Malaysian Journal of Mathematical Sciences 13(S) December: 39–50 (2019)
Special Issue: Conference on Mathematics, Informatics and Statistics (CMIS2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

The Performance of Higher Moments Estimators: An Empirical Study

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ABSTRACT

This study investigates the performance of higher order moments, realised from the model-free Bakshi-Kapadia-Madan (MFBKM). We concentrate on investigating higher order option-implied moments â variance, skewness and kurtosis, chosen in relation to contracts defined in MFBKM, i.e. volatility, cubic, and quartic contract. The three approaches adopted in order to estimate the integrals of the defined MFBKM contracts are the *basic* (trapezoidal-rule), *adapted* (single-combined) and *advanced* method (cubic-spline). The sample data is extracted from DJIA index options data, which covers the period from January 2009 until December 2015. The results show that the *advanced* method performs poorly in estimating the MFBKM, especially in the case of skewness and kurtosis integrals estimation. The *advanced* method outperforms the other approaches in the case of the variance estimation. In estimating both model-free skewness and kurtosis, the *adapted* method is found to perform the best, instead.

Keywords: cubic-spline, higher order moments, model-free, options and trapezoidal-rule.

1. Introduction

The research on using option-implied moments as estimators to improve the performance of a portfolio selection strategy is experiencing a growing interest these recent years. The moments inferred from options can be variance, covariance, skewness, volatility risk premium, beta, etc. For instance, Aït-Sahalia and Brandt (2008) utilised the option-implied state prices in improving the performance of portfolio selection and intertemporal consumption using implied probability density functions and the martingale representation theory. They recorded the different performances induced by adopting the option-implied moments compared to that of the historical moments. However they made no attempt in investigating the optimal portfolio strategy based on the results.

The fact that option information is proven to efficiently encapsulate derivative market perception has triggered many others on studying the optimal selection of portfolio by exploiting the option moments. A wide spectrum of study tends to utilise historical return data in estimating the option moments. However the portfolio that is based on historical-data estimation has been found to be poorly performed out-of-sample (DeMiguel et al. (2009)). Echoing to this concern, this research utilises option moments implied by option prices, rather than focusing on the use of historical data in improving option moments estimated in constructing an optimal portfolio strategy.

Option-implied information is inferred from the option prices, hitherto is referred as forward-looking option-implied moments. This approach can be perceived as an alternative to the backward-looking historical data. Owing to its forward-looking nature, these option-implied moments able to comprehensively capture the derivative market perception better than that of the historical data (See Kempf et al. (2014)). It is then expected that the estimation done based on these forward-looking implied moments to perform superiorly in constructing optimal portfolio. There are several aspects of study on the option-implied moments used in selecting portfolios. One can either consider option-implied volatility, correlation, skewness, risk premium, beta or covariance. This is evident in a plethora of empirical study that attempt to estimate these option-implied moments in a number of ways (Kostakis et al. (2011), Aït-Sahalia and Brandt (2008), and DeMiguel et al. (2013)).

It is documented in DeMiguel et al. (2013) that the option-implied volatility, risk premium and skewness perform extremely well in enhancing the portfolio selection strategy. The authors considered another option-implied moments in their paper, i.e. option-implied correlation. They showed that using the option-implied correlation does not significantly lead to an improved portfolio selection policy. They justified this claim based on the lack of stability in the covariance matrix when the historical time-series correlation is replaced with option-implied correlation. This induces the magnitude of the off-diagonal elements inside the matrix to be larger than they should be.

So far in the literature there is not much attention paid on improving the option-implied moments as an estimator in dealing with a portfolio selection policy. The option-implied correlation was utilised by Buss and Vilkov (2012) in solving the optimal portfolio problem. The estimation of the option-implied correlation was then used in obtaining the beta coefficient predictors. The implied information was found to deliver better predictor. In similar vein, the strategy was utilised in French et al. (1983). New family of estimators of the covariance matrix was considered by Kempf et al. (2014). They constructed the new estimators family and examined the performance power of the new estimators in contrast with the historical benchmark. The authors pointed out that the use of the new estimators developed in this study significantly lead to a better outperformed estimator compared to that of the historically estimated benchmark. However they worked in the domain of fully-implied covariance matrix and not on the option-implied covariance matrix. Related findings were sought by Siegel (1995), Skintzi and Refenes (2005), Husmann and Stephan (2007), Chang et al. (2011), as well as Baule et al. (2016). Thus, we manifest the hybrid-estimators strategy as used in Buss and Vilkov (2012), by considering both model-free and Black-Scholes-Merton (BSM) option pricing model, which was developed by Black and Scholes (1973) and Merton (1973).

