e}^{2\pi \mathrm{i}(rm-k)\cdot},F(\cdot)\right\rangle_{L^2(0,1)}\right|^2\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\left|\left\langle \sum_{k\in\mathbb{Z}}\left(\overline{\mathcal{L}_j a(rm-k+\epsilon_{mj})}-\overline{\mathcal{L}_j a(rm-k)}\right)\mathrm{e}^{2\pi \mathrm{i}k\cdot},F(\cdot)\right\rangle_{L^2(0,1)}\right|^2\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\left|\sum_{k\in\mathbb{Z}}\left(\overline{\mathcal{L}_j a(rm-k+\epsilon_{mj})}-\overline{\mathcal{L}_j a(rm-k)}\right)\overline{a}_k\right|^2\\
&=\sum_{j=1}^{s}\|D_{\epsilon,j}\{a_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})}\leqslant\|D_{\boldsymbol{\epsilon}}\|^2\|\{a_l\}_{l\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})}=\|D_{\boldsymbol{\epsilon}}\|^2\|F\|^2_{L^2(0,1)}\,.
\end{aligned}
\tag{4.43}
$$

By using Lemma A.8, about perturbation of frames, we obtain that the sequence $\left\{\overline{g_{m,j}(w)}\,\mathrm{e}^{2\pi \mathrm{i} rmw}\right\}_{m\in\mathbb{Z};\,j=1,2,\ldots s}$ is a frame for $L^2(0,1)$ (a Riesz basis if $r=s$). Let $\{h^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\,j=1,2,\ldots,s}$ be its canonical dual frame. Hence, for any $F\in L^2(0,1)$

$$
\begin{aligned}
F&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\left\langle F(\cdot),\overline{g_{m,j}(\cdot)}\,\mathrm{e}^{2\pi \mathrm{i} rm\cdot}\right\rangle_{L^2(0,1)}h^{\boldsymbol{\epsilon}}_{j,m}\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\mathcal{L}_j x(rm+\epsilon_{mj})\,h^{\boldsymbol{\epsilon}}_{j,m}\quad\text{in } L^2(0,1)\,.
\end{aligned}
$$

Applying the isomorphism $\mathcal{T}_{U,a}$, one gets (4.42), where $C^{\boldsymbol{\epsilon}}_{j,m}:=\mathcal{T}_{U,a}\big(h^{\boldsymbol{\epsilon}}_{j,m}\big)$ for $m\in\mathbb{Z}$ and $j=1,2,\ldots,s$. Since $\mathcal{T}_{U,a}$ is an isomorphism between $L^2(0,1)$ and $\mathcal{A}_a$, the sequence $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\,j=1,2,\ldots,s}$ is a frame for $\mathcal{A}_a$ (a Riesz basis if $r=s$). The interpolatory property in the case $r=s$ follows from the uniqueness of the coefficients with respect to a Riesz basis. □

As we pointed out at the end of the subsection 2.4.2 sampling formulae like (4.42) are useless from a practical point of view: it is impossible to determine the involved frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\,j=1,2,\ldots,s}$. However, in order to recover $x\in\mathcal{A}_a$ from the sequence of samples $\big\{(\mathcal{L}_j x)(rm+\epsilon_{mj})\big\}_{m\in\mathbb{Z};\,j=1,2,\ldots,s}$ we should implement a frame algorithm in $\ell^2(\mathbb{Z})$, like we already did in Subsection 2.4.3. The interested reader can also check Ref. [42]; another possibility is given in Ref. [2].

In order to prove the existence of sequences $\boldsymbol{\epsilon}:=\{\epsilon_{mj}\}_{m\in\mathbb{Z};\,j=1,\ldots,s}$ such that $\|D_{\boldsymbol{\epsilon}}\|^2<\alpha_{\mathbb{G}}/r$ we need some results from the group of unitary operators theory (see Appendix B).

On the existence of sequences $\boldsymbol{\epsilon}$ such that $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$

Assuming that $b_j \in D_T$, $j = 1, 2, \ldots, s$, the functions $\mathcal{L}_j a(t)$, $j = 1, 2, \ldots, s$, are continuously differentiable on $\mathbb{R}$. According to Stone's theorem, T is the self-adjoint operator such that $U^t = \mathrm{e}^{\mathrm{i}tT}$ with domain D_T. If, for instance, we demand in addition that, for each $j = 1, 2, \ldots, s$, there exists $\eta_j > 0$ such that

$$(\mathcal{L}_j a)'(t) = O(|t|^{-(1+\eta_j)}) \quad \text{whenever } |t| \to \infty\,, \tag{4.44}$$

then we can find a finite bound for the norm $\|D_{\boldsymbol{\epsilon}}\|^2$. Indeed, for $j = 1, 2, \ldots, s$ and $n, m \in \mathbb{Z}$ denote

$$d^{(j)}_{m,k} := \mathcal{L}_j a(rm - k + \epsilon_{m,j}) - \mathcal{L}_j a(rm - k)\,.$$

Taking into account (4.41), for any sequence $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}
\|D_\epsilon c\|^2_{\ell^2_s(\mathbb{Z})} &= \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} d^{(j)}_{m,k}\, c_k \Big|^2 \leqslant \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \sum_{l,k\in\mathbb{Z}} |d^{(j)}_{m,l} c_l \overline{d}^{(j)}_{m,k} \overline{c}_k| \\
&= \sum_{j=1}^{s} \sum_{l,k\in\mathbb{Z}} |c_l|\,|c_k| \sum_{m\in\mathbb{Z}} |d^{(j)}_{m,l} d^{(j)}_{m,k}| \\
&\leqslant \sum_{j=1}^{s} \sum_{l,k\in\mathbb{Z}} \frac{|c_l|^2 + |c_k|^2}{2} \sum_{m\in\mathbb{Z}} |d^{(j)}_{m,l} d^{(j)}_{m,k}| \\
&= \sum_{j=1}^{s} \sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{k,m\in\mathbb{Z}} |d^{(j)}_{m,l} d^{(j)}_{m,k}|\,.
\end{aligned} \tag{4.45}$$

Under the decay conditions (4.44), for $|\gamma| \leqslant 1/2$ we define the continuous functions,

$$M_{(\mathcal{L}_j a)'}(\gamma) := \sum_{k\in\mathbb{Z}} \max_{t\in[k-\gamma,k+\gamma]} |(\mathcal{L}_j a)'(t)|\,,$$

and

$$N_{(\mathcal{L}_j a)'}(\gamma) := \max_{k=0,1,\ldots,r-1} \sum_{m\in\mathbb{Z}} \max_{t\in[rm+k-\gamma,rm+k+\gamma]} |(\mathcal{L}_j a)'(t)|\,.$$

Notice that $N_{(\mathcal{L}_j a)'}(\gamma) \leqslant M_{(\mathcal{L}_j a)'}(\gamma)$ and for $r = 1$ the equality holds.

Theorem 4.9. *Given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\ldots,s}$, define the constant $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}|$ for each $j = 1, 2, \ldots, s$. Then, the inequality*

$$\|D_{\boldsymbol{\epsilon}}\|^2 \leqslant \sum_{j=1}^{s} \gamma_j^2\, N_{(\mathcal{L}_j a)'}(\gamma_j) M_{(\mathcal{L}_j a)'}(\gamma_j)$$

holds. As a consequence, condition

$$\sum_{j=1}^{s} \gamma_j^2\, N_{(\mathcal{L}_j a)'}(\gamma_j) M_{(\mathcal{L}_j a)'}(\gamma_j) < \frac{\alpha_{\mathbf{G}}}{r} \tag{4.46}$$

ensures the hypothesis $\|D_{\boldsymbol{\epsilon}}\|^2 < \alpha_{\mathbb{G}}/r$ *on Theorem 4.8.*

Proof. For each $j = 1, 2, \ldots, s$, the mean value theorem gives

$$\sup_{d \in [-\gamma_j, \gamma_j]} \sum_{n \in \mathbb{Z}} |\mathcal{L}_j a(n+d) - \mathcal{L}_j a(n)| \leqslant \gamma_j \, M_{(\mathcal{L}_j a)'}(\gamma_j)\,, \tag{4.47}$$

and

$$\sup_{\substack{k=0,1,\ldots,r-1 \\ \{d_n\} \subset [-\gamma_j, \gamma_j]}} \sum_{n \in \mathbb{Z}} |\mathcal{L}_j a(rn+k+d_n) - \mathcal{L}_j a(rn+k)| \leqslant \gamma_j \, N_{(\mathcal{L}_j a)'}(\gamma_j)\,. \tag{4.48}$$

Thus, using (4.47) and (4.48), inequality (4.45) becomes

$$\begin{aligned}
\|D_\epsilon c\|^2_{\ell^2_s(\mathbb{Z})} &\leqslant \sum_{j=1}^{s} \sum_{l \in \mathbb{Z}} |c_l|^2 \sum_{k,m \in \mathbb{Z}} |d^{(j)}_{m,l} d^{(j)}_{m,k}| \\
&\leqslant \sum_{j=1}^{s} \sum_{l \in \mathbb{Z}} |c_l|^2 \sum_{m \in \mathbb{Z}} |d^{(j)}_{m,l}| \, \gamma_j \, M_{(\mathcal{L}_j a)'}(\gamma_j) \\
&\leqslant \sum_{j=1}^{s} \sum_{l \in \mathbb{Z}} |c_l|^2 \, (\gamma_j)^2 \, M_{(\mathcal{L}_j a)'}(\gamma_j) N_{(\mathcal{L}_j a)'}(\gamma_j) \\
&= \|c\|^2_{\ell^2(\mathbb{Z})} \sum_{j=1}^{s} \gamma_j^2 \, N_{(\mathcal{L}_j a)'}(\gamma_j) M_{(\mathcal{L}_j a)'}(\gamma_j)\,,
\end{aligned} \tag{4.49}$$

which concludes the proof. □

Condition (4.46) can be improved in the following sense. Define for $|\gamma| < 1/2$ the following functions:

$$\widetilde{M}_{a,b_j}(\gamma) := \sum_{n \in \mathbb{Z}} \max_{t \in [-\gamma, \gamma]} |\mathcal{L}_j a(n+t) - \mathcal{L}_j a(n)|\,,$$

and

$$\widetilde{N}_{a,b_j}(\gamma) := \max_{k=0,1,\ldots,r-1} \sum_{n \in \mathbb{Z}} \max_{t \in [-\gamma, \gamma]} |\mathcal{L}_j a(rn+k+t) - \mathcal{L}_j a(rn+k)|\,.$$

