PhD THESIS

# Fourier series and orthogonal polynomials in Sobolev spaces

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Leganés, de de

*A abuelita Rosario*

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# Summary

In the last 30 years, the study of orthogonal polynomials in Sobolev spaces has obtained an increasing attention from the research community. The first work on Sobolev orthogonal polynomials [6] was published in 1962 by Althammer, who studied the Legendre-Sobolev polynomials orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx + \lambda \int_{-1}^{1} f'(x)g'(x)dx, \quad \lambda > 0.$$

The motivation for such a study was attributed to the paper of Lewis [50]. Later on, Schäfke (see [92]) simplified and generalized some of the results in [6].

Since then this area has paid an increasing attention by many researchers. In Chapter 1 we only point out works which are important for understanding the material contained in this Thesis and we apologize if some title is missing in the bibliography. In any case, the interested reader is referred to [67], a recent survey on Sobolev orthogonal polynomials where the reader can find an overview in this subject as well as an huge number of publications in the bibliography. This contribution has been very helpful for the final form of this memory.

The study of this family of orthogonal polynomials is not only interesting for a comparison with the standard theory of orthogonal polynomials (see [13, 95]), but these polynomials also arise in a natural way in a variety of contexts:

- In Approximation Theory, Sobolev polynomials constitute a natural basis for the approximating subspaces in a certain least square approximation problem (see [50]).

- Spectral theory of ordinary and partial differential equations.

- Analysis of spectral numerical methods for boundary value problems on partial differential equations.

- Search of algorithms for the computation of Fourier Series in terms of Sobolev orthogonal polynomials. The numerical experiments made by A. Iserles et al. (see [40]) point out that the analysis of Gibbs phenomena can be performed in a successful way using Sobolev orthogonal polynomials.

In this thesis, we analyze the properties of polynomials orthogonal with respect to a discrete Sobolev inner product. More precisely, we will focus our attention on the study of connection formulas relating Sobolev orthogonal polynomials with the corresponding ordinary ones. Indeed, we deal with some problems on asymptotic behavior of Sobolev orthogonal polynomials as well as we obtain some results on convergence of Fourier-Sobolev series.

The present Thesis is organized as follows:

- In Chapter 1 we introduce the theory of Sobolev orthogonal polynomials and the notation that we will use along this Thesis. We summarize two main differences between the standard orthogonal polynomials and the Sobolev case: recurrence relations and the location of zeros of orthogonal polynomials. Here, we also include a thorough study about the known connection formulas. Finally, for a better understanding of our work, we give the state of the art about asymptotics and Fourier series of orthogonal polynomials, analyzing both the cases of measures with bounded and unbounded support, respectively.
- In Chapter 2 we study some algebraic and analytic aspects of certain family of Sobolev polynomials orthogonal with respect to a measure with a bounded support on the real line. In Section 2.1 we present an alternative proof for a known result about Outer Relative Asymptotics of Sobolev orthogonal polynomials. In Section 2.2 we also include a new matrix connection relating the matrix associated to the higher order recurrence relation for Sobolev polynomials and the corresponding Jacobi matrix associated to the standard ones. In Section 2.3 we show a result about pointwise convergence of Fourier-Sobolev series in the case of measures with bounded support.
- In Chapter 3 we summarize some known properties of polynomials orthogonal with respect to a modification of the Laguerre measure, the $k$-iterated Christoffel one. Later on, we obtain estimates for the norm of such polynomials as well as a generalized Christoffel formula for them. Finally, we present a detailed study about the diagonal Christoffel kernels associated to the Gamma distribution. In particular, we obtain the asymptotic behavior of these kernel polynomials both inside and outside the support of the measure.
- In Chapter 4 we deal with the Outer and Inner Relative Asymptotics of Sobolev-type orthogonal polynomials when the mass points are located inside the support of the measure, the oscillatory region for such polynomials. Finally, we obtain the asymptotic behavior of the coefficients appearing in the higher order recurrence relation that Sobolev polynomials satisfy.

- In Chapter 5 we show the divergence of a certain Fourier-Sobolev series. The main tool for this purpose will be a Cohen type inequality. This problem is dealing for the first time for a Sobolev-type inner product with a mass point outside the support of the measure.

Some of the original results contained in this Thesis have published in the following scientific journals and we list them within the bibliography at the end of this work:


[38] E. J. Huertas, F. Marcellán, M. F. Pérez-Valero, and Y. Quintana, *Asymptotics for Laguerre-Sobolev type orthogonal polynomials modified within their oscillatory regime*. Appl. Math. Comput. **236** (2014), 260–272.

[39] E. J. Huertas, F. Marcellán, M. F. Pérez-Valero, and Y. Quintana, *A Cohen type inequality for Laguerre-Sobolev expansions with a mass point outside their oscillatory regime*. Turkish J. Math. **38** (2014), 994–1006.

[59] F. Marcellán, M. F. Pérez-Valero, Y. Quintana, and A. Urieles, *Recurrence relations and outer relative asymptotics of orthogonal polynomials with respect to a discrete Sobolev type inner product*, Bull. Math. Sci. **4** (1) (2014), 83–97.

[60] F. Marcellán, M. F. Pérez-Valero and Y. Quintana, *Asymptotic behavior of derivatives of Laguerre kernels and some applications*, J. Math. Anal. Appl. **421** (2015), 314–328.

# Resumen

El desarrollo del estudio de polinomios ortogonales en espacios de Sobolev ha tenido lugar a lo largo de los últimos 30 años. El primer artículo sobre polinomios ortogonales de Sobolev, [6], fue publicado en 1962 por Althammer, quien estudió los polinomios de Sobolev-Legendre ortogonales respecto al producto interno

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx + \lambda \int_{-1}^{1} f'(x)g'(x)dx, \quad \lambda > 0.$$

El autor atribuyó la motivación para este análisis al artículo de Lewis [50] sobre aproximación suave de mínimos cuadrados. Más tarde, algunos de los resultados obtenidos en [6] fueron simplificados y generalizados por Schäfke en [92].

Desde entonces han aparecido multitud de publicaciones en esta área, de forma que preferimos en el capítulo 1 centrarnos sólo en aquellas aportaciones importantes para la comprensión y justificación del material contenido en esta memoria y pedimos disculpas si llegara a haber omisión de algún trabajo relevante. No obstante, el lector interesado puede consultar [67], un estudio sobre polinomios ortogonales en espacios de Sobolev aparecido en arXiv y recientemente aceptado para publicación donde se ofrece una visión general sobre esta materia, además de un resumen bastante completo de la bibliografía y que ha sido de gran ayuda en la forma final que ha adquirido este trabajo.

El estudio de estas nuevas familias de polinomios ortogonales es interesante, no sólo por la comparación entre las propiedades y comportamiento de estos polinomios frente a los polinomios ortogonales estándar (véanse [13, 95]), sino por las múltiples aplicaciones que tienen en diferentes contextos:

- En teoría de aproximación, los polinomios de Sobolev constituyen una base natural para los subespacios de aproximación de cierto problema de mínimos cuadrados (véase [50]).
- Teoría espectral de ecuaciones diferenciales ordinarias así como para ecuaciones en derivadas parciales.

- Análisis de métodos espectrales en el tratamiento numérico de problemas de valores en la frontera para ecuaciones diferenciales en derivadas parciales.
- Búsqueda de algoritmos para el cálculo de Series de Fourier en términos de polinomios ortogonales de Sobolev. Los experimentos numéricos llevados a cabo por A. Iserles et al. (véase [40]) obtuvieron resultados satisfactorios en el análisis del fenómeno de Gibbs en desarrollos de Fourier usando polinomios de Sobolev en comparación con los polinomios estándar.

En esta memoria analizaremos el comportamiento y las propiedades de polinomios ortogonales respecto a productos internos de Sobolev discretos. Más concretamente, nuestro interés será el estudio de fórmulas de conexión entre polinomios ortogonales estándar y polinomios ortogonales de Sobolev. De esta forma, podremos abordar algunos problemas de asintótica de polinomios ortogonales de Sobolev, así como obtener resultados de convergencia de series de Fourier asociadas a tales polinomios.

Estos contenidos se dividen en los siguientes capítulos:

- En el capítulo 1 presentamos una introducción a la teoría de polinomios de Sobolev, introduciendo la notación que se utilizará a lo largo de esta tesis. Se resumirán las principales diferencias entre el caso estándar y el caso Sobolev. En concreto nos centraremos en el estudio de relaciones de recurrencia y localización de ceros de dichos polinomios ortogonales. Se incluirá un estudio bastante completo de los diferentes tipos de fórmulas de conexión existentes. Finalmente, daremos una panorámica de los resultados conocidos en asintótica y desarrollos en series de Fourier de polinomios ortogonales de Sobolev tanto en el caso de soporte acotado como en el no acotado, que permitirá una mejor comprensión de nuestro trabajo.
- En el capítulo 2 se estudian aspectos analíticos y algebraicos de cierta familia de polinomios de Sobolev ortogonales respecto a una medida de soporte acotado. En la sección 2.1 se presenta una demostración alternativa de un resultado conocido sobre asintótica relativa exterior de ciertos polinomios de Sobolev. Demostraremos una nueva relación matricial entre la matriz asociada a la relación de recurrencia que satisfacen los polinomios de Sobolev y la matriz de Jacobi de los correspondientes polinomios ordinarios en la sección 2.2. En la última sección presentaremos un resultado sobre convergencia puntual de series de Fourier asociadas a ciertos polinomios de Sobolev.
- En el capítulo 3 resumiremos algunas propiedades conocidas de polinomios ortogonales respecto a una medida de Laguerre modificada, una $k$-iteración de Christoffel de la medida de Laguerre. A continuación, obtendremos estimaciones para la norma de estos polinomios y proporcionaremos una fórmula generalizada de Christoffel para tal familia de polinomios. Finalmente, presentaremos un estudio completo y detallado de los núcleos de Christoffel diagonales

asociados a la distribución Gamma, obteniendo la asintótica de los mismos tanto dentro como fuera del soporte de la medida.

- En el capítulo 4 se hace un estudio de asintótica relativa de polinomios ortogonales de Sobolev discreto cuando las masas en la parte discreta del producto interno están situadas dentro el soporte de la medida, la región de oscilación de dichos polinomios. El comportamiento de dichos polinomios será estudiado tanto dentro como fuera del soporte de la medida. Finalmente, obtendremos el comportamiento asintótico de los coeficientes que aparecen en la relación de recurrencia que satisfacen los polinomios de Sobolev.

- En el capítulo 5 abordaremos el problema de convergencia de series de Fourier-Sobolev. Mostraremos la divergencia de la serie de Fourier asociada a cierta familia de polinomios ortogonales de Sobolev y la principal herramienta para ello serán las desigualdades de tipo Cohen. Este problema es tratado por primera vez para un producto de Sobolev discreto con una masa fuera del soporte de la medida.

Todo el material original contenido en esta memoria ha sido aceptado para publicación en revistas científicas bajo los siguientes títulos y el número que les asignamos es el que ocuparán dentro de la bibliografía:


[38] E. J. Huertas, F. Marcellán, M. F. Pérez-Valero, and Y. Quintana, *Asymptotics for Laguerre-Sobolev type orthogonal polynomials modified within their oscillatory regime*. Appl. Math. Comput. **236** (2014), 260–272.

[39] E. J. Huertas, F. Marcellán, M. F. Pérez-Valero, and Y. Quintana, *A Cohen type inequality for Laguerre-Sobolev expansions with a mass point outside their oscillatory regime*. Turkish J. Math. **38** (2014), 994–1006.

[59] F. Marcellán, M. F. Pérez-Valero, Y. Quintana, and A. Urieles, *Recurrence relations and outer relative asymptotics of orthogonal polynomials with respect to a discrete Sobolev type inner product*, Bull. Math. Sci. **4** (1) (2014), 83–97.

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# Contents

# 1

# Introduction: Sobolev type orthogonal polynomials

From now on, we will denote by $\mathbb{R}$ the set of real numbers, $\mathbb{P}$ will denote the vector space of polynomials with real coefficients and $\mathbb{P}_n$ is the vector subspace of polynomials with real coefficients and degree less than or equal to $n$.

Let $E \subseteq \mathbb{R}$ be a bounded or unbounded infinite set. If we consider a nontrivial positive Borel measure $\mu$ supported on $E$, then a sequence of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is said to be an *orthogonal polynomial sequence* with respect to $\mu$ if for all nonnegative integers $n$ and $m$,

(i) $p_n(x)$ is a polynomial of degree $n$,

(ii) $\int_E p_n(x)p_m(x)d\mu(x) = K_n\delta_{nm}, \quad K_n \neq 0,$

where

$$\delta_{nm} = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m, \end{cases}$$

is the Kronecker delta.

If $\{p_n(x)\}_{n=0}^{\infty}$ is an orthogonal polynomial sequence and, in addition, we also have $K_n = 1$, then it will be called an *orthonormal polynomial sequence.*

In what follows, we will refer to this kind of sequences as sequences of *standard* orthogonal polynomials. As a starting point, the main properties of orthogonal polynomials on the real line are assumed to be known and we refer to the interested reader to [13] for a review on this subject.

The aim of this chapter is to introduce another family of orthogonal polynomials, the so-called *Sobolev-type orthogonal polynomials.* Without any intention of plagiarism, we summarize the properties of such sequences of orthogonal polynomials by comparison with those of the standard ones, specially some very well known results that we will need for the development of the ideas in the subsequent chapters.

A Sobolev-type or discrete Sobolev inner product on the linear space $\mathbb{P}$ of polynomials with real coefficients is defined by

$$\langle f, g\rangle_S = \int f(x)g(x)d\mu(x) + \sum_{k=1}^{K} \mathbb{F}(c_k)A_k\mathbb{G}(c_k)^T,\ K \in \mathbb{Z}_+, \tag{1.0.1}$$

where $\mu$ is a nontrivial positive Borel measure supported on the real line, $f, g \in \mathbb{P}$, and for $k = 1, \dots, K$, $K \in \mathbb{Z}_+$, the matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{R}^{(1+N_k)(1+N_k)}$ are positive semi-definite.
We denote by $\mathbb{F}(c_k)$ and $\mathbb{G}(c_k)$ the vectors $\mathbb{F}(c_k) = \big(f(c_k), f'(c_k), \dots, f^{(N_k)}(c_k)\big)$ and $\mathbb{G}(c_k) = \big(g(c_k), g'(c_k), \dots, g^{(N_k)}(c_k)\big)$, respectively, with $c_k \in \mathbb{R}$, $N_k \in \mathbb{Z}_+$ where, as usual, $v^T$ denotes the transpose of the row vector $v$. This notion was initially introduced in [25] for diagonal matrices $A_k$, which is the case we are dealing in this work, and, in such a case, we can express the previous formula as follows

$$\langle f, g\rangle = \langle f, g\rangle_\mu + \sum_{k=1}^{K}\sum_{i=0}^{N_k} M_{k,i} f^{(i)}(c_k) g^{(i)}(c_k), \tag{1.0.2}$$

$$\langle f, g\rangle_\mu = \int f(x)g(x)d\mu(x),$$

where $M_{k,i} = a_{ii}^{(k)} \geqslant 0$ for $i = 0, \dots, N_k - 1$, and $M_{k,N_k} > 0$, when $k = 1, \dots, K$.

The Gram-Schmidt process applied to the canonical basis of $\mathbb{P}$ generates the orthonormal sequence of polynomials $\{B_n(x)\}_{n=0}^{\infty}$ for (1.0.2), i.e.,

$$\langle B_n, B_k\rangle = \delta_{n,k}, \quad k, n = 0, 1, \dots,$$

where

$$B_n(x) = \lambda_n x^n + \text{lower degree terms}, \quad \lambda_n > 0.$$

We denote the corresponding monic polynomials by $\{\widehat{B}_k(x)\}_{k=0}^{\infty}$. Let $\{p_n(x)\}_{n=0}^{\infty}$ and $\{\widehat{p}_n(x)\}_{n=0}^{\infty}$ be the sequences of orthonormal and monic polynomials with respect to $\mu$, respectively.

Throughout this thesis, the notation $u_n \sim v_n$ or $u_n \sim_n v_n$ means that the sequence $\{\frac{u_n}{v_n}\}_{n=0}^{\infty}$ converges to certain non zero constant as $n \to \infty$ while the notation $u_n \cong v_n$ means that there exist positive real numbers $C_1$ and $C_2$ such that $C_1 u_n \leqslant v_n \leqslant C_2 u_n$ for $n$ large enough. Any other standard notation will be properly introduced whenever needed.

## 1.1 Recurrence relations

It is well-known that a sequence of standard orthonormal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with respect to some measure $\mu$ supported on the real line satisfies a three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0, \tag{1.1.3}$$

with initial conditions $p_{-1}(x) = 0$ and $p_0(x) = \frac{1}{(\int d\mu(x))^{1/2}}$. The recurrence coefficients are

$$a_n = \int xp_{n-1}(x)p_n(x)d\mu(x) > 0, \quad b_n = \int xp_n^2(x)d\mu(x) \in \mathbb{R}.$$

The converse result was proved by Favard in 1935 and the interested reader could find more detail about this fact in [13, 26]. This is equivalent to the symmetry of the multiplication by $x$ with respect to an inner product as a characterization of standard inner products.

Due to the fact that Sobolev-type inner products we are dealing are non-standard, i.e.

$$\langle xf, g\rangle \neq \langle f, xg\rangle, \quad f, g \in \mathbb{P},$$

Sobolev orthogonal polynomials no longer satisfy a three term recurrence relation. However, it can be proved that they satisfy a higher-order recurrence relation.

Let $h_N(x)$ be the monic polynomial of least degree verifying $h_N^{(i)}(c_k) = 0$ for $k = 1, \ldots, K, i = 0, \ldots, N_k$, that is,

$$h_N(x) = \prod_{k=1}^{K} (x - c_k)^{N_k+1},$$

where $N = \sum_{k=1}^{K} N_k + K$, then we have

**Theorem 1.1.1.** *The Sobolev polynomials* $\{B_n(x)\}_{n=0}^{\infty}$ *orthonormal with respect to* (1.0.2) *satisfy a* $(2N+1)$*-term recurrence relation*

$$h_N(x)B_n(x) = \sum_{k=n-N}^{n+N} \alpha_{n,k} B_k(x), \tag{1.1.4}$$

*with* $\alpha_{n,n-N} \neq 0$.

*Proof.* We can expand the polynomial $h_N(x)B_n(x)$ in the basis of Sobolev orthogonal polynomials obtaining

$$h_N(x)B_n(x) = \sum_{k=0}^{n+N} \alpha_{n,k} B_k(x)$$

with

$$\alpha_{n,k} = \langle h_N B_n, B_k \rangle = \int h_N(x)B_n(x)B_k(x)d\mu(x) = \langle B_n, h_N B_k \rangle = 0, \quad \text{if } k < n-N,$$

but

$$\alpha_{n,n-N} = \langle h_N B_n, B_{n-N} \rangle = \langle B_n, h_N B_{n-N} \rangle = \frac{\lambda_{n-N}}{\lambda_n} \langle B_n, B_n \rangle > 0.$$

□

Along this thesis, we will study different matrix interpretations of this higher order recurrence relation as well as the asymptotic behavior of the coefficients appearing in such a formula for some particular cases of the measure $\mu$.

On the other hand, it can be proven that polynomials defined by (1.1.4) are closely related to matrix polynomials satisfying a three term recurrence relation. For more details about these relations, you can see [24].

## 1.2 Connection formulas

In this Section, we study different formulas involving Sobolev-type orthogonal polynomials and the standard ones. We will call these kind of relations *connection formulas* and they will be a key tool in order to obtain properties for Sobolev polynomials from those of the standard ones.

1. The technique for obtaining the first connection formula we are going to study is due to Marcellán and Ronveaux and it can be found in [64]. Since then, many authors have used or generalized this kind of expansion.

   Recall that $\{p_n(x)\}_{n=0}^{\infty}$ and $\{\widehat{p}_n(x)\}_{n=0}^{\infty}$ are the sequences of orthonormal and monic polynomials with respect to $\mu$, respectively.

   We denote by $K_n(x,y)$ the $n$-th Christoffel-Darboux kernel

$$K_n(x,y) = \sum_{k=0}^{n} \frac{\widehat{p}_k(x)\widehat{p}_k(y)}{\langle \widehat{p}_k, \widehat{p}_k \rangle_\mu}, \tag{1.2.5}$$

and we use the following notation for its partial derivatives

$$\frac{\partial^{j+k} K_n(x,y)}{\partial x^j \partial y^k} = K_n^{(j,k)}(x,y), \quad 0 \leqslant j,k \leqslant n. \tag{1.2.6}$$

**Proposition 1.2.1** (Connection formula type I)**.** *With the previous notation,*

$$\widehat{B}_n(x) = \widehat{p}_n(x) - \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} \widehat{B}_n^{(i)}(c_k) K_{n-1}^{(0,i)}(x, c_k). \tag{1.2.7}$$

*Proof.* The Fourier expansion of the polynomial $\widehat{B}_n$ in terms of the orthogonal basis $\{\widehat{p}_n(x)\}_{n=0}^{\infty}$ leads to

$$\widehat{B}_n(x) = \widehat{p}_n(x) + \sum_{j=0}^{n-1} \sigma_{n,j} \widehat{p}_j(x). \tag{1.2.8}$$

In the usual way, we find the coefficients $\sigma_{n,j}$, $0 \leqslant j \leqslant n-1$, as follows

$$\sigma_{n,j} = \frac{\langle \widehat{B}_n, \widehat{p}_j \rangle_\mu}{\langle \widehat{p}_j, \widehat{p}_j \rangle_\mu} = \frac{-\sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} \widehat{B}_n^{(i)}(c_k) \widehat{p}_j^{(i)}(c_k)}{\langle \widehat{p}_j, \widehat{p}_j \rangle_\mu}.$$

Then (1.2.8) becomes

$$\widehat{B}_n(x) = \widehat{p}_n(x) - \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} \widehat{B}_n^{(i)}(c_k) K_{n-1}^{(0,i)}(x, c_k). \tag{1.2.9}$$

In order to obtain an explicit expression for $\widehat{B}_n^{(i)}(c_k)$ when $k = 1, \ldots, K$, $i = 0, \ldots, N_k$, we can write (1.2.9) and its corresponding derivatives with respect to $x$ evaluated at $x = c_1, \ldots, c_K$, in a matrix form as follows,

$$A\mathbf{B} = \mathbf{P},$$

where $A$ is the block matrix

$$A = I + \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1K} \\ A_{21} & A_{22} & \ldots & A_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ A_{K1} & A_{K2} & \ldots & A_{KK} \end{pmatrix},$$

where $I$ is the $N \times N$ identity matrix and the $(N_i + 1) \times (N_j + 1)$ block $A_{ij}$ is given by

$$A_{ij}=\begin{pmatrix} M_{i,0}K_{n-1}(c_i,c_j) & M_{i,1}K_{n-1}^{(0,1)}(c_i,c_j) & M_{i,2}K_{n-1}^{(0,2)}(c_i,c_j) & \dots & M_{i,N_i}K_{n-1}^{(0,N_j)}(c_i,c_j)\\ M_{i,0}K_{n-1}^{(1,0)}(c_i,c_j) & M_{i,1}K_{n-1}^{(1,1)}(c_i,c_j) & M_{i,2}K_{n-1}^{(1,2)}(c_i,c_j) & \dots & M_{i,N_i}K_{n-1}^{(1,N_j)}(c_i,c_j)\\ M_{i,0}K_{n-1}^{(2,0)}(c_i,c_j) & M_{i,1}K_{n-1}^{(2,1)}(c_i,c_j) & M_{i,2}K_{n-1}^{(2,2)}(c_i,c_j) & \dots & M_{i,N_i}K_{n-1}^{(2,N_j)}(c_i,c_j)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ M_{i,0}K_{n-1}^{(N_i,0)}(c_i,c_j) & M_{i,1}K_{n-1}^{(N_i,1)}(c_i,c_j) & M_{i,2}K_{n-1}^{(N_i,2)}(c_i,c_j) & \dots & M_{i,N_i}K_{n-1}^{(N_i,N_j)}(c_i,c_j)\end{pmatrix},$$

with $A_{ij}\in\mathbb{R}^{(N_i+1,N_j+1)}$,

$$\mathbf{P}=\left(\widehat{p}_n(c_1),\widehat{p}_n'(c_1),\dots,\widehat{p}_n^{(N_1)}(c_1),\widehat{p}_n(c_2),\dots,\widehat{p}_n^{(N_2)}(c_2),\dots,\widehat{p}_n(c_K),\dots,\widehat{p}_n^{(N_K)}(c_K)\right)^T,$$

and

$$\mathbf{B}=\left(\widehat{B}_n(c_1),\widehat{B}_n'(c_1),\dots,\widehat{B}_n^{(N_1)}(c_1),\widehat{B}_n(c_2),\dots,\widehat{B}_n^{(N_2)}(c_2),\dots,\widehat{B}_n(c_K),\dots,\widehat{B}_n^{(N_K)}(c_K)\right)^T.$$

Then, applying Cramer's rule we get

$$\widehat{B}_n^{(i)}(c_k)=\frac{\det(\mathbf{A_m})}{\det(\mathbf{A})},\quad \text{for } m=1,\dots,N,$$

where $\widehat{B}_n^{(i)}(c_k)$ is the $m$-th position in the vector **B** and $\mathbf{A_m}$ is the matrix obtained by replacing the $m$-th column in the matrix **A** by the column vector **P**.

□

2. The second connection formula is based on the expansion of the Sobolev polynomial not in terms of the standard ones but in terms of a family of polynomials orthogonal with respect to a suitable polynomial modification of the measure $\mu$. This technique appeared for instance in [57].

   From now on, $k(\Pi_n)$ will denote the leading coefficient of any polynomial $\Pi_n$ with real coefficients and degree $n$ and $\mathbb{P}_n$ will be the linear subspace of all polynomials of degree less than or equal to $n$.

   Let $h_N(x)$ be a polynomial such that $h_N(x)d\mu(x)$ is a positive definite measure on the support of $\mu$ and let $\{p_n^{[h]}(x)\}_{n=0}^{\infty}$ be the sequence of polynomials orthonormal with respect to $h_N(x)d\mu(x)$.

   **Proposition 1.2.2** (Connection formula type II)**.** *The following connection formula holds,*

$$B_n(x)=\sum_{j=0}^{N}A_{n,j}p_{n-j}^{[h]}(x),\quad A_{n,N}\neq 0. \tag{1.2.10}$$

*Moreover,*

$$A_{n,0} = \frac{k(B_n)}{k(p_n^{[h]})},$$
$$A_{n,N} = \frac{k(p_{n-N})}{k(p_{n+N}^{[h]})}\frac{1}{A_{n,0}}.$$

*Proof.* Since each $p_k^{[h]}(x)$ is of degree $k$, $\{p_0^{[h]}(x), p_1^{[h]}(x), \ldots, p_n^{[h]}(x)\}$ is a basis of $\mathbb{P}_n$. Thus, there exist real numbers $\alpha_{n,j}$, $j = 0, 1, \ldots, n$, such that

$$B_n(x) = \sum_{j=0}^{n} \alpha_{n,j} p_j^{[h]}(x)$$

with

$$\alpha_{n,j} = \int h_N(x)B_n(x)p_j^{[h]}(x)d\mu(x) = \langle B_n, h_N p_j^{[h]}\rangle = 0, \quad \text{if } j < n-N.$$

This proves the first statement with $A_{n,j} = \alpha_{n,n-j}$. Furthermore,

$$A_{n,0} = \int h_N(x)B_n(x)p_n^{[h]}d\mu(x) = \frac{k(B_n)}{k(p_n^{[h]})},$$

as well as

$$A_{n,N} = \int h_N(x)B_n p_{n-N}^{[h]}(x)d\mu(x) = \langle B_n, h_N p_{n-N}^{[h]}\rangle = \frac{k(p_{n-N}^{[h]})}{k(B_n)} = \frac{k(p_{n-N}^{[h]})}{k(p_n^{[h]})}\frac{1}{A_{n,0}},$$

and the result holds.

□

3. The last kind of connection formula appears for the first time in [44], where Koekoek and Meijer were working with a Sobolev type inner product such that $\mu$ is the Gamma distribution, $k = 1$, $c_k = 0$ and $N \geqslant 1$. Later on, generalizations for more general inner products have been studied.
   For the sake of simplicity, we consider a particular case of the inner product (1.0.2). More precisely, we study the case of the inner product

$$(p,q)_S = \int p(x)q(x)d\mu(x) + \begin{pmatrix} p(c) & p'(c) & \ldots & p^{(N-2)}(c) \end{pmatrix} M_{N-1} \begin{pmatrix} q(c) \\ q'(c) \\ \vdots \\ q^{(N-2)}(c) \end{pmatrix}, \quad N \geqslant 3, \tag{1.2.11}$$

where $M_{N-1}$ is a $(N-1) \times (N-1)$ positive semi-definite matrix.

Let us denote by $\{\widehat{B}_n(x)\}_{n=0}^{\infty}$, $\{\widehat{p}_n(x)\}_{n=0}^{\infty}$ and $\{p_n^{[2k]}(x)\}_{n=0}^{\infty}$ the monic sequences of polynomials orthogonal with respect to $(\cdot,\cdot)_S$, $d\mu(x)$ and $d\mu_k(x) = (x-c)^{2k}d\mu(x)$, respectively.

A connection formula is said to be of type III when the $n$-th Sobolev polynomial $B_n$ is given as a linear combination of $\widehat{p}_n(x), (x-c)p_{n-1}^{[2]}(x), (x-c)^2 p_{n-2}^{[4]}(x), \ldots, (x-c)^{N-1}p_{n-2}^{[2N-2]}(x)$.

Before dealing with this connection formula, we will need some auxiliary results. Taking into account that

$$\mathbb{P}_n = (x-c)^k\mathbb{P}_{n-k} \oplus \left\{(x-c)^k\mathbb{P}_{n-k}\right\}^{\perp}, \tag{1.2.12}$$

where

$$\left\{(x-c)^k\mathbb{P}_{n-k}\right\}^{\perp} = \left\{p \in \mathbb{P}_n \quad | \quad \int p(x)v(x)d\mu(x) = 0, \forall v \in (x-c)^k\mathbb{P}_{n-k}\right\},$$

we obtain a decomposition for the polynomial $B_n$ as a sum of a polynomial in the vector subspace $(x-c)^k\mathbb{P}_{n-k}$ and his corresponding orthogonal component.

**Proposition 1.2.3.** *(i) The set $\{(x-c)^k\widehat{p}_j^{[2k]}\}_{j=0}^{n-k}$ is an orthonormal basis for the vector subspace $(x-c)^k\mathbb{P}_{n-k}$ with respect to $\mu$.*

*(ii)*

$$\left\{(x-c)^k\mathbb{P}_{n-k}\right\}^{\perp} = \operatorname{span}\{K_n(x,c), \ldots, K_n^{(0,k-1)}(x,c)\}.$$

*(iii) There exist real numbers $a_{n,j}^{(k)}$, $j = 0, \ldots, k-1$, such that*

$$\widehat{p}_n(x) = \alpha_{n,k}(x-c)^k\widehat{p}_{n-k}^{[2k]}(x) + \sum_{j=0}^{k-1} a_{n,j}^{(k)}K_n^{(0,j)}(x,c).$$

*Proof.* (i) It is an immediate consequence of the orthogonality condition for the sequence $\{p_n^{[2k]}(x)\}_{n=0}^{\infty}$.

(ii) It is enough to take into account the reproducing property of the kernel polynomial

$$\int (x-c)^k\widehat{p}_l^{[2k]}(x)K_n^{(0,j)}(x,c)d\mu(x) = 0,$$

for $0 \leqslant l \leqslant n-k$, $0 \leqslant j \leqslant k-1$.

(iii) It follows from (ii) having in mind the orthogonal decomposition (1.2.12) .