Realising that, this research differentiates itself from other existing literature by examining the performance of higher order moments, realised from the model-free Bakshi-Kapadia-Madan (MFBKM). We concentrate on investigating higher order option-implied moments â variance, skewness and kurtosis, chosen in relation to contracts defined in MFBKM, i.e. volatility, cubic, and quartic contract. The three approaches adopted in order to estimate the integrals of the defined MFBKM contracts are the *basic*, *adapted* and *advanced* method. This study intends to empirically investigate the index options data, specifically those that are able to directly proxy the global index options market. For that reason, the Dow Jones Industrial Average (DJIA) index options data traded on The Chicago Board Options Exchange (CBOE) is utilised in this study. DJIA is the most cited and the most extensively accepted of the stock market indices. The options consists of the 30-blue chipped companies index and equity options which represent the most heavily traded and listed in US. Today, the CBOE has became the largest options exchange in US, hence has been acknowledged as the largest options market in the world. Owing to that fact, this data is believed to be the best in reflecting the US; hence the world index options market. The sample data considered in this study covers the period from January 2009 until the end of 2015. The volatility is proxied by the volatility implied by the Black-Scholes-Merton option pricing model.

For a better analysis comprehension, the rest of the research is drawn into a number of sections. A brief background of study is already provided in the first section. The data utilised in this paper is illustrated in Section 2. In Section 3, we detail out the methodology used in assessing the performance of Model-free Bakshi-Kapadia-Madan (MFBKM). The main findings of this study are presented in Section 4. Finally, we conclude in Section 5.

2. Data

This paper utilises all call and put options on the Dow Jones Industrial Index (DJIA) traded daily on the Chicago Board Options Exchange (CBOE) during the period of January 2009 until December 2015. The daily index data retrieved from the DJIA are composed of trading date, expiration date, closing price, exercise price and trading volume for each trading option. The underlying price used in this study will utilise the closing price of the DJIA index, whereas the actual option price is taken from the closing price of the option price. In this study, we utilise the Dow Jones Industrial Average (DJIA) index options data. The options consists of the 30-blue chipped companies index and equity options which represent the most heavily traded and listed in US.

3. Methodology

Generally, this study relies on two core strands of literature, i.e. Bakshi et al. (2003) and Buss and Vilkov (2012). The approaches used in the two studies are mainly adopted in this research with several adjustments and modifications come into consideration for a better MFBKM performance. In order to obtain the option-implied moments values, we adopt the same methodology as in Buss and Vilkov (2012), which is from the estimated moments of the market index return. We control the noise embedded in the MFBKM by considering three approaches. The MFBKM value is then compared against the realized higher moments regressed from the historical values.

3.1 Model-Free Bakshi-Kapadia-Madan

We calculate the option-implied moments based on the extraction approach introduced in Bakshi et al. (2003). The moments include variance contract, cubic contract, quartic contract, model-free implied volatility, as well as model-free option implied skewness. We take into account the model-free framework since the whole information of the BSM implied volatility smile can be considered using this model. Moreover, this model outperforms the BSM volatility in foreseeing realized volatility.

We first compute the option-implied higher moments from the market index data using the same methodology utilised in Bakshi et al. (2003). However, the theoretical foundation behind these model-free higher moments is beyond our scope. We, therefore, will not discuss it in this paper. The respective computation of option-implied moments, as derived by Bakshi et al. (2003) are as follows:

$$R(t,T) \equiv \ln S(t+T) - \ln S(t); \tag{1}$$

$$V(t,T) \equiv E_t^* \left\{ e^{-rt} R(t,T)^2 \right\}; \tag{2}$$

$$W(t,T) \equiv E_t^* \left\{ e^{-rt} R(t,T)^3 \right\}; \tag{3}$$

$$X(t,T) \equiv E_t^* \left\{ e^{-rt} R(t,T)^4 \right\}. \tag{4}$$

We let $S(t)$ be the stock price at time t, r be the risk-free interest rate, $K(t)$ be the strike price at time t, and $R(t,T)$ be the T-log return. $C(t)$ and $P(t)$ are the price of call and put option, respectively, at time t. The model-free option-implied volatility is simply the square root of Equation (2):

$$MFIV(t,T) = \sqrt{V(t,T)}. \tag{5}$$

The model-free option-implied skewness (MFIS) is obtained based from Equations (1) to (4).