Notice that $\widetilde{N}_{a,b_j}(\gamma) \leqslant \widetilde{M}_{a,b_j}(\gamma)$ and for $r = 1$ the equality holds. Moreover, assuming, for instance, that the continuous functions $\mathcal{L}_j a(t) := \langle a, U^t b_j \rangle$, $j = 1, 2, \ldots, s$, satisfy a decay condition like

$$\mathcal{L}_j a(t) = O\big(|t|^{-(1+\eta_j)}\big) \quad \text{when } |t| \to \infty \ \text{ for some } \eta_j > 0\,,$$

we may deduce that the functions $\widetilde{N}_{a,b_j}(\gamma)$ and $\widetilde{M}_{a,b_j}(\gamma)$ are continuous near to 0. Notice that in this case the elements b_j are not necessarily in the domain D_T of T.

Proceeding as above (see (4.49)), one easily proves that

$$\|D_{\boldsymbol{\epsilon}}\|^2 \leqslant \sum_{j=1}^{s} \widetilde{M}_{a,b_j}(\gamma_j)\,\widetilde{N}_{a,b_j}(\gamma_j)$$

where $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{m,j}|$ for each $j = 1, 2, \ldots, s$. Thus, the condition

$$\sum_{j=1}^{s} \widetilde{M}_{a,b_j}(\gamma_j)\,\widetilde{N}_{a,b_j}(\gamma_j) < \frac{\alpha_{\mathbf{G}}}{r}$$

implies that the thesis of Theorem 4.9 also holds.

4.4.1 Studying the perturbed sequence $\big\{U^{rk+\epsilon_{kj}} b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$

In so far of this section we did not study the sequence $\big\{U^{rk+\epsilon_{kj}} b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ directly, this is because the sequence $\big\{U^{rk} b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is not necessarily a frame for the entire Hilbert space $\mathcal{H}$, moreover, its perturbed sequence does not necessarily belong to the subspace $\mathcal{A}_a$, that is the reason that Theorem A.8 cannot be applied to these sequences. To avoid these problems we have used the isomorphism $\mathcal{T}_{U,a}$ to move our analysis to the space $L^2(0,1)$.

Nevertheless, given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{kj}\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$, there is something that we can say about the perturbed sequence $\big\{U^{rk+\epsilon_{kj}} b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$.

Let $\mathrm{i}T$ be the infinitesimal generator of the continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ (see Appendix B). Then b_j will belong to the domain D_T of T whether the condition

$$\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2 < \infty$$

is satisfied (see Theorem B.1 and Theorem B.2). Here $\{E_w\}_{w\in\mathbb{R}}$ is the resolution of the identity associated to the self-adjoint operator T.

Theorem 4.10. *Assume that for certain $b_j \in D_T$, $j = 1, 2, \ldots, r$, the sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a Riesz basis for $\mathcal{A}_a$ with Riesz bounds $0 < A_{\boldsymbol{\Psi}} \leqslant B_{\boldsymbol{\Psi}} < \infty$. For a sequence $\boldsymbol{\epsilon} := \{\epsilon_{kj}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ of errors, let R be the constant given by*

$$R := \|\boldsymbol{\epsilon}\|^2 \max_{j=1,2,\ldots,r} \left\{ \int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2 \right\},$$

where $\|\boldsymbol{\epsilon}\|$ denotes the ℓ^2_r-norm of the sequence $\boldsymbol{\epsilon}$.

If $R < A_{\boldsymbol{\Psi}}$, then the perturbed sequence $\{U^{kr+\epsilon_{kj}} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,r}$ is a Riesz sequence in $\mathcal{H}$ with Riesz bounds $A_{\boldsymbol{\Psi}}\big(1-\sqrt{R/A_{\boldsymbol{\Psi}}}\big)^2$ and $B_{\boldsymbol{\Psi}}\big(1+\sqrt{R/B_{\boldsymbol{\Psi}}}\big)^2$.

Proof. By using inequality (B.2) we have

$$
\begin{aligned}
\left|\langle x, U^{kr}b_j - U^{kr+\epsilon_{kj}}b_j\rangle\right| &= \Big|\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}krw} d\langle E_w x, b_j\rangle - \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}krw - i\epsilon_{kj}w} d\langle E_w x, b_j\rangle\Big| \\
&= \Big|\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}krw}(1-\mathrm{e}^{-\mathrm{i}\epsilon_{kj}w}) d\langle E_w x, b_j\rangle\Big| \\
&= \Big|\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}krw}(1-\mathrm{e}^{\mathrm{i}\epsilon_{kj}w}) d\langle E_w b_j, x\rangle\Big| \\
&\leqslant \|x\| \sqrt{\int_{-\infty}^{\infty} |1-\mathrm{e}^{\mathrm{i}\epsilon_{kj}w}|^2 d\|E_w b_j\|^2} \\
&\leqslant \|x\| \sqrt{\int_{-\infty}^{\infty} w^2|\epsilon_{kj}|^2 d\|E_w b_j\|^2} \\
&= |\epsilon_{kj}|\|x\| \sqrt{\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} |\langle x, U^{kr}b_j - U^{kr+\epsilon_{kj}}b_j\rangle|^2 \leqslant \|x\|^2 \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} \Big(\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2\Big)|\epsilon_{kj}|^2 \\
&\leqslant \|x\|^2 \max_{j=1,2,\dots,r}\Big\{\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2\Big\} \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} |\epsilon_{kj}|^2
\end{aligned}
$$

Hence, Lemma A.8 and Theorem 15.3.2 in [25] give the desired result. □

4.5 The case of multiple generators

The case of L generators can be analogously derived. Indeed, consider the U-invariant subspace generated by $\mathbf{a} := \{a_1, a_2, \dots, a_L\} \subset \mathcal{H}$, i.e.,

$$\mathcal{A}_{\mathbf{a}} := \overline{\operatorname{span}}\big\{U^n a_l,\ n\in\mathbb{Z};\ l = 1, 2.\dots, L\big\}\,.$$

Assuming that the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz sequence in $\mathcal{H}$, the U-invariant subspace $\mathcal{A}_{\mathbf{a}}$ can be expressed as

$$\mathcal{A}_{\mathbf{a}} = \Big\{\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}} \alpha_n^l\, U^n a_l : \{\alpha_n^l\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z});\ l = 1, 2.\dots, L\Big\}\,.$$

The sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ can be thought as an L-dimensional stationary sequence. Its *covariance matrix* $\mathbf{R}_{\mathbf{a}}(k)$ is the $L\times L$ matrix

$$\mathbf{R}_{\mathbf{a}}(k) = \Big[\langle U^k a_m, a_n\rangle_{\mathcal{H}}\Big]_{m,n=1,2,\dots,L}\,, \qquad k\in\mathbb{Z}\,.$$

Its admits the spectral representation [72]:

$$\mathbf{R}_{\mathbf{a}}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\boldsymbol{\mu}_{\mathbf{a}}(\theta)\,, \quad k \in \mathbb{Z}\,.$$

The *spectral measure* $\boldsymbol{\mu}_{\mathbf{a}}$ is an $L \times L$ matrix; its entries are the spectral measures associated with the cross-correlation functions $R_{m,n}(k) := \langle U^k a_m, a_n\rangle_{\mathcal{H}}$. It can be decomposed into an absolute continuous part and its singular part. Thus we can write

$$d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \boldsymbol{\Phi}_{\mathbf{a}}(\theta)d\theta + d\boldsymbol{\mu}^s_{\mathbf{a}}(\theta)\,.$$

In case that the singular part $\boldsymbol{\mu}^s_{\mathbf{a}} \equiv 0$, the hermitian $L \times L$ matrix $\boldsymbol{\Phi}_{\mathbf{a}}(\theta)$ is called the *spectral density* of the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$. The following theorem holds:

Theorem 4.11. *Let $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ be a sequence obtained from an unitary operator in a separable Hilbert space $\mathcal{H}$ with spectral measure $d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \boldsymbol{\Phi}_{\mathbf{a}}(\theta)d\theta + d\boldsymbol{\mu}^s_{\mathbf{a}}(\theta)$, and let $\mathcal{A}_{\mathbf{a}}$ be the closed subspace spanned by $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$. Then the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ if and only if the singular part $\boldsymbol{\mu}^s_{\mathbf{a}} \equiv 0$ and*

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \leqslant \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] < \infty\,. \tag{4.50}$$

Proof. For a fixed ℓ^2_L-sequence $c := \{c^l_n\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ we have

$$\begin{aligned}
\Big\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} c^l_k U^k a_l\Big\|^2 &= \sum_{i,j=1}^{L}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c^i_m \bar{c}^j_m \langle U^m a_i, U^n a_j\rangle \\
&= \sum_{i,j=1}^{L}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c^i_m \bar{c}^j_n \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}m\theta}\mathrm{e}^{-\mathrm{i}n\theta} d\mu_{a_i,a_j}(\theta) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} (\mathbf{c}_m\, \mathrm{e}^{\mathrm{i}m\theta})^\top d\boldsymbol{\mu}_{\mathbf{a}}(\theta)\bar{\mathbf{c}}_n\, \mathrm{e}^{-\mathrm{i}n\theta}\,,
\end{aligned} \tag{4.51}$$

where $\mathbf{c}_k = (c^1_k, c^2_k, \dots, c^L_k)^\top$ for every $k \in \mathbb{Z}$.