□

If we put $k = 1, \ldots, N-1$, in (iii) of the previous proposition, we obtain

$$\begin{aligned}
\widehat{p}_n(x) &= \alpha_{n,1}(x-c)\widehat{p}^{[2]}_{n-1}(x) + a^{(1)}_{n,0}K^{(0,0)}_n(x,c),\\
\widehat{p}_n(x) &= \alpha_{n,2}(x-c)^2\widehat{p}^{[4]}_{n-2}(x) + a^{(2)}_{n,0}K^{(0,0)}_n(x,c) + a^{(2)}_{n,1}K^{(0,1)}_n(x,c),\\
&\;\;\vdots\\
\widehat{p}_n(x) &= \alpha_{n,N-1}(x-c)^{N-1}\widehat{p}^{[2N-2]}_{n-N+1}(x) + a^{(N-1)}_{n,0}K^{(0,0)}_n(x,c) + \ldots + a^{(N-1)}_{n,N-2}K^{(0,N-2)}_n(x,c).
\end{aligned}$$

This can be written in a matrix form as follows,

$$A^{(n)}_{N-1}\begin{pmatrix} K_n(x,c)\\ K^{(0,1)}_n(x,c)\\ \vdots\\ K^{(0,N-2)}_n(x,c)\end{pmatrix} = \begin{pmatrix} \widehat{p}_n(x) - \alpha_{n,1}(x-c)\widehat{p}^{[2]}_{n-1}(x)\\ \widehat{p}_n(x) - \alpha_{n,2}(x-c)^2\widehat{p}^{[4]}_{n-2}(x)\\ \vdots\\ \widehat{p}_n(x) - \alpha_{n,N-1}(x-c)^{N-1}\widehat{p}^{[2N-2]}_{n-N+1}(x)\end{pmatrix},$$

where

$$A^{(n)}_{N-1} = \begin{pmatrix} a^{(1)}_{n,0} & 0 & \ldots & 0\\ a^{(2)}_{n,0} & a^{(2)}_{n,1} & \ldots & 0\\ \vdots & & \ddots & \vdots\\ a^{(N-1)}_{n,0} & a^{(N-1)}_{n,1} & \ldots & a^{(N-1)}_{n,N-2}\end{pmatrix},$$

and

$$\alpha_{n,1} = \frac{||\widehat{p}_n||^2}{||\widehat{p}^{[2]}_{n-1}||^2_{[2]}}, \quad \alpha_{n,2} = \frac{||\widehat{p}_n||^2}{||\widehat{p}^{[4]}_{n-2}||^2_{[4]}}, \quad \ldots, \quad \alpha_{n,N-1} = \frac{||\widehat{p}_n||^2}{||\widehat{p}^{[2N-2]}_{n-N+1}||^2_{[2N-2]}}.$$

Now, we want to prove the nonsingularity of the matrix $A^{(n)}_{N-1}$. For that purpose, we evaluate the above expressions and their successive derivatives until order $(N-2)$ at $x = c$ to obtain

$$A^{(n)}_{N-1}\mathbb{K}^{(N-2)}_n(c,c) = \begin{pmatrix} \hat{p}_n(c) & \hat{p}'_n(c) - \alpha_{n,1}\hat{p}^{[2]}_{n-1}(c) & \ldots & \hat{p}^{(N-2)}_n(c) - \binom{N-2}{1}\alpha_{n,1}\left(\hat{p}^{[2]}_{n-1}\right)^{(N-3)}(c)\\ \hat{p}_n(c) & \hat{p}'_n(c) & \ldots & \hat{p}^{(N-2)}_n(c) - \binom{N-2}{2}2\alpha_{n,2}\left(\hat{p}^{[4]}_{n-2}\right)^{(N-4)}(c)\\ \hat{p}_n(c) & \hat{p}'_n(c) & \ldots & \hat{p}^{(N-2)}_n(c) - \binom{N-2}{3}3!\alpha_{n,3}\left(\hat{p}^{[6]}_{n-3}\right)^{(N-5)}(c)\\ \vdots & \vdots & \ddots & \vdots\\ \hat{p}_n(c) & \hat{p}'_n(c) & \ldots & \hat{p}^{(N-2)}_n(c)\end{pmatrix}, \tag{1.2.13}$$

where $A^{(n)}_{N-1}$ is a lower triangular matrix and

$$\begin{aligned}\mathbb{K}_n^{(N-2)}(c,c) &= \begin{pmatrix} K_n(c,c) & K_n^{(1,0)}(c,c) & \dots & K_n^{(N-2,0)}(c,c) \\ K_n^{(0,1)}(c,c) & K_n^{(1,1)}(c,c) & \dots & K_n^{(N-2,1)}(c,c) \\ \vdots & \vdots & \ddots & \vdots \\ K_n^{(0,N-2)}(c,c) & K_n^{(1,N-2)}(c,c) & \dots & K_n^{(N-2,N-2)}(c,c) \end{pmatrix} \\ &= \begin{pmatrix} p_0(c) & p_1(c) & \dots & p_n(c) \\ p_0'(c) & p_1'(c) & \dots & p_n'(c) \\ \vdots & \vdots & \ddots & \vdots \\ p_0^{(N-2)}(c) & p_1^{(N-2)}(c) & \dots & p_n^{(N-2)}(c) \end{pmatrix} \begin{pmatrix} p_0(c) & p_0'(c) & \dots & p_0^{(N-2)}(c) \\ p_1(c) & p_1'(c) & \dots & p_1^{(N-2)}(c) \\ \vdots & \vdots & \ddots & \vdots \\ p_n(c) & p_n'(c) & \dots & p_n^{(N-2)}(c) \end{pmatrix} \\ &= \mathcal{P}_{(N-1)\times(n+1)}\mathcal{P}^T_{(n+1)\times(N-1)}.\end{aligned}$$

On the one hand, let notice that $\operatorname{rank}\mathcal{P}_{(N-1)\times(n+1)} = N-1$. Thus, $\operatorname{rank}\mathbb{K}_n^{(N-2)}(c,c) = N-1$, i.e. $\mathbb{K}_n^{(N-2)}(c,c)$ is a full rank matrix and, as a consequence, $\det\mathbb{K}_n^{(N-2)}(c,c) \neq 0$.

On the other hand, for the determinant of the matrix of the right hand side of (1.2.13), we have a matrix in an echelon form when you subtract to any row the last one. Indeed, the determinant becomes

$$\begin{aligned}&\begin{vmatrix} 0 & -\alpha_{n,1}\widehat{p}^{[2]}_{n-1}(c) & -2\alpha_{n,1}\left(\widehat{p}^{[2]}_{n-1}\right)^{(\prime)}(c) & \dots & -\binom{N-2}{1}\alpha_{n,1}\left(\widehat{p}^{[2]}_{n-1}\right)^{(N-3)}(c) \\ 0 & 0 & -2!\alpha_{n,2}\widehat{p}^{[4]}_{n-2}(c) & \dots & -\binom{N-2}{2}2!\alpha_{n,2}\left(\widehat{p}^{[4]}_{n-2}\right)^{(N-4)}(c) \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & -\binom{N-2}{N-2}(N-2)!\alpha_{n,N-2}\widehat{p}^{[2N-4]}_{n-N+2}(c) \\ \widehat{p}_n(c) & \widehat{p}_n'(c) & \widehat{p}_n''(c) & \dots & \widehat{p}_n^{(N-2)}(c) \end{vmatrix} \\ &= (-1)^N(-1)^{N-2}\widehat{p}_n(c)\widehat{p}^{[2]}_{n-1}(c)\widehat{p}^{[4]}_{n-2}(c)\dots\widehat{p}^{[2N-4]}_{n-(N-2)}(c)1!2!\dots(N-2)!\alpha_{n,1}\dots\alpha_{n,N-2} \\ &= \alpha_{n,1}\alpha_{n,2}\dots\alpha_{n,N-2}1!2!\dots(N-2)!\widehat{p}_n(c)\widehat{p}^{[2]}_{n-1}(c)\widehat{p}^{[4]}_{n-2}(c)\dots\widehat{p}^{[2N-4]}_{n-(N-2)}(c) \neq 0.\end{aligned}$$

Thus we get the following result.

**Proposition 1.2.4.** *The lower triangular matrix $A^{(n)}_{N-1}$ is nonsingular. As a consequence*

$$\begin{pmatrix} K_n(x,c) \\ K_n^{(0,1)}(x,c) \\ \vdots \\ K_n^{(0,N-2)}(x,c) \end{pmatrix} = \{A^{(n)}_{N-1}\}^{-1} \begin{pmatrix} \widehat{p}_n(x) - \alpha_{n,1}(x-c)\widehat{p}^{[2]}_{n-1}(x) \\ \widehat{p}_n(x) - \alpha_{n,2}(x-c)^2\widehat{p}^{[4]}_{n-2}(x) \\ \vdots \\ \widehat{p}_n(x) - \alpha_{n,N-1}(x-c)^{N-1}\widehat{p}^{[2N-2]}_{n-N+1}(x) \end{pmatrix},$$

*or, equivalently,*

$$\begin{pmatrix} K_{n-1}(x,c) \\ K_{n-1}^{(0,1)}(x,c) \\ \vdots \\ K_{n-1}^{(0,N-2)}(x,c) \end{pmatrix} = \left[ \begin{pmatrix} \frac{-\widehat{p}_n(c)}{||\widehat{p}_n||^2} \\ \frac{-\widehat{p}_n'(c)}{||\widehat{p}_n||^2} \\ \vdots \\ \frac{-\widehat{p}_n^{(N-2)}(c)}{||\widehat{p}_n||^2} \end{pmatrix} + \{A_{N-1}^{(n)}\}^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right] \widehat{p}_n(x)$$
$$- \alpha_{n,1}\{A_{N-1}^{(n)}\}^{-1}(x-c)\vec{e}_1 \widehat{p}_{n-1}^{[2]}(x) - \alpha_{n,2}\{A_{N-1}^{(n)}\}^{-1}(x-c)^2 \vec{e}_2 \widehat{p}_{n-2}^{[4]}(x)$$
$$- \ldots - \alpha_{n,N-1}\{A_{N-1}^{(n)}\}^{-1}(x-c)^{N-1}\vec{e}_{N-1}\widehat{p}_{n-N+1}^{[2N-2]}(x), \tag{1.2.14}$$

*where*

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \vec{e}_{N-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Now, we are ready to formulate the result we was looking for.

**Proposition 1.2.5.** *(Connection formula type III.) With the previous notation, there exist real numbers $\{A_{n,j}\}_{j=0}^{N-1}$ such that*

$$B_n(x) = A_{n,0}\widehat{p}_n(x) + A_{n,1}(x-c)\widehat{p}_{n-1}^{[2]}(x)$$
$$+ A_{n,2}(x-c)^2\widehat{p}_{n-2}^{[4]}(x) + \ldots + A_{n,N-1}(x-c)^{N-1}\widehat{p}_{n-N+1}^{[2N-2]}(x).$$

*In fact, for $1 \leqslant j \leqslant N-1$,*

$$A_{n,j} = \alpha_{n,j} \begin{pmatrix} \widehat{p}_n(c) & \ldots & \widehat{p}_n^{(N-2)}(c) \end{pmatrix} (A_{N-1}^{(n)} M_{N-1}^{-1} + A_{N-1}^{(n)} \mathbb{K}_{n-1}^{(N-2)}(c,c))^{-1} \vec{e}_j,$$

*and*

$$A_{n,0} = 1 + \begin{pmatrix} \widehat{p}_n(c) & \ldots & \widehat{p}_n^{(N-2)}(c) \end{pmatrix} \left( A_{N-1}^{(n)} M_{N-1}^{-1} + A_{N-1}^{(n)} \mathbb{K}_{n-1}^{(N-2)}(c,c) \right)^{-1}$$
$$\left( A_{N-1}^{(n)} \begin{pmatrix} \frac{-\widehat{p}_n(c)}{||\widehat{p}_n||^2} \\ \frac{-\widehat{p}_n'(c)}{||\widehat{p}_n||^2} \\ \vdots \\ \frac{-\widehat{p}_n^{(N-2)}(c)}{||\widehat{p}_n||^2} \end{pmatrix} + \{A_{N-1}^{(n)}\}^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right).$$

*Proof.* The set $\{\widehat{p}_0(x),\ldots,\widehat{p}_n(x)\}$ is a basis of $\mathbb{P}_n$. Then, we can expand the Sobolev orthogonal polynomial $B_n$ in terms of them as follows

$$\widehat{B}_n(x)=\widehat{p}_n(x)+\sum_{j=0}^{n-1}\beta_{n,j}\widehat{p}_j(x), \tag{1.2.15}$$

where

$$\begin{aligned}\beta_{n,j}&=\frac{\int\widehat{B}_n(x)\widehat{p}_j(x)d\mu(x)}{||\widehat{p}_j||^2}\\&=-\frac{1}{||\widehat{p}_j||^2}\begin{pmatrix}\widehat{B}_n(c) & \widehat{B}_n'(c) & \ldots & \widehat{B}_n^{(N-2)}(c)\end{pmatrix}M_{N-1}\begin{pmatrix}\widehat{p}_j(c)\\ \widehat{p}_j'(c)\\ \vdots\\ \widehat{p}_j^{(N-2)}(c)\end{pmatrix}.\end{aligned}$$

Putting the previous expression into (1.2.15), we obtain

$$\widehat{B}_n(x)=\widehat{p}_n(x)-\begin{pmatrix}\widehat{B}_n(c) & \ldots & \widehat{B}_n^{(N-2)}(c)\end{pmatrix}M_{N-1}\begin{pmatrix}K_{n-1}(x,c)\\ K_{n-1}'(x,c)\\ \vdots\\ K_{n-1}^{(0,N-2)}(x,c)\end{pmatrix}.$$

Taking derivatives up to order $N-2$ and evaluating at $x=c$ we obtain a system that can be written in matrix form as follows,

$$\begin{pmatrix}\widehat{B}_n(c) & \ldots & \widehat{B}_n^{(N-2)}(c)\end{pmatrix}\left[I+M_{N-1}\mathbb{K}_{n-1}^{(N-2)}(c,c)\right]=\begin{pmatrix}\widehat{p}_n(c) & \ldots & \widehat{p}_n^{(N-2)}(c)\end{pmatrix}.$$

Finally, by using (1.2.14) we get

$$\begin{aligned}\widehat{B}_n(x)&=\widehat{p}_n(x)-\begin{pmatrix}\widehat{p}_n(c) & \ldots & \widehat{p}_n^{(N-2)}(c)\end{pmatrix}\left[I+M_{N-1}\mathbb{K}_{n-1}^{(N-2)}(c,c)\right]^{-1}M_{N-1}\begin{pmatrix}K_{n-1}(x,c)\\ \vdots\\ K_{n-1}^{(0,N-2)}(x,c)\end{pmatrix}\\&=\widehat{p}_n(x)-\begin{pmatrix}\widehat{p}_n(c) & \ldots & \widehat{p}_n^{(N-2)}(c)\end{pmatrix}[M_{N-1}^{-1}+\mathbb{K}_{n-1}^{(N-2)}(c,c)]^{-1}\left[\begin{pmatrix}\frac{-\widehat{p}_n(c)}{||\widehat{p}_n||^2}\\ \frac{-\widehat{p}_n'(c)}{||\widehat{p}_n||^2}\\ \vdots\\ \frac{-\widehat{p}_n^{(N-2)}(c)}{||\widehat{p}_n||^2}\end{pmatrix}+\{A_{N-1}^{(n)}\}^{-1}\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}\right]\widehat{p}_n(x)\\&\quad-\alpha_{n,1}\{A_{N-1}^{(n)}\}^{-1}(x-c)\vec{e}_1\widehat{p}_{n-1}^{[2]}(x)-\alpha_{n,2}\{A_{N-1}^{(n)}\}^{-1}(x-c)^2\vec{e}_2\widehat{p}_{n-2}^{[4]}(x)\\&\quad-\ldots-\alpha_{n,N-1}\{A_{N-1}^{(n)}\}^{-1}(x-c)^{N-1}\vec{e}_{N-1}\widehat{p}_{n-N+1}^{[2N-2]}(x)\\&=A_{n,0}\widehat{p}_n(x)+A_{n,1}(x-c)\widehat{p}_{n-1}^{[2]}(x)+A_{n,2}(x-c)^2\widehat{p}_{n-2}^{[4]}(x)+\ldots+A_{n,N-1}(x-c)^{N-1}\widehat{p}_{n-N+1}^{[2N-2]}(x)\end{aligned}$$

where $\{A_{n,j}\}_{j=0}^{N-1}$ can be given in an explicit way.

□

## 1.3 Asymptotics

Our aim in this section is to describe some known results about asymptotics of Sobolev polynomials to provide a historical framework for our contributions. As in the standard case, three types of asymptotics for Sobolev polynomials have been studied: strong asymptotics, $n$th root asymptotics and ratio asymptotics. We will focus on the so called Outer Relative Asymptotics of orthogonal polynomials with respect to the inner product (1.0.2). More precisely, we compare the asymptotic behaviour of the Sobolev-type orthogonal polynomials with that of the standard ones by studying the limit

$$\lim_{n\to\infty} \frac{B_n(x)}{p_n(x)}, \tag{1.3.16}$$

where $x$ does not belong to the support of the measure $\mu$.

The key idea for obtaining Outer Relative Asymptotics is to use the connection formulas type I, II or III for the Sobolev-type orthogonal polynomials in the numerator of (1.3.16). Then, the problem has been reduced to the study of the ratio asymptotics

$$\lim_{n\to\infty} \frac{\tilde{p}_n(x)}{p_{n-1}(x)}, \quad \mathbb{C}\backslash \mathit{supp}\,(d\mu),$$

where $\tilde{p}_n(x)$ could be either the $n$-th standard polynomial or another polynomial sequence orthogonal with respect to perturbations of the measure $\mu$ appearing in the connection formulas.

The techniques for the study of the problem change depending on the fact of the support of the measure $\mu$ is a bounded or unbounded set of the real line. For this reason, we will divide our analysis in two subsections.

### 1.3.1 Bounded support

As we already mention, we will be interested in the study of families of polynomials having ratio asymptotics. In the bounded case, a widely studied class of orthogonal polynomials is the Nevai class of measures $M(0,1)$ (see [77]). By simplicity, we assume that the support of the measure $\mu$ is $[-1,1]$. More precisely, $\mu$ will be in the *Nevai class of measures* $M(0,1)$, i.e. the recurrence coefficients in the three term recurrence formula (1.1.3) satisfy

$$\lim_{n\to\infty} b_n = 0, \quad \lim_{n\to\infty} a_n = \frac{1}{2}.$$

The Outer Ratio Asymptotics for measures $\mu$ in the Nevai class reads as follows.

**Theorem 1.3.1.** *(Outer Ratio Asymptotics) Let $\mu \in M(0,1)$. Then*

$$\lim_{n\to\infty} \frac{p_n(x)}{p_{n-1}(x)} = \rho(x),$$

*uniformly on compact subsets of $\mathbb{C}\backslash supp(d\mu)$, where for $z \in \mathbb{C}\backslash[-1,1]$ we define $\rho(z)$ by*

$$\rho(z) = z + \sqrt{z^2-1},$$

*where we take the branch of $\sqrt{z^2-1}$ for which $|\rho(z)| > 1$ whenever $z \in \mathbb{C}\backslash[-1,1]$.*

As for the case of bounded support is concerned, we restrict ourselves to the case of asymptotics for Sobolev polynomials orthogonal with respect to discrete Sobolev inner products. As far as we know, the first paper in relative asymptotics for Sobolev polynomials was [66], where the authors worked with an inner product in which first derivatives appear. Some years later, in [47], G. López Lagomasino, F. Marcellán and W. Van Assche investigated the asymptotic properties for a class of polynomials orthogonal with respect to a family of inner products that includes the discrete Sobolev inner products with a finite number of complex masses located outside the support of the measure. Since then many authors have made contributions in this area and for a historical review of this period the reader is referred to [71]. In Chapter 2, we study a result on Relative Asymptotics of Sobolev-type orthogonal polynomials on bounded support. Now, we only summarize the results necessary for understanding this contribution.

It is well known that the Outer Relative Asymptotics changes according to the location of the mass points with respect to the support of the measure. We distinguish two cases:

1. The case of several mass points located inside the support of the measure was studied in [90]. Indeed, when one adds mass points inside the support of the measure, the asymptotic behavior of the orthogonal polynomials does not change. The result reads as follows.

   **Theorem 1.3.2.** *If $\mu' > 0$ a.e. then*

   $$\lim_{n\to\infty} \frac{B_n(x)}{p_n(x)} = 1$$

   *uniformly on compact subsets of $\mathbb{C}\backslash[-1,1]$.*

2. The situation is very different when we add mass points outside the support of the measure. The parallel study was done in [57] for the particular case $K = 1$, $N_1 = 1$. In this case, we will refer to $c_1$ with $c$.

We will denote by $\{\tilde{q_n}(x)\}_{n=0}^{\infty}$ and $\{q_n(x)\}_{n=0}^{\infty}$ the orthonormal sequences with respect to $(c-x)d\mu(x)$ and $(c-x)^2 d\mu(x)$ respectively.

The Outer Relative Asymptotics depending on the positiveness of the coefficients in the discrete part of the inner product is the following:

**Theorem 1.3.3.** *Under the previous conditions we have*

- *For* $M > 0, N > 0$,
$$\lim_{n\to\infty} \frac{B_n(x)}{q_n(x)} = \frac{\rho^{-1}(c)}{2}\left(1 - \frac{\rho(c)}{\rho(x)}\right)^2$$
*uniformly in compact sets of* $\mathbb{C}\backslash[-1,1]$.
- *For* $N = 0$,
$$\lim_{n\to\infty} \frac{B_n(x)}{\tilde{q}_n(x)} = \left(\frac{\rho^{-1}(c)}{2}\right)^{1/2}\left(1 - \frac{\rho(c)}{\rho(x)}\right)$$
*uniformly in compact sets of* $\mathbb{C}\backslash[-1,1]$.
- *For* $M = 0$,
$$\lim_{n\to\infty} \frac{B_n(x)}{\tilde{q}_n(x)} = \frac{1}{2}\left(1 - \frac{\rho(c)}{\rho(x)}\right)\left(1 - \frac{\rho^{-1}(c)}{\rho(x)}\right)$$
*uniformly on compact sets of* $\mathbb{C}\backslash[-1,1]$.

As we already mention, we will study Relative Asymptotics for certain Sobolev type orthogonal polynomials. We will work with an inner product where the measure $\mu$ is assumed to belong to the Nevai class $M(0,1)$. It is well known that the support of measures in that class is bounded and consists of the interval $[-1,1]$ plus a denumerable set of isolated points whose accumulation points are $\pm 1$. Then, using techniques given in [90], [57] and [91], we study the case of an inner product with mass points located inside the support of the measure but outside $[-1,1]$.

### 1.3.2 Unbounded support

As far as we concern, there are no so many results in asymptotics for Sobolev-type inner products when the measure $\mu$ has an unbounded support. Despite the examples related to Laguerre weights or Hermite weights very few examples are known. For the historical context in this subject, we will consider a more general family of Sobolev inner products. More precisely, we will deal with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)d\mu_0(x) + \sum_{k=1}^{m} \int_{\mathbb{R}} f^{(k)}(x)g^{(k)}(x)d\mu_k(x), \tag{1.3.17}$$

where $d\mu_k$, $k = 0, 1, \dots, m$, are positive Borel measures supported on $\mathbb{R}$. We distinguish the following cases:

- All the measures $d\mu_k$ have a non-discrete set as support. In such a case we speak of *continuous Sobolev inner product* or just continuous case.
- $d\mu_0$ has continuous support and $d\mu_1, \dots, d\mu_m$ are supported in finite sets. In this case, we are dealing with a *discrete Sobolev inner product* or *Sobolev type inner product*.

In the case of continuous Sobolev inner products, for the case $m = 1$ we can find three contributions [55, 74, 81] which can be situated in the context of coherence of measures, all of them with Laguerre and Hermite measures. In [12], the authors studied the case of $d\mu_0(x) = e^{-x^4}dx$ and $d\mu_1(x) = \lambda e^{-x^4}dx$. In particular, Relative Asymptotics for Sobolev orthogonal polynomials in terms of Freud polynomials associated with the weight function $\omega(x) = e^{-x^4}$ was deduced. In a more general context, in the case $m = 1$ different types of asymptotic behavior of Sobolev orthogonal polynomials with exponential weights were analyzed (see [34]). For a more detailed description on the subject, the reader can see [56].

However, we will be interested in the case of discrete Sobolev inner products. As a representative example, we focus our attention on the particular case of inner product (1.0.2) when the measure $\mu$ is the classical Laguerre measure, i.e., the Gamma distribution.

In [18], H. Dueñas and F. Marcellán proved that the addition of a mass point at zero does not modify the asymptotic behavior of these Sobolev orthogonal polynomials. More precisely, they work with a non-diagonal inner product with only one mass point at zero and their main results read:

**Theorem 1.3.4.** *With the previous notation,*

$$\lim_{n\to\infty} \frac{\widehat{B}_n(x)}{\widehat{p}_n(x)} = 1$$

*uniformly on compact subsets of* $\mathbb{C}\setminus[0,\infty)$.

In the same way as happens in bounded support, the addition of mass points outside the support of the measure $\mu$ modifies the asymptotic behavior of the Sobolev orthogonal polynomials. This case was studied for the first time in [37] and with our notation, the results in outer relative asymptotics read as follows.

**Theorem 1.3.5.** *[37, Theorem 5.6] Let us denote by* $L_n^{(\alpha)}(x)$ *and* $B_n^{(\alpha)}(x)$ *the Laguerre and Laguerre Sobolev type polynomials with leading coefficient equal to* $\frac{(-1)^n}{n!}$*, respectively. For an inner product with* $K = 1$ *and* $N_1 = 1$*, we get*

- *If* $M > 0$ *and* $N > 0$, *then*

$$\lim_{n\to\infty} \frac{B_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = \left(\frac{\sqrt{-x}-\sqrt{|c|}}{\sqrt{-x}+\sqrt{|c|}}\right)^2,$$

*uniformly on compact subsets of* $\mathbb{C}\backslash[0,\infty)$.

- *If* $M = 0$ *and* $N > 0$ *or* $M > 0$ *and* $N = 0$, *then*

$$\lim_{n\to\infty} \frac{B_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = \frac{\sqrt{-x}-\sqrt{|c|}}{\sqrt{-x}+\sqrt{|c|}},$$

*uniformly on compact subsets of* $\mathbb{C}\backslash[0,\infty)$.

The case of addition of mass points inside the support of the Laguerre measure was unknown. In Chapter 3 we will deal with the study of asymptotics for Sobolev type polynomials orthogonal with respect to (1.0.2) when the mass points $c_k$ are located inside the oscillatory regime of the polynomials.

## 1.4 Fourier series

### 1.4.1 Pointwise convergence

The pointwise convergence of Fourier series in terms of standard orthogonal polynomials has been studied, for instance, in the particular case when $\mu$ is the Jacobi measure. The problem now is to determine under which conditions a function $f$ satisfies

$$\sum_{n=0}^{\infty} \langle f, B_n\rangle B_n(x_0) = f(x_0), \text{ with } x_0 \in (-1,1), \quad \text{(Pointwise convergence)}$$

where $\{B_n(x)\}_{n=0}^{\infty}$ is the sequence of orthonormal polynomials with respect to (1.0.2).

Moreover, we will study the convergence of

$$\sum_{n=0}^{\infty} \langle f, B_n\rangle B_n^{(i)}(c_k) = f^{(i)}(c_k)$$

when $M_{k,i} > 0$.

For the general case of the inner product (1.0.2), when $\mu$ is the Jacobi measure and all the mass points are located inside the support of the measure $\mu$, the problem was solved by I. A. Rocha, F. Marcellán and L. Salto in [90].

For the case of only one mass point ($K = 1$) located outside the support of the Jacobi measure, the same result can be found in [57].

In both cases, the authors found two kind of conditions about the function $f$ in order to obtain the pointwise convergence of the Fourier-Sobolev series. The result reads as follows.

**Theorem 1.4.1.** *Let* $x_0 \in (-1,1)\backslash \cup_{i=1}^{K} \{c_i\}$ *and let* $f$ *be a function with derivatives at the points* $c_k$ *such that* $\frac{f(x_0)-f(t)}{x_0-t}$ *belongs to* $L^2(\mu_{\alpha,\beta})$. *Then*

$$\sum_{n=0}^{\infty} \langle f, B_n \rangle B_n(x_0) = f(x_0).$$

*If* $M_{k,i} > 0$, *then*

$$\sum_{n=0}^{\infty} \langle f, B_n \rangle B_n^{(i)}(c_k) = f^{(i)}(c_k)$$

**Theorem 1.4.2.** *Let* $f(x)$ *be a function with derivatives at the points* $a_k$ *satisfying a Lipschitz condition of order* $0 < \eta < 1$ *uniformly in* $[-1,1]$, *i.e.* $|f(x+h) - f(x)| \leqslant M|h|^{\eta}$ *for* $|h| < \delta$ *and for some* $\delta > 0$. *Then*

$$\sum_{n=0}^{\infty} \langle f, B_n \rangle B_n(x) = f(x), \quad x \in (-1,1),$$

*and the convergence is uniform in compact subsets of* $(-1,1)\backslash \cup_{k=1}^{K} \{c_k\}$. *Moreover, at the mass points,*

$$\sum_{n=0}^{\infty} \langle f, B_n \rangle B_n^{(i)}(c_k) = f^{(i)}(c_k)$$

*provided that* $M_{k,i} > 0$.

### 1.4.2 Cohen-type inequalities

In this section, we present Cohen-type inequalities as a tool in order to prove divergence of the Fourier series of orthogonal polynomials in weighted $L^p$ spaces.

The study of Cohen type inequalities began in the sixties of the previous century and its seminal goal was to prove a conjecture stated by Littlewood in 1948 (see [36]) related to find an estimate from below for the $L^1$ norm of a trigonometric polynomial. More precisely, Littlewood conjectured that for any trigonometric polynomial $F_K(x) = \sum_{k=1}^{K} a_k e^{in_k x}$, where $0 < n_1 < \cdots < n_K$, $K \geqslant 2$, and $|a_k| \geqslant 1$ for $1 \leqslant k \leqslant K$, then the following estimate holds

$$\int_0^{2\pi} |F_K(x)| dx \geqslant C \log K. \tag{1.4.18}$$

Here $C$ is a universal constant. Cohen [14] proved that $\frac{C}{8}\frac{\log K}{\log\log K}$ is a lower bound for the norm in (1.4.18).

Since then Cohen type inequalities have been investigated by many authors in various contexts and forms (cf. [22, 23, 35, 69, 70, 80].) Even though Dreseler and Soardi [22, 23] seem to be the first people who found Cohen type inequalities in the setting of Jacobi expansions, it is worthwhile to point out that is due to Markett [70] the presentation of an approach admitting a simpler proof of Dreseler and Soardi result for Jacobi expansions and stating the corresponding Cohen type inequalities for Laguerre and Hermite expansions.

In the sequel, we will be interested in the Laguerre case, so we are going to focus this section to analyze the Markett approach for Fourier series associated with classical Laguerre polynomials. For more details, see [70].

This author considers two weighted $L^p$ spaces in his works, namely classical weighted Lebesgue spaces:

- $$L^p_{w(\alpha)} = \begin{cases} \{f : \{\int_0^\infty |f(x)e^{-x/2}|^p x^\alpha dx\}^{1/p} < \infty\}, & 1 \leqslant p < \infty, \\ \left\{f : \text{ess sup}_{x>0}|f(x)e^{-x/2}| < \infty\right\}, & p = \infty, \end{cases}$$

  for $\alpha > -1$. Furthermore,

- $$L^p_{u(\alpha)} = \{f : ||f(x)u(x,\alpha)||_{L^p(0,\infty)} < \infty, u(x,\alpha) = e^{-x/2}x^{\alpha/2}\},$$

  where $\alpha > -\frac{2}{p}$ if $1 \leqslant p < \infty$ and $\alpha \geqslant 0$ if $p = \infty$.

Also, we use the notation $L^p_{f(\alpha)}$, where the subscript $f(\alpha)$ means either $w(\alpha)$ or $u(\alpha)$.

For $\alpha > -1$, let $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ be the sequences of normalized Laguerre polynomials with leading coefficient equal to $\frac{(-1)^n}{n!}$.

For $f \in L^p_{f(\alpha)}$ and $\{c_{k,n}\}_{k=0}^n, n \in \mathbb{N} \cup \{0\}$, a family of complex numbers with $|c_{n,n}| > 0$, let introduce the generalized partial sum operators $T_n^\alpha$

$$T_n^\alpha(f) := \sum_{k=0}^n c_{k,n}\hat{f}(k)L_k^{(\alpha)},$$

where $\hat{f}(k) = \left\{\int_0^\infty \left(L_k^{(\alpha)}(x)\right)^2 x^\alpha e^{-x}dx\right\}^{-1} \int_0^\infty f(x)L_k^{(\alpha)}(x)x^\alpha e^{-x}dx, \quad k = 0, 1, \ldots$

Let us denote $q_0 = \frac{4\alpha+4}{2\alpha+1}$, when $\beta = \alpha$, and $q_0 = 4$, when $\beta = p\alpha/2$, and let $p_0$ be the conjugate of $q_0$. The Cohen-type inequality for the family of classical Laguerre polynomials is given by the following.