$$MFIS(t,T) = \frac{e^{rt}W(t,T) - 3\mu(t,T)e^{rt}V(t,T) + 2(\mu(t,T))^3}{(e^{rt}V(t,T) - (\mu(t,T))^2)^{3/2}}. \tag{6}$$

Besides, Bakshi et al. (2003) show that the three defined contracts can attain the following forms:

$$\begin{aligned} V(t,\tau) = &\int_{S_t}^{\infty} \frac{2\left(1 - ln\left[\frac{K}{S_t}\right]\right)}{K^2} C(t,\tau;K)dK \\ &+ \int_{0}^{S_t} \frac{2\left(1 - ln\left[\frac{K}{S_t}\right]\right)}{K^2} P(t,\tau;K)dK; \end{aligned} \tag{7}$$

$$\begin{aligned} W(t,\tau) = &\int_{S_t}^{\infty} \frac{6ln\left[\frac{K}{S_t}\right] - 3\left(ln\left[\frac{K}{S_t}\right]\right)^2}{K^2} C(t,\tau;K)dK \\ &- \int_{0}^{S_t} \frac{6ln\left[\frac{K}{S_t}\right] - 3\left(ln\left[\frac{K}{S_t}\right]\right)^2}{K^2} P(t,\tau;K)dK; \end{aligned} \tag{8}$$

$$\begin{aligned} X(t,\tau) = &\int_{S_t}^{\infty} \frac{12\left(ln\left[\frac{K}{S_t}\right]\right)^2 - 4\left(ln\left[\frac{K}{S_t}\right]\right)^3}{K^2} C(t,\tau;K)dK \\ &+ \int_{0}^{S_t} \frac{12\left(ln\left[\frac{K}{S_t}\right]\right)^2 - 4\left(ln\left[\frac{K}{S_t}\right]\right)^3}{K^2} P(t,\tau;K)dK. \end{aligned} \tag{9}$$

The risk-neutral variance is depicted as:

$$VAR(t,\tau) \equiv E^q\left\{\left(R_{t,\tau} - E^q\left[R_{t,\tau}\right]\right)^2\right\}; \tag{10}$$

$$VAR(t,\tau) = e^{r\tau}V(t,\tau) - \mu(t,\tau)^2. \tag{11}$$

Recall that in Equation (6) the risk-neutral skewness is shown as

$$\begin{aligned} &MFIS(t,\tau) \\ &\equiv \frac{E^q\left\{R_{t,\tau} - E^q[R_{t,\tau}]^3\right\}}{E^q\left\{R_{t,\tau} - E^q[R_{t,\tau}]^2\right\}^{3/2}} \\ &= \frac{e^{r\tau}W(t,\tau) - 3e^{r\tau}\mu(t,\tau)V(t,\tau) + 2\mu\left(t,\tau\right)^3}{\left[e^{r\tau}V(t,\tau) - \mu\left(t,\tau\right)^2\right]^{3/2}}. \end{aligned} \tag{12}$$

Whereas, the risk-neutral kurtosis is as follows:

$$MFIK(t,\tau) \equiv \frac{E^q\left\{(R_{t,\tau}-E^q[R_{t,\tau}])^4\right\}}{E^q\left\{(R_{t,\tau}-E^q[R_{t,\tau}])^2\right\}^2}; \tag{13}$$

$$\begin{aligned} MFIK(t,\tau) = & \frac{e^{r\tau}X(t,\tau) - 4e^{r\tau}\mu(t,\tau)W(t,\tau)}{\left[e^{r\tau}V(t,\tau) - \mu\left(t,\tau\right)^2\right]^2} \\ & + \frac{6e^{r\tau}\mu(t,\tau)^2V(t,\tau) - 3\mu(t,\tau)^4}{\left[e^{r\tau}V(t,\tau) - \mu\left(t,\tau\right)^2\right]^2}, \end{aligned} \tag{14}$$

in which μ-expectation is

$$\mu(t,\tau) = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}V(t,\tau) - \frac{e^{r\tau}}{6}W(t,\tau) - \frac{e^{r\tau}}{24}X(t,\tau). \tag{15}$$

For record, Equations (1) and (2) are simply representing the variance contract. The cubic contract is depicted by Equation (3); while Equation (4) represents the quartic contract. This study focuses on the variance contract, i.e. the model-free variance (MFV).