First we show that if the measure $\boldsymbol{\mu}_{\mathbf{a}}$ is not absolutely continuous with respect to Lebesgue measure then $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is not a Riesz basis for $\mathcal{A}_{\mathbf{a}}$. Indeed, if the spectral measure $\boldsymbol{\mu}_{\mathbf{a}}$ is not absolutely continuous with respect to Lebesgue measure λ then there exists $i \in \{1, 2, \dots, L\}$ such that the positive spectral measure μ_{a_i,a_i} is not absolutely continuous with respect to Lebesgue measure; this comes from the fact that, if any spectral measure in the diagonal μ_{a_j,a_j} is absolutely continuous with respect to Lebesgue measure, the same occurs for each measure μ_{a_j,a_k} with $k \neq j$ (see [14, p. 137]). Then, $\mu_{a_i,a_i}(B) > 0$ for a (Lebesgue) measurable set $B \subset (-\pi, \pi)$ of Lebesgue measure zero. Bearing in mind that every measurable set is included in a Borel set, actually an intersection of a countable collection of open sets, having the same Lebesgue measure (see [88, p. 63]), we take B to be a Borel set. Moreover, since every finite Borel measure on $(-\pi,\pi)$ is inner regular (see [88, p. 340]) we may also assume that B is a compact set. For any $\varepsilon>0$ there exists a sequence of disjoint open intervals $I_j\subset(-\pi,\pi)$ such that

$$B\subset\bigcup_{j=1}^{\infty}I_j \quad\text{and}\quad \sum_{j=1}^{\infty}\lambda(I_j)\leqslant\lambda(B)+\varepsilon=\varepsilon$$

(see [88, pp. 58 and 42]). Since B is compact we may take the sequence to be finite. Hence, for every $N\in\mathbb{N}$ there exist open disjoint intervals $I_1^N,I_2^N,\ldots,I_{j_N}^N$ in $(-\pi,\pi)$ such that

$$B\subset\bigcup_{j=1}^{j_N}I_j^N \quad\text{and}\quad \sum_{j=1}^{j_N}\lambda(I_j^N)\leqslant\frac{1}{3^N}\,.$$

Besides, $\sum_{j=1}^{j_N}\mu_{a_i,a_i}(I_j^N)\geqslant\mu_{a_i,a_i}(B)$. Consider the function $g_N\colon(-\pi,\pi)\to\mathbb{R}$, where $g_N=2^{N/2}\chi_{\bigcup_{j=1}^{j_N}I_j^N}$, that satisfies

$$\|g_N\|_2^2=2^N\sum_{j=1}^{j_N}\lambda(I_j^N)\leqslant\frac{2^N}{3^N}<1\,.$$

We modify and extend each g_N to obtain a 2π-periodic function $f_N:\mathbb{R}\longrightarrow\mathbb{R}$ such that f_N and its derivative are continuous on $\mathbb{R}$, $\|f_N\|_2^2\leqslant 1$ and $f_N(\theta)=g_N(\theta)$ for every $\theta\in\bigcup_{j=1}^{j_N}I_j^N$. Let $\sum_k c_k^N\mathrm{e}^{\mathrm{i}k\theta}$ be the Fourier series of f_N. First, by using Parseval's identity we have

$$\|c_k^N\|_2^2=\frac{1}{2\pi}\|f_N\|_2^2\leqslant\frac{1}{2\pi}\quad\text{for every } N\in\mathbb{N}\,,$$

so that $\{c^N\}_{N=1}^{\infty}$ is a bounded sequence in $\ell^2(\mathbb{Z})$. Besides, the regularity of each f_N ensures that each Fourier series converges uniformly to f_N. Therefore each series $\sum_k c_k^N\,\mathrm{e}^{\mathrm{i}k\theta}$ converges to f_N in $L^2_{\mu_{a_i,a_i}(-\pi,\pi)}$ and consequently,

$$\begin{aligned}\Big\|\sum_k c_k^N\,\mathrm{e}^{\mathrm{i}k\theta}\Big\|^2_{L^2_{\mu_{a_i,a_i}}(-\pi,\pi)} &= \int_{-\pi}^{\pi}|f_N|^2d\mu_{a_i,a_i}\geqslant\int_{-\pi}^{\pi}|g_N|^2d\mu_{a_i,a_i}\\ &= 2^N\sum_{j=1}^{j_N}\mu_{a_i,a_i}(I_j^N)\geqslant 2^N\mu_{a_i,a_i}(B)\,.\end{aligned}$$

For every $c^N\in\ell^2(\mathbb{Z})$ we consider the ℓ^2_L-sequence $\{c_n^{Nl}\}_{n\in\mathbb{Z};\,l=1,2,\ldots,L}$ given by $c_n^{Ni}=c_n^N$ and $c_n^{Nl}=0$ if $l\neq i$. Substituting each $\{c_n^{Nl}\}_{n\in\mathbb{Z};\,l=1,2,\ldots,L}$ in (4.51) we have that

$$\Big\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}}c_k^{Nl}U^ka_l\Big\|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}}c_k^N\,\mathrm{e}^{\mathrm{i}k\theta}\Big|^2d\mu_{a_i,a_i}(\theta)$$

tends to infinity with N, so $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ cannot be a Bessel sequence, therefore, not a Riesz basis.

For the remainder of the proof we assume that the singular part $\boldsymbol{\mu}_{\mathbf{a}}^s \equiv 0$ and that $d\boldsymbol{\mu}_{\mathbf{a}}(\theta) = \boldsymbol{\Phi}_{\mathbf{a}}(\theta)d\theta$. Then (4.51) yields that

$$\Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \big(\sum_{m\in\mathbb{Z}} \mathbf{c}_m \,\mathrm{e}^{\mathrm{i}m\theta} \big)^\top \boldsymbol{\Phi}_{\mathbf{a}}(\theta) \overline{\sum_{n\in\mathbb{Z}} \mathbf{c}_n \,\mathrm{e}^{\mathrm{i}n\theta}} d\theta \,. \tag{4.52}$$

We have to show that $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ if and only if (4.50) holds. The Rayleigh-Ritz theorem (see [64, p. 176]) provides the inequalities

$$\begin{aligned} \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2 &\leqslant \big(\sum_{m\in\mathbb{Z}} \mathbf{c}_m \,\mathrm{e}^{\mathrm{i}m\theta} \big)^\top \boldsymbol{\Phi}_{\mathbf{a}}(\theta) \overline{\sum_{n\in\mathbb{Z}} \mathbf{c}_n \,\mathrm{e}^{\mathrm{i}n\theta}} \\ &\leqslant \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2 \end{aligned}$$

and taking into account (4.52) we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2 d\theta \quad &\leqslant \quad \Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 \\ &\leqslant \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \big| \sum_{k\in\mathbb{Z}} \mathbf{c}_k \,\mathrm{e}^{\mathrm{i}k\theta} \big|^2 d\theta \,, \end{aligned}$$

so that

$$\begin{aligned} \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 \quad &\leqslant \quad \Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 \\ &\leqslant \quad \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 \,. \end{aligned}$$

Therefore (4.50) implies that $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$.

Conversely, if $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ is a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ then there exist constants $0 < A \leqslant B < \infty$ such that

$$A \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 \leqslant \Big\| \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} c_k^l U^k a_l \Big\|^2 \leqslant B \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |c_k^l|^2 \tag{4.53}$$

for every ℓ_L^2-sequence $c := \{c_n^l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$. Let us prove that

$$A \leqslant \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \leqslant \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \leqslant B \,. \tag{4.54}$$

Proceeding by contradiction, if (4.54) would not hold, then

$$A \leqslant \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \leqslant \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] \leqslant B$$

does not hold on a subset of $(-\pi,\pi)$ with positive Lebesgue measure. In case the set $\Gamma_B := \{\theta \in (-\pi,\pi)\colon \lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] > B\}$ has positive Lebesgue measure we introduce the Fourier expansion of the function $F \in L^2_L(-\pi,\pi)$ ($L^2_L(-\pi,\pi)$ denotes the usual product Hilbert space $L^2(-\pi,\pi)\times\cdots\times L^2(-\pi,\pi)$ (L times)) in (4.52), where $F(\theta) = \mathbf{X}(\theta)\,\chi_{\Gamma_B}(\theta)$ and $\mathbf{X}(\theta)$ is an eigenvector of norm 1 associated with the biggest eigenvalue of $\boldsymbol{\Phi}_{\mathbf{a}}(\theta)$. We get

$$\Big\|\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} c_k^l U^k a_l\Big\|^2 = \frac{1}{2\pi}\int_{\Gamma_B}\lambda_{\max}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big]d\theta > \frac{1}{2\pi}\int_{\Gamma_B} B d\theta$$

which contradicts the right inequality in (4.53) for such a Fourier expansion. Whenever Lebesgue measure of the set Γ_B is zero then we proceed in a similar way with the set of positive Lebesgue measure $\Gamma_A := \{\theta \in (-\pi,\pi)\colon \lambda_{\min}\big[\boldsymbol{\Phi}_{\mathbf{a}}(\theta)\big] < A\}$. □

The above proof is similar to that of Lemma 2 in [86], except we do not exclude the case in which the singular measure is atomless. Another characterization for being $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ a Riesz basis for $\mathcal{A}_{\mathbf{a}}$ can be found in [5].