**Theorem 1.4.3.** *Let* $1 \leqslant p \leqslant \infty$. *For* $\alpha > -1/2$,

$$||T_n^\alpha||_{[L^p_{w(\alpha)}]} \geqslant C|c_{n,n}| \begin{cases} n^{\frac{2\alpha+2}{p}-\frac{2\alpha+3}{2}} & \text{if } a \leqslant p < p_0, \\ (\log n)^{\frac{2\alpha+1}{4\alpha+4}} & \text{if } p = p_0, p = q_0, \\ n^{\frac{2\alpha+1}{2}-\frac{2\alpha+2}{p}} & \text{if } q_0 < p \leqslant b. \end{cases}$$

*For* $\alpha > -2/p$ *if* $1 \leqslant p < \infty$ *and* $\alpha \geqslant 0$ *if* $p = \infty$,

$$||T_n^\alpha||_{[L^p_{u(\alpha)}]} \geqslant C|c_{n,n}| \begin{cases} n^{\frac{2}{p}-\frac{3}{2}} & \text{if } a \leqslant p < p_0, \\ (\log n)^{\frac{1}{4}} & \text{if } p = p_0, p = q_0, \\ n^{\frac{1}{2}-\frac{2}{p}} & \text{if } q_0 < p \leqslant b. \end{cases}$$

The main ideas for the proof of this result are the following:

- We can choose an appropriate test function $g_n^{\alpha,j}$ in order to obtain the inequality
$$||T_n^\alpha||_{[L^p_{f(\alpha)}]} = \sup_{0\neq f\in L^p_{f(\alpha)}} \frac{||T_n^\alpha(f)||_{L^p_{f(\alpha)}}}{||f||_{L^p_{f(\alpha)}}} \geqslant \left(||g_n^{\alpha,j}||_{L^p_{f(\alpha)}}\right)^{-1} ||T_n^\alpha(g_n^{\alpha,j})||_{L^p_{f(\alpha)}}.$$
The idea consists to pick out a test function that will allow to reach the Cohen-type inequality.
- The choice of Markett was
$$g_n^{\alpha,j}(x) := n^{-\alpha/2}\left[x^j L_n^{(\alpha+j)}(x) - \left(\frac{(n+1)(n+2)}{(n+\alpha+j+1)(n+\alpha+j+2)}\right)^{1/2} x^j L_{n+2}^{(\alpha+j)}(x)\right],$$
and the important characteristics of these functions are:
  - They simplify the expression of the generalized partial sum operators in the following sense
  $$T_n^\alpha(g_n^{\alpha,j}) = c_{n,n}\hat{g_n^{\alpha,j}}(n)L_n^{(\alpha)},$$
  i.e., only the last term of the sum survives.
  - We can estimate $||g_n^{\alpha,j}||_{L^p_{f(\alpha)}}$.

In the setting of Sobolev orthogonality the study of Cohen type inequalities is most recent and it has attracted considerable attention, mainly when it is possible to use the same (up to constant factor) previous test functions. For instance, the authors of [30, 83] have obtained Cohen type inequalities for Laguerre orthonormal expansions with respect to discrete Sobolev inner products with only one mass point at $c = 0$. Similar results for Laguerre orthonormal expansions with respect to a non-discrete Sobolev inner product appear in [27].

## 1.5 Zeros

It is well known that the zeros of $\{p_n(x)\}_{n=0}^{\infty}$ are real, simple and are located in the interior of the convex hull of the support of $\mu$. They will be denoted by $x_{nk} : x_{n1} < x_{n2} < \ldots < x_{nn}$.

Due to the fact we are working with a non standard inner product, the Sobolev orthogonal polynomials lose this property. For instance, for polynomials orthogonal with respect to the Sobolev-type inner product (1.0.2) with $[-1, 1]$ as support of the measure $\mu$, it has been proved that $n-N$ zeros of $B_n(x)$ belong to $[-1, 1]$ and the other $N$ zeros accumulate in $[-1, 1]$. For the case of unbounded support, parallel results can be found. For more details on this subject see [67].

## 1.6 Classical Laguerre polynomials

In the study of all aspects of Sobolev polynomials is quite frequent to explore for the first time the case when $\mu$ is a classical measure. There are three families of classical polynomials: Jacobi, Hermite and Laguerre polynomials. For the computations in the following chapters we will need to review the properties of Laguerre polynomials.

Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle f, g \rangle_\alpha = \int_0^\infty f(x)g(x)\, x^\alpha e^{-x} dx, \quad \alpha > -1, \quad f, g \in \mathbb{P}. \tag{1.6.19}$$

The expression of these polynomials as an ${}_1F_1$ hypergeometric function is very well known in the literature (see for instance, [41, 79, 95]). The connection between these two facts follows from a characterization of such orthogonal polynomials as eigenfunctions of a second order linear differential operator with polynomial coefficients.

For $\alpha > -1$, let $\{\widehat{L}_n^\alpha(x)\}_{n=0}^\infty$, $\{L_n^\alpha(x)\}_{n=0}^\infty$, and $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$ be the sequences of monic, orthonormal and normalized Laguerre polynomials with leading coefficient equal to $\frac{(-1)^n}{n!}$, respectively. The following proposition summarizes some structural and asymptotic properties of the classical Laguerre polynomials (see [38, 39, 68] and the references therein.)

**Proposition 1.6.1.** *The following statements hold.*

1. *Three-term recurrence relation. For every* $n \geqslant 1$

$$x\widehat{L}_n^\alpha(x) = \widehat{L}_{n+1}^\alpha(x) + \beta_n \widehat{L}_n^\alpha(x) + \gamma_n \widehat{L}_{n-1}^\alpha(x), \tag{1.6.20}$$

*with initial conditions* $\widehat{L}_0^\alpha(x) = 1$, $\widehat{L}_1^\alpha(x) = x - (\alpha + 1)$, *and* $\beta_n = 2n + \alpha + 1$, $\gamma_n = n(n + \alpha)$.

2. *[95, p. 102] They satisfy the structure relation:*

$$L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x).$$

3. *For every $n \in \mathbb{N}$,*

$$||\widehat{L}_n^{\alpha}||_{\alpha}^2 = \Gamma(n+1)\Gamma(n+\alpha+1). \tag{1.6.21}$$

4. *Hahn's condition. For every $n \in \mathbb{N}$,*

$$[\widehat{L}_n^{\alpha}(x)]' = n\widehat{L}_{n-1}^{\alpha+1}(x). \tag{1.6.22}$$

5. *The $n$-th Dirichlet kernel $K_n(x,y)$, given by*

$$K_n(x,y) = \sum_{k=0}^{n} \frac{\widehat{L}_k^{\alpha}(x)\widehat{L}_k^{\alpha}(y)}{||\widehat{L}_k^{\alpha}||_{\alpha}^2}, \tag{1.6.23}$$

*satisfies the Christoffel-Darboux formula (cf. [95, Theorem 3.2.2]):*

$$K_n(x,y) = \frac{1}{||\widehat{L}_n^{\alpha}||_{\alpha}^2} \left( \frac{\widehat{L}_{n+1}^{\alpha}(x)\widehat{L}_n^{\alpha}(y) - \widehat{L}_n^{\alpha}(x)\widehat{L}_{n+1}^{\alpha}(y)}{(x-y)} \right), \quad n \geqslant 0. \tag{1.6.24}$$

6. *The so called confluent form of the above kernel is given by*

$$K_n(x,x) = \frac{1}{||\widehat{L}_n^{\alpha}||_{\alpha}^2} \left\{ [\widehat{L}_{n+1}^{\alpha}]'(x)\widehat{L}_n^{\alpha}(x) - [\widehat{L}_n^{\alpha}]'(x)\widehat{L}_{n+1}^{\alpha}(x) \right\}, \quad n \geqslant 0. \tag{1.6.25}$$

7. *[95, Theorem 8.22.3] (Outer strong asymptotics or Perron asymptotics formula on $\mathbb{C}\backslash\mathbb{R}_+$). Let $\alpha \in \mathbb{R}$, then*

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{2}\pi^{-1/2} e^{x/2} (-x)^{-\alpha/2-1/4} n^{\alpha/2-1/4} \exp\left(2(-nx)^{1/2}\right) \\ &\times \left\{ \sum_{k=0}^{p-1} C_k(\alpha;x) n^{-k/2} + \mathcal{O}(n^{-p/2}) \right\}. \end{aligned} \tag{1.6.26}$$

*Here $C_k(\alpha;x)$ is independent of $n$. This relation holds for $x$ in the complex plane with a cut along the positive real semiaxis, and it also holds if $x$ is in the cut plane mentioned. $(-x)^{-\alpha/2-1/4}$ and $(-x)^{1/2}$ must be taken real and positive if $x < 0$. The bound for the remainder holds uniformly in every compact subset of the complex plane with empty intersection with $\mathbb{R}_+$.*

8. *[95, Theorem 8.22.2] (Perron generalization of Fejér formula on $\mathbb{R}_+$. Let $\alpha \in \mathbb{R}$). Then for $x > 0$ we have*

$$\begin{aligned} L_n^{(\alpha)}(x) &= \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \cos\{2\,(nx)^{1/2} - \alpha\pi/2 - \pi/4\} \\ &\quad \cdot \left\{ \sum_{k=0}^{p-1} A_k(x) n^{-k/2} + \mathcal{O}(n^{-p/2}) \right\} \\ &+ \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \sin\{2\,(nx)^{1/2} - \alpha\pi/2 - \pi/4\} \\ &\quad \cdot \left\{ \sum_{k=0}^{p-1} B_k(x) n^{-k/2} + \mathcal{O}(n^{-p/2}) \right\}, \end{aligned} \tag{1.6.27}$$

*where $A_k(x)$ and $B_k(x)$ are certain functions of $x$ independent of $n$ and regular for $x > 0$. The bound for the remainder holds uniformly in $[\epsilon, \omega]$. For $k = 0$ we have $A_0(x) = 1$ and $B_0(x) = 0$.*

9. *[15, Theorem 2] (Alternative outer strong asymptotics). Let $\alpha > -1$. The Laguerre polynomial $L_n^{(\alpha)}(x)$ admits the following asymptotic expansion as $n \to \infty$ :*

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{2}\pi^{-1/2} \frac{\Gamma(n+\alpha+1)}{n!} e^{x/2} (-\kappa x)^{-\alpha/2-1/4} \exp\left(2\,(-\kappa x)^{1/2}\right) \\ &\quad \times \left\{ \sum_{k=0}^{d-1} \hat{B}_m(\alpha, x) n^{-m/2} + O(n^{-d/2}) \right\}, \end{aligned}$$

*for some coefficients $\hat{B}_m(\alpha, x)$ independent of $n$ and $\kappa = \kappa(n, \alpha) = \frac{\alpha+1}{2} + n$. This modified expansion holds for $x$ in the complex plane with a cut along the positive real semiaxis, and it also holds if $x$ is in the cut plane mentioned. The bound for the remainder holds uniformly in every compact subset of the complex plane with empty intersection with $\mathbb{R}_+$.*

10. *For every $n \in \mathbb{N}$,*

$$h_n^{(\alpha)} := \int_0^\infty [L_n^{(\alpha)}(x)]^2 d\mu(x) \sim n^\alpha.$$

11. *[95, Theorem 8.1.3] Mehler-Heine type formula. For a fixed $j$, with $j \in \mathbb{N} \cup \{0\}$, if $J_\alpha$ denotes the Bessel function of the first kind, then*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x/(n+j))}{n^\alpha} = x^{-\alpha/2} J_\alpha\left(2\sqrt{x}\right), \tag{1.6.28}$$

*uniformly on compact subsets of $\mathbb{C}$.*

12. *[4, formula (1.10)] Ratio asymptotics for scaled Laguerre polynomials:*

$$\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = -\frac{1}{\phi((x-2)/2)}$$

*holds uniformly on compact subsets of* $\mathbb{C}\backslash[0,4]$ *and uniformly on* $j \in \mathbb{N} \cup \{0\}$, *where* $\phi$ *is conformal mapping of* $\mathbb{C}\backslash[-1,1]$ *onto the exterior of the unit circle given by*

$$\phi(x) = x + \sqrt{x^2 - 1}, \quad x \in \mathbb{C}\backslash[-1,1],$$

*with* $\sqrt{x^2-1} > 0$ *when* $x > 1$.

# 2

# Sobolev type polynomials on bounded support

In this Chapter we study some analytic and algebraic properties of Sobolev-type polynomials with respect to nontrivial probability measures with a bounded support on the real line.

In Section 2.1 we analyze the outer relative symptotics for a family of Sobolev-polynomials orthogonal with respect to an inner product of the form (1.0.2) with the measure $\mu$ in the Nevai class and the mass points located outside the support of the measure. In Section 2.2 we deduce a new matrix interpretation of the recurrence relation satisfied by the Sobolev orthogonal polynomial sequence in terms of a matrix polynomial of the Jacobi matrix associated with the sequence of orthonormal polynomials $\{p_n(x)\}_{n=0}^{\infty}$. The analysis of the connection coefficients for such sequences constitutes a basic tool for such an approach. Finaly, in Section 2.3 we study the pointwise convergence of the Fourier series associated with a family of Sobolev polinomials orthogonal with respect to a Jacobi-Sobolev inner product with several mass points outside the support of the measure.

## 2.1 Some background on asymptotics

Let $\mu$ be a finite positive Borel measure supported on the interval $[-1, 1]$ with infinitely many points at the support and let $b_k, k = 1, \ldots, K$, be real numbers located outside $[-1, 1]$. For $f$ and $g$ in $L^2(\mu) \cap C^{\infty}[-1, 1]$ such that there exist the derivatives at $b_k$, we can introduce the Sobolev-type inner product

$$\langle f, g\rangle = (f, g) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(b_k) g^{(i)}(b_k), \tag{2.1.1}$$

with $(f, g) = \int f(x)g(x)d\mu(x)$, $M_{k,i} \in \mathbb{C}$, and $M_{k,N_k} \neq 0$. For convenience, we work with another normalization of the Sobolev orthogonal polynomials. Let $\{\tilde{B}_n(x)\}_{n=0}^{\infty}$ be the sequence of monic polynomials of least degree such that

$$\langle \tilde{B}_n, p\rangle = 0, \quad p \in \mathbb{P}_{n-1},$$

where $\mathbb{P}_{n-1}$ is the linear space of all polynomials with complex coefficients of degree less than or equal to $n-1$. The existence of $\tilde{B}_n \in \mathbb{P}_n$ for each $n \in \mathbb{Z}_+$ follows from the solution of a system of $n$ linear homogeneous equations and $n+1$ unknowns. Since $\int |\tilde{B}_n(x)|^2 d\mu(x) = 1/\tau_n^2 > 0$, we can define $\hat{B}_n(x) = \tau_n \tilde{B}_n(x)$ and we have a sequence $\{\hat{B}_n(x)\}_{n=0}^{\infty}$ such that

$$deg(\hat{B}_n) \leqslant n, \quad \langle \hat{B}_n, p\rangle = 0, \quad p \in \mathbb{P}_{n-1}, \quad \int |\hat{B}_n(x)|^2 d\mu(x) = 1. \tag{2.1.2}$$

It is clear that the polynomials $\hat{B}_n$ are not orthonormal with respect to (4.1.11), but it is possible to prove that, for $\mu$ belonging to the Nevai class $M(0,1)$ and $n$ large enough, they are equal up to constant factors $\alpha_n$, with $\lim_{n\to\infty} \alpha_n = 1$. More precisely,

**Lemma 2.1.1.** *[7, Lemma 2.2] For $\mu \in M(0,1)$, the polynomials $\hat{B}_n$ satisfy the conditions*

*(i) If $M_{k,i} \neq 0$, then $\lim_{n\to\infty} \hat{B}_n^{(i)}(b_k) = 0$.*

*(ii) $\lim_{n\to\infty} \langle \hat{B}_n, \hat{B}_n\rangle = 1$.*

*(iii) There exists a positive integer $n_0$ such that $deg(\hat{B}_n) = n$ for all $n \geqslant n_0$.*

Recall that we denote by $\{p_n(x)\}_{n=0}^{\infty}$ the sequence of orthonormal polynomials with respect to $\mu$. In what follows, we assume that either $\mu' > 0$ a.e. on the support of $\mu$ or $\mu \in M(0,1)$ with the additional assumption that none of the mass points $b_k$ belong to the support of the measure $\mu$ (cf. [77, 87, 88]). Let us consider $N = \sum_{k=1}^{K}(N_k+1)$ and the polynomial

$$\omega_N(x) := \prod_{k=1}^{K} (x - b_k)^{N_k+1}. \tag{2.1.3}$$

Let $I_k$ be the number of coefficients $M_{k,j}, j = 0, \ldots, N_k$, different from 0 in (4.1.10) and let $J_k$ be such that $I_k + J_k = N_k + 1$.

In order to deduce asymptotic properties for the polynomials $\hat{B}_n$, a successful strategy is to find orthogonality relations involving the polynomials $\hat{B}_n$, $p_n$, $\omega_N$, and the monomials $(x-b_k)^m$, for $m \in \{1,\dots,I_k\} \cup \{1,\dots,J_k\}$. Concerning this issue, we have the following result.

**Lemma 2.1.2.** *(i) For* $m = 1,\dots,I_k$,

$$\lim_{n\to\infty} \int \frac{\hat{B}_n(x)p_n(x)}{(x-b_k)^m} d\mu(x) = 0.$$

*(ii) If* $J_k > 0$, *then*

$$\lim_{n\to\infty} \int \frac{\omega_N(x)\hat{B}_n(x)p_{n-N}(x)}{(x-b_k)^m} d\mu(x) = 0, \quad m = 1,\dots,J_k.$$

*Proof.* It suffices to follow the proof given in [7, Lemma 3.1], with the corresponding modifications.

□

**Lemma 2.1.3.** *For* $n \geqslant n_0$, *the polynomial* $\omega_N \hat{B}_n$ *has the following representation in terms of the sequence* $\{p_n(x)\}_{n=0}^{\infty}$ *of orthonormal polynomials with respect to* $\mu$.

$$\omega_N(x)\hat{B}_n(x) = \sum_{j=0}^{2N} A_{n,j} p_{n+N-j}(x), \quad A_{n,0} \neq 0. \tag{2.1.4}$$

*Moreover,* $A_{n,j}$ *are bounded and* $A_{n,2N} = \frac{\kappa(p_{n-N})}{\kappa(p_{n+N})} \frac{1}{A_{n,0}} \langle \hat{B}_n, \hat{B}_n \rangle \neq 0$.

*Proof.* (2.1.4) is an immediate consequence of

$$\int \omega_N(x)\hat{B}_n(x)p_i(x)d\mu(x) = \langle \hat{B}_n, \omega_N p_i \rangle.$$

Thus,

$$\sum_{i=0}^{2N} |A_{n,i}|^2 = \int \omega_N^2(x)|\hat{B}_n(x)|^2 d\mu(x) \leqslant \max_{x \in supp\, \mu} \omega_N^2(x),$$

$$A_{n,0} = \int \omega_N(x)\hat{B}_n(x)p_{n+N}(x)d\mu(x) = \frac{\kappa(\hat{B}_n)}{\kappa(p_{n+N})},$$

$$A_{n,2N} = \int \omega_N(x)\hat{B}_n(x)p_{n-N}(x)d\mu(x) = \langle \hat{B}_n, \omega_N p_{n-N} \rangle$$

$$= \frac{\kappa(p_{n-N})}{\kappa(\hat{B}_n)} \langle \hat{B}_n, \hat{B}_n \rangle, \quad n \geqslant n_0.$$

Therefore, the coefficients $A_{n,j}$, $0 \leqslant j \leqslant 2N$, are bounded, and $A_{n,0}$, $A_{n,2N}$ satisfy

$$A_{n,0}A_{n,2N} = \frac{\kappa(p_{n-N})}{\kappa(p_{n+N})}\langle \hat{B}_n, \hat{B}_n \rangle.$$

□

We can already provide an alternative (and more simple) proof of a well-known result about the outer relative asymptotics for the polynomials $\hat{B}_n$, which is a special case of [47, Theorem 4]. We denote by $T_j$ the $j$th Chebyshev polynomial of the first kind and by $\varphi^{\pm}(x) := x \pm \sqrt{x^2-1}$ with the assumption that the square root is positive for $x > 1$.

**Theorem 2.1.1.** *[47, formula (1.10)] Let $\mu$ be a finite positive Borel measure in the Nevai class $M(0,1)$, such that all the mass points $b_k \notin$ supp $\mu$. Then the polynomials $\{\hat{B}_n(x)\}_{n=0}^{\infty}$ satisfy*

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{p_n(x)} = \prod_{k=1}^{K} \left( \frac{1}{|\varphi^+(b_k)|} \frac{(\varphi^+(x) - \varphi^+(b_k))^2}{2\varphi^+(x)(x-b_k)} \right)^{I_k}, \tag{2.1.5}$$

*uniformly on compact sets of $\bar{\mathbb{C}} \backslash$supp $\mu$.*

*Proof.* Since $\mu \in M(0,1)$,

$$\lim_{n\in\Lambda} \frac{\omega_N(x)\hat{B}_n(x)}{p_{n+N}(x)} = \sum_{j=0}^{2N} A_j(\varphi^-(x))^j \tag{2.1.6}$$

uniformly on compact sets of $\bar{\mathbb{C}} \backslash$*supp* $\mu$.

Now, we are going to show that the $A_j$'s are completely determined for any sequence of nonnegative integers $\Lambda$. In order to do it, we need to obtain a factorization of the polynomial $\sum_{j=0}^{2N} A_j z^j$.

Since $b_k \notin$ *supp* $\mu$, $1/(x-b_k)^i$ are continuous functions on *supp* $\mu$. Hence, for a fixed $k \in \{1,\ldots,K\}$, by orthogonality, (2.4), and the weak asymptotic property (see [7, formula (2)]) we get for $i = 1,\ldots,N_k+1$,

$$\lim_{n\in\Lambda} \int \frac{\omega_N(x)}{(x-b_k)^i}\hat{B}_n(x)p_{n+N}(x)d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sum_{j=0}^{2N} A_j T_j(x)}{(x-b_k)^i} \frac{dx}{\sqrt{1-x^2}} = 0. \tag{2.1.7}$$

According to the residue's theorem, (2.1.7) means that the polynomial $\sum_{j=0}^{2N} A_j z^j$ has a zero of multiplicity at least $N_k+1$ at $\varphi^-(b_k)$.

On the other hand,

$$\lim_{n\to\infty}\int \frac{\omega_N(x)}{(x-b_k)^{N_k+1}}\hat{B}_n(x)\frac{p_{n+N}(x)}{(x-b_k)^m}d\mu(x)$$
$$=\lim_{n\to\infty}\int\left\{\sum_{j=0}^{m-1}\frac{1}{j!}\left(\frac{\omega_N}{(x-b_k)^{N_k+1}}\hat{B}_n\right)^{(j)}(b_k)(x-b_k)^j+(x-b_k)^m\pi_{n+N-N_k-1-m}(x)\right\}\frac{p_{n+N}(x)}{(x-b_k)^m}d\mu(x)$$
$$=\lim_{n\to\infty}\sum_{j=0}^{m-1}\frac{1}{j!}\left(\frac{\omega_N}{(x-b_k)^{N_k+1}}\hat{B}_n\right)^{(j)}(b_k)\int\frac{p_{n+N}(x)}{(x-b_k)^{m-j}}d\mu(x)=0,$$

where $\pi_{n+N-N_k-1-m}$ is a polynomial of degree $n+N-N_k-1-m$ and the last equality holds as a consequence of the following two facts (cf. [7, the proof of statement (i) of Lemma 3.1. and Lemma 2.3]).

(i)

$$\lim_{n\to\infty}\frac{n^{I_k-1-j_1}}{j_1!p_{n+N}(b_k)}\left(\frac{\omega_N}{(x-b_k)^{N_k+1}}\hat{B}_n\right)^{(j_1)}(b_k)$$
$$=\cdots=\lim_{n\to\infty}\frac{n^{I_k-1-j_{J_k}}}{j_{J_k}!p_{n+N}(b_k)}\left(\frac{\omega_N}{(x-b_k)^{N_k+1}}\hat{B}_n\right)^{(j_{J_k})}(b_k)=0,$$

where $j_1<j_2<\cdots<j_{J_k}$ are the non negative integers corresponding to the masses $M_{k,j}=0$.

(ii)

$$\lim_{n\to\infty}\frac{1}{n^{m-j-1}}\int\frac{p_{n+N}(x)p_{n+N}(b_k)}{(x-b_k)^{m-j}}d\mu(x)=\frac{(-1)^{m-j}}{(m-j-1)!}\left(\frac{1}{\sqrt{b_k^2-1}}\right)^{m-j}, \tag{2.1.8}$$

for each $b_k$ and $m-j>0$.

Then $\varphi^-(b_k)$ is a zero of the polynomial $\sum_{j=0}^{2N}A_jz^j$ of multiplicity at least $N_k+1+I_k$.

From statement (ii) of Lemma 2.1.2, we have

$$\lim_{n\to\infty}\int\frac{\omega_N(x)\hat{B}_n(x)p_{n-N}(x)}{(x-b_k)^m}d\mu(x)=0,\quad m=1,\dots,N_k+1-I_k.$$

As a consequence,

$$\lim_{n\in\Lambda}\int\frac{\omega_N(x)\hat{B}_n(x)p_{n-N}(x)}{(x-b_k)^m}d\mu(x)=\frac{1}{\pi}\int_{-1}^{1}\frac{\sum_{j=0}^{2N}A_jT_{2N-j}(x)}{(x-b_k)^m}\frac{dx}{\sqrt{1-x^2}}=0, \tag{2.1.9}$$

for $m = 1, \ldots, N_k + 1 - I_k$. Hence, (2.1.9) means that $\varphi^+(b_k)$ is a zero of the polynomial $\sum_{j=0}^{2N} A_j z^j$ of multiplicity at least $N_k + 1 - I_k$.

Therefore, $\sum_{j=0}^{2N} A_j z^j$ has the following factorization

$$\sum_{j=0}^{2N} A_j z^j = A_{2N} \prod_{k=1}^{K} (z - \varphi^-(b_k))^{N_k+1+I_k} (z - \varphi^+(b_k))^{N_k+1-I_k},$$

and for $z = \varphi^-(x)$ we obtain

$$\sum_{j=0}^{2N} A_j (\varphi^-(x))^j = A_{2N} \prod_{k=1}^{K} (\varphi^-(x) - \varphi^-(b_k))^{N_k+1+I_k} (\varphi^-(x) - \varphi^+(b_k))^{N_k+1-I_k}. \tag{2.1.10}$$

If $x$ tends to infinity, then we find $A_0 = A_{2N} \prod_{k=1}^{K} (\varphi^-(b_k))^{2I_k}$ and having in mind that $A_0 A_{2N} = \dfrac{1}{2^{2N}}$, we can deduce that

$$A_{2N} = \frac{\prod_{k=1}^{K} |\varphi^+(b_k)|^{I_k}}{2^N},$$

and this last equation determines completely $A_j$ for any sequence of nonnegative integers $\Lambda$. Also, from (2.1.6) and (2.1.10) we can deduce that

$$\lim_{n\to\infty} \frac{\omega_N(x) \hat{B}_n(x)}{p_{n+N}(x)} = \frac{\prod_{k=1}^{K} |\varphi^+(b_k)|^{I_k}}{2^N} \prod_{k=1}^{K} (\varphi^-(x) - \varphi^-(b_k))^{N_k+1+I_k} (\varphi^-(x) - \varphi^+(b_k))^{N_k+1-I_k}$$
$$= \frac{1}{2^N} \prod_{k=1}^{K} |\varphi^+(b_k)|^{I_k} (\varphi^-(x) - \varphi^-(b_k))^{N_k+1+I_k} (\varphi^-(x) - \varphi^+(b_k))^{N_k+1-I_k}$$
$$= \frac{1}{2^N} \prod_{k=1}^{K} \frac{|\varphi^+(b_k)|^{I_k} (\varphi^-(x) - \varphi^-(b_k))^{I_k}}{(\varphi^-(x) - \varphi^+(b_k))^{I_k}} ((\varphi^-(x) - \varphi^-(b_k))(\varphi^-(x) - \varphi^+(b_k)))^{N_k+1}.$$

Finally, taking into account the outer ratio asymptotics for orthonormal polynomials associated with measures in the Nevai class as well as the fact that

$$\omega_N(x) = \frac{1}{2^N} (\varphi^+(x))^N \prod_{k=1}^{K} ((\varphi^-(x))^2 - 2b_k \varphi^-(x) + 1)^{N_k+1} \tag{2.1.11}$$

on compact subsets of $\bar{\mathbb{C}} \backslash \mathit{supp}\, \mu$, we get

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{p_n(x)} = \prod_{k=1}^{K} \left( \frac{1}{|\varphi^+(b_k)|} \frac{(\varphi^+(x) - \varphi^+(b_k))^2}{2\varphi^+(x)(x - b_k)} \right)^{I_k}.$$

□ □

**Remark 2.1.1.** *When* $b_k \in \operatorname{supp}\mu$, $k = 1, \ldots, K$, *for instance,* $\{a_k\}_{k=1}^{K_1}$ *and* $\{b'_j\}_{j=1}^{K_2}$ *are the mass points of* $\mu$ *on* $[-1, 1]$ *and the mass points of* $\mu$ *on* $\operatorname{supp}\mu\backslash[-1, 1]$*, respectively, for* $k = 1, \ldots, K_1$ *and* $j = 1, \ldots, K_2$, $K = K_1 + K_2$, *it was proved in [7, Theorem 3.1, (i)] that for* $\mu \in M(0, 1)$ *the following outer relative asymptotics holds uniformly on compact subsets of* $\bar{\mathbb{C}}\backslash\operatorname{supp}\mu$.

$$\lim_{n\to\infty} \frac{\hat{B}_n(x)}{q_n(x)} = \prod_{k=1}^{K_2} \left( \frac{1}{|\varphi^+(b'_k)|} \frac{(\varphi^+(x) - \varphi^+(b'_k))^2}{2\varphi^+(x)(x - b'_k)} \right)^{I_k}, \tag{2.1.12}$$

*where* $\{q_n(x)\}_{n=0}^{\infty}$ *is the sequence of orthonormal polynomials with respect to the measure* $\nu \in M(0, 1)$ *defined by* $\nu = \mu - \sum_{k=1}^{K_1} \mu(\{a_k\})\delta_{a_k} - \sum_{k=1}^{K_2} \mu(\{b'_k\})\delta_{b'_k}$.

*Even though in the previous proof we follow the ideas of the proof given in [7, Theorem 3.1], it is worthwhile to point out that the existing difference between the arguments of both proofs is the use of Lemma 2.1.2, which is necessary for us since we do not consider the mass points of* $\mu$ *inside* $\operatorname{supp}\mu$.

## 2.2 Matrix interpretation

Next, we will assume that the values $M_{k,i}$ in the inner product (4.1.11) are nonnegative real numbers. In such a way, $n_0 = 0$ in Lemma 2.1.1 and we can define the sequence of orthogonal polynomials $\{\hat{B}_n(x)\}_{n=0}^{\infty}$ with $deg(\hat{B}_n) = n$. Thus, it constitutes a basis of the linear space $\mathbb{P}$.

**Lemma 2.2.1.** *The polynomial* $\hat{B}_n$ *has the following representation in terms of the sequence* $\{p_n(x)\}_{n=0}^{\infty}$ *of orthonormal polynomials with respect to* $\mu$.

$$\hat{B}_n(x) = \sum_{j=0}^{n} \alpha_{n,j} p_j(x) \tag{2.2.13}$$

*where*

$$\alpha_{n,n} = \frac{\tau_n}{\kappa(p_n)},$$
$$\alpha_{n,j} = -\sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} \hat{B}_n^{(i)}(b_k) p_j^{(i)}(b_k), \ \textit{for}\ 0 \leqslant j \leqslant n - 1.$$

*Proof.* By the orthonormality of $p_n$, we have

$$\alpha_{n,n} = \int \hat{B}_n(x)p_n(x)d\mu(x) = \tau_n \int \tilde{B}_n(x)p_n(x)d\mu(x) = \frac{\tau_n}{\kappa(p_n)}.$$

For $0 \leqslant j \leqslant n-1$, using (2.1.2) we have

$$\begin{aligned}
\alpha_{n,j} &= \int \hat{B}_n(x) p_j(x) d\mu(x) \\
&= \langle \hat{B}_n, p_j \rangle - \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} \hat{B}_n^{(i)}(b_k) p_j^{(i)}(b_k) \\
&= - \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} \hat{B}_n^{(i)}(b_k) p_j^{(i)}(b_k).
\end{aligned}$$

□ □

**Lemma 2.2.2.** *The polynomial* $p_n$ *has the following representation in terms of the sequence* $\{\hat{B}_n(x)\}_{n=0}^{\infty}$.

$$p_n(x) = \sum_{j=0}^{n} \beta_{n,j} \hat{B}_j(x), \tag{2.2.14}$$

*where*

$$\begin{aligned}
\beta_{n,n} &= \frac{\tau_n}{\langle \hat{B}_n, \hat{B}_n \rangle \kappa(p_n)} + \frac{1}{\langle \hat{B}_n, \hat{B}_n \rangle} \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} p_n^{(i)}(b_k) \hat{B}_n^{(i)}(b_k), \\
\beta_{n,j} &= \frac{1}{\langle \hat{B}_j, \hat{B}_j \rangle} \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} p_n^{(i)}(b_k) \hat{B}_j^{(i)}(b_k), \ \textit{for} \ 0 \leqslant j \leqslant n-1.
\end{aligned}$$

*Proof.* This is a straightforward result that follows by using the same arguments as in the previous lemma.