3.2 Basic Approach

As the name depicts, this *basic* approach is the simplest and the easiest method that can be used in approximating the integrals. Its bottom line is drawn based on the left-point rule. In that sense, it is quite straightforward in obtaining the value for each higher order option-implied moment. In Bakshi et al. (2003), the integrals approximation is done using the summation equations. This is equivalent to the left-point rule. For an illustration purpose, the followings explain how the cubic contact is approximated.

We estimate the out-of-the-money calls taking the long position as:

$$\sum_{j=1}^{(K-S_t)/\Delta K} w\left[S_t + j\Delta K\right] C\left(t,\tau;S_t + j\Delta K\right)\Delta K, \tag{16}$$

where the highest value that can be taken by call strike price is denoted as K and $w[K]$ is defined as:

$$w\left[K\right] \equiv \frac{6\ln\left[\frac{K}{S_t}\right] - 3\left(\ln\left[\frac{K}{S_t}\right]\right)^2}{K^2}, \tag{17}$$

while

$$v\left[K\right]\equiv\frac{2\left(1-\ln\left[\frac{K}{S_t}\right]\right)}{K^2}, \tag{18}$$

and

$$x\left[K\right]\equiv\frac{12\left(\ln\left[\frac{K}{S_t}\right]\right)^2-4\left(\ln\left[\frac{K}{S_t}\right]\right)^3}{K^2}. \tag{19}$$

In similar sense, the estimation of out-of-the-money puts taking the long position can be approximated as:

$$\sum_{j=1}^{(S_t-K)/\Delta K} w\left[j\Delta K\right]P\left(t,\tau;j\Delta K\right)\Delta K, \tag{20}$$

in which

$$v\left[K\right]\equiv\frac{2\left(1+\ln\left[\frac{K}{S_t}\right]\right)}{K^2}, \tag{21}$$

$$w\left[K\right]\equiv-\frac{6\ln\left[\frac{K}{S_t}\right]+3\left(\ln\left[\frac{K}{S_t}\right]\right)^2}{K^2}, \tag{22}$$

and

$$x\left[K\right]\equiv\frac{12\left(\ln\left[\frac{K}{S_t}\right]\right)^2+4\left(\ln\left[\frac{K}{S_t}\right]\right)^3}{K^2}. \tag{23}$$

In this study, instead of relying on this easy-yet-too-basic method, we rely most on the trapezoidal-rule instead, adopting from methodology Dennis and Mayhew (2002). The approximation based on this integration technique is much more accurate and is not that complex to apply.

3.3 Adapted Approach

The contracts defined in MFBKM involve two separate integrals in order to cater the different option price of both calls and puts. However, this imbues to a sequence of problem, especially in estimating the prices of all three contracts. The *adapted* approach provides an initiative to this issue. Instead of considering the integrals of calls and puts as two separate integrals, this method only considers one integral. The two separate integrals are combined and are treated as one. This *adapted* approach treats the single-combined integrand into whether the strike price falls at above or below the spot price.

3.4 Advanced Approach

This approach is the addition to the *adapted* approach, in which on top of treating the contract separated integrals as one, it involves the use of smoothing method. The smoothing technique is adopted from Jiang and Tian (2005). Interpolation and extrapolation is applied on the implied volatility curve based on the cubic-spline method. The new interpolated and extrapolated implied volatility is then converted back into the option prices, adjusted based on their relative strike price value compared to the spot price. Based on the new retrieved option prices, the higher order MFBKM moments are estimated.

4. Results and Discussions

In this section, the performance of how each moment is approximated against the true values is compared based on three approaches considered: the *basic* method, the *adapted* method; and the *advanced* method. By hypothesis, the *advanced* method is supposedly to deliver the most accurate estimation of the three methods.