As a final remark we can also mention that authors in [86] also stated a neccesary and sufficient condition in order to be the sequence $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ a frame sequence in $\mathcal{H}$. Namely, the singular measure $\boldsymbol{\mu}_{\mathbf{a}}^s \equiv 0$ and there exist positive constants A, B such that

$$A \leqslant \lambda_m(\theta) \leqslant B, \qquad \text{for } \theta \in (-\pi,\pi)\backslash\mathcal{I}_m$$

for each $m = 1,2,\dots,L$, where $\lambda_m(\theta)$ is the m-th eigenvalue of $\boldsymbol{\Phi}_{\mathbf{a}}(\theta)$ and $\mathcal{I}_m := \{\theta \in (-\pi,\pi) : \lambda_m(\theta) = 0\}$.

The resulting regular sampling formulas

As in the one-generator case, the space $\mathcal{A}_{\mathbf{a}}$ is the image of the usual product Hilbert space $L^2_L(0,1)$ by means of the isomorphism $\mathcal{T}_{U,\mathbf{a}} : L^2_L(0,1) \longrightarrow \mathcal{A}_{\mathbf{a}}$, which maps the orthonormal basis $\{\mathrm{e}^{-2\pi i n w}\mathbf{e}_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ for $L^2_L(0,1)$ (here, $\{\mathbf{e}_l\}_{l=1}^L$ denotes the canonical basis for $\mathbb{C}^L$) onto the Riesz basis $\{U^n a_l\}_{n\in\mathbb{Z};\, l=1,2,\dots,L}$ for $\mathcal{A}_{\mathbf{a}}$, i.e.,

$$\mathcal{T}_{U,\mathbf{a}}\mathbf{F} := \sum_{l=1}^{L}\sum_{n\in\mathbb{Z}}\big\langle F_k, \mathrm{e}^{2\pi i n\cdot}\big\rangle_{L^2(0,1)} U^n a_l = \sum_{l=1}^{L}\sum_{n\in\mathbb{Z}} \alpha_n^l\, U^n a_l\,, \tag{4.55}$$

where $\mathbf{F} = (F_1, F_2, \dots, F_L)^\top \in L^2_L(0,1)$.

Here, for $\mathbf{F} \in L^2_L(0,1)$ and $N \in \mathbb{Z}$ the U-shift property reads:

$$\mathcal{T}_{U,\mathbf{a}}\big(\mathbf{F}\mathrm{e}^{2\pi i N w}\big) = U^N\big(\mathcal{T}_{U,\mathbf{a}}\mathbf{F}\big)\,. \tag{4.56}$$

Concerning the representation of an U-system $\mathcal{L}_b$, for $x \in \mathcal{A}_{\mathbf{a}}$ we have

$$\begin{aligned}\mathcal{L}_b x(t) = \langle x, U^t b\rangle_{\mathcal{H}} &= \sum_{l=1}^{L}\sum_{n\in\mathbb{Z}} \alpha_n^l \,\overline{\langle U^t b, U^n a_l\rangle}_{\mathcal{H}} \\ &= \sum_{l=1}^{L}\Big\langle F_l, \sum_{n\in\mathbb{Z}}\langle U^t b, U^n a_l\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i n w}\Big\rangle_{L^2(0,1)} = \big\langle \mathbf{F}, \mathbf{K}_t\big\rangle_{L^2_L(0,1)},\end{aligned}$$

where $\mathcal{T}_{U,a}\mathbf{F} = x$, $\mathbf{F} = \big(F_1, F_2, \ldots, F_L\big)^\top \in L^2_L(0,1)$, and the function

$$\mathbf{K}_t(w) := \Big(\sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a_1(t-n)}\, \mathrm{e}^{2\pi i n w}, \ldots, \sum_{n\in\mathbb{Z}} \overline{\mathcal{L}_b a_L(t-n)}\, \mathrm{e}^{2\pi i n w}\Big)^\top$$

belongs to $L^2_L(0,1)$. In particular, given s U-systems $\mathcal{L}_j := \mathcal{L}_{b_j}$ associated with $b_j \in \mathcal{H}$, $j = 1, 2, \ldots, s$, we get the expression for the samples:

$$\mathcal{L}_j x(rm) = \big\langle \mathbf{F}, \overline{\mathbf{g}_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\rangle_{L^2_L(0,1)} \quad \text{for } m \in \mathbb{Z} \text{ and } j = 1, 2, \ldots, s, \tag{4.57}$$

where $\mathcal{T}_{U,a}\mathbf{F} = x$ and

$$\mathbf{g}_j(w) := \Big(\sum_{k\in\mathbb{Z}} \mathcal{L}_j a_1(k)\, \mathrm{e}^{2\pi i k w}, \ldots, \sum_{k\in\mathbb{Z}} \mathcal{L}_j a_L(k)\, \mathrm{e}^{2\pi i k w}\Big)^\top \in L^2_L(0,1).$$

As in the one-generator case the sequence $\big\{\overline{\mathbf{g}_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s}$ should be studied in $L^2_L(0,1)$. Consider the $s \times rL$ matrix of functions in $L^2(0,1)$

$$\mathbb{G}(w) := \begin{bmatrix} \mathbf{g}_1^\top(w) & \mathbf{g}_1^\top(w+\frac{1}{r}) & \cdots & \mathbf{g}_1^\top(w+\frac{r-1}{r}) \\ \mathbf{g}_2^\top(w) & \mathbf{g}_2^\top(w+\frac{1}{r}) & \cdots & \mathbf{g}_2^\top(w+\frac{r-1}{r}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_s^\top(w) & \mathbf{g}_s^\top(w+\frac{1}{r}) & \cdots & \mathbf{g}_s^\top(w+\frac{r-1}{r}) \end{bmatrix} \tag{4.58}$$

$$= \Big[\mathbf{g}_j^\top\Big(w + \frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\ldots,s\\ k=1,2,\ldots,r}}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)].$$

In [45, Lemma 2] one can find the proof of the following lemma (see also Lemma 2.3):

Lemma 4.7. *Let $\mathbf{g}_j$ be in $L^2_L(0,1)$ for $j = 1,2,\dots,s$ and let $\mathbb{G}(w)$ be its associated matrix given in* (4.58). *Then, the following results hold:*

(a) The sequence $\big\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a complete system for $L^2_L(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is rL a.e. in $(0,1/r)$.

(b) The sequence $\big\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Bessel sequence for $L^2_L(0,1)$ if and only if $\mathbf{g}_j \in L^\infty_L(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.

(c) The sequence $\big\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2_L(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \leqslant \beta_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.

(d) The sequence $\big\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz basis for $L^2_L(0,1)$ if and only if is a frame and $s = rL$.

In case that the sequence $\big\{\overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2_L(0,1)$ (here, necessarily $s \geqslant rL$), a dual frame is given by

$$\big\{r\mathbf{h}_j(w)\,\mathrm{e}^{2\pi \mathrm{i} rnw}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

where the functions $\mathbf{h}_j$, $j = 1,2,\dots,s$, form an $L \times s$ matrix

$$\mathbf{h}(w) := \big[\mathbf{h}_1(w), \mathbf{h}_2(w), \dots, \mathbf{h}_s(w)\big]$$

with entries in $L^\infty(0,1)$, and satisfying

$$\big[\mathbf{h}_1(w), \mathbf{h}_2(w), \dots, \mathbf{h}_s(w)\big]\,\mathbb{G}(w) = \big[\mathbb{I}_L, \mathbb{O}_{L\times(r-1)L}\big] \quad \text{a.e. in } (0,1)$$

(see Ref. [45] for the details). That is, the matrix $\mathbf{h}(w)$ is formed with the first L rows of a left-inverse of the matrix $\mathbb{G}(w)$ having essentially bounded entries in $(0,1)$. In other words, all the dual frames of $\big\{\overline{\mathbf{g}_j(\mathrm{e}^{2\pi \mathrm{i} rnw})}\big\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ with the above property are obtained by taking the first L rows of the $rL \times s$ matrices given by

$$\mathbb{H}_{\mathbb{U}}(w) := \mathbb{G}^\dagger(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^\dagger(w)\big]\,,$$

where $\mathbb{G}^\dagger(w)$ denotes the Moore-Penrose pseudo inverse, and $\mathbb{U}(w)$ denotes any $rL\times s$ matrix with entries in $L^\infty(0,1)$.
Thus, any $\mathbf{F} \in L^2_L(0,1)$ can be expanded as

$$\mathbf{F} = \sum_{j=1}^{s}\sum_{n\in\mathbb{Z}} \big\langle \mathbf{F}, \overline{\mathbf{g}_j(w)}\,\mathrm{e}^{2\pi \mathrm{i} rnw}\big\rangle_{L^2_L(0,1)}\, r\mathbf{h}_j(w)\,\mathrm{e}^{2\pi \mathrm{i} rnw} \quad \text{in } L^2_L(0,1)\,.$$

Applying the isomorphism $\mathcal{T}_{U,a}$ and taken into account (4.57), for each $x = \mathcal{T}_{U,a}\mathbf{F} \in \mathcal{A}_\mathbf{a}$ we get the sampling expansion

$$\begin{aligned} x &= \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} \mathcal{L}_j x(rn)\, U^{rn}\big[\mathcal{T}_{U,a}(r\mathbf{h}_j)\big] \\ &= \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} \mathcal{L}_j x(rn)\, U^{rn} c_{j,\mathbf{h}} \quad \text{in } \mathcal{H}\,, \end{aligned}$$

where, for each $j = 1, 2, \dots, s$, the element $c_{j,\mathbf{h}} = \mathcal{T}_{U,a}(r\mathbf{h}_j) \in \mathcal{A}_\mathbf{a}$, and the sampling sequence $\{U^{rn}c_{j,\mathbf{h}}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_\mathbf{a}$.

Proceeding as in Section 4.3, it is straighforward to state and prove, in a similar way, the corresponding results.

A

Frames in Hilbert spaces

This appendix is devoted to state the main definitions and results concerning frame theory in a separable Hilbert space. Most of them have been used along the memory, for the proofs and details the reader can check, for instance, Refs. [23, 25, 26, 49, 57, 58, 122].