□

In order to write in matrix form, we introduce the following notation.

$$\begin{aligned}
\hat{\mathbb{B}} &= \left(\hat{B}_0(x), \quad \dots \quad, \hat{B}_n(x), \quad \dots\right)^T, \\
\mathcal{P} &= \left(p_0(x), \quad \dots \quad, p_n(x), \quad \dots\right)^T,
\end{aligned}$$

and $\Lambda$ is the following lower triangular infinite matrix

$$\Lambda = \begin{pmatrix} \alpha_{0,0} & 0 & 0 & \dots \\ \alpha_{1,0} & \alpha_{1,1} & 0 & \dots \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\hat{\mathbb{B}} = \Lambda\mathcal{P}$ and (2.2.13), (2.2.14) and (2.1.4), respectively, can be written in matrix form as follows.

$$\hat{B}_n(x) = \left(\alpha_{n,0}, \quad \dots \quad, \alpha_{n,n}\right)\left(p_0(x), \quad \dots \quad, p_n(x)\right)^T,$$

$$p_n(x) = \left(\beta_{n,0}, \quad \dots \quad, \beta_{n,n}\right)\left(\hat{B}_0(x), \quad \dots \quad, \hat{B}_n(x)\right)^T,$$

and

$$\begin{aligned}\omega_N(x)\hat{B}_n(x) &= \left(A_{n,2N}, \quad \dots \quad, A_{n,0}\right)\left(p_{n-N}(x), \quad \dots \quad, p_{n+N}(x)\right)^T \\ &= \left(A_{n,2N}, \quad \dots \quad, A_{n,0}\right)\tilde{H}\left(\hat{B}_0(x), \quad \dots \quad, \hat{B}_{n+N}(x)\right)^T,\end{aligned}$$

where $\tilde{H} \in \mathbb{R}^{(2N+1)\times(n+N+1)}$ is the Hessenberg matrix

$$\tilde{H} = \begin{pmatrix} \beta_{n-N,0} & \dots & \beta_{n-N,n-N} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n,0} & \dots & \beta_{n,n-N} & \dots & \beta_{n,n} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n+N,0} & \dots & \beta_{n+N,n-N} & \dots & \beta_{n+N,n} & \beta_{n+N,n+1} & \dots & \beta_{n+N,n+N} \end{pmatrix}.$$

Now, we decompose

$$\tilde{H}\left[\begin{pmatrix} \hat{B}_0(x) \\ \vdots \\ \hat{B}_{n-N-1}(x) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hat{B}_{n-N}(x) \\ \vdots \\ \hat{B}_{n+N}(x) \end{pmatrix}\right] =$$

$$\begin{pmatrix} \beta_{n-N,0} & \dots & \beta_{n-N,n-N-1} \\ \vdots & & \vdots \\ \beta_{n+N,0} & \dots & \beta_{n+N,n-N-1} \end{pmatrix}\begin{pmatrix} \hat{B}_0(x) \\ \vdots \\ \hat{B}_{n-N-1}(x) \end{pmatrix} + \hat{H}\begin{pmatrix} \hat{B}_{n-N}(x) \\ \vdots \\ \hat{B}_{n+N}(x) \end{pmatrix},$$

where $\hat{H}$ is the matrix obtained from $\tilde{H}$ deleting its first $n-N$ columns, i.e.

$$\hat{H} = \begin{pmatrix} \beta_{n-N,n-N} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \beta_{n+N,n-N} & \dots & \beta_{n+N,n+N} \end{pmatrix}.$$

Having in mind that $\omega_N\hat{B}_n(x) \in \operatorname{span}(\hat{B}_{n-N}, \dots, \hat{B}_{n+N})$ it follows that

$$\left(A_{n,2N}, \quad \dots \quad, A_{n,0}\right)\begin{pmatrix} \beta_{n-N,0} & \dots & \beta_{n-N,n-N-1} \\ \vdots & & \vdots \\ \beta_{n+N,0} & \dots & \beta_{n+N,n-N-1} \end{pmatrix} = \left(0, \quad \dots \quad, 0\right).$$

Finally, we obtain

$$\omega_N(x)\hat{B}_n(x) = \left(A_{n,2N}, \quad \dots \quad, A_{n,0}\right) \hat{H} \left(\hat{B}_{n-N}(x), \quad \dots \quad, \hat{B}_{n+N}(x)\right)^T .$$

These remarks can be summarized as follows.

**Proposition 2.2.1.** *The sequence of polynomials* $\{\hat{B}_n(x)\}_{n=0}^{\infty}$ *satisfies the following recurrence relation.*

$$\omega_N(x)\hat{B}_n(x) = \sum_{j=0}^{2N} c_{n,j}\hat{B}_{n+N-j}(x)$$

*where*

$$\left(c_{n,2N}, \quad \dots \quad, c_{n,0}\right) = \left(A_{n,2N}, \quad \dots \quad, A_{n,0}\right) \begin{pmatrix} \beta_{n-N,n-N} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \beta_{n+N,n-N} & \cdots & \beta_{n+N,n+N} \end{pmatrix}.$$

On the other hand, according to Lemma 2.1.3 and Lemma 2.2.1

$$\begin{aligned} A_{n,j} &= \int \omega_N(x)\hat{B}_n(x)p_{n+N-j}(x)d\mu(x) \\ &= \langle \omega_N\hat{B}_n, p_{n+N-j}\rangle = \langle \hat{B}_n, \omega_N p_{n+N-j}\rangle \\ &= \left\langle \sum_{l=0}^{n} \alpha_{n,l}p_l, \omega_N p_{n+N-j} \right\rangle \\ &= \sum_{l=0}^{n} \alpha_{n,l}\langle p_l, \omega_N p_{n+N-j}\rangle. \end{aligned}$$

But,

$$\langle \omega_N p_{n+N-j}, p_l\rangle = [\omega_N(J)]_{n+N-j,l},$$

i.e., $\langle \omega_N p_{n+N-j}, p_l\rangle$ is the $(n+N-j, l)$ entry in the $(2N+1)$-diagonal matrix $\omega_N(J)$, where $J$ is the Jacobi matrix associated with the measure $\mu$, i.e. the matrix associated with the multiplication operator in terms of the orthonormal basis $\{p_n(x)\}_{n=0}^{\infty}$.

Notice that

$$
\begin{aligned}
A_{n,2N} &= \sum_{l=0}^{n} \alpha_{n,l}[\omega_N(J)]_{n-N,l} = \sum_{l=n-2N}^{n} \alpha_{n,l}[\omega_N(J)]_{n-N,l} \\
A_{n,2N-1} &= \sum_{l=0}^{n} \alpha_{n,l}[\omega_N(J)]_{n-N+1,l} = \sum_{l=n-2N+1}^{n} \alpha_{n,l}[\omega_N(J)]_{n-N+1,l} \\
&\vdots \\
A_{n,0} &= \sum_{l=0}^{n} \alpha_{n,l}[\omega_N(J)]_{n+N,l} = \alpha_{n,n}[\omega_N(J)]_{n+N,n}.
\end{aligned}
$$

Thus,

$$
\begin{pmatrix} A_{n,2N}, & \dots & , A_{n,0} \end{pmatrix} = \begin{pmatrix} \alpha_{n,n-2N}, & \dots & , \alpha_{n,n} \end{pmatrix} \begin{pmatrix} [\omega_N(J)]_{n-N,n-2N} & \cdots & 0 \\ \vdots & \ddots & \\ [\omega_N(J)]_{n-N,n} & \dots & [\omega_N(J)]_{n+N,n} \end{pmatrix}.
$$

As a conclusion, from Proposition 2.2.1 we obtain

$$
\begin{pmatrix} c_{n,2N}, & \dots & , c_{n,0} \end{pmatrix} =
$$

$$
\begin{pmatrix} \alpha_{n,n-2N}, & \dots & , \alpha_{n,n} \end{pmatrix} \begin{pmatrix} [\omega_N(J)]_{n-N,n-2N} & \cdots & 0 \\ \vdots & \ddots & \\ [\omega_N(J)]_{n-N,n} & \dots & [\omega_N(J)]_{n+N,n} \end{pmatrix} \begin{pmatrix} \beta_{n-N,n-N} & \cdots & 0 \\ \vdots & \ddots & \\ \beta_{n+N,n-N} & \cdots & \beta_{n+N,n+N} \end{pmatrix}.
$$

This yields the relation between the parameters of the recurrence formula for $\{\hat{B}_n(x)\}_{n=0}^{\infty}$ in terms of $\{\alpha_{n,j}\}_{j=n-2N}^{n}$ and $\{\beta_{n-N+k,j}\}_{k=0}^{2N}$.

On the other hand, from Lemma 2.1.3

$$
\omega_N \hat{\mathbb{B}} = H\mathcal{P}.
$$

Here $H$ denotes the $2N+1$ banded infinite matrix with entries $h_{k,j} = A_{k,k+N-j}$ , $k - N \leqslant j \leqslant k + N$, and 0, otherwise.

Given $\mathcal{C} = (c_0(x), \dots, c_n(x), \dots)^T$ and $\mathcal{D} = (d_0(x), \dots, d_n(x), \dots)^T$, we will denote by $(\mathcal{C}, \mathcal{D}^T)$ and $\langle \mathcal{C}, \mathcal{D}^T \rangle$ the infinite matrices whose entries are $(c_i(x), d_j(x))$ and $\langle c_i(x), d_j(x) \rangle$, respectively.

Since $(\mathcal{P}, \mathcal{P}^T) = I$, we have

$$
\begin{aligned}
H &= (\omega_N \hat{\mathbb{B}}, \mathcal{P}^T) = (\hat{\mathbb{B}}, \omega_N \mathcal{P}^T) = (\hat{\mathbb{B}}, \mathcal{P}^T \omega_N(J)) \\
&= (\Lambda \mathcal{P}, \mathcal{P}^T)\omega_N(J) = \Lambda \omega_N(J).
\end{aligned}
$$

From Proposition 2.2.1 we get

$$\omega_N \hat{\mathbb{B}} = S\hat{\mathbb{B}},$$

hence $S\langle\hat{\mathbb{B}}, \hat{\mathbb{B}}^T\rangle = \langle\omega_N\hat{\mathbb{B}}, \hat{\mathbb{B}}^T\rangle$. Here $S$ is the $2N + 1$ banded infinite matrix with entries $s_{k,j} = c_{k,k+N-j}$ , $k - N \leqslant j \leqslant k + N$, and 0 otherwise. But $\langle\omega_N\hat{\mathbb{B}}, \hat{\mathbb{B}}^T\rangle = (\omega_N\hat{\mathbb{B}}, \hat{\mathbb{B}}^T)= (H\mathcal{P}, \mathcal{P}^T\Lambda^T) = H\Lambda^T$. Taking into account that $\langle\hat{\mathbb{B}}, \hat{\mathbb{B}}^T\rangle = D$ is a diagonal matrix according to the orthogonality of the polynomial sequence $\{\hat{B}_n(x)\}_{n=0}^{\infty}$ with respect to our Sobolev inner product, then $SD = H\Lambda^T$ and we get the following result.

**Proposition 2.2.2.** *The matrices $S$ and $\omega_N(J)$ satisfy the following connection relation*

$$SD = \Lambda\omega_N(J)\Lambda^T. \tag{2.2.15}$$

For an alternative approach based on $LU$ and $UL$ factorization see [17].

## 2.3 Fourier series

We are interested in the study of the pointwise convergence of the Fourier series expansions in terms of the polynomials $\{B_n(x)\}_{n=0}^{\infty}$ orthonormal with respect to (4.1.11) when $\mu$ is now the Jacobi measure. Our idea is to generalize some results given in [57] for the case of only one mass point outside *supp* $\mu$. In order to do this, we need some pointwise estimates for the polynomials $B_n$ and its derivatives at the mass points $b_k$.

We will say that a measure $\mu$ belongs to the Szegő class and we will denote it by $\mu \in \mathcal{S}$ if *supp* $(d\mu) = [-1, 1]$ and

$$\int_{-1}^{1} \frac{\ln \mu'(x)}{\sqrt{1-x^2}} > -\infty.$$

It is well known that if $\{p_n(x)\}_{n=0}^{\infty}$ is the sequence of orthonormal polynomials with respect to some measure $\lambda \in \mathcal{S}$, then

$$\frac{p_n'(x)}{p_n(x)} = O(n)$$

outside the support of the measure.

The following result generalizes the above property to derivatives of higher order.

**Lemma 2.3.1.** *Let $\lambda \in \mathcal{S}$. Then, for each $1 \leqslant k < n$ we have*

$$\frac{p_n^{(k)}(x)}{p_n(x)} = O\left(n^k\right), \tag{2.3.16}$$

*uniformly on compact subsets of* $\mathbb{C}\backslash[-1, 1]$.

*Proof.* Since $\lambda \in \mathcal{S}$, the polynomials $p_n$ satisfy the following outer strong asymptotics (see [95]):

$$p_n(x) = \frac{D(\lambda, 0)\Phi^n(x)}{D(\lambda, \frac{1}{2\Phi(x)})}(1 + O(1)),$$

uniformly in compact subsets of $\mathbb{C}\backslash[-1, 1]$, where $D(\lambda, x)$ denotes the Szegő function

$$D(\lambda, z) = \exp\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log \lambda'(\cos t)\frac{1 + ze^{-it}}{1 - ze^{-it}}dt\}$$

for $|z| < 1$, and

$$\Phi(x) = \frac{\varphi^+(x)}{2}.$$

Finally, (2.3.16) follows by estimating the $k$th derivative of $\Phi^n(x)$. □

Let

$$\tilde{\omega}_N(x) = \prod_{k=1}^{K}(x - b_k)^{N_k^*},$$

where $N = \sum_{k=1}^{K} N_k^*$ and

$$N_k^* = \begin{cases} N_k + 1, & \text{if } N_k \text{ is odd} \\ N_k + 2, & \text{if } N_k \text{ is even.} \end{cases}$$

and let us denote $\tilde{\omega}_{N,k}(x) = \frac{\tilde{\omega}_N(x)}{(x-b_k)^{N_k^*}}$.

**Lemma 2.3.2.** *Let $\{q_n(x)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials with respect to $\tilde{\omega}_N(x)d\mu(x)$, where $\mu \in \mathcal{S}$. Then, for $0 < m \leqslant N_k^*$ we get*

$$\int_{-1}^{1} q_n(x)(x - b_k)^{N_k^*-m}\tilde{\omega}_{N,k}(x)d\mu(x) = O\left(n^{m-1}(\varphi^-(b_k))^n\right). \tag{2.3.17}$$

*In particular,*

$$\int_{-1}^{1} q_n(x)d\mu(x) = O\left(n^{N_k^*-1}(\varphi^-(b_k))^n\right). \tag{2.3.18}$$

*Proof.* We prove (2.3.17) by induction on $m$. For $m = 1$ we have

$$\begin{aligned}
&\int_{-1}^{1} q_n(x)(x - b_k)^{N_k^*-1}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&= \frac{1}{q_n(b_k)}\int_{-1}^{1} q_n(x)\left(q_n(b_k) + \Pi_{n-1}(x)(x - b_k)\right)(x - b_k)^{N_k^*-1}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&= \frac{1}{q_n(b_k)}\int_{-1}^{1} q_n^2(x)(x - b_k)^{N_k^*-1}\tilde{\omega}_{N,k}(x)d\mu(x) = O\left((\varphi^-(b_k))^n\right),
\end{aligned}$$

where $\Pi_{n-1}(x)$ is a polynomial of degree at most $n-1$. For $m=2$, using the previous case and Lemma 2.3.1 we get

$$\begin{aligned}
&\int_{-1}^{1} q_n(x)(x-b_k)^{N_k^*-2}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&= \frac{1}{q_n(b_k)}\int_{-1}^{1} q_n(x)(q_n(b_k)+q_n'(b_k)(x-b_k))(x-b_k)^{N_k^*-2}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&- \frac{q_n'(b_k)}{q_n(b_k)}\int_{-1}^{1} q_n(x)(x-b_k)^{N_k^*-1}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&= \frac{1}{q_n(b_k)}\int_{-1}^{1} q_n^2(x)(x-b_k)^{N_k^*-2}\tilde{\omega}_{N,k}(x)d\mu(x) + O\left(n(\varphi^-(b_k))^n\right) \\
&= O\left(n(\varphi^-(b_k))^n\right).
\end{aligned}$$

Now, let assume that (2.3.17) holds for every positive integer $i<m$, i.e.

$$\int_{-1}^{1} q_n(x)(x-b_k)^{N_k^*-i}\tilde{\omega}_{N,k}(x)d\mu(x) = O\left(n^{i-1}(\varphi^-(b_k))^n\right).$$

Then, using again Lemma 2.3.1, we obtain

$$\begin{aligned}
&\int_{-1}^{1} q_n(x)(x-b_k)^{N_k^*-m}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&= \frac{1}{q_n(b_k)}\int_{-1}^{1} q_n(x)\left(q_n(b_k)+\sum_{i=1}^{m-1}\frac{q_n^{(i)}(b_k)}{i!}(x-b_k)^i\right)(x-b_k)^{N_k^*-m}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&- \sum_{i=1}^{m-1}\frac{q_n^{(i)}(b_k)}{i!q_n(b_k)}\int_{-1}^{1} q_n(x)(x-b_k)^{N_k^*-m+i}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&= \frac{1}{q_n(b_k)}\int_{-1}^{1} q_n^2(x)(x-b_k)^{N_k^*-m}\tilde{\omega}_{N,k}(x)d\mu(x) \\
&+ \sum_{i=1}^{m-1} O\left(n^i\right) O\left(n^{m-i-1}(\varphi^-(b_k))^n\right) \\
&= \left((\varphi^-(b_k))^n\right) + \sum_{i=1}^{m-1} O\left(n^{m-1}(\varphi^-(b_k))^n\right) = O\left(n^{m-1}(\varphi^-(b_k))^n\right).
\end{aligned}$$

Finally, taking $m=N_k^*$ we obtain (2.3.18). □

Let $\Pi_{k,i}(x)$ be the polynomial of least degree such that the following conditions hold

$$\begin{aligned}
\Pi_{k,i}^{(j)}(b_t) &= 0, \quad t \neq k, \quad j = 0, \ldots, N_t, \\
\Pi_{k,i}^{(j)}(b_k) &= 0, \text{ if } j \neq i, \\
\Pi_{k,i}^{(i)}(b_k) &= 1.
\end{aligned}$$

If $M_{k,i} > 0$, then, for $n$ large enough, if $\{B_n(x)\}_{n=0}^{\infty}$ is the sequence of orthonormal polynomials with respect to (4.1.11) we have

$$0 = \langle B_n, \Pi_{k,i} \rangle = \int_{-1}^{1} B_n(x) \Pi_{k,i}(x) d\mu(x) + M_{k,i} B_n^{(i)}(b_k),$$

and, as a consequence,

$$|B_n^{(i)}(b_k)| = \frac{1}{M_{k,i}} \left| \int_{-1}^{1} B_n(x) \Pi_{k,i}(x) d\mu(x) \right|.$$

Defining $C := \max_{x \in [-1,1]} |\Pi_{k,i}(x)|$, we obtain the following estimate

$$\left| B_n^{(i)}(b_k) \right| \leqslant \frac{C}{M_{k,i}} \left| \int_{-1}^{1} B_n(x) d\mu(x) \right|. \tag{2.3.19}$$

Then, using the same arguments as in the proof of Lemma 2.1.3, we can deduce that

$$B_n(x) = \sum_{j=0}^{N} D_{n,j} q_{n-j}(x), \quad \text{with } \{D_{n,j}\}_{n=0}^{\infty} \text{ bounded sequences for } j = 1, \ldots, N. \tag{2.3.20}$$

Therefore, in order to estimate $B_n^{(i)}(b_k)$, we only need to use (2.3.18) in the following way

$$\begin{aligned}
&\left| B_n^{(i)}(b_k) \right| \leqslant \frac{C}{M_{k,i}} \left| \int_{-1}^{1} B_n(x) d\mu(x) \right| = \frac{C}{M_{k,i}} \left| \int_{-1}^{1} \sum_{j=0}^{N} A_{n,j} q_{n-j}(x) d\mu(x) \right| \\
&\leqslant \frac{C'}{M_{k,i}} \left| \sum_{j=0}^{N} \int_{-1}^{1} q_{n-j}(x) d\mu(x) \right| = \frac{C'}{M_{k,i}} \left| \sum_{j=0}^{N} O\left( (n-j)^{N_k^*-1} (\varphi^-(b_k))^{n-j} \right) \right| \\
&= O\left( n^{N_k^*-1} (\varphi^-(b_k))^n \right).
\end{aligned}$$

These remarks can be summarized as follows (see also Corollary 3.4 in [57]).

**Theorem 2.3.1.** *If $\mu \in \mathcal{S}$ and the index $k \in \{1, \dots, K\}$ is such that $M_{k,i} > 0$, then*

$$\left|B_n^{(i)}(b_k)\right| = O\left(n^{N_k^*-1}(\varphi^-(b_k))^n\right). \tag{2.3.21}$$

A straightforward pointwise estimate for the orthonormal Jacobi-Sobolev type polynomials $B_n^{(\alpha,\beta)}$ in the interval $(-1, 1)$ is the following.

**Lemma 2.3.3.** *Let $\mu_{\alpha,\beta}$ be the Jacobi measure with parameters $\alpha, \beta > -1$. Then the orthonormal Jacobi-Sobolev type polynomials $B_n^{(\alpha,\beta)}$ satisfy*

$$\left|B_n^{(\alpha,\beta)}(x)\right| \leqslant Ch(x), \quad \textit{for } x \in (-1,1) \textit{ and for all } n, \tag{2.3.22}$$

*where $h(x)$ is a function depending on the parameters $\alpha, \beta$.*

*Proof.* Using the same arguments as in the proof of Lemma 2.1.3, we get a connection formula between Jacobi-Sobolev type polynomials $B_n^{(\alpha,\beta)}$ and the orthonormal Jacobi polynomials $p_n^{(\alpha,\beta)}$

$$\omega_N(x) B_N^{(\alpha,\beta)}(x) = \sum_{j=0}^{2N} A_{n,j} p_{n+N-j}^{(\alpha,\beta)}(x),$$

with bounded coefficients. Then, there exists a positive constant $C$ such that, for every $x \in \mathbb{R}$,

$$\left|\omega_N(x) B_N^{(\alpha,\beta)}(x)\right| \leqslant C \sum_{j=-N}^{N} \left|p_{n+j}^{(\alpha,\beta)}(x)\right|. \tag{2.3.23}$$

Moreover, it is well known that the orthonormal Jacobi polynomials $p_n^{(\alpha,\beta)}$ satisfy the pointwise estimates [80, 95]:

$$(1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}\left|p_n^{(\alpha,\beta)}(x)\right| \leqslant C, \quad \alpha > -\frac{1}{2}, \quad \beta > -\frac{1}{2}, \tag{2.3.24}$$

$$\left|p_n^{(\alpha,\beta)}(x)\right| \leqslant C, \quad -1 < \alpha \leqslant -\frac{1}{2}, \quad -1 < \beta \leqslant -\frac{1}{2}, \tag{2.3.25}$$

for $x \in (-1, 1)$.

From (2.3.23)-(2.3.25) the pointwise estimate (2.3.22) follows. □

Finally, regarding Fourier series in this setting, for an appropriate function $f$ the pointwise convergence of the Jacobi-Sobolev Fourier series to $f$ on the interval $(-1, 1)$ is standard and the corresponding results are a straightforward consequence of those given in [58]. We refer the interested reader to [58, Theorems 4.1–4.3 and Lemma 4.1], more precisely.

# 3

# Modified Laguerre measures and kernel polynomials. Asymptotics.

In this chapter we study asymptotic properties of the polynomials orthogonal with respect to modified Laguerre weights.

Let $d\mu$ be a nontrivial probability measure supported on a subset of the real line. Several examples of modifications of the measure $\mu$ have been studied in the literature. In particular, it is worthwhile to point out the three canonical cases studied in [97, 98]:

- Christoffel transformations:

$$d\hat{\mu}(x) = \prod_{i=1}^{N}(x-\xi_i)d\mu(x), \quad \xi_i \notin \textit{supp}\,(d\mu). \tag{3.0.1}$$

- Geronimus transformations:

$$d\hat{\mu}(x) = \frac{d\mu(x)}{\prod_{j=1}^{M}(x-\eta_j)} + \sum_{j=1}^{M} M_j\delta(x-\eta_j), \quad \eta_j \notin \textit{supp}\,(d\mu). \tag{3.0.2}$$

- Modification by a rational factor:

$$d\hat{\mu}(x) = \frac{\prod_{i=1}^{N}(x-\xi_i)}{\prod_{j=1}^{M}(x-\eta_j)}d\mu(x), \quad \xi_i,\eta_j \notin \textit{supp}\,(d\mu). \tag{3.0.3}$$

Now, we are going to focus our attention on the first type of perturbation. We will modify the measure by the multiplication by the polynomial $\prod_{k=1}^{K}(x-a_k)^{N_k}$, where $\sum_{k=1}^{K} N_k = N$.

Let $\{L_n^{[\alpha,N]}(x)\}_{n=0}^{\infty}$ denote the sequence of orthogonal polynomials with respect to the modified Laguerre measure $d\mu_{\alpha,N}(x) = \prod_{k=1}^{K}(x-a_k)^{N_k}x^{\alpha}e^{-x}dx$, $\alpha > -1$ and $a_k < 0$, normalized by the condition that $L_n^{[\alpha,N]}(x)$ have the same leading coefficient as the classical Laguerre orthogonal polynomials $L_n^{(\alpha)}(x) = L_n^{[\alpha,0]}(x)$, i.e. with leading coefficient equal to $\frac{(-1)^n}{n!}$.

Some structure formulas for this family of polynomials have been studied in [28]. Indeed,

**Proposition 3.0.1.** *[28, Proposition 2.1.] For $N \geqslant 1$, the following relation holds:*

$$(x-\xi_N)L_n^{[\alpha,N]}(x) = -(n+1)L_{n+1}^{[\alpha,N-1]}(x) + (n+1)\frac{L_{n+1}^{[\alpha,N-1]}(\xi_N)}{L_n^{[\alpha,N-1]}(\xi_N)}L_n^{[\alpha,N-1]}(x), \quad n \geqslant 1.$$

**Proposition 3.0.2.** *[28, Proposition 2.2.] We have*

$$L_n^{[\alpha-1,N]}(x) = L_n^{[\alpha,N]}(x) - A_n^N L_{n-1}^{[\alpha,N]}(x),$$

*where*

$$A_n^N = \left(\frac{n+1}{n}\right)^N \prod_{i=1}^{N} \frac{L_{n+1}^{[\alpha-1,i-1]}(\xi_i)L_{n-1}^{[\alpha,i-1]}(\xi_i)}{L_n^{[\alpha-1,i-1]}(\xi_i)L_n^{[\alpha,i-1]}(\xi_i)}.$$

**Corollary 3.0.1.** *[28, Corollary 2.3.] For $N \geqslant 1$, we have*

$$(x-\xi_N)L_n^{[\alpha,N]}(x) = -(n+1)L_{n+1}^{[\alpha-1,N-1]}(x) + (n+1)\frac{L_{n+1}^{[\alpha-1,N-1]}(\xi_N)}{L_n^{[\alpha,N-1]}(\xi_N)}L_n^{[\alpha,N-1]}(x), \quad n \geqslant 1.$$

With the previous relations B. Xh. Fejzullahu obtained some asymptotic properties that we summarize in the following:

**Proposition 3.0.3.** *[28, Proposition 2.4.–2.5]*

*(a) Uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$,*

$$\lim_{n\to\infty} n^{1/2}\frac{L_{n+1}^{[\alpha-1,N]}(x)}{L_n^{[\alpha,N]}(x)} = \sqrt{-x}.$$

*(b) Uniformly on compact subsets of* $\mathbb{C}\backslash[0,\infty)$,

$$\lim_{n\to\infty}\frac{L_n^{[\alpha,N]}(x)}{n^{N/2}L_n^{(\alpha)}(x)}=\prod_{k=1}^{K}\frac{1}{\left(\sqrt{-x}+\sqrt{-a_k}\right)^{N_k}}.$$

*(c) Uniformly on compact subsets of* $\mathbb{C}\backslash[0,\infty)$,

$$\lim_{n\to\infty}\frac{L_n^{[\alpha,N]}(x/(n+j))}{n^{\alpha+N/2}}=\frac{1}{\prod_{k=1}^{K}\left(\sqrt{-a_k}\right)^{N_k}}x^{-\alpha/2}J_\alpha(2\sqrt{x}),$$

*where* $j\in\mathbb{N}\cup\{0\}$ *and* $J_\alpha$ *is the Bessel function of the first kind.*

*(d) Plancherel-Rotach type outer asymptotics for* $L_n^{(\alpha,N)}$ :

$$\lim_{n\to\infty}\frac{L_n^{(\alpha,N)}((n+j)x)}{L_n^{(\alpha)}((n+j)x)}=\left(\frac{\phi((x-2)/2)+1}{x}\right)^N,$$

*where*

$$\phi(x)=x+\sqrt{x^2-1},\quad x\in\mathbb{C}\backslash[-1,1],$$

*with* $\sqrt{x^2-1}>0$ *when* $x>1$. *This asymptotic holds uniformly on compact subsets of* $\mathbb{C}\backslash[0,4)$ *and uniformly on* $j\in\mathbb{N}\cup\{0\}$.

In this chapter, we restrict ourselves to the case $K=1$, i.e. we will work with iterations of Christoffel perturbations. For this modification of the Laguerre measure, we obtain estimates for the norm of the perturbed polynomials as well as a generalized Christoffel representation formula for them. Finally, we focus our attention on the study of the asymptotics of kernel polynomials associated with the Gamma distribution as well as the asymptotics for the partial derivatives of such polynomials.

## 3.1 $k$-iterated Laguerre polynomials

Using a $k$-iterated Christoffel transform of the measure $\mu$, to the best of our knowledge, a fifth type of Laguerre expansions can be introduced. This family of functions is called $k$-iterated Laguerre polynomials, and it is constituted essentially by polynomials orthogonal with respect to the modified Laguerre measure $(x-c)^k d\mu(x)$, for $k\in\mathbb{N}$ fixed (see [13, 95].) Note that the modified Laguerre measure $(x-c)^k d\mu(x)$ is positive when either $k$ is an even integer number or $k$ is an odd integer number and $c$ is a real number located outside the support of $\mu$. Furthermore, it is very well known that, when $k=1$ and $c$ is outside *supp* $\mu$, these polynomials are actually the kernel polynomials corresponding to the moment functional associated with $\mu$ and the $K$-parameter $c$ [13, Sec. I.7].

In the sequel we will denote by $\{L_n^{\alpha,[k]}(x)\}_{n=0}^{\infty}$ and $\{L_n^{(\alpha),[k]}(x)\}_{n=0}^{\infty}$ the sequences of orthonormal and normalized $k$-iterated Laguerre polynomials with leading coefficient equal to $\frac{(-1)^n}{n!}$, respectively. It is clear that for $k = 0$ these sequences coincide with the orthonormal and normalized Laguerre polynomials with leading coefficient $\frac{(-1)^n}{n!}$, respectively.

The next Proposition gives the ratio asymptotics for $k$-iterated Laguerre polynomials with consecutive indexes of iteration.

**Proposition 3.1.1.** *[28, page 79] The limit*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha),[k]}(x)}{n^{1/2} L_n^{(\alpha),[k-1]}(x)} = \frac{1}{\sqrt{-x}+\sqrt{-c}} \tag{3.1.4}$$

*holds uniformly on compact subsets of* $\mathbb{C}\backslash[0,\infty)$.

The Mehler-Heine formula for $k$-iterated polynomials is just a particular case of Proposition 3.0.3 (c).

**Proposition 3.1.2.** *Uniformly on compact subsets of* $\mathbb{C}\backslash[0,\infty)$,

$$\lim_{n\to\infty} \frac{L_n^{(\alpha),[k]}(x/(n+j))}{n^{\alpha+k/2}} = \frac{1}{\left(\sqrt{|c|}\right)^k} x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

*where* $j \in \mathbb{N} \cup \{0\}$ *and* $J_\alpha$ *is the Bessel function of the first kind.*

### 3.1.1 Estimates for the norm of $k$-iterated polynomials

In this section, we obtain some estimates for the norm of the $k$-iterated Laguerre orthogonal polynomials and Laguerre-Sobolev type polynomials, respectively. In addition, we complete our study by deducing a connection formula involving different families of $k$-iterated Laguerre orthogonal polynomials. It is worth to mention that this is a result of independent interest.

**Proposition 3.1.3.** *For* $\alpha > -1$ *we have*

$$h_n^{(\alpha),[k]} := \int_0^\infty [L_n^{(\alpha),[k]}(x)]^2 (x-c)^k d\mu(x) \sim n^{\alpha+k}, \quad k \geqslant 0. \tag{3.1.5}$$

*Proof.* First of all, we proceed by induction on $k$ in order to prove

$$L_{n+1}^{(\alpha),[k]}(c) \sim L_n^{(\alpha),[k]}(c), \quad k \geqslant 0. \tag{3.1.6}$$

For $k = 0$, from Perron asymptotics formula (1.6.26), we obtain $L_{n+1}^{(\alpha)}(c) \sim L_n^{(\alpha)}(c)$. Assuming that (3.1.6) is true for $i \leqslant k-1$, we use this induction hypothesis for $i = k-1$ and (3.1.4), as follows

$$\frac{L_{n+1}^{(\alpha),[k]}(c)}{L_n^{(\alpha),[k]}(c)} = \frac{L_{n+1}^{(\alpha),[k]}(c)}{L_{n+1}^{(\alpha),[k-1]}(c)} \frac{L_n^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k]}(c)} \frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} \sim 1.$$

Finally, the estimate (3.1.6) together with [68, equation (9)] yields (3.1.5).