The skewness and kurtosis in this study are set to be always $\lambda_{1T} = 0$ and $\lambda_{2T} = 3$, respectively based on the normal distribution. The option prices of both calls and puts, in which inclusive for both out-of-the-money (OTM) and at-the-money (ATM) moneyness are estimated using the BSM option pricing functions developed. The tick size of the strike prices is chosen to be \$1, ranging from $K = \$54$ to \$150. The model-free estimates generated by the three methods - *basic*, *adapted*, and *advanced* - are reported in Table 1.

Table 1: Estimated Values of Model-Free Moments

Model-Free Moments	True Values	Estimated Values		
		Basic	Adapted	Advanced
T-Period Variance (${\sigma_T}^2$)	0.0225	0.0150	0.0154	0.0185
T-Period Skewness (λ_{1T})	0.0000	-2.6522	-2.5748	-2.7093
T-Period Kurtosis (λ_{2T})	3.0000	39.9404	38.3627	48.0483

In order to better analyse how each estimated values generated from the three methods deviate from the true values, the approximation error based on the absolute method is presented in Table 2. It can be observed that the *basic* method performs poorly in estimating all model-free moments in all three methods. However, our hypothesis does not hold in the case of variance and kurtosis estimation. It occurs in this study that the *adapted* approach is more accurate compared to the *advanced* method.

Table 2: Absolute Error for Model-Free Moments Estimates

Model-Free Moments	True Values	Absolute Error		
		Basic	Adapted	Advanced
T-Period Variance (${\sigma_T}^2$)	0.0225	0.0075	0.0071	0.0040
T-Period Skewness (λ_{1T})	0.0000	2.6522	2.5748	2.7093
T-Period Skewness (λ_{2T})	3.0000	36.9404	35.3627	45.0483

Table 3: Relative Percentage Error for Model-Free Moments Estimates

Model-Free Moments	True Values	Relative Percentage Error (%)		
		Basic	Adapted	Advanced
T-Period Variance (${\sigma_T}^2$)	0.0225	0.33	0.32	0.18
T-Period Skewness (λ_{1T})	0.0000	26.52*	25.75*	27.09*
T-Period Skewness (λ_{2T})	3.0000	12.31	11.79	15.02

Nevertheless, looking at the number per se is quite unreliable. The relative error by percentage is found to be much relevant since the different in the absolute error can be quite negligible by number per se. The relative percentage error for model-free moments estimates is reported in Table 3. Special condition is applied in the case of skewness, in which the true value is assumed to be 0.100 to cater the zero-denominator problem in finding the percentage value of the absolute error.

Based on both Table 2 and Table 3, it is obvious that the *advanced* approach fails to accurately estimate the model-free moments as stated in the hypothesis in the case of skewness and kurtosis estimation. The percentage different between the *adapted* and *advanced* method itself is quite small, i.e. 1.35% different in estimating skewness and 3.23% in estimating kurtosis. Thus, the *adapted* method performs the best in estimating the model-free skewness and kurtosis. With the small different in percentage, however, a clear line can be established in drawing conclusion that the *advanced* method somehow is reliably quite accurate in all the cases of estimation

5. Conclusions

This research differentiates itself from other existing literature by investigating the performance of higher order moments, realised from the model-free Bakshi-Kapadia-Madan (MFBKM). In that sense, this study focuses on investigating three kinds of higher order option-implied moments - variance, skewness and kurtosis. This study finds that the *advanced* method unable to accurately estimate the MFBKM considered as stated in the hypothesis, especially in the case of skewness and kurtosis integrals estimation.

The hypothesis only holds for the variance estimation. The *adapted* method is found to perform the best in estimating the model-free skewness and kurtosis. This finding is may be due to the error resulted from the choice of discrete strike prices of $1. However, the expectation error is only 0.13%, which is far deviated in the case of skewness and kurtosis estimation. It is suggested that further denoising treatment to be done on the available data. This provides a future avenue to fill into. However, based on the percentage error analysis, the margin error is quite insignificant. Thus, it is concluded that the *advanced* method is consistently quite accurate in all cases of estimation done on the MFBKM.

Acknowledgement

The authors gratefully acknowledge the support from Fundamental Research Grant Scheme (FGRS15-191-0432) research grant by the Ministry of Higher Education (MOHE). The authors are immensely grateful to the anonymous reviewers for their useful insights and comments.

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