Let $\mathcal{X}$ be a normed vector space, with norm denoted by $\|\cdot\|$. A sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{X}$

- converges to $x \in \mathcal{X}$ if
$$\|x - x_k\| \longrightarrow 0 \ \ for \ \ k \longrightarrow \infty;$$
- is a Cauchy sequence if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that
$$\|x_k - x_l\| < \epsilon \ \text{ whenever } \ k, l \geqslant N.$$

A convergent sequence is always a Cauchy sequence, but the opposite is not true in general. The spaces in which these two properties are equivalent are called Banach spaces. An important class of Banach spaces is the L^p-spaces, $1 \leqslant p \leqslant \infty$. $L^\infty(\mathbb{R})$ is the space of essentially bounded measurable functions $f : \mathbb{R} \to \mathbb{C}$, equipped with the supremum-norms, for $1 \leqslant p < \infty$, $L^p(\mathbb{R})$ is the space of functions f for which $|f|^p$ is integrable with respect to the Lebesgue measure:

$$L^p(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} | f \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^p dx < \infty \right\}$$

The norm on $L^p(\mathbb{R})$ is

$$\|f\| = \Big(\int_{-\infty}^{\infty} |f(x)|^p dx\Big)^{1/p}$$

A vector space $\mathcal{X}$ with an inner product $\langle \cdot, \cdot \rangle$ can be equipped with the norm

$$\|x\| := \sqrt{\langle x, x\rangle}, \quad x \in \mathcal{X}$$

A vector space with inner product, which is a Banach space with respect to the induced norm, is called a Hilbert space. The standard examples are the spaces $L^2(\mathbb{R})$ and $\ell^2(\mathbb{Z})$, $L^2(\mathbb{R})$ is defined as the space of complex-valued functions, defined on $\mathbb{R}$ which are square integrable with respect to Lebesgue measure:

$$L^2(\mathbb{R}) := \Big\{ f : \mathbb{R} \to \mathbb{C} | f \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \Big\}$$

This space is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx, \quad f, g \in L^2(\mathbb{R}).$$

The discrete version of $L^2(\mathbb{R})$ is $\ell^2(\mathbb{Z})$, the space of square summable scalar sequeces on $\mathbb{Z}$:

$$\ell^2(\mathbb{Z}) := \Big\{ \{x_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C} | \sum_{k\in\mathbb{Z}} |x_k|^2 < \infty \Big\}.$$

with the inner product

$$\langle \{x_k\}, \{y_k\}\rangle = \sum_{k\in\mathbb{Z}} x_k \overline{y_k}.$$

Perhaps the most important concept in the analysis of vector spaces is the concept of basis. The idea is to consider a family of elements such that any vector on the given space can be expressed in a unique way as a linear combination of these elements.

Definition A.1. *Let $\mathcal{X}$ be a Banach space. A sequence of vectors $\{e_k\}_{k\in\mathbb{Z}}$ belonging to $\mathcal{X}$ is a (Schauder) basis for $\mathcal{X}$ if, for each $f \in \mathcal{X}$, there exist unique scalar coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ such that*

$$f = \sum_{k\in\mathbb{Z}} c_k(f) e_k. \tag{A.1}$$

Henceforth, we are going to focus our attention on Hilbert spaces. Let us start with the definition of a sort of sequences with an important role in the memoir.

Definition A.2. *A sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that*

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2 \quad \textit{for all } f \in \mathcal{H}\,. \tag{A.2}$$

Every number B satisfying (A.2) is called a Bessel bound for $\{f_k\}_{k\in\mathbb{Z}}$.

Theorem A.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$. Then $\{f_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence with Bessel bound B if and only if*

$$T : \{c_k\}_{k\in\mathbb{Z}} \longmapsto \sum_{k\in\mathbb{Z}} c_k f_k$$

is a well-defined bounded operator from $\ell^2(\mathbb{Z})$ into $\mathcal{H}$ and $\|T\| \leqslant \sqrt{B}$.

Among all the bases for a Hilbert space, the most famous ones are the so called orthonormal bases. They are the abstract counterpart of canonical bases on $\mathbb{C}^n$ and they have been widely used in many branches of mathematics and physics.

Definition A.3. *A sequence $\{e_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ is an orthonormal system if*

$$\langle e_k, e_j\rangle = \delta_{k,j}.$$

An orthonormal basis is an orthonormal system $\{e_k\}_{k\in\mathbb{Z}}$ which is a basis for $\mathcal{H}$.

The next theorem gives equivalent conditions for an orthonormal system to be an orthonormal basis.

Theorem A.2. *For an orthonormal system $\{e_k\}_{k\in\mathbb{Z}}$, the following are equivalent:*

(i) $\{e_k\}_{k\in\mathbb{Z}}$ *is an orthonormal basis.*

(ii) $f = \sum_{k\in\mathbb{Z}}\langle f, e_k\rangle$, *for all* $f \in \mathcal{H}$.

(iii) $\langle f, g\rangle = \sum_{k\in\mathbb{Z}}\langle f, e_k\rangle\langle e_k, g\rangle$, *for all* $f, g \in \mathcal{H}$.

(iv) $\sum_{k\in\mathbb{Z}} |\langle f, e_k\rangle|^2 = \|f\|^2$, *for all* $f \in \mathcal{H}$.

(v) $\overline{\operatorname{span}}\{e_k\}_{k\in\mathbb{Z}} = \mathcal{H}$.

(vi) *If* $\langle f, e_k\rangle = 0$, *for all* $k \in \mathbb{Z}$, $then$ $f = 0$.

It is well known that having one orthonormal basis, the rest of them can be obtained by applying an unitary operator to the given basis; the following definition appears by weakening this unitary condition on the operator:

Definition A.4. *A Riesz basis for $\mathcal{H}$ is a family of the form $\{Re_k\}_{k\in\mathbb{Z}}$, where $\{e_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{H}$ and $R : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.*

It is easy to check that Riesz bases are actually bases.

Theorem A.3. *If $\{f_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$, there exists a unique sequence $\{g_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ such that*

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k, \quad \text{for all } f \in \mathcal{H}.$$

$\{g_k\}_{k\in\mathbb{Z}}$ is also a Riesz basis, and $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are biorthogonal.

The next theorem gives equivalent conditions for a sequence being a Riesz basis.

Theorem A.4. *For a sequence $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$, the following conditions are equivalent:*

(i) $\{f_k\}_{k\in\mathbb{Z}}$ *is a Riesz basis.*

(ii) $\{f_k\}_{k\in\mathbb{Z}}$ *is complete in $\mathcal{H}$, and there exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_k\}$ one has*

$$A \sum |c_k|^2 \leqslant \left\| \sum c_k f_k \right\|^2 \leqslant B \sum |c_k|^2. \tag{A.3}$$

(iii) $\{f_k\}_{k\in\mathbb{Z}}$ *is complete and its Gram matrix $[\langle f_k, f_j\rangle]_{j,k\in\mathbb{Z}}$ defines a bounded, invertible operator on $\ell^2(\mathbb{Z})$.*

(iv) $\{f_k\}_{k\in\mathbb{Z}}$ *is a complete Bessel sequence, and it has a complete biorthogonal sequence $\{g_k\}_{k\in\mathbb{Z}}$ which is also a Bessel sequence.*

A sequence satisfying condition (A.3) for all finite sequences $\{c_k\}_{k\in\mathbb{Z}}$ is called a Riesz sequence. A Riesz sequence $\{f_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\overline{\text{span}}\{f_k\}_{k\in\mathbb{Z}}$.

If $\{f_k\}_{k\in\mathbb{Z}}$ is a Riesz basis, numbers $A, B > 0$ which satisfy (A.3) are called lower Riesz bounds and upper Riesz bounds respectively. They are clearly not unique, and we define the optimal Riesz bounds as the largest possible value for A and the smallest possible value for B. The optimal Riesz bounds are characterized in the following theorem:

Theorem A.5. *Let $\{f_k\}_{k\in\mathbb{Z}} = \{Re_k\}_{k\in\mathbb{Z}}$ be a Riesz basis for $\mathcal{H}$, and let G be the Gram matrix. Then the optimal Riesz bounds are*

$$A = \frac{1}{\|R^{-1}\|^2} = \frac{1}{\|G^{-1}\|} \quad \text{and} \quad B = \|R\|^2 = \|G\|.$$

The main property of a basis in a Hilbert space is that every vector $f \in \mathcal{H}$ can be expressed as an infinite linear combination of the elements of the basis, that is, an expansion of the form (A.1). We are ready to introduce the concept of frame.

A frame is again a family of vectors which allows also to write every $f \in \mathcal{H}$ as (A.1), however, the corresponding coefficients are not necessarily unique. This fact, instead of being a drawback is a very useful property, both for practical and theoretical purposes.

Definition A.5. *A sequence $\{f_k\}_{k\in\mathbb{Z}}$ of elements in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leqslant \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2, \quad \text{for all } f \in \mathcal{H}. \tag{A.4}$$

The numbers A, B are called frame bounds. Obviously they are not unique, the optimal upper frame bound is the infimum over all upper frame bounds, and the optimal lower frame bound is the supremum over all lower frame bounds.

It is easy to see that every frame is a Bessel sequence and a complete system in $\mathcal{H}$. If we can choose $A = B$ in (A.4), then the frame is called tight. A frame is said to be exact if it ceases to be a frame when an arbitrary element is removed.

Theorem A.1 assures us that

$$T : \ell^2(\mathbb{Z}) \longrightarrow \mathcal{H}, \qquad T\{c_k\}_{k\in\mathbb{Z}} = \sum_{k\in\mathbb{Z}} c_k f_k \tag{A.5}$$

is well-defined and bounded operator with $\|T\| \leqslant \sqrt{B}$; T is called the pre-frame operator or the synthesis operator. The adjoint of T is given by

$$T^* : \mathcal{H} \longrightarrow \ell^2(\mathbb{Z}), \qquad T^* f = \{\langle f, f_k\rangle\}_{k\in\mathbb{Z}}. \tag{A.6}$$

T^* is called the analysis operator. By composing T and T^*, we obtain the frame operator

$$S : \mathcal{H} \longrightarrow \mathcal{H}, \qquad Sf = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k. \tag{A.7}$$

The most important properties of the operator S are collected in the following proposition:

Proposition A.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a frame for $\mathcal{H}$, then we have*

(i) S is positive, self-adjoint, invertible and bounded with $\|S\| \leqslant \sqrt{B}$.