□

### 3.1.2 Representation formula for $k$-iterated Laguerre polynomials

We complete our study of $k$-iterated Laguerre orthogonal polynomials by giving a representation formula. The following lemma has been used repeatedly in order to obtain representation formulas involving different families of Laguerre orthogonal polynomials (up to multiplication for the corresponding weight functions):

**Lemma 3.1.1.** *[8, p. 1192] (Askey inversion formula). Let $w$ and $w_1$ be positive functions on $[0,\infty)$ such that $w^2/w_1 \in L^1[0,\infty)$. Let $\{p_n(x)\}_{n=0}^{\infty}$ and $\{q_n(x)\}_{n=0}^{\infty}$ be the orthonormal polynomials associated with $w$ and $w_1$, respectively. Then if*

$$q_n(x) = \sum_{k=0}^{n} c_{k,n} p_k(x),$$

*we have*

$$w(x)p_k(x) = \sum_{n=k}^{\infty} c_{k,n} q_n(x) w_1(x), \qquad (3.1.7)$$

*where the above convergence of the series is taken in the appropriate $L^2$ space.*

However, this method can not be applied in order to obtain a connection formula for $k$-iterated polynomials as that for classical Laguerre ones given in [70, equation (2.15)] due to the fact that the function $\frac{(x-c)^{2j}}{x^j}$ does not belong to $L^1(x^\alpha e^{-x}dx)$. An alternative method is presented in the following proposition and, in addition, we obtain estimates for the coefficients appearing therein.

**Proposition 3.1.4.** *The following connection formula holds.*

$$(x-c)^j L_n^{(\alpha),[k]}(x) = \sum_{m=0}^{j} a_{m,j,k}(\alpha,n) L_{n+m}^{(\alpha),[k-j]}(x), \quad \textit{for } 1 \leqslant j \leqslant k, \qquad (3.1.8)$$

*where $a_{m,j,k}(\alpha,n) \sim (-1)^m \binom{j}{m} n^j$. In particular, for $j = k$ we have (generalized Christoffel representation formula)*

$$(x-c)^j L_n^{(\alpha),[j]}(x) = \sum_{m=0}^{j} a_{m,j}(\alpha,n) L_{n+m}^{(\alpha)}(x).$$

*Proof.* We proceed by induction on $j$. For the case $j = 1$, the Christoffel formula reads (see [13, Sec. I.7])

$$(x-c)L_n^{(\alpha),[k]}(x) = -(n+1)L_{n+1}^{(\alpha),[k-1]}(x) + (n+1)\frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} L_n^{(\alpha),[k-1]}(x), \tag{3.1.9}$$

and, using (3.1.6), we obtain

$$a_{0,1,k}(\alpha,n) = (n+1)\frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} \sim n.$$

For $j = 2$, it is enough to note that $(x-c)^2 L_n^{(\alpha),[k]}(x) = (x-c)\left[(x-c)L_n^{(\alpha),[k]}(x)\right]$ and, according to (3.1.9), we have

$$(x-c)^2 L_n^{(\alpha),[k]}(x) = a_{2,2,k}(\alpha,n)L_{n+2}^{(\alpha),[k-2]}(x) + a_{1,2,k}(\alpha,n)L_{n+1}^{(\alpha),[k-2]}(x) + a_{0,2,k}(\alpha,n)L_n^{(\alpha),[k-2]}(x),$$

where

$$\begin{aligned}
a_{1,2,k}(\alpha,n) &= (n+1)(n+2) \sim n^2 \\
a_{1,2,k}(\alpha,n) &= -(n+1)(n+2)\frac{L_{n+2}^{(\alpha),[k-2]}(c)}{L_{n+1}^{(\alpha),[k-2]}(c)} - (n+1)^2 \frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} \sim -2n^2, \\
a_{0,2,k}(\alpha,n) &= (n+1)^2 \frac{L_{n+1}^{(\alpha),[k-1]}(c)}{L_n^{(\alpha),[k-1]}(c)} \frac{L_{n+1}^{(\alpha),[k-2]}(c)}{L_n^{(\alpha),[k-2]}(c)} \sim n^2.
\end{aligned}$$

Let assume that

$$(x-c)^{j-1} L_n^{(\alpha),[k]}(x) = \sum_{m=0}^{j-1} a_{m,j-1,k}(\alpha,n) L_{n+m}^{(\alpha),[k-j+1]}(x),$$

where $a_{m,j-1,k}(\alpha,n) \sim (-1)^m \binom{j-1}{m} n^{j-1}$. Then,

$$
\begin{aligned}
(x-c)^j L_n^{(\alpha),[k]}(x) &= \sum_{m=0}^{j-1} a_{m,j-1,k}(\alpha,n)(x-c)L_{n+m}^{(\alpha),[k-j+1]}(x) \\
&= \sum_{m=0}^{j-1} a_{m,j-1,k}(\alpha,n)\left(-(n+1)L_{n+m+1}^{(\alpha),[k-j]}(x) + (n+1)\frac{L_{n+m+1}^{(\alpha),[k-j]}(c)}{L_{n+m}^{(\alpha),[k-j]}(c)} L_{n+m}^{(\alpha),[k-j]}(x)\right) \\
&= \sum_{m=0}^{j} a_{m,j,k}(\alpha,n)L_{n+m}^{(\alpha),[k-j]}(x),
\end{aligned}
$$

with

$$
\begin{aligned}
a_{0,j,k}(\alpha,n) &= (n+1)a_{0,j-1,k}(\alpha,n)\frac{L_{n+1}^{(\alpha),[k-j]}(c)}{L_{n}^{(\alpha),[k-j]}(c)} \sim n^j, \\
a_{m,j,k}(\alpha,n) &= -(n+1)a_{m-1,j-1,k}(\alpha,n) + (n+1)a_{m,j-1,k}(\alpha,n)\frac{L_{n+m+1}^{(\alpha),[k-j]}(c)}{L_{n+m}^{(\alpha),[k-j]}(c)} \\
&\sim (-1)^m\binom{j-1}{m-1}n^j + (-1)^m\binom{j-1}{m}n^j = (-1)^m\binom{j}{m}n^j, \quad 1 \leqslant m \leqslant j-1, \\
a_{j,j,k}(\alpha,n) &= -(n+1)a_{j-1,j-1,k}(\alpha,n) \sim (-1)^j n^j,
\end{aligned}
$$

and this proves (3.1.8).

□

## 3.2 Kernel polynomials associated to the Gamma distribution

As we already mention, when $k = 1$ and $c$ is outside *supp* $\mu$, the $k$-iterated polynomials are actually the kernel polynomials corresponding to the moment functional associated with $\mu$ and the $K$-parameter $c$ [13, Sec. I.7].

Recall that we denote

$$K_n(x,y) = \sum_{k=0}^{n} \frac{\widehat{L}_k^{\alpha}(x)\widehat{L}_k^{\alpha}(y)}{\left\langle \widehat{L}_k^{\alpha}, \widehat{L}_k^{\alpha} \right\rangle_{\alpha}}, \tag{3.2.10}$$

and its partial derivatives

$$\frac{\partial^{j+k} K_n(x,y)}{\partial x^j \partial y^k} = K_n^{(j,k)}(x,y), \quad 0 \leqslant i,j \leqslant n. \tag{3.2.11}$$

This kernel function satisfies important properties such as the reproducing property:

**Proposition 3.2.1.** *If $q$ is a polynomial of degree less than or equal to $n$, then*

$$q(y) = \int K_n(x,y)q(x)d\mu(x).$$

*In particular, since $K_n$ is a polynomial in $y$ of degree $n$, we have*

$$K_n(x,z) = \int K_n(x,y)K_n(y,z)d\mu(y).$$

In addition, kernel polynomials appear in a natural way in the expression of the $n$-th partial sum of a Fourier expansion. If we denote by $\{p_n(x)\}_{n=0}^{\infty}$ a system of orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle$, the Fourier orthogonal expansion of a function $f$ in terms of the orthogonal polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ is defined by

$$f(x) = \sum_{n=0}^{\infty} \widehat{f}_n p_n(x), \quad \widehat{f}_n = \frac{1}{\langle p_n, p_n \rangle}\langle f, p_n \rangle.$$

The $n$-th partial sum $S_n f$ is defined by

$$S_n f(x) := \sum_{k=0}^{n} \widehat{f}_k p_k(x) = \langle f, K_n(x, \cdot) \rangle.$$

### 3.2.1 Asymptotics for the partial derivatives of kernels

Our goal here will be to analyze the asymptotic behavior of the partial derivatives of the diagonal Christoffel-Darboux kernels corresponding to classical Laguerre orthogonal polynomials (in short, the diagonal Laguerre kernels).

Then, for $c \in \mathbb{R}_+$ we will study the asymptotic behavior of $K_n^{(j,k)}(c,c), \quad 0 \leqslant j,k \leqslant n$.

To the best of our knowledge, asymptotic properties of the Laguerre kernels $K_n^{(j,k)}(c,c)$, $0 \leqslant j,k \leqslant n$, are not available in the literature, except possibly for those cases in which some of the following situations have been considered.

- *Case 1:* $c \geqslant 0$ and $j = k = 0$ or $0 \leqslant j,k \leqslant 1$ (cf. [38, 42].)
- *Case 2:* $c = 0$ and $0 \leqslant j,k \leqslant 1$ or $0 \leqslant j,k \leqslant n$ (cf. [18, 83].)

Here, we will describe the asymptotic behavior of this kernel functions by analyzing the following cases:

#### Case $c = 0$

This is a very well known case. The result reads as follows.

**Proposition 3.2.2.** *[83, Equation (6)] For $0 \leqslant j, k \leqslant n-1$, the following asymptotic behavior holds:*

$$K_{n-1}^{(k,j)}(0,0) \sim C_1 n^{\alpha+k+j+1}, \tag{3.2.12}$$

*where $C_1$ is a constant independent of $n$.*

### Case $c \notin [0,\infty)$

The following result gives the asymptotic behavior of the partial derivatives of the diagonal Laguerre kernels when $x = y = c \notin [0,\infty)$.

**Proposition 3.2.3.** *For $c \notin [0,\infty)$ and $0 \leqslant j, k \leqslant n-1$, the following asymptotic behavior holds*

$$K_{n-1}^{(k,j)}(c,c) \sim C_1 n^{\frac{k+j}{2}} e^{4\sqrt{-nc}}, \tag{3.2.13}$$

*where $C_1$ is a positive real number independent of $n$.*

*Proof.* Suppose that $c \notin [0,\infty)$ and let us denote

$$f(x) = x^{\frac{k+j}{2}} e^{4\sqrt{-xc}}, \quad x > 0.$$

Applying the Stolz criterion (see e.g. [43])

$$\frac{K_{n-1}^{(k,j)}(c,c)}{f(n)} \sim \frac{\left(L_{n-1}^{(\alpha)}\right)^{(k)}(c)\left(L_{n-1}^{(\alpha)}\right)^{(j)}(c)}{||L_{n-1}^{(\alpha)}||_\alpha^2\,(f(n)-f(n-1))}, \tag{3.2.14}$$

and using the mean value theorem, there exists $\xi \in (n-1,n)$ such that

$$f(n)-f(n-1) = f'(\xi) = \frac{k+j}{2}\xi^{\frac{k+j}{2}-1}e^{4\sqrt{-\xi c}} + 2\sqrt{-c}\xi^{\frac{k+j-1}{2}}e^{4\sqrt{-\xi c}} \sim C_1 n^{\frac{k+j-1}{2}} e^{4\sqrt{-nc}}. \tag{3.2.15}$$

Finally, from (1.6.22), (1.6.26), (1.6.21), (3.2.14) and (3.2.15) the result follows.

□

### Case $c \in (0,\infty)$

For the sake of simplicity, we study first the cases $K_{n-1}^{(0,1)}(c,c)$ and $K_{n-1}^{(1,1)}(c,c)$. Later on, we could extend this technique to the general case $K_{n-1}^{(k,j)}(c,c)$.

Taking derivatives with respect to $y$ in (3.2.10) and considering $x = y = c$ we get

$$K_{n-1}^{(0,1)}(c,c) = \frac{1}{2}\frac{\widehat{L}_{n-1}^{\alpha}(c)[\widehat{L}_{n}^{\alpha}]''(c) - \widehat{L}_{n}^{\alpha}(c)[\widehat{L}_{n-1}^{\alpha}]''(c)}{\Gamma(n)\Gamma(n+\alpha)}. \tag{3.2.16}$$

On the other hand,

$$K_{n-1}^{(1,1)}(c,c) = \frac{1}{3!}\frac{1}{\Gamma(n)\Gamma(n+\alpha)}\times$$

$$\left\{\widehat{L}_{n-1}^{\alpha}(c)[\widehat{L}_{n}^{\alpha}]'''(c) + 3[\widehat{L}_{n-1}^{\alpha}]'(c)[\widehat{L}_{n}^{\alpha}]''(c) - \widehat{L}_{n}^{\alpha}(c)[\widehat{L}_{n-1}^{\alpha}]'''(c) - 3[\widehat{L}_{n}^{\alpha}]'(c)[\widehat{L}_{n-1}^{\alpha}]''(c)\right\}. \tag{3.2.17}$$

The asymptotic behavior as $n \to \infty$ of the above Laguerre kernels at $x = c$, $c \in \mathbb{R}_+$, that is, within the oscillatory regime of the classical Laguerre orthogonal polynomials reads as follows.

**Lemma 3.2.1.** *For every $c > 0$, we have*

$$\begin{aligned} K_{n-1}(c,c) &\sim \pi^{-1}e^{c}c^{-\frac{1}{2}-\alpha}\, n^{1/2}, \\ K_{n-1}^{(0,1)}(c,c) &\sim \pi^{-1}e^{c}c^{-\frac{1}{2}-\alpha}\, n^{1/2}, \\ K_{n-1}^{(1,1)}(c,c) &\sim \frac{1}{3}\pi^{-1}e^{c}c^{-\frac{3}{2}-\alpha}\, n^{3/2}. \end{aligned}$$

*Proof.* Taking $p = 1$ in (1.6.27), we have $A_0(x) = 1$ and $B_0(x) = 0$. Thus, when $x \in \mathbb{R}_+$ we obtain the behavior of $\widehat{L}_n^{(\alpha)}(x)$ for $n$ large enough,

$$\widehat{L}_n^{\alpha}(x) = (-1)^n\Gamma(n+1)\pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}n^{\alpha/2-1/4}$$
$$\cdot \cos\{2\,(nx)^{1/2} - \alpha\pi/2 - \pi/4\} \cdot (1 + \mathcal{O}(n^{-1/2})).$$

We can rewrite the above expression as

$$\widehat{L}_n^{\alpha}(x) = (-1)^n\Gamma(n+1)n^{\frac{\alpha}{2}-\frac{1}{4}}\sigma^{\alpha}(x)\cos\varphi_n^{\alpha}(x)(1+\mathcal{O}(n^{-1/2})) \tag{3.2.18}$$

where

$$\varphi_n^{\alpha}(x) = 2(nx)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4},$$

and

$$\sigma^{\alpha}(x) = \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4} \tag{3.2.19}$$

is a function independent of $n$. Combining (1.6.22) with (3.2.32), we get

$$K_{n-1}(c,c) \sim \frac{\Gamma(n+1)}{\Gamma(n+\alpha)}n^{\alpha}\Theta_n(c;\alpha),$$

where

$$\Theta_n(c;\alpha) = \sigma^{\alpha}(c)\sigma^{\alpha+1}(c)\left[\cos\varphi_{n-1}^{\alpha+1}(c)\cos\varphi_{n-1}^{\alpha}(c) - \cos\varphi_{n-2}^{\alpha+1}(c)\cos\varphi_{n}^{\alpha}(c)\right]. \tag{3.2.20}$$

Let us deal with the above expression. From

$$\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2},$$

we have

$$\frac{\Theta_n(c;\alpha)}{\sigma^\alpha(c)\sigma^{\alpha+1}(c)} = \frac{1}{2}\cos\left(4\sqrt{c\,(n-1)}-\pi\alpha-\pi\right) - \frac{1}{2}\cos\left(2\sqrt{nc}-\pi\alpha-\pi+2\sqrt{c\,(n-2)}\right) - \frac{1}{2}\cos\left(2\sqrt{c\,(n-2)}-2\sqrt{nc}-\frac{\pi}{2}\right). \tag{3.2.21}$$

The last term on the right hand side is

$$\frac{-1}{2}\cos\left(2\sqrt{(n-2)c}-2\sqrt{nc}-\frac{\pi}{2}\right) = \frac{1}{2}\sin\left(2\sqrt{nc}-2\sqrt{(n-2)c}\right),$$

which behaves with $n$ as follows

$$\lim_{n\to\infty}\frac{\sqrt{n}}{2}\sin\left(2\sqrt{nc}-2\sqrt{(n-2)c}\right) = \lim_{n\to\infty}\frac{\sqrt{n}}{2}\frac{\sin\left(2\sqrt{nc}-2\sqrt{(n-2)c}\right)}{2\sqrt{nc}-2\sqrt{(n-2)c}}(2\sqrt{nc}-2\sqrt{(n-2)c}) = \sqrt{c},$$

and, therefore,

$$\frac{1}{2}\sin\left(2\sqrt{(n-2)c}-2\sqrt{nc}\right)\sim\sqrt{\frac{c}{n}}. \tag{3.2.22}$$

Next we study

$$\frac{1}{2}\cos\left(4\sqrt{c\,(n-1)}-\pi\alpha-\pi\right)-\frac{1}{2}\cos\left(2\sqrt{nc}-\pi\alpha-\pi+2\sqrt{c\,(n-2)}\right) \tag{3.2.23}$$

in (3.2.21). Using

$$\cos a-\cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),$$

(3.2.23) becomes

$$-\sin\left(\sqrt{cn}-\pi\alpha-\pi+2\sqrt{c\,(n-1)}+\sqrt{c\,(n-2)}\right) \cdot\sin\left(2\sqrt{c\,(n-1)}-\sqrt{cn}-\sqrt{c\,(n-2)}\right)$$

where the first factor is bounded, and the second one verifies

$$\lim_{n\to\infty}\sqrt{n}\sin\left(2\sqrt{c\,(n-1)}-\sqrt{cn}-\sqrt{c\,(n-2)}\right)=0. \tag{3.2.24}$$

From (3.2.22) and (3.2.24), we conclude

$$\Theta_n(c;\alpha)\sim\pi^{-1}e^c c^{-\frac{1}{2}-\alpha}\,n^{-1/2}.$$

On the other hand, from the Stirling's formula for the Gamma function, we deduce

$$\frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \sim n^{1-\alpha}, \tag{3.2.25}$$

under the above assumptions we get

$$K_{n-1}(c,c) \sim \pi^{-1} e^{c} c^{-\frac{1}{2}-\alpha}\, n^{1/2}, \quad c \in \mathbb{R}_{+}.$$

Next, we can proceed as above and we obtain the asymptotic behavior given in (3.2.16). For $n$ large enough, we get

$$K_{n-1}^{(0,1)}(c,c) \sim \frac{1}{2}\frac{\Gamma(n+1)}{\Gamma(n+\alpha)} n^{\alpha+\frac{1}{2}} \Psi_n(c;\alpha), \tag{3.2.26}$$

where

$$\Psi_n(c;\alpha) = \sigma^{\alpha}(c)\sigma^{\alpha+2}(c)\left[\cos\varphi_n^{\alpha}(c)\cos\varphi_{n-3}^{\alpha+2}(c) - \cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n-2}^{\alpha+2}(c)\right].$$

The expression in square brackets can be rewritten as

$$\begin{aligned} &-\sin\left(\sqrt{cn} - \pi\alpha - \frac{3}{2}\pi + \sqrt{c\,(n-1)} + \sqrt{c\,(n-2)} + \sqrt{c\,(n-3)}\right) \\ &\qquad \cdot \sin\left(\sqrt{cn} - \sqrt{c\,(n-1)} - \sqrt{c\,(n-2)} + \sqrt{c\,(n-3)}\right) \\ &-\sin\left(\pi + \sqrt{cn} + \sqrt{c\,(n-1)} - \sqrt{c\,(n-2)} - \sqrt{c\,(n-3)}\right) \\ &\qquad \cdot \sin\left(\sqrt{cn} - \sqrt{c\,(n-1)} + \sqrt{c\,(n-2)} - \sqrt{c\,(n-3)}\right), \end{aligned}$$

where

$$\begin{aligned} \lim_{n\to\infty} \Big[ -n \sin\left(\sqrt{cn} - \pi\alpha - \tfrac{3}{2}\pi + \sqrt{c\,(n-1)} + \sqrt{c\,(n-2)} + \sqrt{c\,(n-3)}\right) \\ \cdot \sin\left(\sqrt{cn} - \sqrt{c\,(n-1)} - \sqrt{c\,(n-2)} + \sqrt{c\,(n-3)}\right)\Big] = 0, \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} \Big[ -n \sin\left(\sqrt{cn} - \sqrt{c\,(n-1)} + \sqrt{c\,(n-2)} - \sqrt{c\,(n-3)}\right) \\ \cdot \sin\left(\pi + \sqrt{cn} + \sqrt{c\,(n-1)} - \sqrt{c\,(n-2)} - \sqrt{c\,(n-3)}\right)\Big] = 2c. \end{aligned}$$

As a consequence, taking into account (3.2.33), we get

$$\Psi_n(c;\alpha) \sim \pi^{-1} e^{c} c^{-\alpha-\frac{3}{2}} \cdot 2c\, n^{-1}.$$

Replacing the above expression in (3.2.26) and using again (3.2.25), we conclude

$$K_{n-1}^{(0,1)}(c,c) \sim \pi^{-1} e^{c} c^{-\frac{1}{2}-\alpha}\, n^{1/2}.$$

Finally,

$$K_{n-1}^{(1,1)}(c,c) \sim \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} n^{\alpha+1} \left( \frac{1}{3!} \Lambda_{1,n}(c;\alpha) + \frac{1}{2!} \Lambda_{2,n}(c;\alpha) \right), \tag{3.2.27}$$

where

$$\Lambda_{1,n}(c;\alpha) = \sigma^{\alpha}(c)\sigma^{\alpha+3}(c) \left[ \cos\varphi_{n-3}^{\alpha+3}(c)\cos\varphi_{n-1}^{\alpha}(c) - \cos\varphi_{n-4}^{\alpha+3}(c)\cos\varphi_{n}^{\alpha}(c) \right], \tag{3.2.28}$$

$$\Lambda_{2,n}(c;\alpha) = \sigma^{\alpha+1}(c)\sigma^{\alpha+2}(c) \left[ \cos\varphi_{n-2}^{\alpha+2}(c)\cos\varphi_{n-2}^{\alpha+1}(c) - \cos\varphi_{n-3}^{\alpha+2}(c)\cos\varphi_{n-1}^{\alpha+1}(c) \right]. \tag{3.2.29}$$

The two expressions in square brackets of (3.2.28) and (3.2.29) can be rewritten, respectively, as follows

$$\begin{aligned} &-\sin\left(\sqrt{nc} - \pi\alpha - 2\pi + \sqrt{c\,(n-1)} + \sqrt{c\,(n-3)} + \sqrt{c\,(n-4)}\right) \\ &\qquad \cdot \sin\left(\sqrt{c\,(n-1)} - \sqrt{nc} + \sqrt{c\,(n-3)} - \sqrt{c\,(n-4)}\right) \\ &-\sin\left(\sqrt{c\,(n-3)} - \sqrt{nc} - \sqrt{c\,(n-1)} - \frac{3}{2}\pi + \sqrt{c\,(n-4)}\right) \\ &\qquad \cdot \sin\left(\sqrt{nc} - \sqrt{c\,(n-1)} + \sqrt{c\,(n-3)} - \sqrt{c\,(n-4)}\right), \end{aligned}$$

$$\begin{aligned} &-\sin\left(\sqrt{c\,(n-1)} - \pi\alpha - 2\pi + 2\sqrt{c\,(n-2)} + \sqrt{c\,(n-3)}\right) \\ &\cdot \sin\left(2\sqrt{c\,(n-2)} - \sqrt{c\,(n-1)} - \sqrt{c\,(n-3)}\right) \\ &-\frac{1}{2}\cos\left(2\sqrt{c\,(n-3)} - 2\sqrt{c\,(n-1)} - \frac{1}{2}\pi\right), \end{aligned}$$

where the terms of each sumand in the above expresions have the following behavior

$$\begin{aligned} \lim_{n\to\infty} \Big[ &-n^{\frac{1}{2}} \sin\left(\sqrt{nc} - \pi\alpha - 2\pi + \sqrt{c\,(n-1)} + \sqrt{c\,(n-3)} + \sqrt{c\,(n-4)}\right) \\ &\cdot \sin\left(\sqrt{c\,(n-1)} - \sqrt{nc} + \sqrt{c\,(n-3)} - \sqrt{c\,(n-4)}\right) \Big] = 0, \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \Big[ &-n^{\frac{1}{2}} \sin\left(\sqrt{c\,(n-3)} - \sqrt{nc} - \sqrt{c\,(n-1)} - \tfrac{3}{2}\pi + \sqrt{c\,(n-4)}\right) \\ &\cdot \sin\left(\sqrt{nc} - \sqrt{c\,(n-1)} + \sqrt{c\,(n-3)} - \sqrt{c\,(n-4)}\right) \Big] = -\sqrt{c}, \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} \Big[ &-n^{\frac{1}{2}} \sin\left(\sqrt{c\,(n-1)} - \pi\alpha - 2\pi + 2\sqrt{c\,(n-2)} + \sqrt{c\,(n-3)}\right) \\ &\cdot \sin\left(2\sqrt{c\,(n-2)} - \sqrt{c\,(n-1)} - \sqrt{c\,(n-3)}\right) \Big] = 0, \end{aligned}$$

and

$$\lim_{n\to\infty}\left(n^{\frac{1}{2}}\left(-\frac{1}{2}\cos\left(2\sqrt{c\,(n-3)}-2\sqrt{c\,(n-1)}-\frac{1}{2}\pi\right)\right)\right)=\sqrt{c}.$$

Hence, using again (3.2.33), we have

$$\begin{aligned}\Lambda_{1,n}(c;\alpha) &\sim -\pi^{-1}e^{c}c^{-2-\alpha}\sqrt{c}\,n^{-1/2},\\ \Lambda_{2,n}(c;\alpha) &\sim \pi^{-1}e^{c}c^{-2-\alpha}\sqrt{c}\,n^{-1/2}.\end{aligned}$$

Therefore

$$\left(\frac{1}{3!}\Lambda_{1,n}(c;\alpha)+\frac{1}{2!}\Lambda_{2,n}(c;\alpha)\right)\sim\frac{1}{3}\pi^{-1}e^{c}c^{-\alpha-\frac{3}{2}}\,n^{-1/2}.$$

Replacing in (3.2.27) we conclude,

$$K_{n-1}^{(1,1)}(c,c)\sim\frac{1}{3}\pi^{-1}e^{c}c^{-\frac{3}{2}-\alpha}\,n^{3/2}.$$

□

Now, we are ready to approach the general case. The results will be obtained just by generalizing the previous technique. In the next result, we show a confluent form for the partial derivatives of the kernel polynomial $K_{n-1}(x,y)$ at the point $x=y=c$.

**Proposition 3.2.4.** *For every $n\in\mathbb{N}$ and $0\leqslant j,k\leqslant n-1$ we have*

$$\begin{aligned}K_{n-1}^{(k,j)}(c,c)=\frac{j!k!}{(j+k+1)!||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2}}\Big[\sum_{l=0}^{j}\binom{j+k+1}{l}\big([\widehat{L}_{n-1}^{\alpha}]^{(l)}(c)\,[\widehat{L}_{n}^{\alpha}]^{(j+k+1-l)}(c)\\ -[\widehat{L}_{n}^{\alpha}]^{(l)}(c)\,[\widehat{L}_{n-1}^{\alpha}]^{(j+k+1-l)}(c)\big)\Big].\end{aligned}\quad (3.2.30)$$

*Proof.* For $k=0$ and $0\leqslant j\leqslant n-1$ it suffices to follow a standard technique in literature (see, for instance [2, p. 269]) by taking derivatives in (1.6.24) with respect to the variable $y$ and then to evaluate it at $y=c$. Thus we obtain

$$K_{n-1}^{(0,j)}(x,c)=\frac{j!}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2}(x-c)^{j+1}}\left(T_j(x,c;\widehat{L}_{n-1}^{\alpha})\widehat{L}_{n}^{\alpha}(x)-T_j(x,c;\widehat{L}_{n}^{\alpha})\widehat{L}_{n-1}^{\alpha}(x)\right),\quad (3.2.31)$$

where $T_j(x,c;f)$ is the $j$-th Taylor polynomial of $f$ in $c$.

Using the Taylor expansion of $\widehat{L}_{n}^{\alpha}(x)$ and $\widehat{L}_{n-1}^{\alpha}(x)$ in (3.2.31), we only need to look for the coefficients of $(x-c)^{j+k+1}$ there for finding $K_{n-1}^{(k,j)}(c,c)$.

□

Now, we continue by recalling that when $p = 1$ in (1.6.27), we have $A_0(x) = 1$ and $B_0(x) = 0$. Thus, we obtain the behavior of $\widehat{L}_n^\alpha(x)$ for $n$ large enough, when $x \in \mathbb{R}_+$

$$\widehat{L}_n^\alpha(x) = (-1)^n \Gamma(n+1) \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4}$$
$$\cdot \cos\{2\,(nx)^{1/2} - \alpha\pi/2 - \pi/4\} \cdot (1 + \mathcal{O}(n^{-1/2})).$$

Then, we can rewrite the above expression as

$$\widehat{L}_n^\alpha(x) = (-1)^n \Gamma(n+1) n^{\frac{\alpha}{2}-\frac{1}{4}} \sigma^\alpha(x) \cos\varphi_n^\alpha(x)(1 + \mathcal{O}(n^{-1/2})), \tag{3.2.32}$$

where

$$\varphi_n^\alpha(x) = 2(nx)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4},$$

and

$$\sigma^\alpha(x) = \pi^{-1/2} e^{x/2} x^{-\alpha/2-1/4}, \tag{3.2.33}$$

is a function independent of $n$.

Now our task is to find the asymptotic behavior of the diagonal Laguerre kernels. In order to do this we have to estimate expressions of the following kind:

$$\cos\varphi_{n-n_1}^\alpha(c)\cos\varphi_{n-n_2}^{\alpha+i}(c) - \cos\varphi_{n-n_3}^{\alpha+i}(c)\cos\varphi_n^\alpha(c).$$

Under some conditions on the parameters $i, n_1, n_2$, and $n_3$ we can prove that the above expression tends to zero when $n$ tends to infinity and, moreover, we can compute its speed of convergence. The result reads as follows.

**Lemma 3.2.2.** *Let* $\underline{m} = (i, n_1, n_2, n_3) \in \mathbb{N}^4$ *be a multi-index such that* $n_3 = n_1 + n_2$. *For* $\alpha > -1$ *and* $c \in \mathbb{R}_+$ *let us consider the function*

$$F_{\underline{m}}^{\alpha,c}(n) := \cos\varphi_{n-n_1}^\alpha(c)\cos\varphi_{n-n_2}^{\alpha+i}(c) - \cos\varphi_{n-n_3}^{\alpha+i}(c)\cos\varphi_n^\alpha(c). \tag{3.2.34}$$

*Then, the following asymptotic behavior holds.*

$$F_{\underline{m}}^{\alpha,c}(n) \sim \begin{cases} \frac{-1}{4}(n_2 - n_1 - n_3)(n_2 - n_1 + n_3)c\,n^{-1} & \text{if } i \equiv 0 \mod 4, \\ \frac{-1}{2}(n_2 - n_1 - n_3)\sqrt{c}\,n^{-1/2} & \text{if } i \equiv 1 \mod 4, \\ \frac{1}{4}(n_2 - n_1 - n_3)(n_2 - n_1 + n_3)c\,n^{-1} & \text{if } i \equiv 2 \mod 4, \\ \frac{1}{2}(n_2 - n_1 - n_3)\sqrt{c}\,n^{-1/2} & \text{if } i \equiv 3 \mod 4. \end{cases}$$

*Proof.* From

$$\cos(a)\cos(b) = \frac{\cos(a+b) + \cos(a-b)}{2},$$

and

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),$$

we obtain

$$F_{\underline{m}}^{\alpha,c}(n) =$$
$$\frac{1}{2}\cos\left(2\sqrt{c(n-n_1)} + 2\sqrt{c(n-n_2)} - \alpha\pi - \frac{(i+1)\pi}{2}\right) + \frac{1}{2}\cos\left(2\sqrt{c(n-n_1)} - 2\sqrt{c(n-n_2)} + \frac{i\pi}{2}\right)$$
$$-\frac{1}{2}\cos\left(2\sqrt{c(n-n_3)} + 2\sqrt{cn} - \alpha\pi - \frac{(i+1)\pi}{2}\right) - \frac{1}{2}\cos\left(2\sqrt{c(n-n_3)} - 2\sqrt{cn} - \frac{i\pi}{2}\right)$$
$$= f_{\underline{m}}^{\alpha,c}(n) + g_{\underline{m}}^{c}(n),$$

where

$$f_{\underline{m}}^{\alpha,c}(n) = -\sin\left(\sqrt{cn} + \sqrt{c(n-n_1)} + \sqrt{c(n-n_2)} + \sqrt{c(n-n_3)} - \alpha\pi - \frac{(i+1)\pi}{2}\right)$$
$$\times \sin\left(\sqrt{c(n-n_2)} + \sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} - \sqrt{cn}\right),$$

and

$$g_{\underline{m}}^{c}(n) = -\sin\left(\sqrt{c(n-n_1)} - \sqrt{c(n-n_2)} + \sqrt{c(n-n_3)} - \sqrt{cn}\right)$$
$$\times \sin\left(\sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} + \sqrt{cn} - \sqrt{c(n-n_2)} - \frac{i\pi}{2}\right).$$

Our first technical step will be to show that

$$\lim_{n\to\infty} n^{3/2}\sin\left(\sqrt{c(n-n_2)} + \sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} - \sqrt{cn}\right) = \frac{n_1 n_2\sqrt{c}}{4} \neq 0. \tag{3.2.35}$$

Notice that the function

$$h_{\underline{m}}(n) = \left(\sqrt{n} - \sqrt{n-n_2}\right) - \left(\sqrt{n-n_1} - \sqrt{n-n_3}\right)$$

can be written as

$$h_{\underline{m}}(n) = k(n) - k(n-n_1), \quad \text{with } k(n) = \sqrt{n} - \sqrt{n-n_2}.$$

Next, using the mean value theorem, we obtain

$$h_{\underline{m}}(n) = n_1 k'(\xi_n) = \frac{n_1}{2}\left(\frac{1}{\sqrt{\xi_n}} - \frac{1}{\sqrt{\xi_n - n_2}}\right), \quad \text{where } n - n_1 \leqslant \xi_n \leqslant n.$$

Denoting $l(n) = \frac{1}{\sqrt{n}}$, we can apply again the mean value theorem in order to obtain

$$h_{\underline{m}}(n) = \frac{n_1}{2}\left(l(\xi_n) - l(\xi_n - n_2)\right) = \frac{n_1}{2} n_2 l'(\delta_n) = \frac{-n_1 n_2}{2}\delta_n^{-3/2},$$

where $n - n_1 - n_2 \leqslant \xi_n - n_2 \leqslant \delta_n \leqslant \xi_n \leqslant n$.

Taking into account that $\lim_{n\to\infty} \frac{\delta_n}{n} = 1$, we get (3.2.35). Since the first factor in $f_{\underline{m}}^{\alpha,c}(n)$ is bounded, we obtain

$$\lim_{n\to\infty} n^{1/2} f_{\underline{m}}^{\alpha,c}(n) = 0, \tag{3.2.36}$$

$$\lim_{n\to\infty} n f_{\underline{m}}^{\alpha,c}(n) = 0. \tag{3.2.37}$$

Our second technical step will be to show that the speed of convergence of the first factor in $g_{\underline{m}}^{c}(n)$ is $n^{-1/2}$:

$$\begin{aligned}
&\lim_{n\to\infty} \sqrt{n}\sin\left(\sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} + \sqrt{cn} - \sqrt{c(n-n_2)}\right) = \\
&\lim_{n\to\infty} \sqrt{c}\sqrt{n}\left(\sqrt{n-n_1} - \sqrt{n-n_3} + \sqrt{n} - \sqrt{n-n_2}\right) = \\
&\lim_{n\to\infty} \sqrt{c}\left(\frac{\sqrt{n}(n-n_1-(n-n_3))}{\sqrt{n-n_1}+\sqrt{n-n_3}} + \frac{\sqrt{n}(n-(n-n_2))}{\sqrt{n-n_2}+\sqrt{n}}\right) = \\
&\lim_{n\to\infty} \sqrt{c}\left(\frac{(n_3-n_1)}{\sqrt{1-\frac{n_1}{n}}+\sqrt{1-\frac{n_3}{n}}} + \frac{n_2}{\sqrt{1-\frac{n_2}{n}+1}}\right) = \\
&= \frac{1}{2}\sqrt{c}\,(n_3 - n_1 + n_2) \neq 0.
\end{aligned}$$

Then, in order to deduce the speed of convergence of $g_{\underline{m}}^{c}(n)$ we will analyze the following four cases:

(i) If $i \equiv 0 \mod 4$, using that $\sin(x - 2\pi) = \sin(x)$, then

$$\lim_{n\to\infty} n g_{\underline{m}}^{c}(n) = \frac{-1}{4}c(n_2 - n_1 - n_3)(n_2 - n_1 + n_3) \neq 0.$$

(ii) If $i \equiv 1 \mod 4$, using that $\sin(x - \frac{\pi}{2}) = \cos(x)$, then

$$\lim_{n\to\infty} n^{1/2} g_{\underline{m}}^{c}(n) = \frac{-1}{2}\sqrt{c}(n_2 - n_1 - n_3) \neq 0.$$

(iii) If $i \equiv 2 \mod 4$, using that $\sin(x - \pi) = -\sin(x)$, then

$$\lim_{n\to\infty} n g_{\underline{m}}^{c}(n) = \frac{1}{4}c(n_2 - n_1 - n_3)(n_2 - n_1 + n_3) \neq 0.$$

(iv) If $i \equiv 3 \mod 4$, using that $\sin(x - \frac{3\pi}{2}) = -\cos(x)$, then

$$\lim_{n\to\infty} n^{1/2} g^c_{\underline{m}}(n) = \frac{1}{2}\sqrt{c}(n_2 - n_1 - n_3) \neq 0.$$

The above analysis together (3.2.36) yields the statement of Lemma. □

**Theorem 3.2.1.** *For $c \in \mathbb{R}_+$, the partial derivatives of the diagonal Laguerre kernels satisfy the following asymptotics.*

$$K^{(k,j)}_{n-1}(c,c) \sim \begin{cases} C_0\, n^{\frac{j+k+1}{2}} & \text{if } j+k \equiv 0 \mod 2, \\ C_1\, n^{\frac{j+k}{2}} & \text{if } j+k \equiv 1 \mod 2, \end{cases}$$

*where $0 \leqslant j, k \leqslant n-1$ and*

$$C_0 = (-1)^{\frac{j+k}{2}} \frac{j!k!}{(k+j+1)!} \sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c)\sqrt{c} \sum_{l=0}^{j} \binom{j+k+1}{l} (-1)^l,$$
$$C_1 = (-1)^{\frac{j+k-1}{2}} \frac{j!k!}{(k+j+1)!} \sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c)\, c \sum_{l=0}^{j} \binom{j+k+1}{l} (j+k+1-2l)(-1)^{l+1}.$$

*Proof.* Without loss of generality, we can suppose that $j \leqslant k$. From (1.6.22) and (3.2.30), we obtain

$$K^{(k,j)}_{n-1}(c,c) = \frac{j!k!\, n^{j+k+1}}{(j+k+1)!\, ||\widehat{L}^{\alpha}_{n-1}||^2_{\alpha}} \times \sum_{l=0}^{j} \binom{j+k+1}{l} \left( \widehat{L}^{\alpha+l}_{n-1-l}(c) \widehat{L}^{\alpha+j+k+1-l}_{n-j-k-1+l}(c) - \widehat{L}^{\alpha+l}_{n-l}(c) \widehat{L}^{\alpha+j+k+1-l}_{n-j-k-2+l}(c) \right). \tag{3.2.38}$$

Now, using (1.6.21) and (3.2.32), we get

$$K^{(k,j)}_{n-1}(c,c) \sim \sum_{l=0}^{j} \binom{j+k+1}{l} (-1)^{j+k} \frac{j!k!}{(j+k+1)!} \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} \sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c) n^{\alpha+\frac{j+k}{2}} \{\cos\varphi^{\alpha+l}_{n-1-l}(c)\cos\varphi^{\alpha+j+k+1-l}_{n-j-k-1+l}(c) - \cos\varphi^{\alpha+l}_{n-l}(c)\cos\varphi^{\alpha+j+k+1-l}_{n-j-k-2+l}(c)\}.$$

From Lemma 3.2.2, we can express the above formula as follows.

$$K^{(k,j)}_{n-1}(c,c) \sim \sum_{l=0}^{j} \binom{j+k+1}{l} (-1)^{j+k} \frac{j!k!}{(j+k+1)!} \sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c) n^{\frac{j+k}{2}+1} F^{\alpha+l,c}_{\underline{m}}(n-l),$$

where $\underline{m} = (j+k+1-2l; 1, j+k+1-2l, j+k+2-2l)$.

Then, for all $l = 0, \ldots, j$, we get

$$F_{\underline{m}}^{\alpha+l,c}(n-l) \sim \begin{cases} (j+k+1-2l)cn^{-1} & \text{if } j+k+1-2l \equiv 0 \mod 4, \\ \sqrt{c}n^{-1/2} & \text{if } j+k+1-2l \equiv 1 \mod 4, \\ -(j+k+1-2l)cn^{-1} & \text{if } j+k+1-2l \equiv 2 \mod 4, \\ -\sqrt{c}n^{-1/2} & \text{if } j+k+1-2l \equiv 3 \mod 4, \end{cases}$$

or, equivalently,

$$F_{\underline{m}}^{\alpha+l,c}(n-l) \sim \begin{cases} (-1)^l(j+k+1-2l)cn^{-1} & \text{if } j+k+1 \equiv 0 \mod 4, \\ (-1)^l\sqrt{c}n^{-1/2} & \text{if } j+k+1 \equiv 1 \mod 4, \\ (-1)^{l+1}(j+k+1-2l)cn^{-1} & \text{if } j+k+1 \equiv 2 \mod 4, \\ (-1)^{l+1}\sqrt{c}n^{-1/2} & \text{if } j+k+1 \equiv 3 \mod 4. \end{cases}$$

Since the above relation can be reduced as follows,

$$F_{\underline{m}}^{\alpha+l,c}(n-l) \sim \begin{cases} (-1)^{l+\frac{j+k}{2}}\sqrt{c}n^{-1/2} & \text{if } j+k \equiv 0 \mod 2, \\ (-1)^{l+\frac{j+k-1}{2}}(j+k+1-2l)cn^{-1} & \text{if } j+k \equiv 1 \mod 2, \end{cases}$$

we get the statement of Theorem.

□

**Remark 3.2.1.** *Notice that Theorem 3.2.1 generalizes the asymptotic behavior of the diagonal Laguerre kernels given in [38], where only the case* $0 \leqslant j, k \leqslant 1$ *has been analyzed. The interested reader can find the analogous of Theorem 3.2.1 when* $c = 0$, $0 \leqslant j, k \leqslant 1$, *and* $c = 0$, $0 \leqslant j, k \leqslant n-1$, *in [18, 83], respectively. Also, it is worthwhile to point out that, with a different approach, the authors of [42] obtained a lower bound for the Christoffel functions in the case* $c \geqslant 0$.

# 4

# Unbounded support: asymptotics for Laguerre-Sobolev type polynomials

As it was already mentioned, recent works have focused the attention on the study of asymptotic properties of sequences of orthogonal polynomials with respect to specific cases of the inner product (1.0.2) with ‘mass points outside’ or ‘mass points inside’ of *supp*$\mu$, being *supp*$\mu$ a bounded interval of the real line. However, to the best of our knowledge, asymptotic properties of the sequences of orthogonal polynomials associated with (1.0.2) in the case of nontrivial probability measures with an unbounded support on the real line and mass points inside the support of the measure are not available in the literature.

In this Chapter, we carry out a wide study of asymptotic properties of a representative family of Sobolev polynomials orthogonal with respect to an inner product with unbounded support, the Laguerre-Sobolev type polynomials. Taking into account the results of 3.2.1 concerning the asymptotic behavior of the diagonal Laguerre Kernels, in Section 4.1 we prove the outer relative asymptotic of the Laguerre-Sobolev type orthogonal polynomials modified into the *positive* real semiaxis, i.e. a family of Laguerre-Sobolev polynnomials orthogonal with respect to an inner product with mass points located inside the support of the measure. In Section 4.2 we deduce the limit behavior of the coefficients of the corresponding five-term recurrence relation. Finally, in Section 4.3 we study the inner relative asymptotics of Laguerre-Sobolev type orthogonal polynomials when the mass points are also inside the support of the measure.

## 4.1 Outer relative asymptotics

In this section we deal with sequences of polynomials orthogonal with respect to a particular case of (1.0.2). Indeed, $\mu$ is the Gamma measure corresponding to classical Laguerre orthogonal polynomials and

$$\langle f, g \rangle_S = \int_0^\infty f(x)g(x)x^\alpha e^{-x}dx + \mathbb{F}(c)A\mathbb{G}(c)^t, \alpha > -1, \tag{4.1.1}$$

$f, g \in \mathbb{P}$. The matrix $A$ and the vectors $\mathbb{F}(c)$, $\mathbb{G}(c)$ are

$$A = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}, \quad \mathbb{F}(c) = (f(c), f'(c)) \text{ and } \mathbb{G}(c) = (g(c), g'(c)), \text{ respectively,}$$

$M, N \in \mathbb{R}_+$, and the mass point $c$ is located inside the oscillatory region for the classical Laguerre polynomials, i.e., $c > 0$.

The main result of this section will be the outer relative asymptotics for the Laguerre-Sobolev type polynomials $\widehat{S}_n^{M,N}(x)$, orthogonal with respect to (4.1.1), when $c \in \mathbb{R}_+$. The proof will naturally falls in several parts, which will be established through an appropriate sequence of Lemmas.

First, we will present a well known expansion of the monic polynomials $\widehat{S}_n^{M,N}(x)$ in terms of classical Laguerre polynomials $\widehat{L}_n^\alpha(x)$. The most usual way to represent the Laguerre-Sobolev type orthogonal polynomials $\widehat{S}_n^{M,N}(x)$ is using the Laguerre kernel and its derivatives as follows (see [64] and Theorem 5.1 in [37]).

$$(x-c)^2 \widehat{S}_n^{M,N}(x) = A(n;x)\widehat{L}_n^\alpha(x) + B(n;x)\widehat{L}_{n-1}^\alpha(x), \tag{4.1.2}$$

where

$$\begin{aligned} A(n;x) &= (x-c)^2 + (x-c)A_1(n;c) + A_0(n;c), \\ B(n;x) &= (x-c)B_1(n;c) + B_0(n;c), \end{aligned} \tag{4.1.3}$$

with

$$\begin{aligned} A_1(n;c) &= -\frac{M\widehat{S}_n^{M,N}(c)\widehat{L}_{n-1}^\alpha(c)}{||\widehat{L}_{n-1}^\alpha||_\alpha^2} - \frac{N[\widehat{S}_n^{M,N}]'(c)[\widehat{L}_{n-1}^\alpha]'(c)}{||\widehat{L}_{n-1}^\alpha||_\alpha^2}, \\ A_0(n;c) &= -\frac{N[\widehat{S}_n^{M,N}]'(c)\widehat{L}_{n-1}^\alpha(c)}{||\widehat{L}_{n-1}^\alpha||_\alpha^2}, \\ B_1(n;c) &= \frac{M\widehat{S}_n^{M,N}(c)\widehat{L}_n^\alpha(c)}{||\widehat{L}_{n-1}^\alpha||_\alpha^2} + \frac{N[\widehat{S}_n^{M,N}]'(c)[\widehat{L}_n^\alpha]'(c)}{||\widehat{L}_{n-1}^\alpha||_\alpha^2}, \\ B_0(n;c) &= \frac{N[\widehat{S}_n^{M,N}]'(c)\widehat{L}_n^\alpha(c)}{||\widehat{L}_{n-1}^\alpha||_\alpha^2}. \end{aligned} \tag{4.1.4}$$

Notice that

$$\widehat{S}_n^{M,N}(c) = \frac{\begin{vmatrix} \widehat{L}_n^{\alpha}(c) & NK_{n-1}^{(0,1)}(c,c) \\ [\widehat{L}_n^{\alpha}]'(c) & 1+NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}{\begin{vmatrix} 1+MK_{n-1}(c,c) & NK_{n-1}^{(0,1)}(c,c) \\ MK_{n-1}^{(1,0)}(c,c) & 1+NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}, \tag{4.1.5}$$

$$[\widehat{S}_n^{M,N}]'(c) = \frac{\begin{vmatrix} 1+MK_{n-1}(c,c) & \widehat{L}_n^{\alpha}(c) \\ MK_{n-1}^{(1,0)}(c,c) & [\widehat{L}_n^{\alpha}]'(c) \end{vmatrix}}{\begin{vmatrix} 1+MK_{n-1}(c,c) & NK_{n-1}^{(0,1)}(c,c) \\ MK_{n-1}^{(1,0)}(c,c) & 1+NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}. \tag{4.1.6}$$

We will analyze the polynomial coefficients in the above expansion in order to obtain the desired results. If we replace (4.1.5) and (4.1.6) in (4.1.4), we obtain

$$\begin{aligned} A_1(n;c) = {} & \frac{-M\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_n^{\alpha}(c)-MN\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_n^{\alpha}(c)K_{n-1}^{(1,1)}(c,c)+MNn\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_{n-1}^{\alpha+1}(c)K_{n-1}^{(0,1)}(c,c)}{\|\widehat{L}_{n-1}^{\alpha}\|_{\alpha}^2\left(1+MK_{n-1}(c,c)+NK_{n-1}^{(1,1)}(c,c)+MNK_{n-1}(c,c)K_{n-1}^{(1,1)}(c,c)-MNK_{n-1}^{(0,1)}(c,c)K_{n-1}^{(1,0)}(c,c)\right)} \\ & + \frac{\left(-Nn^2\widehat{L}_{n-2}^{\alpha+1}(c)\widehat{L}_{n-1}^{\alpha+1}(c)-MNn^2\widehat{L}_{n-2}^{\alpha+1}(c)\widehat{L}_{n-1}^{\alpha+1}(c)K_{n-1}(c,c)+MNn\widehat{L}_{n-2}^{\alpha+1}(c)\widehat{L}_n^{\alpha}(c)K_{n-1}^{(1,0)}(c,c)\right)}{\|\widehat{L}_{n-1}^{\alpha}\|_{\alpha}^2\left(1+MK_{n-1}(c,c)+NK_{n-1}^{(1,1)}(c,c)+MNK_{n-1}(c,c)K_{n-1}^{(1,1)}(c,c)-MNK_{n-1}^{(0,1)}(c,c)K_{n-1}^{(1,0)}(c,c)\right)}, \end{aligned}$$

$$A_0(n;c) = \frac{-Nn\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_{n-1}^{\alpha+1}(c)-MNn\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_{n-1}^{\alpha+1}(c)K_{n-1}(c,c)+MN\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_n^{\alpha}(c)K_{n-1}^{(1,0)}(c,c)}{\|\widehat{L}_{n-1}^{\alpha}\|_{\alpha}^2\left(1+MK_{n-1}(c,c)+NK_{n-1}^{(1,1)}(c,c)+MNK_{n-1}(c,c)K_{n-1}^{(1,1)}(c,c)-MNK_{n-1}^{(0,1)}(c,c)K_{n-1}^{(1,0)}(c,c)\right)},$$

$$\begin{aligned} B_1(n;c) = {} & \frac{M\widehat{L}_n^{\alpha}(c)\widehat{L}_n^{\alpha}(c)+MN\widehat{L}_n^{\alpha}(c)\widehat{L}_n^{\alpha}(c)K_{n-1}^{(1,1)}(c,c)-MNn\widehat{L}_n^{\alpha}(c)\widehat{L}_{n-1}^{\alpha+1}(c)K_{n-1}^{(0,1)}(c,c)}{\|\widehat{L}_{n-1}^{\alpha}\|_{\alpha}^2\left(1+MK_{n-1}(c,c)+NK_{n-1}^{(1,1)}(c,c)+MNK_{n-1}(c,c)K_{n-1}^{(1,1)}(c,c)-MNK_{n-1}^{(0,1)}(c,c)K_{n-1}^{(1,0)}(c,c)\right)} \\ & + \frac{Nn^2\widehat{L}_{n-1}^{\alpha+1}(c)\widehat{L}_{n-1}^{\alpha+1}(c)+MNn^2\widehat{L}_{n-1}^{\alpha+1}(c)\widehat{L}_{n-1}^{\alpha+1}(c)K_{n-1}(c,c)-MNn\widehat{L}_{n-1}^{\alpha+1}(c)\widehat{L}_n^{\alpha}(c)K_{n-1}^{(1,0)}(c,c)}{\|\widehat{L}_{n-1}^{\alpha}\|_{\alpha}^2\left(1+MK_{n-1}(c,c)+NK_{n-1}^{(1,1)}(c,c)+MNK_{n-1}(c,c)K_{n-1}^{(1,1)}(c,c)-MNK_{n-1}^{(0,1)}(c,c)K_{n-1}^{(1,0)}(c,c)\right)}, \end{aligned}$$

$$B_0(n;c) = \frac{Nn\widehat{L}_n^{\alpha}(c)\widehat{L}_{n-1}^{\alpha+1}(c)+MNn\widehat{L}_n^{\alpha}(c)\widehat{L}_{n-1}^{\alpha+1}(c)K_{n-1}(c,c)-MN\widehat{L}_n^{\alpha}(c)\widehat{L}_n^{\alpha}(c)K_{n-1}^{(1,0)}(c,c)}{\|\widehat{L}_{n-1}^{\alpha}\|_{\alpha}^2\left(1+MK_{n-1}(c,c)+NK_{n-1}^{(1,1)}(c,c)+MNK_{n-1}(c,c)K_{n-1}^{(1,1)}(c,c)-MNK_{n-1}^{(0,1)}(c,c)K_{n-1}^{(1,0)}(c,c)\right)}.$$

Using (3.2.32) and the estimates in Lemma 3.2.1, we can compute the asymptotic behavior of the previous expressions as follows.

$$A_1(n;c) \sim \frac{1}{Nc\sigma^{\alpha+1}(c)\sigma^{\alpha+3}(c)} n^{-3/2}\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_n^{\alpha}(c) + \cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_n^{\alpha}(c)$$
$$+2\sqrt{c}n^{-1/2}\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c) + \frac{1}{M\sigma^{\alpha}(c)\sigma^{\alpha}(c)} n^{-1/2}\cos\varphi_{n-2}^{\alpha+1}(c)\cos\varphi_{n-1}^{\alpha+1}(c)$$
$$+\cos\varphi_{n-2}^{\alpha+1}(c)\cos\varphi_{n-1}^{\alpha+1}(c) + 2n^{-1/2}\cos\varphi_{n-2}^{\alpha+1}(c)\cos\varphi_n^{\alpha}(c),$$

$$A_0(n;c) \sim \frac{-1}{Mc\sigma^{\alpha}(c)\sigma^{\alpha+3}(c)} n^{-1}\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c) - c^{1/2}n^{-1/2}\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c)$$
$$-2n^{-1}\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_n^{\alpha}(c),$$

$$B_1(n;c) \sim \frac{1}{Nc\sigma^{\alpha+1}(c)\sigma^{\alpha+3}(c)} n^{-1/2}\cos\varphi_n^{\alpha}(c)\cos\varphi_n^{\alpha}(c) + n\cos\varphi_n^{\alpha}(c)\cos\varphi_n^{\alpha}(c)$$
$$+2\sqrt{c}n^{1/2}\cos\varphi_n^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c) + \frac{1}{M\sigma^{\alpha}(c)\sigma^{\alpha}(c)} n^{1/2}\cos\varphi_{n-1}^{\alpha+1}(c)\cos\varphi_{n-1}^{\alpha+1}(c)$$
$$+n\cos\varphi_{n-1}^{\alpha+1}(c)\cos\varphi_{n-1}^{\alpha+1}(c) + 2n^{1/2}\cos\varphi_{n-1}^{\alpha+1}(c)\cos\varphi_n^{\alpha}(c),$$

$$B_0(n;c) \sim \frac{-1}{Mc\sigma^{\alpha}(c)\sigma^{\alpha+3}(c)} \cos\varphi_n^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c) - c^{1/2}n^{1/2}\cos\varphi_n^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c)$$
$$-2\cos\varphi_n^{\alpha}(c)\cos\varphi_n^{\alpha}(c). \quad (4.1.7)$$

Due to the oscillatory behaviour of the cosine functions appearing in the preceding formulas, there are no real numbers $\beta_0$ and $\beta_1$ such that

$$A_0(n;c) \sim C_0 n^{\beta_0},$$
$$B_0(n;c) \sim C_1 n^{\beta_1},$$

for some $C_0$ and $C_1$.

However, we can describe the asymptotic behaviour of our coefficients functions in the following way:

**Proposition 4.1.1.** *Let* $A_0(n;c), A_1(n;c), B_0(n;c)$ *and* $B_1(n;c)$ *the functions defined by* (4.1.4). *Then, we have*

$$A_1(n;c) \sim 1, \qquad \lim_{n\to\infty} n^{\beta} A_0(n;c) = \begin{cases} 0 & \text{if } \beta < \frac{1}{2}, \\ \nexists & \text{if } \beta \geqslant \frac{1}{2}, \end{cases}$$

$$B_1(n;c) \sim n, \qquad \lim_{n\to\infty} n^{\beta} B_0(n;c) = \begin{cases} 0 & \text{if } \beta < -\frac{1}{2}, \\ \nexists & \text{if } \beta \geqslant -\frac{1}{2}. \end{cases}$$

*Proof.* The asymptotic behaviour of $A_0(n;c)$ and $B_0(n;c)$ is a straightforward consequence of the estimates in (4.1.7).

In order to obtain the asymptotics for $A_1(n;c)$ and $B_1(n;c)$, we joint up the terms

$$\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n}^{\alpha}(c)+\cos\varphi_{n-2}^{\alpha+1}(x)\cos\varphi_{n-1}^{\alpha+1}(c)=$$
$$\cos\left(2\sqrt{c(n-1)}+\sqrt{cn}+\sqrt{c(n-2)}-\alpha\pi-\pi\right)\cos\left(\sqrt{cn}-\sqrt{c(n-2)}+\frac{\pi}{2}\right)$$
$$+\frac{1}{2}\cos\left(2\sqrt{c(n-1)}-2\sqrt{cn}\right)+\frac{1}{2}\cos\left(2\sqrt{c(n-2)}-2\sqrt{c(n-1)}\right),$$

and

$$\cos\varphi_{n}^{\alpha}(c)\cos\varphi_{n}^{\alpha}(c)+\cos\varphi_{n-1}^{\alpha+1}(c)\cos\varphi_{n-1}^{\alpha+1}(c)=$$
$$\cos\left(2\sqrt{cn}+2\sqrt{c(n-1)}-\alpha\pi-\pi\right)\cos\left(2\sqrt{cn}-2\sqrt{c(n-1)}+\frac{\pi}{2}\right)+1.$$

Taking into account that the previous expressions tend to $1$ when $n$ tends to infinity, we obtain the desired result.

□

We can now state our main result.

**Theorem 4.1.1.** *The outer relative asymptotics for Laguerre Sobolev-type polynomials $\widehat{S}_n^{M,N}(x)$, orthogonal with respect to the discrete Sobolev inner product (4.1.1), is*

$$\lim_{n\to\infty}\frac{\widehat{S}_n^{M,N}(x)}{\widehat{L}_n^{\alpha}(x)}=1,$$

*uniformly on compact subsets of* $\mathbb{C}\backslash\mathbb{R}_+$.

*Proof.* Replacing (4.1.3) in (4.1.2)

$$\frac{\widehat{S}_n^{M,N}(x)}{\widehat{L}_n^{\alpha}(x)}=\left\{1+\frac{A_1(n;c)}{(x-c)}+\frac{A_0(n;c)}{(x-c)^2}\right\}+\left\{\frac{B_1(n;c)}{(x-c)}+\frac{B_0(n;c)}{(x-c)^2}\right\}\frac{\widehat{L}_{n-1}^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)}, \tag{4.1.8}$$

From the Perron's formula (1.6.26) (for more details we refer the reader to [15]) we get

$$\frac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}=1-\frac{\sqrt{-x}}{\sqrt{n}}+\mathcal{O}(n^{-1}).$$

For monic polynomials the above relation becomes

$$\frac{\widehat{L}_{n-1}^{\alpha}(x)}{\widehat{L}_n^{\alpha}(x)}=\frac{-1}{n}\left(1-\frac{\sqrt{-x}}{\sqrt{n}}+\mathcal{O}(n^{-1})\right). \tag{4.1.9}$$

By using (4.1.9) we can rewrite (4.1.8) as

$$\frac{\widehat{S}_n^{M,N}(x)}{\widehat{L}_n^{\alpha}(x)} \sim \left\{1+\frac{A_1(n;c)}{(x-c)}+\frac{A_0(n;c)}{(x-c)^2}\right\} - \left\{\frac{\frac{B_1(n;c)}{n}}{(x-c)}+\frac{\frac{B_0(n;c)}{n}}{(x-c)^2}\right\}.$$

Then, in order to conclude our proof, we only need to check that

$$\lim_{n\to\infty}\left(A_1(n;c)-\frac{B_1(n;c)}{n}\right) = 0, \tag{4.1.10}$$

$$\lim_{n\to\infty}\left(A_0(n;c)-\frac{B_0(n;c)}{n}\right) = 0. \tag{4.1.11}$$

By applying Proposition 4.1.1, we obtain (4.1.10). From (4.1.7), we get

$$\begin{aligned} A_0(n;c)-\frac{B_0(n;c)}{n} \sim \frac{-1}{Mc\sigma^{\alpha}(c)\sigma^{\alpha+3}(c)} & n^{-1}\left(\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c)-\cos\varphi_{n}^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c)\right)\\ & - c^{1/2}n^{-1/2}\left(\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c)-\cos\varphi_{n}^{\alpha}(c)\cos\varphi_{n-1}^{\alpha+1}(c)\right)\\ & 2n^{-1}\left(\cos\varphi_{n-1}^{\alpha}(c)\cos\varphi_{n}^{\alpha}(c)-\cos\varphi_{n}^{\alpha}(c)\cos\varphi_{n}^{\alpha}(c)\right). \end{aligned}$$

Since this expression tends to zero when $n$ tends to infinity, then (4.1.11) hold.

□

## 4.2 The five-term recurrence relation

This section is focused on the five-term recurrence relation that the sequence of discrete Laguerre–Sobolev orthogonal polynomials $\{\widehat{S}_n^{M,N}(x)\}_{n\geqslant 0}$ satisfies. Next, we will estimate the coefficients of such a recurrence relation for $n$ large enough and $c \in \mathbb{R}_+$. To this end, we will use the remarkable fact, which is a straightforward consequence of (4.1.1), that the multiplication operator by $(x-c)^2$ is a symmetric operator with respect to such a discrete Sobolev inner product. Indeed, for any $f(x), g(x) \in \mathbb{P}$

$$\langle (x-c)^2 f(x), g(x)\rangle_S = \langle f(x), (x-c)^2 g(x)\rangle_S. \tag{4.2.12}$$

Notice that

$$\langle (x-c)^2 f(x), g(x)\rangle_S = \langle f(x), g(x)\rangle_{[2]}. \tag{4.2.13}$$

An equivalent formulation of (4.2.13) is

$$\langle (x-c)^2 f(x), g(x)\rangle_S = \langle (x-c)^2 f(x), g(x)\rangle_{\alpha}. \tag{4.2.14}$$

We will need some preliminary results that will be stated as Lemmas 4.2.1, and 4.2.2.

**Lemma 4.2.1.** *For every* $n \geqslant 1$ *and initial conditions* $\widehat{L}_{-1}^{\alpha}(x) = 0$, $\widehat{L}_{0}^{\alpha}(x) = 1$, $\widehat{L}_{1}^{\alpha}(x) = x - (\alpha+1)$, *the connection formula (4.1.2) reads as*

$$(x-c)^2 \widehat{S}_n^{M,N}(x) =$$

$$\widehat{L}_{n+2}^{\alpha}(x) + \tilde{b}_n \widehat{L}_{n+1}^{\alpha}(x) + \tilde{c}_n \widehat{L}_{n}^{\alpha}(x) + \tilde{d}_n \widehat{L}_{n-1}^{\alpha}(x) + \tilde{e}_n \widehat{L}_{n-2}^{\alpha}(x),$$

*where*

$$\tilde{b}_n = \beta_{n+1} + \beta_n - 2c + A_1(n;c) \sim 4n,$$

$$\tilde{c}_n = \gamma_{n+1} + \gamma_n + (\beta_n - c)^2 + A_1(n;c)\,[\beta_n - c] + A_0(n;c) + B_1(n;c) \sim 6n^2,$$

$$\tilde{d}_n = \gamma_n(\beta_n + \beta_{n-1} - 2c) + \gamma_n A_1(n;c) + (\beta_{n-1} - c)B_1(n;c) + B_0(n;c) \sim 4n^3,$$

$$\tilde{e}_n = \gamma_n \gamma_{n-1} + \gamma_{n-1} B_1(n;c) \sim n^4.$$

*Proof.* We begin with the expression

$$(x-c)^2 \widehat{L}_n^{\alpha}(x) =$$

$$\widehat{L}_{n+2}^{\alpha}(x) + b_n \widehat{L}_{n+1}^{\alpha}(x) + c_n \widehat{L}_{n}^{\alpha}(x) + d_n \widehat{L}_{n-1}^{\alpha}(x) + e_n \widehat{L}_{n-2}^{\alpha}(x), \tag{4.2.15}$$

where

$$b_n = \beta_{n+1} + \beta_n - 2c \sim 4n, \qquad c_n = \gamma_{n+1} + \gamma_n + (\beta_n - c)^2 \sim 6n^2,$$
$$d_n = \gamma_n(\beta_n + \beta_{n-1} - 2c) \sim 4n^3, \quad e_n = \gamma_n \gamma_{n-1} \sim n^4,$$

according to (1.6.20) and the definition of $\beta_n$ and $\gamma_n$ in (1.6.20).