(ii) The sequence $\{S^{-1}f_k\}_{k\in\mathbb{Z}}$ is also a frame for $\mathcal{H}$ with bounds B^{-1}, A^{-1}. This frame is called canonical dual frame of $\{f_k\}_{k\in\mathbb{Z}}$, and its frame operator is S^{-1}.

(iii) For any $f \in \mathcal{H}$ we have

$$f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle S^{-1} f_k = \sum_{k\in\mathbb{Z}} \langle f, S^{-1} f_k\rangle f_k. \tag{A.8}$$

Given a frame for $\mathcal{H}$ the so called frame algorithm allow us to recover any $f \in \mathcal{H}$ from the operator $\mathcal{A}f := \frac{2}{A+B} Sf$.

Proposition A.2. *Any $f \in \mathcal{H}$ can be approximate by the sequence $\{g_n\}_{n=1}^{\infty}$ generated in the following recursive way:*

$$\begin{cases} g_1 = \mathcal{A}f \\ g_n = g_{n-1} + \mathcal{A}(f - g_{n-1}) \qquad n \geqslant 2. \end{cases}$$

The speed of convergence depends on $B - A$; thus, the closer frame bounds the faster is the algorithm.

The equation (A.8) assures us that the bounded operator T is also surjective, this actually characterizes frames:

Theorem A.6. *A sequence $\{f_k\}_{k\in\mathbb{Z}}$ is a frame in $\mathcal{H}$ if and only if the synthesis operator T is bounded and surjective.*

Given a frame $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$, it is known that the sequence $\{\langle f, S^{-1}f_k\rangle\}_{k\in\mathbb{Z}}$ has the minimal ℓ^2-norm among all the sequeces $\{c_k\}_{k\in\mathbb{Z}}$ such that $f = \sum_{k\in\mathbb{Z}} c_k f_k$.

Proposition A.3. *Assume that $\{f_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$, is an overcomplete (not exact) frame. Then there exist frames $\{g_k\}_{k\in\mathbb{Z}} \neq \{S^{-1}f_k\}_{k\in\mathbb{Z}}$ for which*

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k, \qquad \textit{for all } f \in \mathcal{H}.$$

The sequence $\{g_k\}_{k\in\mathbb{Z}}$ is called a dual frame of $\{f_k\}_{k\in\mathbb{Z}}$.

The following important proposition has been widely used on the work:

Proposition A.4. *Assume that $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are Bessel sequences in $\mathcal{H}$. Then the following are equivalent:*

(i) $f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k$, *for all* $f \in \mathcal{H}$.

(ii) $f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle g_k$, *for all* $f \in \mathcal{H}$.

(iii) $\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle \langle g_k, g\rangle$, *for all* $f, g \in \mathcal{H}$.

In case that equivalent conditions are satisfied, $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are dual frames for $\mathcal{H}$.

Next result collects several conditions for a frame being a Riesz basis:

Theorem A.7. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a frame for $\mathcal{H}$. Then the following are equivalent.*

(i) $\{f_k\}_{k\in\mathbb{Z}}$ *is a Riesz basis for* $\mathcal{H}$.

(ii) $\{f_k\}_{k\in\mathbb{Z}}$ *is an exact frame.*

(iii) $\{f_k\}_{k\in\mathbb{Z}}$ *is minimal.*

(iv) $\{f_k\}_{k\in\mathbb{Z}}$ *has a biorthogonal sequence.*

(v) $\{f_k\}_{k\in\mathbb{Z}}$ *and* $\{S^{-1}f_k\}_{k\in\mathbb{Z}}$ *are biorthogonal.*

(vi) $\{f_k\}_{k\in\mathbb{Z}}$ *is* ω*-independent.*

(vii) If $\sum_{k\in\mathbb{Z}} c_k f_k = 0$ *for some* $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2$*, then* $c_k = 0,$ *for all* $k \in \mathbb{Z}$*.*

(viii) $\{f_k\}_{k\in\mathbb{Z}}$ *is a basis.*

So far we have not defined minimality and ω-independence properties:

Definition A.6. *Let* $\{f_k\}_{k\in\mathbb{Z}}$ *be a sequence in* $\mathcal{H}$*. We say that*

(i) $\{f_k\}_{k\in\mathbb{Z}}$ *is linearly independent if every finite subset of* $\{f_k\}_{k\in\mathbb{Z}}$ *is linearly independent.*

(ii) $\{f_k\}_{k\in\mathbb{Z}}$ *is* ω*-independent if whenever the series* $\sum_{k\in\mathbb{Z}} c_k f_k$ *is convergent and equal to zero for some scalar coefficients* $\{c_k\}_{k\in\mathbb{Z}}$*, then necessarily* $c_k = 0$ *for all* $k \in \mathbb{Z}$*.*

(iii) $\{f_k\}_{k\in\mathbb{Z}}$ *is minimal if* $f_j \notin \overline{\operatorname{span}}\{f_k\}_{k\neq j}$*, for all* $j \in \mathbb{Z}$*.*

It can be prove that minimality implies w-independence and w-independence implies linear independence, but the opposite implications are not valid.

Finally, we include an important result concerning perturbation of frame, which indeed was used more than once in the work:

Theorem A.8. *Let* $\{f_k\}_{k\in\mathbb{Z}}$ *be a frame for* $\mathcal{H}$ *with bounds* A, B*, and let* $\{g_k\}_{k\in\mathbb{Z}}$ *be a sequence in* $\mathcal{H}$*. If there exists a constant* $R < A$ *such that*

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k - g_k\rangle|^2 \leqslant R\|f\|^2, \qquad \text{for all } f \in \mathcal{H},$$

then $\{g_k\}_{k\in\mathbb{Z}}$ *is a frame for* $\mathcal{H}$ *with bounds*

$$A\Big(1 - \sqrt{\frac{R}{A}}\Big)^2, \qquad B\Big(1 + \sqrt{\frac{R}{A}}\Big)^2$$

If $\{f_k\}_{k\in\mathbb{Z}}$ *is a Riesz basis, then* $\{g_k\}_{k\in\mathbb{Z}}$ *is a Riesz basis.*

B

Continuous group of unitary operators

In Chapter 4 we use properties of the continuos group of unitary operators; we include here some of the main definitions and theorem from the theory of self-adjoint operators which naturally lead us to the fundamental result: the Stone's theorem [95]. For the details the reader can check, for instance, Refs.[4, 14, 89, 114, 120].

Definition B.1. *A resolution of the identity is a one-parameter family of projection operators* $\{E_t\}_{t\in\mathbb{R}}$ *in* $\mathcal{H}$ *such that*

(i) $E_{-\infty} := \lim_{t\to-\infty} E_t = O_{\mathcal{H}}, \quad E_\infty := \lim_{t\to\infty} E_t = I_{\mathcal{H}}$,

(ii) $E_{t^+} = E_t$ *for any* $-\infty < t < \infty$,

(iii) $E_r\, E_s = E_t$ *where* $t = \min\{r, s\}$.

For every $f \in \mathcal{H}$ define

$$\rho_f(t) = \langle f, E_t f\rangle = \|E_t f\|^2, \qquad t \in \mathbb{R}$$

The function $\rho : \mathbb{R} \to \mathbb{R}$ is obviously bounded, non-decreasing and right continuous; $\lim_{t\to-\infty} \rho_f(t) = 0,\ \lim_{t\to\infty} \rho_f(t) = \|f\|^2$.

A function $u : \mathbb{R} \to \mathbb{K}(\mathbb{R}\,\text{or}\,\mathbb{C})$ is said to be E-measurable if it is ρ_f-measurable for every $f \in \mathcal{H}$. Non-trivial examples of E-measurable functions are all continuous functions, all step functions, and all functions that are pointwise limits of step functions; all Borel measurable functions are E-measurable.

Theorem B.1. *Let $\{E_t\}_{t\in\mathbb{R}}$ be a resolution of the identity on the Hilbert space $\mathcal{H}$, and let $u : \mathbb{R} \to \mathbb{K}$ be an E-measurable function. Then the formulae*

$$D(\hat{E}(u)) = \{f \in \mathcal{H} : \int_{-\infty}^{\infty} |u(t)|^2 d\rho_f(t) < \infty\}$$

$$\hat{E}(u) = \int_{-\infty}^{\infty} u(t) d\rho_f(t) \qquad for \qquad f \in D(\hat{E}(u))$$

define a normal operator $\hat{E}(u)$ on $\mathcal{H}$, the last equation justifies the notation

$$\hat{E}(u) = \int_{-\infty}^{\infty} u(t) dE_t \tag{B.1}$$

Next result shows that every self-adjoint operator can be expressed as B.1 and there exists exactly one such representation with the identity function id i.e., $u(t) = t$.

Theorem B.2. *For every self adjoint operator T on the Hilbert spaces $\mathcal{H}$ there exist exactly one resolution of the identity $\{E_t\}_{t\in\mathbb{R}}$ for which $T = \hat{E}(id)$; in another notation, $T = \int_{-\infty}^{\infty} t\, dE_t$.*

Henceforth we shall see one of the main consequences of the theory of self-adjoint operators, the Stone's theorem [95].

Definition B.2. *In a Hilbert space $\mathcal{H}$, a family $\{B^t\}_{t\in\mathbb{R}}$ of bounded operators is called a one-parameter group if*

(i) $B^0 = I$,

(ii) $B^{s+t} = B^s B^t$ *for all* $s, t \in \mathbb{R}$.