From the expression of $A(n;x)$ in (4.1.3), the next step is to expand the polynomial $[A_1(n;x)(x-c) + A_0(n;x)]\,\widehat{L}_n^{\alpha}(x)$ in terms of $\{\widehat{L}_n^{\alpha}\}_{n=0}^{\infty}$. Indeed, from (1.6.20)

$$[A_1(n;x)(x-c) + A_0(n;x)]\,\widehat{L}_n^{\alpha}(x) =$$

$$A_1(n;x)\widehat{L}_{n+1}^{\alpha}(x) + [(\beta_n - c)A_1(n;x) + A_0(n;x)]\,\widehat{L}_n^{\alpha}(x) + A_1(n;x)\gamma_n \widehat{L}_{n-1}^{\alpha}(x).$$

Adding these coefficients to those of (4.2.15), we obtain

$$A(n;x)\widehat{L}_n^{\alpha}(x) = \widehat{L}_{n+2}^{\alpha}(x) + \bar{b}_n \widehat{L}_{n+1}^{\alpha}(x) + \bar{c}_n \widehat{L}_{n}^{\alpha}(x) + \bar{d}_n \widehat{L}_{n-1}^{\alpha}(x) + \bar{e}_n \widehat{L}_{n-2}^{\alpha}(x),$$

with

$$\bar{b}_n = b_n + A_1(n;c) \sim 4n, \qquad \bar{c}_n = c_n + A_1(n;c)\,(\beta_n - c) + A_0(n;c) \sim 6n^2,$$
$$\bar{d}_n = d_n + \gamma_n A_1(n;c) \sim 4n^3, \quad \bar{e}_n = e_n \sim n^4,$$

where we have used Proposition 4.1.1. In a similar way, for $B(n;x)$ in (4.1.3) we get

$$B(n;x)\widehat{L}_{n-1}^{\alpha}(x) = \breve{c}_n \widehat{L}_{n}^{\alpha}(x) + \breve{d}_n \widehat{L}_{n-1}^{\alpha}(x) + \breve{e}_n \widehat{L}_{n-2}^{\alpha}(x),$$

where

$$\breve{c}_n = B_1(n;c) \sim n,$$
$$\breve{d}_n = (\beta_{n-1} - c)B_1(n;c) + B_0(n;c) \sim 2n^2,$$
$$\breve{e}_n = \gamma_{n-1}B_1(n;c) \sim n^3.$$

As a conclusion,

$$\begin{aligned}(x-c)^2\widehat{S}_n^{M,N}(x) &= A(n;x)\widehat{L}_n^{\alpha}(x) + B(n;x)\widehat{L}_{n-1}^{\alpha}(x) \\ &= \widehat{L}_{n+2}^{\alpha}(x) + \bar{b}_n\widehat{L}_{n+1}^{\alpha}(x) + (\bar{c}_n + \breve{c}_n)\widehat{L}_n^{\alpha}(x) \\ &\quad +(\bar{d}_n + \breve{d}_n)\widehat{L}_{n-1}^{\alpha}(x) + (\bar{e}_n + \breve{e}_n)\widehat{L}_{n-2}^{\alpha}(x).\end{aligned}$$

This completes the proof. □

**Lemma 4.2.2.** *For every $\alpha > -1$, $n \geqslant 1$, and $c \in \mathbb{R}_+$ the norm of the Laguerre-Sobolev type polynomials $\widehat{S}_n^{M,N}$, orthogonal with respect to (4.1.1) is*

$$||\widehat{S}_n^{M,N}||_S^2 = ||\widehat{L}_n^{\alpha}||_{\alpha}^2 + B_1(n;c)||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 \sim \Gamma(n+1)\Gamma(n+\alpha+1).$$

*where $B_1(n;c)$ is the polynomial coefficient defined in* (4.1.4).

*Proof.* First, let notice that

$$||\widehat{S}_n^{M,N}||_S^2 = \langle \widehat{S}_n^{M,N}(x), (x-c)^2\widehat{\Pi}_{n-2}(x)\rangle_S,$$

for every monic polynomial $\widehat{\Pi}_{n-2}$ of degree $n-2$ . From (4.2.14)

$$\begin{aligned}\langle \widehat{S}_n^{M,N}(x), (x-c)^2\widehat{\Pi}_{n-2}(x)\rangle_S &= \langle (x-c)^2\widehat{S}_n^{M,N}(x), \widehat{\Pi}_{n-2}(x)\rangle_S \\ &= \langle (x-c)^2\widehat{S}_n^{M,N}(x), \widehat{\Pi}_{n-2}(x)\rangle_{\alpha}.\end{aligned}$$

Next we use the connection formula (4.1.2). Taking into account that $A(n;x)$ is a monic quadratic polynomial and $B(n;x)$ is a linear polynomial with leading coefficient $B_1(n;c)$,

$$\begin{aligned}||\widehat{S}_n^{M,N}||_S^2 &= \langle (x-c)^2\widehat{S}_n^{M,N}(x), \widehat{\Pi}_{n-2}(x)\rangle_{\alpha} \\ &= \langle A(n;x)\widehat{L}_n^{\alpha}(x), \widehat{\Pi}_{n-2}(x)\rangle_{\alpha} + \langle B(n;x)\widehat{L}_{n-1}^{\alpha}(x), \widehat{\Pi}_{n-2}(x)\rangle_{\alpha} \\ &= \langle \widehat{L}_n^{\alpha}(x), x^n\rangle_{\alpha} + B_1(n;c)\langle \widehat{L}_{n-1}^{\alpha}(x), x^{n-1}\rangle_{\alpha}.\end{aligned}$$

The first term in the above expression is the norm of the monic Laguerre polynomial of degree $n$ and the second one is the norm of the Laguerre polynomial of degree $n-1$ times $B_1(n;c)$, which is given in (4.1.4). This means

$$||\widehat{S}_n^{M,N}||_S^2 = ||\widehat{L}_n^{\alpha}||_{\alpha}^2 + B_1(n;c)||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2.$$

Using the estimates (1.6.21) and Proposition 4.1.1, we obtain

$$||\widehat{S}_n^{M,N}||_S^2 \sim \Gamma(n+1)\Gamma(n+\alpha+1),$$

which completes the proof. □

We are ready to find the five-term recurrence relation satisfied by $\widehat{S}_n^{M,N}(x)$, and the asymptotic behavior of the corresponding coefficients. Next, we will focus our attention on its proof.

Let consider the Fourier expansion of $(x-c)^2\widehat{S}_n^{M,N}(x)$ in terms of $\{\widehat{S}_n^{M,N}(x)\}_{n=0}^{\infty}$

$$(x-c)^2\widehat{S}_n^{M,N}(x) = \sum_{k=0}^{n+2} \lambda_{n,k}\widehat{S}_k^{M,N}(x),$$

where

$$\lambda_{n,k} = \frac{\langle (x-c)^2\widehat{S}_n^{M,N}(x), \widehat{S}_k^{M,N}(x)\rangle_S}{||\widehat{S}_k^{M,N}||_S^2}, \quad k = 0, \ldots, n+2. \tag{4.2.16}$$

Thus, $\lambda_{n,k} = 0$ for $k = 0, \ldots, n-3$. We are dealing with monic polynomials, so the leading coefficient $\lambda_{n,n+2} = 1$.

To obtain $\lambda_{n,n+1}$, we use the connection formula (4.1.2), with coefficients $A(n;x)$ and $B(n;x)$ as in (4.1.3). Thus,

$$\begin{aligned}\lambda_{n,n+1} &= \frac{1}{||\widehat{S}_{n+1}^{M,N}||_S^2}\langle A(n;x)\widehat{L}_n^{\alpha}(x), \widehat{S}_{n+1}^{M,N}(x)\rangle_S + \frac{1}{||\widehat{S}_{n+1}^{M,N}||_S^2}\langle B(n;x)\widehat{L}_{n-1}^{\alpha}(x), \widehat{S}_{n+1}^{M,N}(x)\rangle_S \\ &= \frac{1}{||\widehat{S}_{n+1}^{M,N}||_S^2}\langle (x-c)^2\widehat{L}_n^{\alpha}(x), \widehat{S}_{n+1}^{M,N}(x)\rangle_S + A_1(n;c).\end{aligned}$$

Let us study the discrete Sobolev inner product $\langle (x-c)^2\widehat{L}_n^{\alpha}(x), \widehat{S}_{n+1}^{M,N}(x)\rangle_S$ above. Applying (4.2.12), (4.2.14), (1.6.21) and Lemma 4.2.1, we obtain

$$\begin{aligned}\langle (x-c)^2\widehat{L}_n^{\alpha}(x), \widehat{S}_{n+1}^{M,N}(x)\rangle_S &= \langle \widehat{L}_n^{\alpha}(x), (x-c)^2\widehat{S}_{n+1}^{M,N}(x)\rangle_{\alpha} \\ &= \tilde{d}_{n+1}\,||\widehat{L}_n^{\alpha}||_{\alpha}^2.\end{aligned}$$

From (3.2.25), Lemma 4.2.2 and Proposition 4.1.1

$$\lambda_{n,n+1} = \frac{\tilde{d}_{n+1}\,||\widehat{L}_n^{\alpha}||_{\alpha}^2 + A_1(n;c)}{||\widehat{S}_{n+1}^{M,N}||_S^2} \sim 4n.$$

In order to compute $\lambda_{n,n}$, from (4.1.2) and (4.1.3) we get

$$\begin{aligned}\lambda_{n,n} &= \frac{\langle (x-c)^2\widehat{L}_n^{\alpha}(x), \widehat{S}_n^{M,N}(x)\rangle_S}{||\widehat{S}_n^{M,N}||_S^2} + A_1(n;c)\frac{\langle (x-c)\widehat{L}_n^{\alpha}(x), \widehat{S}_n^{M,N}(x)\rangle_S}{||\widehat{S}_n^{M,N}||_S^2} \\ &\quad +A_0(n;c) + B_1(n;c).\end{aligned}$$

But, according to (4.2.12), (4.2.14) and Lemma 4.2.1, the first term is

$$\frac{\langle (x-c)^2\widehat{L}_n^{\alpha}(x), \widehat{S}_n^{M,N}(x)\rangle_S}{||\widehat{S}_n^{M,N}||_S^2} = \tilde{c}_n\frac{||\widehat{L}_n^{\alpha}||_{\alpha}^2}{||\widehat{S}_n^{M,N}||_S^2}.$$

After some algebraic manipulations, from (1.6.20) we get

$$(x-c)\widehat{L}_n^{\alpha}(x) = (x-c)^2\widehat{L}_{n-1}^{\alpha}(x)-(\beta_{n-1}-c)(x-c)\widehat{L}_{n-1}^{\alpha}(x)-\gamma_{n-1}(x-c)\widehat{L}_{n-2}^{\alpha}(x),$$

Using this expression, we obtain

$$\begin{aligned}\frac{\langle (x-c)\widehat{L}_n^{\alpha}(x), \widehat{S}_n^{M,N}(x)\rangle_S}{||\widehat{S}_n^{M,N}||_S^2} &= \frac{\langle \widehat{L}_{n-1}^{\alpha}(x), (x-c)^2\widehat{S}_n^{M,N}(x)\rangle_{\alpha}}{||\widehat{S}_n^{M,N}||_S^2} - (\beta_{n-1}-c) \\ &= \tilde{d}_n \frac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2}{||\widehat{S}_n^{M,N}||_S^2} - (\beta_{n-1}-c).\end{aligned}$$

As a consequence,

$$\begin{aligned}\lambda_{n,n} &= \frac{\tilde{c}_n\,||\widehat{L}_n^{\alpha}||_{\alpha}^2 + \tilde{d}_n\,||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 + (\beta_{n-1}-c) + A_0(n;c) + B_1(n;c)}{||\widehat{S}_n^{M,N}||_S^2} \\ &\sim 6n^2.\end{aligned}$$

A similar analysis yields

$$\begin{aligned}\lambda_{n,n-1} &= \frac{\tilde{d}_n\,||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 + A_1(n-1;c)||\widehat{S}_n^{M,N}||_S^2}{||\widehat{S}_{n-1}^{M,N}||_S^2} \sim 4n^3, \\ \lambda_{n,n-2} &= \frac{||\widehat{S}_n^{M,N}||_S^2}{||\widehat{S}_{n-2}^{M,N}||_S^2} \sim n^4.\end{aligned}$$

We can summarize the results of this Section in the following theorem.

**Theorem 4.2.1.** *[Five-term recurrence relation] For every $n \geqslant 1$, $\alpha > -1$, and $c \in \mathbb{R}_+$, the monic Laguerre-Sobolev type polynomials $\{\widehat{S}_n^{M,N}(x)\}_{n=0}^{\infty}$, orthogonal with respect to (4.1.1) satisfy the following five-term recurrence relation*

$$(x-c)^2\widehat{S}_n^{M,N}(x) =$$

$$\widehat{S}_{n+2}^{M,N}(x) + \lambda_{n,n+1}\widehat{S}_{n+1}^{M,N}(x) + \lambda_{n,n}\widehat{S}_n^{M,N}(x) + \lambda_{n,n-1}\widehat{S}_{n-1}^{M,N}(x) + \lambda_{n,n-2}\widehat{S}_{n-2}^{M,N}(x),$$

*with*

$$\lambda_{n,n+1} = \frac{\tilde{d}_{n+1}\,||\widehat{L}_n^{\alpha}||_{\alpha}^2 + A_1(n;c)}{||\widehat{S}_{n+1}^{M,N}||_S^2} \sim 4n = \binom{4}{1}n,$$

$$\lambda_{n,n} = \frac{\tilde{c}_n\,||\widehat{L}_n^{\alpha}||_{\alpha}^2 + \tilde{d}_n\,||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 - (\beta_{n-1}-c) + A_0(n;c) + B_1(n;c)}{||\widehat{S}_n^{M,N}||_S^2} \sim 6n^2 = \binom{4}{2}n^2,$$

$$\lambda_{n,n-1} = \frac{\tilde{d}_n\,||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 + A_1(n-1;c)||\widehat{S}_n^{M,N}||_S^2}{||\widehat{S}_{n-1}^{M,N}||_S^2} \sim 4n^3 = \binom{4}{3}n^3,$$

$$\lambda_{n,n-2} = \frac{||\widehat{S}_n^{M,N}||_S^2}{||\widehat{S}_{n-2}^{M,N}||_S^2} \sim n^4 = \binom{4}{4}n^4.$$

## 4.3 Inner $L^2$-asymptotics

As an application of Theorem 3.2.1, we will study the inner asymptotics for a certain family of Laguerre-Sobolev type orthogonal polynomials. More precisely, we compare the behavior of the Sobolev and standard Laguerre polynomials on $(0, \infty)$ for $n$ large enough. The main result in this section guarantees the norm convergence of the Laguerre-Sobolev polynomials to the Laguerre ones in the Laguerre $L^2$-norm. Before to deal with the general case, we are going to analyze a more simple framework. For example, let us consider the Sobolev type inner product

$$\langle f, g\rangle_S = \langle f, g\rangle_\alpha + M f'(c) g'(c), \tag{4.3.17}$$

where $\alpha > -1$, $c > 0$ and $M > 0$. Notice that this is just a particular case of the family of inner products defined in [64]. Let $\{\widehat{L}_n^{M,\alpha}(x)\}_{n=0}^{\infty}$ be the monic Laguerre-Sobolev polynomials orthogonal with respect to (4.3.17). We also consider the normalization

$$\widetilde{L}_n^{M,\alpha}(x) = \frac{\widehat{L}_n^{M,\alpha}(x)}{||\widehat{L}_n^{\alpha}||_\alpha},$$

i.e., the normalized Laguerre-Sobolev type orthogonal polynomials with the same leading coefficient as the classical orthonormal Laguerre polynomial of degree $n$. Then, (see [64, equation (2.8)])

$$\widetilde{L}_n^{M,\alpha}(x) - L_n^{\alpha}(x) = \frac{M\left(L_n^{\alpha}\right)'(c)}{1 + M K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(x,c).$$

Let consider the standard $L^2$-Laguerre norm of the previous expression, i.e.

$$||\widetilde{L}_n^{M,\alpha} - L_n^{\alpha}||_\alpha^2 = \frac{M^2\left[\left(L_n^{\alpha}\right)'(c)\right]^2}{\left(1 + M K_{n-1}^{(1,1)}(c,c)\right)^2} K_{n-1}^{(1,1)}(c,c) \leqslant \frac{\left[\left(L_n^{\alpha}\right)'(c)\right]^2}{K_{n-1}^{(1,1)}(c,c)}.$$

Now, from Proposition 3.2.1 we obtain

$$K_{n-1}^{(1,1)}(c,c) \sim_n C\, n^{\frac{3}{2}},$$

and, on the other hand,

$$\left(L_n^{\alpha}\right)'(c) = \frac{n\widehat{L}_{n-1}^{\alpha+1}(c)}{||\widehat{L}_n^{\alpha}||_\alpha} = \frac{n!(-1)^{n-1}}{\left(\Gamma(n+1)\Gamma(n+\alpha+1)\right)^{1/2}} L_{n-1}^{(\alpha+1)}(c),$$

from which it follows that

$$\left[(L_n^\alpha)'(c)\right]^2 = \frac{n!}{\Gamma(n+\alpha+1)}|L_{n-1}^{(\alpha+1)}(c)|^2 \leqslant Cn^{1/2}.$$

As a consequence,

$$||\widetilde{L}_n^{M,\alpha} - L_n^\alpha||_\alpha^2 \leqslant Cn^{-1},$$

so, we have proved the norm convergence of the $n$-th Laguerre-Sobolev type orthogonal polynomial to the $n$-th Laguerre one:

$$\lim_{n\to\infty} ||\widetilde{L}_n^{M,\alpha} - L_n^\alpha||_\alpha = 0.$$

### 4.3.1 The multi-index case

Let us consider the Sobolev type inner product (4.1.1) and $\widehat{L}_n^{\alpha,\underline{M}}(x)$ the corresponding monic orthogonal polynomial of degree $n$. Also, we consider the normalization

$$\widetilde{L}_n^{\alpha,\underline{M}}(x) = \frac{\widehat{L}_n^{\alpha,\underline{M}}(x)}{||\widehat{L}_n^\alpha||_\alpha},$$

i.e., the Laguerre-Sobolev polynomials with the same leading coefficient as the orthonormal Laguerre ones.

From now on, we will denote by $j_1 < \cdots < j_q$ the indexes such that $M_{j_1-1} = \cdots = M_{j_q-1} = 0$.

**Theorem 4.3.1.** *With the above notation, the inner $L^2$-asymptotics for the Laguerre-Sobolev polynomials orthogonal with respect to (4.1.1) reads*

$$||\tilde{L}_n^{\alpha,\underline{M}} - L_n^\alpha||_\alpha \leqslant Cn^{-1}, \tag{4.3.18}$$

*where $C$ is a positive constant independent of $n$. In particular,*

$$\lim_{n\to\infty} ||\tilde{L}_n^{\alpha,\underline{M}} - L_n^\alpha||_\alpha = 0. \tag{4.3.19}$$

*Proof.* Following a standard technique we can expand the Laguerre-Sobolev type orthogonal polynomials in terms of the Laguerre classical ones to obtain

$$\begin{aligned}\widetilde{L}_n^{\alpha,\underline{M}}(x) =& L_n^\alpha(x) - \sum_{k=0}^{n-1}\sum_{j=0}^{N} M_j \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c)\,(L_k^\alpha)^{(j)}(c) L_k^\alpha(x)\\ =& L_n^\alpha(x) - \sum_{j=0}^{N} M_j \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c) K_{n-1}^{(j,0)}(c,x).\end{aligned} \tag{4.3.20}$$

At this point, estimations for $\left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c)$ when $j = 0,\dots,N$, $j \neq j_1 - 1,\dots,j_q - 1$, are needed.

In order to do that, we can write (4.3.20) evaluated at $x = c$ in a matrix form as follows,

$$A\mathbf{L}^{\alpha,\underline{M}} = \mathbf{L}^{\alpha},$$

where

$$A = \begin{pmatrix} 1 + M_0 K_{n-1}(c,c) & M_1 K_{n-1}^{(1,0)}(c,c) & M_2 K_{n-1}^{(2,0)}(c,c) & \dots & M_N K_{n-1}^{(N,0)}(c,c) \\ M_0 K_{n-1}^{(0,1)}(c,c) & 1 + M_1 K_{n-1}^{(1,1)}(c,c) & M_2 K_{n-1}^{(2,1)}(c,c) & \dots & M_N K_{n-1}^{(N,1)}(c,c) \\ M_0 K_{n-1}^{(0,2)}(c,c) & M_1 K_{n-1}^{(1,2)}(c,c) & 1 + M_2 K_{n-1}^{(2,2)}(c,c) & \dots & M_N K_{n-1}^{(N,2)}(c,c) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_0 K_{n-1}^{(0,N)}(c,c) & M_1 K_{n-1}^{(1,N)}(c,c) & M_2 K_{n-1}^{(2,N)}(c,c) & \dots & 1 + M_N K_{n-1}^{(N,N)}(c,c) \end{pmatrix},$$

$$\mathbf{L}^{\alpha} = \left(L_n^{\alpha}(c), \left(L_n^{\alpha}\right)'(c), \dots, \left(L_n^{\alpha}\right)^{(N)}(c)\right)^T,$$

and

$$\mathbf{L}^{\alpha,\underline{M}} = \left(\widetilde{L}_n^{\alpha,\underline{M}}(c), \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)'(c), \dots, \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(N)}(c)\right)^T.$$

Here, $v^T$ denotes the transpose of the vector $v$. Then, applying Cramer's rule we get

$$\left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(m-1)}(c) = \frac{\det(\mathbf{A_m})}{\det(\mathbf{A})}, \quad \text{for } m = 1,\dots,N+1,$$

where $\mathbf{A_m}$ is the matrix obtained by replacing the $m$-th column in the matrix $\mathbf{A}$ by the column vector $\mathbf{L}^{\alpha}$.

Thus, by using Lemmas 4.3.2 and 4.3.3, for $n$ large enough we obtain

$$\left|\left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(m-1)}(c)\right| \leqslant C n^{\frac{-2m-1}{4}}, \tag{4.3.21}$$

where $C$ is a positive constant which does not depend on $n$.

Finally, in order to obtain (4.3.19) we take norm in (4.3.20). Thus

$$\begin{aligned} ||\widetilde{L}_n^{\alpha,\underline{M}} - L_n^{\alpha}||_{\alpha}^2 \leqslant & || \sum_{j=0}^{N} M_j \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c) K_{n-1}^{(j,0)}(c,x)||_{\alpha}^2 \\ \leqslant & (N+1) \sum_{j=0}^{N} M_j^2 \left[\left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c)\right]^2 K_{n-1}^{(j,j)}(c,c). \end{aligned}$$

From Theorem 3.2.1 and (4.3.21) we get

$$\begin{aligned}||\widetilde{L}_n^{\alpha,\underline{M}} - L_n^{\alpha}||_{\alpha}^2 \leqslant& (N+1)\sum_{j=0}^{N} C_j M_j^2 n^{\frac{-4(j+1)-2}{4}} n^{\frac{2j+1}{2}}\\ \leqslant& C n^{-1}.\end{aligned}$$

□ □

**Remark 4.3.1.** *Notice that in [34] estimates in the weighted $L^2$-norm for the difference between continuous Sobolev orthogonal polynomials associated with a vector of measures $(\psi W, W)$ and standard orthogonal polynomials associated with $W$, where $W$ is an exponential weight $W(x) = e^{-2Q(x)}$ and $\psi$ is a measurable and positive function on a set of positive measure, such that the moments of the Sobolev product are finite, have been obtained in terms of the Mhaskar-Rakhmanov-Saff number. The authors assume that $Q$ is an even and convex function on the real line such that $Q''$ is continuous in $(0,\infty)$ and $Q' > 0$ in $(0,\infty)$, as well as for some $0 < \alpha < \beta$, $\alpha \leqslant \frac{xQ''(x)}{Q'(x)} \leqslant \beta$, $x \in (0,\infty)$ holds. The study of analogue estimates as above for general exponential weights constitutes an interesting problem in which we are working.*

### Estimates for $\det(\mathbf{A})$ and $\det(\mathbf{A_m})$

First of all we will need the following well-known result, see for instance, [76, vol. III, p. 311].

**Lemma 4.3.1** (Cauchy's double alternant)**.** *Let $x_1,\ldots,x_n,y_1,\ldots,y_n$ be real numbers. Then,*

$$\det\left[\frac{1}{x_i+y_j}\right]_{1\leqslant i,j\leqslant n} = \frac{\prod\limits_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_i-y_j)}{\prod\limits_{1\leqslant i,j\leqslant n}(x_i+y_j)}.$$

Let us denote

$$M := \prod_{\substack{l=1\\ l\neq j_1,\ldots,j_q}}^{N+1} M_{l-1}, \qquad Q := \sum_{\substack{l=1\\ l\neq j_1,\ldots,j_q}}^{N+1} l.$$

**Lemma 4.3.2.** *With the notation introduced in Section 4.3, we have*

$$\det(\boldsymbol{A}) \sim_n C_1 n^{\frac{2Q-(N+1)+q}{2}}, \tag{4.3.22}$$

*where $C_1$ is a positive constant independent of $n$. In particular, there exists a constant $C_2 > 0$ such that*

$$\det(\boldsymbol{A}) > C_2 n^{\frac{2Q-(N+1)+q}{2}}$$

*for $n$ large enough.*

*Proof.* We denote by $a_{ij}$, $1 \leqslant i, j \leqslant N+1$, the $(i, j)$ entry of the matrix $\mathbf{A}$. Notice that these entries verify

$$a_{ij} \sim_n \begin{cases} M_{j-1} K_{n-1}^{(j-1,i-1)}(c, c) & \text{for } j \text{ such that } M_{j-1} > 0, \\ 1 & \text{if } i = j \text{ and } M_{j-1} = 0, \\ 0 & \text{if } i \neq j \text{ and } M_{j-1} = 0. \end{cases}$$

Then, from Theorem 3.2.1, we obtain

$$a_{ij} \sim_n \begin{cases} M_{j-1} C_{0,j-1,i-1} n^{\frac{i+j-1}{2}}, & \text{if } i+j \equiv 0 \mod 2 \text{ and } M_{j-1} > 0, \\ M_{j-1} C_{1,j-1,i-1} n^{\frac{i+j-2}{2}}, & \text{if } i+j \equiv 1 \mod 2 \text{ and } M_{j-1} > 0, \\ 1 & \text{if } i = j \text{ and } M_{j-1} = 0, \\ 0 & \text{if } i \neq j \text{ and } M_{j-1} = 0. \end{cases} \tag{4.3.23}$$

Using the definition of determinant and (4.3.23), we get

$$\begin{aligned} \det(\mathbf{A}) &= \sum_{\delta \in \mathcal{S}_{N+1}} \operatorname{sgn}(\delta) a_{1,\delta(1)} \cdots a_{N+1,\delta(N+1)} \\ &\sim_n \sum_{\delta \in \mathcal{S}_{N+1}} \operatorname{sgn}(\delta) C_\delta n^{\frac{-p_1(\delta)}{2}} n^{-p_2(\delta)} \prod_{\substack{l=1 \\ l \neq j_1,\dots,j_q}}^{N+1} n^{\frac{l+\delta(l)}{2}}, \end{aligned} \tag{4.3.24}$$

where $\mathcal{S}_{N+1}$ is the group of permutations of the set $\{1, \dots, N+1\}$, $p_1(\delta)$ (resp. $p_2(\delta)$) is the number of indexes $l$ in $\{1, \dots, N+1\} \backslash \{j_1, \dots, j_q\}$ such that $l + \sigma(l)$ is even (resp. odd) and

$$C_\delta = \prod_{\substack{l=1 \\ l \neq j_1,\dots,j_q}}^{N+1} M_{\delta(l)-1} C_{0,\delta(l)-1,l-1}.$$

Let us define the set

$$\Delta = \left\{ \delta \in \mathcal{S}_{N+1} \quad : \quad \begin{matrix} l + \delta(l) \text{ is even for all } l = 1, \dots, N+1, \\ \text{and } \delta(l) = l, \text{ for } l = j_1, \dots, j_q \end{matrix} \right\}.$$

Notice that $\frac{p_1(\delta)}{2}+p_2(\delta)$ attains a minimum when $p_2(\delta)=0$. Then, the asymptotic behavior of (4.3.24) will be given by the terms corresponding to permutations in $\Delta$, if they do not vanish. Thus, we have to check that $\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)C_\delta$ is not zero.

$$\begin{aligned}\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)C_\delta &= \sum_{\delta\in\Delta}\operatorname{sgn}(\delta)\prod_{\substack{l=1\\ l\neq j_1,\ldots,j_q}}^{N+1} M_{\delta(l)-1}C_{0,\delta(l)-1,l-1}\\ &= \sum_{\delta\in\Delta}\operatorname{sgn}(\delta)\prod_{\substack{l=1\\ l\neq j_1,\ldots,j_q}}^{N+1} M_{\delta(l)-1}(-1)^{\frac{l+\sigma(l)}{2}+l}\sigma^{\alpha}(c)\sigma^{\alpha+l+\sigma(l)-1}(c)\sqrt{c}\frac{1}{l+\delta(l)-1}.\end{aligned} \tag{4.3.25}$$

Recalling that $\sigma^{\alpha+l+\delta(l)-1}(c)=\pi^{-1/2}e^{c/2}c^{\frac{-\alpha+l+\delta(l)-1}{2}}c^{-1/4}$, we get

$$\begin{aligned}&\prod_{\substack{l=1\\ l\neq j_1,\ldots,j_q}}^{N+1}\sigma^{\alpha+l+\delta(l)-1}(c)=\pi^{-\frac{N+1-q}{2}}e^{\frac{(N+1-q)c}{2}}c^{-\frac{\alpha(N+1-q)+2Q-(N+1-q)}{2}}\\ &=\left(\sigma^{\alpha-1}(c)\right)^{N+1-q}c^{-Q}=\left(\sigma^{\alpha}(c)\right)^{N+1-q}c^{\frac{N+1-q}{2}}c^{-Q}.\end{aligned}$$

After some computations, (4.3.25) becomes

$$\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)C_\delta = M(-1)^{2Q}\left(\sigma^{\alpha}(c)\right)^{2(N+1-q)}c^{N+1-q-Q}\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)\prod_{\substack{l=1\\ l\neq j_1,\ldots,j_q}}^{N+1}\frac{1}{l+\delta(l)-1},$$

Now, let consider

$$\{1,2,\ldots,N+1\}\backslash\{j_1,j_2,\ldots,j_q\}=\{r_1,r_2,\ldots,r_{K_1}\}\cup\{s_1,s_2,\ldots,s_{K_2}\}$$

where $r_i$ is odd for $i=1,2,\ldots,K_1$ and $s_i$ is even for $i=1,2,\ldots,K_2$. Notice that $K_1+K_2=N+1-q$. Then, we have

$$\sum_{\delta\in\Sigma}\operatorname{sgn}(\delta)\prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1}\frac{1}{l+\delta(l)-1}=$$

$$=\sum_{\delta\in S_{K_1}}\sum_{\xi\in S_{K_2}}\operatorname{sgn}(\delta)\operatorname{sgn}(\xi)\prod_{i=1}^{K_1}\frac{1}{r_i+r_{\delta(i)}-1}\prod_{j=1}^{K_2}\frac{1}{s_j+s_{\xi(j)}-1}$$

$$=\sum_{\delta\in S_{K_1}}\operatorname{sgn}(\delta)\prod_{i=1}^{K_1}\frac{1}{r_i+r_{\delta(i)}-1}\sum_{\xi\in S_{K_2}}\operatorname{sgn}(\xi)\prod_{j=1}^{K_2}\frac{1}{s_j+s_{\xi(j)}-1}$$

$$=\frac{\prod\limits_{1\leqslant i<j\leqslant K_1}(r_i-r_j)^2}{\prod\limits_{1\leqslant i<j\leqslant K_1}(r_i+r_j-1)}\frac{\prod\limits_{1\leqslant i<j\leqslant K_2}(s_i-s_j)^2}{\prod\limits_{1\leqslant i<j\leqslant K_2}(s_i+s_j-1)},$$

where we have used Lemma 4.3.1 in the sense

$$\det\left[\frac{1}{r_i-\frac{1}{2}+r_j-\frac{1}{2}}\right]_{1\leqslant i,j\leqslant K_1}=\sum_{\delta\in S_{K_1}}\operatorname{sgn}(\delta)\prod_{i=1}^{K_1}\frac{1}{r_i+r_{\delta(i)}-1}.$$

Finally, (4.3.25) becomes

$$\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)C_\delta=M\left(\sigma^\alpha(c)\right)^{2(N+1-q)}c^{N+1-q-Q}\frac{\prod\limits_{1\leqslant i<j\leqslant K_1}(r_i-r_j)^2}{\prod\limits_{1\leqslant i<j\leqslant K_1}(r_i+r_j-1)}\frac{\prod\limits_{1\leqslant i<j\leqslant K_2}(s_i-s_j)^2}{\prod\limits_{1\leqslant i<j\leqslant K_2}(s_i+s_j-1)}$$

which is, as desired, different from zero. Then, we can state that

$$\det(\mathbf{A})\sim_n Cn^{\frac{2Q-(N+1)+q}{2}},\tag{4.3.26}$$

where $C$ is a positive constant independent of $n$. This concludes the proof.