The one-parameter group $\{B^t\}_{t\in\mathbb{R}}$ is said to be strongly continuous if the function

$$\begin{array}{rccc} B^{(\cdot)} f : & \mathbb{R} & \longrightarrow & \mathcal{H} \\ & t & \longmapsto & B^t f \end{array}$$

is continuous for every $f \in \mathcal{H}$. Let $\{B^t\}_{t\in\mathbb{R}}$ be a one-parameter group of operators in $\mathcal{H}$. The operator A defined by the formulae

$$D(A) = \left\{ f \in \mathcal{H} : \lim_{t\to 0} \frac{1}{t}(B^t - I)f \qquad \text{exists} \right\},$$

$$Af = \lim_{t\to 0} \frac{1}{t}(B^t - I)f \qquad \text{for} \qquad f \in D(A)$$

is called the infinitesimal generator of $\{B^t\}_{t\in\mathbb{R}}$.

Theorem B.3. *Let T be a self-adjoint operator on the Hilbert space $\mathcal{H}$, with spectral family $\{E_t\}_{t\in\mathbb{R}}$, and let*

$$U^t = \mathrm{e}^{\mathrm{i}tT} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ts}\, dE_s \quad \textit{for} \quad t \in \mathbb{R}.$$

Then $\{U^t\}_{t\in\mathbb{R}}$ is a strongly continuous (one-parameter) unitary group. The infinitesimal generator is $\mathrm{i}T$. We have $U^t f \in D(T)$ for all $f \in D(T)$ and $t \in \mathbb{R}$.

Actually, every strongly continuous (one-parameter) unitary group can be represented in this form (Stone's theorem):

Theorem B.4. *$\{U^t\}_{t\in\mathbb{R}}$ be a strongly continuous (one-parameter) unitary group on the Hilbert space $\mathcal{H}$. Then there exists a uniquely determined self-adjoint operator T on $\mathcal{H}$ for which*

$$U^t = \mathrm{e}^{itT} \qquad \textit{for all} \qquad t \in \mathbb{R}.$$

If $\mathcal{H}$ is separable, then strong continuity can be replace by weak measurability, i.e., it is sufficient to require that the function

$$\begin{array}{rccc} \langle f, U^{(\cdot)} g\rangle : & \mathbb{R} & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & \langle f, U^t g\rangle \end{array}$$

is measurable (with respect to Lebesgue measure on $\mathbb{R}$) for all $f, g \in \mathcal{H}$.

Furthermore, for any $f \in D(T)$ we have that $\lim\limits_{t\to 0} \dfrac{U^t f - f}{t} = \mathrm{i}\,Tf$ and the operator iT is the infinitesimal generator of the group $\{U^t\}_{t\in\mathbb{R}}$. For each $f \in D(T)$, $U^t f$ is a continuous differentiable function of t. Here $U^t = \mathrm{e}^{\mathrm{i}tT}$ again means that

$$\langle U^t f, g\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}st} d\langle E_s f, g\rangle\,, \quad t \in \mathbb{R}\,,$$

where $f \in D(T)$ and $g \in \mathcal{H}$.

Finally, we include a result taken from [4, vol.2; p. 24]: For $f \in D_T$ and $g \in \mathcal{H}$, the inequality

$$\Big| \int_{-\infty}^{\infty} \varphi(s) d\langle E_s f, g\rangle \Big| \leqslant \|g\| \sqrt{\int_{-\infty}^{\infty} |\varphi(s)|^2 d\langle E_w f, f\rangle}, \tag{B.2}$$

holds, where φ is a real or complex function which is continuous in $\mathbb{R}$ with the possible exception of a finite number of points.

Conclusions and future work

On the separable Hilbert space of square integrable functions $L^2(\mathbb{R})$ we can define the shift-invariant subspaces V_φ^2 in the following way

$$V_\varphi^2 := \overline{\operatorname{span}}_{L^2(\mathbb{R})}\{\varphi(t-n),\ n \in \mathbb{Z}\}$$

where $\varphi \in L^2(\mathbb{R})$ is a fixed function. If the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is Riesz sequence, i.e. a Riesz basis for V_φ^2 then this space can be expressed as

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n \varphi(t-n) \ :\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

Taking the function $\varphi(t) = \operatorname{sinc}(t)$, the space V_φ^2 coincides with the Paley-Wiener space PW_π of band limited functions to the interval $[-\pi, \pi]$ via Whittaker-Shannon-Kotel'nikov sampling theorem:

$$PW_\pi = \Big\{ \sum_{n\in\mathbb{Z}} a_n \operatorname{sinc}(t-n) \ :\ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

Furthermore, the coefficients $\{a_n\}_{n\in\mathbb{Z}}$ of $f \in PW_\pi$ are precisely the samples of the function at the integers numbers $\{f(n)\}_{n\in\mathbb{Z}}$.

Sampling in shift-invariant spaces V_φ^2 has been profusely studied in the late years, Refs. [8, 6, 11, 13, 24, 109, 113, 118, 119, 124]. One can take into account the case of multiple generators and instead of sampling at $\mathbb{Z}$ consider the several dimensions framework, i.e., signals are functions defined on $\mathbb{R}^d$ with samples taken at a lattice of the form $M\mathbb{Z}^d$, where M is a non singular matrix with integers entries. In Chapter 2 we obtain sampling results in this setting. Our main technique was to consider a Fourier type duality via an isomorphism between the spaces $L_r^2[0,1)^d$ and V_Φ^2, where $\Phi := \{\varphi_1, \varphi_2, \dots, \varphi_r\}$ is the set of generators.

The main results in Chapter 2 were:

- Having as data a sequence of samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha\in\mathbb{Z}^d;\, j=1,2,\dots,s}$ where $\mathcal{L}_j$ are convolution systems acting on the signals; for any $f \in V_\Phi^2$ we obtained a sampling expansion of the form

$$f=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)\,S_j(\cdot-M\alpha)\quad \text{in } L^2(\mathbb{R}^d)\,,$$

where $\{S_j(\cdot-M\alpha)\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$ is a frame for V_Φ^2.

- For practical purposes good properties for the sequence of reconstruction functions $\{S_j\}_{j=1,2,\dots,s}$ are to be compact supported or to have exponential decay. Necessary and sufficients conditions were obtained for both cases.
- We analized the time-jitter error case, i.e., samples affected by an error sequence $\varepsilon:=\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,s}$; in this framework we obtained conditions that make possible the recovery of the signals by means of an expansion like

$$f(t)=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big(\mathcal{L}_j f)(M\alpha+\varepsilon_{j,\alpha})S_{j,\alpha}^{\varepsilon}(t)\,,\quad t\in\mathbb{R}^d\,.$$

- In the above formula the reconstruction functions $S_{j,\alpha}^{\varepsilon}$ are imposible to determine because they depend on the error sequence; to overcome this problem frame algorithm was implemented.

In Chapter 3 we focused our attention to the spaces $L_\nu^p(\mathbb{R}^d)$ composed by functions f such that νf belongs to $L^p(\mathbb{R}^d)$. The weight function ν controls the decay or growth of the signals; weight functions are well-known and widely used in many topics of sampling theory and time-frequency analysis.

We formally define the weighted multiple generated shift-invariant space $V_\nu^p(\Phi)$ as

$$V_\nu^p(\Phi):=\Big\{\sum_{j=1}^{r}\sum_{\alpha\in\mathbb{Z}^d}a_j(\alpha)\,\phi_j(t-\alpha)\;:\;\{a_j(\alpha)\}_{\alpha\in\mathbb{Z}^d}\in\ell_\nu^p(\mathbb{Z}^d),\ j=1,2,\dots,r\Big\}\,.$$

Commonly it is assumed that the sequence $\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,r}$ is a Riesz basis for $V_\nu^p(\Phi)$. Here we assume a more general condition: the sequence $\{\phi_j(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d;\,j=1,2,\dots,r}$ is a p-frame for $V_\nu^p(\Phi)$; this guarantees the closedness of $V_\nu^p(\Phi)$ but we lose the uniqueness on the coefficients $\{a_j(\alpha)\}$ in the above representation.

We obtained the sampling formula

$$f=\sum_{l=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_l f)(M\alpha)S_l(\cdot-M\alpha)\,,$$

valid for any $f\in V_\nu^p(\Phi)$.

Two types of convolution systems were taken into account: those obtained by convolution with functions locally in $L^\infty_\nu(\mathbb{R}^d)$ and globally in $L^1_\nu(\mathbb{R}^d)$ and the ones where the impulse response is a translated Dirac delta.

On the other hand, we can see the mentioned V^2_φ shift-invariant spaces as

$$V^2_\varphi = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, T^n\varphi \,:\, \{a_n\} \in \ell^2(\mathbb{Z}) \Big\}\,,$$

where T is the shift-operator $T : f(t) \mapsto f(t-1)$; this operator is unitary on $L^2(\mathbb{R})$. The replacement of T by an unitary operator U and $L^2(\mathbb{R})$ by an abstract Hilbert space $\mathcal{H}$ lead us to the U-sampling theory, which was the subject of the last chapter.

In this new setting the samples were generalized via $\mathcal{L}_j x(rn) := \langle x, U^{rn} b_j\rangle_{\mathcal{H}}$ where $b_j \in \mathcal{H}$ and $r \in \mathbb{N}$ is the fixed sampling period. In the regular U-sampling case we take in consideration the discrete group of unitary operators $\{U^n\}_{n\in\mathbb{Z}}$ whilst if we want to deal with time-jitter error or asymmetric sampling problems the use of the continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ becomes essential.

We collect here the main results of Chapter 4:

- For a fixed $a \in \mathcal{H}$, provided that $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence we identify the space $\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a,\ n\in\mathbb{Z}\big\}$ as

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}\,.$$

  The characterization of the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ as a Riesz basis for $\mathcal{A}_a$ was done in the multiple generator case.

- We obtained, for any $x \in \mathcal{A}_a$, an expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \mathcal{L}_j x(rk)\, U^{rk} c_j \quad \text{in } \mathcal{H}\,,$$

  where the sequence $\big\{U^{rk} c_j\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$.

- The above expansion was also obtained from a different point of view involving the shift and decimation operators.

- We obtained a sampling exansion in the asymmetric U-sampling framework, i.e., we can recovered any $x \in \mathcal{A}_c$ from the sequence of asymmetric samples (taken with different sampling periods r_j)

$$\big\{\mathcal{L}_j x(\sigma_j + r_j m)\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots s}\,.$$

  Here, the σ_j's are real numbers; as we have mentioned we consider U included in a continuous group of unitary operators.

- Having in mind this later fact, we considered samples affected by a sequence of errors $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$,
$$\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}\,.$$
We obtained conditions on the error sequence to ensure the existence of a frame $\{C^{\boldsymbol{\epsilon}}_{j,m}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x \in \mathcal{A}_a$, the sampling expansion
$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm + \epsilon_{mj})\, C^{\boldsymbol{\epsilon}}_{j,m} \quad \text{in } \mathcal{H}\,,$$
holds.
- Finally, the case of sampling in U invariant subspaces with several generators
$$\mathcal{A}_{\mathbf{a}} = \Big\{ \sum_{l=1}^{L} \sum_{n\in\mathbb{Z}} \alpha_n^l\, U^n a_l \,:\, \{\alpha_n^l\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z});\, l = 1, 2. \dots, L \Big\}\,,$$
was also considered.

Some future work

Now, we propose some possible extensions that we have in mind for the future work:

To carry out a deeper study of the weigthed sampling framework

Concerning the work made in Chapter 3 it is worth to point out that in our opinion there are aspects that can be developed or improved. For instance, we can study the existing reconstruction procedures and adapt them to our framework, also take into account the irregular possibility could provide new results. Furthermore, we could consider different weight functions in both, analysis and synthesis processes.

U-irregular sampling: the general case

Having in mind the results obtained in Chapter 4 we can consider a non-uniform sampling set of points $\{t_n\}_{n\in\mathbb{Z}}$ in $\mathbb{R}$, and try to recover any $x \in \mathcal{A}_a$ from the sequence of non-uniform samples
$$\big\{\mathcal{L}_j\, x(t_n) := \langle x, U^{t_n} b_j\rangle\big\}_{n\in\mathbb{Z};\, j=1,2,\dots s}\,,$$
where $\{b_j\}_{j=1,2,\dots,s}$ are s fixed vectors in $\mathcal{H}$. Conditions on this sequence should be found to make possible the reconstruction.

Suppose that the sequence $\{t_n\}_{n\in\mathbb{Z}}$ satisfies the following stability condition: There exist positive constants c and C such that

$$c\|x\|^2 \leqslant \sum_{j=1} \sum_{n\in\mathbb{Z}} |\langle x, U^{t_n} b_j\rangle|^2 \leqslant C\|x\|^2 \qquad \text{for all } x \in \mathcal{A}_a\,. \tag{B.3}$$

Via the isomorphism $\mathcal{T}_{U,a}$ (our Fourier type duality) given by

$$\begin{array}{rccl} \mathcal{T}_{U,a}: & L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ & F = \sum_{n\in\mathbb{Z}} \alpha_n\, \mathrm{e}^{2\pi i n w} & \longmapsto & x = \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a\,. \end{array}$$

the above inequalities are equivalent to the new inequalities in $L^2(0,1)$:

$$\widetilde{c}\|F\|^2 \leqslant \sum_{j=1} \sum_{n\in\mathbb{Z}} |\langle F, K^j_{t_n}\rangle|^2 \leqslant \widetilde{C}\|F\|^2 \qquad \text{for all } F \in L^2(0,1)\,,$$

where the function $K^j_{t_n}(w) := \sum_{k\in\mathbb{Z}} \langle U^{t_n} b, U^k a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i k w} \in L^2(0,1)$.

The above inequalities imply that the sequence $\{K^j_{t_n}\}_{n\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $L^2(0,1)$; taking for instance its canonical dual frame $\{G^j_{t_n}\}_{n\in\mathbb{Z};\, j=1,2,\ldots s}$ we get

$$F = \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} \mathcal{L}_j\, x(t_n)\, G^j_{t_n} \qquad \text{in } L^2(0,1)\,.$$

By applying $\mathcal{T}^{-1}_{U,a}$ we finally obtain the expansion

$$x = \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} \mathcal{L}_j\, x(t_n)\, \mathcal{T}^{-1}_{U,a}\big(G^j_{t_n}\big) \quad \text{in } \mathcal{A}_a\,.$$

The challenge problem is to find conditions (necessary and sufficient) on the sequence $\{t_n\}_{n\in\mathbb{Z}}$ in order to satisfy inequalities (B.3). A possible strategy to get that is to transfer the non-uniform sampling conditions used in the mathematical literature for shift-invariant spaces V^2_φ to the corresponding functions $\{K^j_{t_n}\}_{n\in\mathbb{Z};\, j=1,2,\ldots s}$ in $L^2(0,1)$.

Sampling in finite U-invariant subspaces with multiple generators

Recently, authors in [51] have derived a sampling theory for finite dimensional U-invariant subspaces of a separable Hilbert space $\mathcal{H}$. In Chapter 4 it was assumed that the stationary sequence $\{U^n a\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ has infinite different elements. It could happen that for some $a \in \mathcal{H}$ there exists $N \in \mathbb{N}$ such that $U^N a = a$, i.e., 1 is an eigenvalue of the unitary operator U^N with eigenvector a. In this case, $\mathcal{A}_a$ is just the finite dimensional subspace of $\mathcal{H}$ spanned by the set $\{a, Ua, U^2a, \ldots, U^{N-1}a\}$. An important example is given by the finite space of N-periodic sequences $\{x(n)\}_{n\in\mathbb{Z}}$ in

$\mathbb{C}$ with the usual inner product $\langle x, y\rangle = \sum_{n=0}^{N-1} x(n)\,\overline{y(n)}$; the unitary operator is the usual shift operator, and the U-systems are periodic convolutions.

Concretely we can consider $\mathbf{a} = (a_1, a_2, \dots, a_L) \in \mathcal{H}^L$ and $N_1, N_2, \dots, N_L \in \mathbb{N}$ such that for the unitary operator U the relations $U^{N_i} a_i = a_i$ for $i = 1, 2, \dots, L$ are satisfied and studied the space

$$\mathcal{A}_{\mathbf{a}} := \operatorname{span}\left\{\begin{array}{c} a_1, Ua_1, \dots, U^{N_1-1}a_1 \\ a_2, Ua_2, \dots, U^{N_2-1}a_2 \\ \vdots \\ a_L, Ua_L, \dots, U^{N_L-1}a_L \end{array}\right\};$$

in case of linear independence of these vectors the space $\mathcal{A}_{\mathbf{a}}$ can be written as

$$\mathcal{A}_{\mathbf{a}} = \left\{ x = \sum_{i=1}^{L} \underbrace{\sum_{k_i=0}^{N_i-1} c_{i,k_i} U^{k_i} a_i}_{x_i} \right\}.$$

Now for each i from 1 to L, choose $r_i \in \mathbb{N}$ a divisor of N_i. Then, we consider the sequence of samples of $x = x_1 + x_2 + \cdots + x_L \in \mathcal{A}_{\mathbf{a}}$ given by

$$\big\{\mathcal{L}_j\, x_i(r_i n_i)\big\}_{j=1,2,\dots,s;\, i=1,2,\dots,L;\, n_i=1,2,\dots,l_{i-1}}$$

where $l_i = N_i/r_i$ for $i = 1, 2, \dots, L$.

The goal is to find vectors $c_1, c_2, \dots, c_s \in \mathcal{A}_{\mathbf{a}}$ such that the sequence

$$\big\{U^{rn_i} c_j\big\}_{j=1,2,\dots,s;\, i=1,2,\dots,L;\, n_i=1,2,\dots,l_{i-1}}$$

is a frame for $\mathcal{A}_{\mathbf{a}}$ and the following expansion is satisfied

$$x = \sum_{j=1}^{s}\sum_{i=1}^{L}\sum_{n_i=0}^{l_i-1} \mathcal{L}_j\, x_i(r_i n_i)\, U^{rn_i} c_j\,, \qquad \text{for any } x \in \mathcal{A}_{\mathbf{a}}\,.$$

The main idea would be identify, via an appropriate isomorphism, the spaces $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \cdots \times \mathbb{C}^{N_L}$ and $\mathcal{A}_{\mathbf{a}}$.

Publications

The work presented along this thesis is reflected in the following publications:

- H. R. Fernández-Morales and A. G. García. *Uniform average sampling in frame-generated weighted shift-invariant spaces*. Preprint 2015.
- H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo. *Generalized sampling: from shift-invariant to U-invariant spaces*. **Anal. Appl., Vol.15(3): 303–329, 2015.**
- H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo. *On some sampling-related frames in U-invariant spaces.* **Abstr. Appl. Anal., Vol. 2013, Article ID 761620, 14 pp., 2013.**
- H. R. Fernández-Morales, A. G. García and M. A. Hernández-Medina *Generalized sampling in U-invariant subspaces.* **Proceedings of the 10th International Conference on Sampling Theory and Applications, Eurasip Open Library, 208-211, 2013.**
- H. R. Fernández-Morales, A. G. García and G. Pérez-Villalón *Generalized sampling in $L^2(\mathbb{R})$ shift-invariant subspaces with multiple stable generators.* **In Multiscale Signal Analysis and Modelling, Lecture Notes in Electrical Engineering, pp. 51–80, Springer, 2013.**

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