□

□

**Lemma 4.3.3.** *For $n$ large enough, there exists a constant $C>0$ such that*

$$|\det(\boldsymbol{A_m})|\leqslant Cn^{\frac{2Q-m-N+q}{2}-\frac{3}{4}}.$$

*Proof.* Notice that for $i\neq m$, the entries of the matrix $\mathbf{A_m}$ are the same as those of the matrix $\mathbf{A}$. Their asymptotic behavior was given in (4.3.23). Let us denote by $\hat{a}_{im}$ the $(i,m)$ entry of the matrix $\mathbf{A_m}$.

According to (3.2.32), we have

$$\hat{a}_{im} = (L_n^{\alpha})^{(i-1)}(c) \sim_n (-1)^{n-i+1}\sigma^{\alpha+i-1}(c)n^{\frac{i}{2}-\frac{3}{4}}\cos\varphi_{n-i+1}^{n+i-1}(c), \tag{4.3.27}$$

for $i = 1, \ldots, N+1$.

We expand $\det(\mathbf{A_m})$ along the $m$-th column:

$$\det(\mathbf{A_m}) = \sum_{i=1}^{N+1}(-1)^{i+m}\hat{a}_{im}\det\mathbf{B_{im}}, \tag{4.3.28}$$

where $\mathbf{B_{im}}$ is the $N \times N$ matrix obtained by deleting of $\mathbf{A}$ the $i$-th row and the $m$-th column.

Using (4.3.27) in (4.3.28), we obtain

$$\det(\mathbf{A_m}) \sim_n \sum_{i=1}^{N+1}(-1)^{n+m+1}\sigma^{\alpha+i-1}(c)n^{\frac{i}{2}-\frac{3}{4}}\cos\varphi_{n-i+1}^{n+i-1}(c)\det\mathbf{B_{im}},$$

where $\det\mathbf{B_{im}}$ can be computed as

$$\det\mathbf{B_{im}} = \sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\prod_{l=1}^{N}b_{l,\sigma(l)} = \sum_{\psi\in\Psi}\operatorname{sgn}(\psi)\prod_{\substack{l=1\\ l\neq i,j_1,\ldots,j_q}}^{N+1}a_{l,\psi(l)}, \tag{4.3.29}$$

with

$$\Psi = \left\{\psi\in S_{N+1} \quad : \quad \begin{array}{c}\psi(l)=l, \text{ for } l=j_1,\ldots,j_q\\ \text{and } \psi(i)=m\end{array}\right\}.$$

Now, we will discuss two cases:

1. **Case $i+m$ even.**

   The highest power of $n$ that can be reached in the sum (4.3.29) appears when $l+\psi(l)$ is even for all $l = 1, \ldots, N+1, l \neq i, j_1, \ldots, j_q$. This means that

   $$\det\mathbf{B_{im}} \sim_n C\left(\sum_{\gamma\in\Gamma}\operatorname{sgn}(\gamma)C'_\gamma\right)n^{\frac{2Q-i-m-N+q}{2}},$$

   with

$$\Gamma = \left\{ \gamma \in S_{N+1} \quad : \quad \begin{array}{c} l+\gamma(l) \text{ is even for all } l=1,\dots,N+1, \\ \gamma(l)=l, \text{ for } l=j_1,\dots,j_q \\ \text{and } \gamma(i)=m \end{array} \right\},$$

whenever

$$\sum_{\gamma\in\Gamma} \operatorname{sgn}(\gamma) C'_\gamma \neq 0. \tag{4.3.30}$$

2. **Case $i+m$ odd.**
In this case, the highest power of $n$ in the sum (4.3.29) could be at most $\frac{2Q-i-m-N+q-1}{2}-1$, when the permutation $\psi$ satisfies that $l+\psi(l)$ is odd for one $l \in \{1,\dots,N+1\}\backslash\{i,j_1,\dots,j_q\}$, and it is even for the remainder indexes.

We obtain the highest power of $n$ for the first case, and after checking (4.3.30), we conclude

$$\det(\mathbf{A_m}) \sim_n \sum_{i=1}^{N+1} (-1)^{n+m+1} \sigma^{\alpha+i-1}(c) n^{\frac{2Q-m-N+q}{2}-\frac{3}{4}} \cos\varphi_{n-i+1}^{n+i-1}(c).$$

Then, for $n$ large enough, there exists a constant $C>0$ such that

$$|\det(\mathbf{A_m})| \leqslant C n^{\frac{2Q-m-N+q}{2}-\frac{3}{4}}.$$

In order to conclude the proof we must check that (4.3.30) holds. Indeed,

$$\begin{aligned} &\sum_{\gamma\in\Gamma} \operatorname{sgn}(\gamma) \prod_{\substack{l=1\\ l\neq i,j_1,\dots,j_q}}^{N+1} M_{\gamma(l)-1} C_{0,\gamma(l)-1,l-1} \\ &= \frac{M}{M_{m-1}} (-1)^{Q-i} \left(\sigma^\alpha(c)\right)^{2(N-q)} c^{\frac{2N+i+m-2q-2Q}{2}} \sum_{\gamma\in\Gamma} \operatorname{sgn}(\gamma) \prod_{\substack{l=1\\ l\neq i,j_1,\dots,j_q}}^{N+1} \frac{1}{l+\gamma(l)-1}. \end{aligned}$$

Let suppose now that $m$ is even. Let

$$\{1,2,\dots,N+1\}\backslash\{i,j_1,j_2,\dots,j_q\} = \{r_1,r_2,\dots,r_{K_1}\} \cup \{m,s_1,s_2,\dots,s_{K_2}\},$$

where $r_i$ is odd for $i=1,2,\dots,K_1$, and $s_i$ is even for $i=1,2,\dots,K_2$. Notice that $K_1+K_2=N-q$. We can write

$$\sum_{\gamma\in\Gamma} \operatorname{sgn}(\gamma) \prod_{\substack{l=1\\ l\neq i,j_1,\dots,j_q}}^{N+1} \frac{1}{l+\gamma(l)-1} = \Big( \sum_{\delta\in S_{K_1}} \operatorname{sgn}(\delta) \prod_{i=1}^{K_1} \frac{1}{r_i+r_{\delta(i)}-1} \Big) \det\mathbf{B},$$

where

$$\mathbf{B} = \begin{pmatrix} \frac{1}{m+i-1} & \frac{1}{m+s_1-1} & \cdots & \frac{1}{m+s_{K_2}-1} \\ \frac{1}{s_1+i-1} & \frac{1}{s_1+s_1-1} & \cdots & \frac{1}{s_1+s_{K_2}-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{s_{K_2}+i-1} & \frac{1}{s_{K_2}+s_1-1} & \cdots & \frac{1}{s_{K_2}+s_{K_2}-1}. \end{pmatrix},$$

Using Lemma 4.3.1,

$$\sum_{\gamma\in\Gamma} \operatorname{sgn}(\gamma) \prod_{\substack{l=1 \\ l\neq i,j_1,\ldots,j_q}}^{N+1} \frac{1}{l+\gamma(l)-1}$$
$$= \frac{\prod_{l=1}^{K_2}(m-s_l)(i-s_l) \prod_{1\leqslant i<j\leqslant K_2}(s_i-s_j)^2}{(m+i-1)\prod_{1\leqslant i<j\leqslant K_2}(s_i+s_j-1)} \frac{\prod_{1\leqslant i<j\leqslant K_1}(r_i-r_j)^2}{\prod_{1\leqslant i<j\leqslant K_1}(r_i+r_j-1)}.$$

In an analogue way, if $m$ is odd, let

$$\{1,2,\ldots,N+1\}\backslash\{i,j_1,j_2,\ldots,j_q\} = \{m,r_1,r_2,\ldots,r_{K_1}\} \cup \{s_1,s_2,\ldots,s_{K_2}\},$$

where $r_i$ is odd for $i = 1,2,\ldots,K_1$, and $s_i$ is even for $i = 1,2,\ldots,K_2$, and

$$\sum_{\gamma\in\Gamma} \operatorname{sgn}(\gamma) \prod_{\substack{l=1 \\ l\neq i,j_1,\ldots,j_q}}^{N+1} \frac{1}{l+\gamma(l)-1}$$
$$= \frac{\prod_{l=1}^{K_1}(m-r_l)(i-r_l) \prod_{1\leqslant i<j\leqslant K_1}(r_i-r_j)^2}{(m+i-1)\prod_{1\leqslant i<j\leqslant K_1}(r_i+r_j-1)} \frac{\prod_{1\leqslant i<j\leqslant K_2}(s_i-s_j)^2}{\prod_{1\leqslant i<j\leqslant K_2}(s_i+s_j-1)}.$$

This is different from zero and we get our statement.

□

# 5

## Divergence of Fourier Series: A Cohen type inequality

The aim of this Chapter is to establish a Cohen type inequality when we deal with the following Sobolev-type inner product on the linear space $\mathbb{P}$ of polynomials with real coefficients

$$\langle f, g\rangle_S = \int_0^{\infty} f(x)g(x)d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c), \tag{5.0.1}$$

where $d\mu(x) = x^{\alpha}e^{-x}dx$, $\alpha > -1$, is the Laguerre measure, $M, N \geqslant 0$, and the mass point $c$ is a real number located outside the support of $\mu$.

The novelty of our approach comes from two directions: First, we consider a Sobolev-type inner product with only a mass point outside the support of the measure $\mu$ and, second, we incorporate new test functions different from those used in [70].

The outline of the chapter is as follows. Section 5.1 provides a basic background dealing with structural and asymptotic properties of $k$-iterated Laguerre orthogonal polynomials, as well as some well known analytic properties of Laguerre-Sobolev type polynomials. Section 5.1 contains some estimates for the norm of Laguerre-Sobolev type polynomials (Propositions 5.1.2). In Section 5.2 we prove our main result (Theorem 5.2.1). We obtain an estimate from below for the $S^p_{f(\alpha)}$-norm of the partial sums of some balanced Fourier expansions in terms of Laguerre-Sobolev type orthonormal polynomials. As an immediate consequence (Corollaries 5.2.1 and 5.2.2) the divergence of such partial sums and Cesàro means of order $\delta$ when $p$ is located outside the Pollard interval is deduced.

## 5.1 Background: structural and asymptotic properties

The following proposition will be useful in the sequel and it summarizes some recent structural and asymptotic properties of Laguerre-Sobolev type polynomials.

**Proposition 5.1.1.** *Let* $\{L_n^{(\alpha,M,N)}(x)\}_{n=0}^{\infty}$ *be the sequence of normalized Laguerre-Sobolev type polynomials with leading coefficient equal to* $\frac{(-1)^n}{n!}$*, associated with the Sobolev-type inner product* (5.0.1)*. Then the following statements hold.*

*(a) [68, Theorem 4] Connection formula for* $L_n^{(\alpha,M,N)}(x)$*.*

$$L_n^{(\alpha,M,N)}(x) = B_{0,n}L_n^{(\alpha)}(x)+B_{1,n}(x-c)L_{n-1}^{(\alpha),[2]}(x)+B_{2,n}(x-c)^2L_{n-2}^{(\alpha),[4]}(x), \tag{5.1.2}$$

*where*

$(i)$ *If* $M>0$ *and* $N>0$*, then*

$$B_{0,n} \sim \frac{8cn^{\alpha}}{M\left(L_n^{(\alpha)}(c)\right)^2}, \qquad B_{1,n} \sim -\frac{32c\sqrt{|c|}\,n^{\alpha-1/2}}{M\left(L_n^{(\alpha)}(c)\right)^2}, \qquad B_{2,n} \sim \frac{1}{n^2}. \tag{5.1.3}$$

$(ii)$ *If* $M=0$ *and* $N>0$*, then*

$$B_{0,n} \sim \frac{1}{4\sqrt{|c|n}}, \qquad B_{1,n} \sim -\frac{1}{n}, \qquad B_{2,n} \sim \frac{1}{4n^2\sqrt{|c|n}}.$$

$(iii)$ *If* $M>0$ *and* $N=0$*, then*

$$B_{0,n} \sim \frac{\sqrt{|c|}}{Mn^{1/2-\alpha}\left(L_{n-1}^{(\alpha)}(c)\right)^2}, \qquad B_{1,n} \sim -\frac{1}{n}, \qquad B_{2,n} = 0.$$

*(b) [68, Theorem 5 (ii)] Mehler-Heine type formula.*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha,M,N)}(\frac{x}{n})}{n^{\alpha}} = \begin{cases} x^{-\alpha/2}J_{\alpha}(2\sqrt{x}), & \text{if } M>0, N>0, \\ -x^{-\alpha/2}J_{\alpha}(2\sqrt{x}), & \text{if } M=0, N>0 \text{ or } M>0, N=0, \end{cases} \tag{5.1.4}$$

*uniformly on compact subsets of* $\mathbb{C}$*.*

Now, we need to estimate the Laguerre-Sobolev type norm

$$h_n^{(\alpha,M,N)} := \langle L_n^{(\alpha,M,N)}, L_n^{(\alpha,M,N)} \rangle_S.$$

The next Proposition states that the estimate of this norm is the same as the estimate obtained for the norms of classical Laguerre polynomials.

**Proposition 5.1.2.** *For every $n \in \mathbb{N}$,*

$$h_n^{(\alpha,M,N)} \sim n^\alpha. \tag{5.1.5}$$

*Proof.* From the Sobolev type orthogonality, we get

$$h_n^{(\alpha,M,N)} = \left\langle L_n^{(\alpha,M,N)}(x), \frac{(-1)^n}{n!}(x-c)^n \right\rangle_S, \quad n \geqslant 0. \tag{5.1.6}$$

Since the non standard component of the Sobolev type inner product on the right side of (5.1.6) is equal to zero for $n \geqslant 2$, according to (5.1.2) we have

$$\left\langle L_n^{(\alpha,M,N)}(x), \frac{(-1)^n}{n!}(x-c)^n \right\rangle_S = \int_0^\infty L_n^{(\alpha,M,N)}(x) \frac{(-1)^n}{n!}(x-c)^n x^\alpha e^{-x} dx$$
$$= B_{0,n} h_n^{(\alpha)} - \frac{B_{1,n}}{n} h_{n-1}^{(\alpha),[2]} + \frac{B_{2,n}}{n(n-1)} h_{n-2}^{(\alpha),[4]}.$$

Finally, analyzing the asymptotic behavior given in (5.1.3) and using (3.1.5) the result follows.

□

Notice that the above estimate for the norm of the Laguerre-Sobolev type orthogonal polynomials together with (5.1.4) (resp. (5.1.2)) allows us to obtain the corresponding Mehler-Heine type formula (resp. a connection formula for the orthonormal Sobolev type polynomials $L_n^{\alpha,M,N}(x)$.)

We conclude this section with the analog of [30, Proposition 5] when $c < 0$.

**Proposition 5.1.3.** *Let $M, N \geqslant 0$ and $\{L_n^{\alpha,M,N}(x)\}_{n=0}^\infty$ be the sequence of orthonormal Laguerre-Sobolev type polynomials. For $\alpha > -1/2$ we have*

$$\left( \int_0^\infty \left| L_n^{\alpha,M,N}(x) e^{-x/2} \right|^p x^\alpha dx \right)^{1/p} \geqslant \begin{cases} C n^{-1/4} (\log n)^{1/p}, & \text{if } p = \frac{4\alpha+4}{2\alpha+1}, \\ C n^{\alpha/2-(\alpha+1)/p}, & \text{if } \frac{4\alpha+4}{2\alpha+1} < p < \infty, \end{cases} \tag{5.1.7}$$

*and for $\alpha > -2/p$, $1 < p < \infty$, we have*

$$\left(\int_0^\infty \left|L_n^{\alpha,M,N}(x)e^{-x/2}\right|^p x^\alpha dx\right)^{1/p} \geqslant \begin{cases} Cn^{-1/4}(\log n)^{1/p} & \textit{if } p=4, \\ Cn^{-1/p}, & \textit{if } 4<p<\infty. \end{cases} \tag{5.1.8}$$

*Proof.* It suffices to follow the proof given in [30, Proposition 5] by making the corresponding modifications and using (5.1.5) as well as (5.1.4) for orthonormal polynomials.

□

## 5.2 Cohen type inequality for Fourier expansions with respect to Laguerre-Sobolev type orthogonal polynomials associated with the inner product (5.0.1)

The goal of this section is to show a Cohen type inequality for Fourier expansions with respect to Laguerre-Sobolev type orthonormal polynomials associated with the Sobolev inner product (5.0.1). To this end, we will follow the Markett approach but, as was mentioned at the beginning, we will incorporate new test functions different from whose used in [70].

Now, we are going to introduce the notation concerning weighted $L^p$ spaces, Sobolev type spaces, test functions, and some usual elements from functional analysis, which will needed in the sequel.

We consider the following weighted $L^p$ spaces.

$$L^p_{w(\alpha)} = \begin{cases} \left\{f : \{\int_0^\infty |f(x)e^{-x/2}|^p x^\alpha dx\}^{1/p} < \infty\right\}, & 1 \leqslant p < \infty, \\ \left\{f : \operatorname{ess\,sup}_{x>0}|f(x)e^{-x/2}| < \infty\right\}, & p=\infty, \end{cases}$$

for $\alpha > -1$. Furthermore,

$$L^p_{u(\alpha)} = \{f : ||f(x)u(x,\alpha)||_{L^p(0,\infty)} < \infty, u(x,\alpha) = e^{-x/2}x^{\alpha/2}\},$$

where $\alpha > -\frac{2}{p}$ if $1 \leqslant p < \infty$ and $\alpha \geqslant 0$ if $p = \infty$.

Also, we use the notation $L^p_{g(\alpha)}$, where the subscript $g(\alpha)$ means either $w(\alpha)$ or

$u(\alpha)$. The Sobolev type spaces are denoted by

$$S^p_{g(\alpha)} = \{f \in L^p_{g(\alpha)} \cap C^\infty : ||f||^p_{S^p_{g(\alpha)}} = ||f||^p_{L^p_{g(\alpha)}} + M|f(c)|^p + N|f'(c)|^p < \infty\}, \quad 1 \leqslant p < \infty, \tag{5.2.9}$$

$$S^\infty_{g(\alpha)} = \left\{f \in L^\infty_{g(\alpha)} \cap C^\infty : ||f||_{S^\infty_{g(\alpha)}} = \max\{||f||_{L^\infty_{g(\alpha)}}, |f(c)|, |f'(c)|\} < \infty\right\}, \quad p = \infty. \tag{5.2.10}$$

Let $[S^p_{g(\alpha)}]$ be the space of all bounded linear operators $T : S^p_{g(\alpha)} \to S^p_{g(\alpha)}$, with the standard operator norm

$$||T||_{[S^p_{g(\alpha)}]} = \sup_{0 \neq f \in S^p_{g(\alpha)}} \frac{||T(f)||_{S^p_{g(\alpha)}}}{||f||_{S^p_{g(\alpha)}}}.$$

For $f \in S^1_{g(\alpha)}$, the Fourier series in terms of the Laguerre-Sobolev type orthonormal polynomials is given by

$$\sum_{k=0}^{\infty} \hat{f}(k) L_k^{\alpha,M,N}(x), \tag{5.2.11}$$

where $\hat{f}(k) = \langle f, L_k^{\alpha,M,N} \rangle_S, \quad k = 0, 1, \ldots$

The Cesáro means of order $\delta$, a nonnegative integer number, of the series (5.2.11) is

$$\sigma_n^\delta f(x) := \sum_{k=0}^{n} \frac{A^\delta_{n-k}}{A^\delta_n} \hat{f}(k) L_k^{\alpha,M,N}(x),$$

where $A_k^\delta = \binom{k+\delta}{k}$.

For $f \in S^p_{g(\alpha)}$ and $\{c_{k,n}\}_{k=0}^n, n \in \mathbb{N} \cup \{0\}$, a family of complex numbers with $|c_{n,n}| > 0$, let introduce the operators $T_n^{\alpha,M,N}$

$$T_n^{\alpha,M,N}(f) := \sum_{k=0}^{n} c_{k,n} \hat{f}(k) L_k^{\alpha,M,N}.$$

The first technical step required for the proof of our main result is the choice of the suitable test functions. For instance, in the setting of Laguerre-Sobolev type expansions, see [30, 70, 83], the authors consider (up to a constant factor) the following test functions.

$$g_n^{\alpha,j}(x) := n^{-\alpha/2}\left[x^j L_n^{(\alpha+j)}(x) - \left(\frac{(n+1)(n+2)}{(n+\alpha+j+1)(n+\alpha+j+2)}\right)^{1/2} x^j L_{n+2}^{(\alpha+j)}(x)\right]. \quad (5.2.12)$$

These functions and their derivatives vanish at 0 and this fact is a key property in the development of the ideas of [30, 70, 83]. Unfortunately, they do not vanish at the mass point $c < 0$. For this reason, it seems to be natural to consider the following slight modification of the functions (5.2.12)

$$G_n^{\alpha,j}(x) := n^{-\alpha/2}\left[(x-c)^2 x^j L_{n+2}^{(\alpha+j)}(x) - A_{n,\alpha}(x-c)^2 x^j L_{n+4}^{(\alpha+j)}(x)\right] \quad (5.2.13)$$

with $A_{n,\alpha} = \left(\frac{(n+3)(n+4)}{(n+\alpha+j+3)(n+\alpha+j+4)}\right)^{1/2}$.

As a consequence, it is well-known that the test polynomials $G_n^{\alpha,j}(x)$ can be expressed as (see [70, equation (2.15)])

$$G_n^{\alpha,j}(x) = n^{-\alpha/2}(x-c)^2 \sum_{m=0}^{j+2} a_{m,j}(\alpha,n) L_{n+2+m}^{(\alpha)}(x), \quad (5.2.14)$$

with

$$a_{0,j}(\alpha,n) \cong n^j.$$

Finally, the last technical step is to estimate the norm of the test functions (5.2.13).

**Lemma 5.2.1.** *For some $j > \alpha - 1/2 - 2(\alpha+1)/p$, we have*

$$||G_n^{\alpha,j}||_{S^p_{g(\alpha)}} \leqslant C \begin{cases} n^{j+2-\alpha/2-1/2+(\alpha+1)/p}, & \text{if } g(\alpha) = w(\alpha), \\ n^{j+2-1/2+1/p}, & \text{if } g(\alpha) = u(\alpha). \end{cases} \quad (5.2.15)$$

*Proof.* Taking into account that the Sobolev norm of $G_n^{\alpha,j}(x)$ coincides with its $L^p_{g(\alpha)}$-norm (for $g(\alpha) = w(\alpha)$ or $g(\alpha) = u(\alpha)$) and also considering the following expression for $G_n^{\alpha,j}(x)$,

$$G_n^{\alpha,j}(x) = g_{n+2}^{\alpha-2,j+2}(x) - 2c g_{n+2}^{\alpha-1,j+1}(x) + c^2 g_{n+2}^{\alpha,j}(x),$$

where $g_n^{\alpha,j}(x)$ is the test polynomial given in (5.2.12), so, we only need to use [70, Lemma 1] in order to obtain the estimates (5.2.15).

□

According to the notation in [30], let us denote $q_0 = \frac{4\alpha+4}{2\alpha+1}$, when $\beta = \alpha$, and $q_0 = 4$, when $\beta = p\alpha/2$, and let $p_0$ be the conjugate of $q_0$. We are ready to state our main result.

**Theorem 5.2.1.** *Let $M, N \geqslant 0$ and $1 \leqslant p \leqslant \infty$. For $\alpha > -1/2$,*

$$||T_n^{\alpha,M,N}||_{[S^p_{w(\alpha)}]} \geqslant C|c_{n,n}| \begin{cases} n^{\frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}} & \text{if } a \leqslant p < p_0, \\ (\log n)^{\frac{2\alpha+1}{4\alpha+4}} & \text{if } p = p_0, p = q_0, \\ n^{\frac{2\alpha+1}{2} - \frac{2\alpha+2}{p}} & \text{if } q_0 < p \leqslant b. \end{cases}$$

*For $\alpha > -2/p$ if $1 \leqslant p < \infty$ and $\alpha \geqslant 0$ if $p = \infty$,*

$$||T_n^{\alpha,M,N}||_{[S^p_{u(\alpha)}]} \geqslant C|c_{n,n}| \begin{cases} n^{\frac{2}{p} - \frac{3}{2}} & \text{if } a \leqslant p < p_0, \\ (\log n)^{\frac{1}{4}} & \text{if } p = p_0, p = q_0, \\ n^{\frac{1}{2} - \frac{2}{p}} & \text{if } q_0 < p \leqslant b, \end{cases}$$

*where*

*(i) if $M = 0$, $N \geqslant 0$, then $a = 1$ and $b = \infty$,*

*(ii) if $M > 0$, $N \geqslant 0$, then $a > 1$, $b < \infty$, and $1/a + 1/b = 1$.*

*Proof.* Applying the operator $T_n^{\alpha,M,N}$ to the test functions $G_n^{\alpha,j}(x)$ we get

$$T_n^{\alpha,N,M}(G_n^{\alpha,j}) = \sum_{k=0}^{n} c_{k,n} (G_n^{\alpha,j})^\wedge(k) L_k^{\alpha,M,N}, \tag{5.2.16}$$

where

$$(G_n^{\alpha,j})^\wedge(k) = \langle G_n^{\alpha,j}, L_k^{\alpha,M,N} \rangle_S, \qquad k = 0, \dots, n.$$

From (5.2.14) and the Sobolev orthogonality it follows in a straightforward way that

$$(G_n^{\alpha,j})^\wedge(k) = 0 \quad \text{if } k < n.$$

When $k = n$, we get

$$\begin{aligned}(G_n^{\alpha,j})^\wedge(n) =& n^{-\alpha/2} a_{0,j}(\alpha, n) \int_0^\infty L_{n+2}^{(\alpha)}(x) L_n^{\alpha,M,N}(x)(x-c)^2 x^\alpha e^{-x} dx \\ &= n^{-\alpha/2} a_{0,j}(\alpha, n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} \int_0^\infty L_{n+2}^{(\alpha)}(x) L_n^{(\alpha,M,N)}(x)(x-c)^2 x^\alpha e^{-x} dx.\end{aligned}$$

We can expand the polynomial $(x-c)^2 L_n^{(\alpha,M,N)}(x)$ in terms of the classical Laguerre polynomials,

$$(x-c)^2 L_n^{(\alpha,M,N)}(x) = \sum_{k=0}^{n+2} \alpha_{n+2,k} L_k^{(\alpha)}(x).$$

The comparison of the leading coefficient of both hand sides yields

$$\alpha_{n+2,n+2} = (n+2)(n+1).$$

On the other hand,

$$\begin{aligned}(G_n^{\alpha,j})^\wedge(n) =& n^{-\alpha/2} a_{0,j}(\alpha,n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} \int_0^\infty L_{n+2}^{(\alpha)}(x) L_n^{(\alpha,M,N)}(x)(x-c)^2 x^\alpha e^{-x} dx \\ &= n^{-\alpha/2} a_{0,j}(\alpha,n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} \alpha_{n+2,n+2} \int_0^\infty \left(L_{n+2}^{(\alpha)}(x)\right)^2 x^\alpha e^{-x} dx \\ &= n^{-\alpha/2} a_{0,j}(\alpha,n) \left(h_n^{(\alpha,M,N)}\right)^{-1/2} (n+2)(n+1) h_{n+2}^{(\alpha)} \\ &\sim n^{j+2}.\end{aligned}$$

As a conclusion,

$$\begin{cases}(G_n^{\alpha,j})^\wedge(k) = 0, & 0 \leqslant k \leqslant n-1, \\ (G_n^{\alpha,j})^\wedge(n) \sim n^{j+2}. & \end{cases}$$

Now, we follow the proof given in [30, Theorem 1], taking into account that

$$\begin{aligned}|L_n^{\alpha,0,0}(c)| &\cong n^{\alpha/2-1/4} e^{2\sqrt{-nc}} \left\{ \sum_{k=0}^{p-1} C_k(\alpha;c) n^{-k/2} + O(n^{-p/2}) \right\}, \\ |L_n^{\alpha,0,N}(c)| &\cong \frac{1}{\sqrt{-cn}} n^{\alpha/2-1/4} e^{2\sqrt{-nc}} \left\{ \sum_{k=0}^{p-1} C_k(\alpha;c) n^{-k/2} + O(n^{-p/2}) \right\}.\end{aligned}$$

□

**Corollary 5.2.1.** *Let $\beta, p_0, q_0$, and $p$ be the same as in Theorem 5.2.1. For $c_{k,n} = 1$, for all $k = 0, \ldots, n$, and for $p$ outside the Pollard interval $(p_0, q_0)$ we get*

$$||S_n||_{[S^p_{g(\beta)}]} \to \infty, \quad n \to \infty,$$

*where $S_n$ denotes the $n$th partial sum of the expansion* (5.2.11).

It is worthwhile to point out that Corollary 5.2.1 says that as for the results of [30, 70, 83], the divergence of Fourier expansions in terms of this kind of Laguerre-Sobolev type orthonormal polynomials remains true.

For $c_{k,n} = \frac{A^{\delta}_{n-k}}{A^{\delta}_{n}}$, $k = 0, \ldots, n$, from Theorem 5.2.1 we also get the divergence of Cesàro means of order $\delta$ when $p$ is located outside the Pollard interval.

**Corollary 5.2.2.** *Let $M, N \geqslant 0$ and $1 \leqslant p \leqslant \infty$. For $\alpha > -1/2$,*

$$\begin{cases} 0 \leqslant \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}, & \text{if } a \leqslant p < p_0, \\ \\ 0 \leqslant \delta < \frac{2\alpha+1}{2} - \frac{2\alpha+2}{p}, & \text{if } q_0 < p \leqslant b, \end{cases}$$

*and $p \notin [p_0, q_0]$, then*

$$||\sigma^{\delta}_{n}||_{[S^p_{w(\alpha)}]} \to \infty, \quad n \to \infty.$$

*For $\alpha > -2/p$ if $1 \leqslant p < \infty$, and $\alpha \geqslant 0$, if $p = \infty$,*

$$\begin{cases} 0 \leqslant \delta < \frac{2}{p} - \frac{3}{2}, & \text{if } a \leqslant p < p_0, \\ \\ 0 \leqslant \delta < \frac{1}{2} - \frac{2}{p}, & \text{if } q_0 < p \leqslant b, \end{cases}$$

*and $p \notin [p_0, q_0]$, then we get*

$$||\sigma^{\delta}_{n}||_{[S^p_{u(\alpha)}]} \to \infty, \quad n \to \infty.$$

**Remark 5.2.1.** *It still remains as an open question the study of Cohen type inequalities for the Laguerre Sobolev type orthonormal polynomials with respect to the inner product*

$$\langle f, g \rangle_S = \int_0^{\infty} f(x)g(x)d\mu(x) + \sum_{j=0}^{N} M_j f^{(j)}(c)g^{(j)}(c), \tag{5.2.17}$$

*where $d\mu(x) = x^{\alpha}e^{-x}dx$ is the Laguerre measure, $c < 0$, and $M_j \geqslant 0$ for $j = 0, \ldots, N$ assuming that $M_N > 0$. The main difficulties in this case would be how to choose suitable test functions as well as the possibility to have gaps in the Sobolev type inner products , i.e. $M_j = 0$ for some $j = 0, \ldots, N-1$. This means that the matrix $diag(M_0, ..., M_N)$ has not full rank (see for instance, [83] and the references therein.)*

# 6

# Conclusions

In this thesis, we have dealt with Sobolev-type orthogonal polynomials. In particular, we have focused our attention on four interesting problems:

(i) Connection formulas between Sobolev type and the associated standard orthogonal polynomials.

(ii) Matrix interpretation of recurrence relations and connection formulas.

(iii) Outer relative asymptotics for Sobolev type orthogonal polynomials.

(iv) Convergence of Fourier series associated to Sobolev-type orthogonal polynomials.

As a result of the research in these directions, the original contributions of this thesis have been the following ones:

- We have made an exhaustive study of connection formulas relating Sobolev type and standard polynomials and we have found a quite general formula included in Proposition 1.2.5.
- As for the case of Sobolev-type orthogonal polynomials with respect to nontrivial probability measures with bounded support on the real line, we have found an alternative proof of a known result on outer relative asymptotics of Sobolev

polynomials (Theorem 2.1.1). We have also obtained a matrix connection between the $(2N+1)$-diagonal matrix associated to the Sobolev-type orthogonal polynomials and the Jacobi matrix of the corresponding standard polynomials (Theorem 2.2.2). Finally, we have generalized the study of the pointwise convergence of Fourier series associated to Jacobi-Sobolev polynomials to the case of inner products with several mass points outside the support of the measure.

- We have worked with a family of polynomials orthogonal with respect to a polynomial perturbation of the classical Laguerre measure, the $k$-iterated Laguerre polynomials. In this direction, we have obtained estimates for the norm of $k$-iterated Laguerre polynomials (Proposition 3.1.3) as well as a generalized Christoffel representation formula for $k$-iterated polynomials (Proposition 3.1.4).

- We have described the asymptotic behavior of the partial derivatives of diagonal Laguerre kernels depending on the location of the mass points in terms of the support of the measure, i.e., if it is inside, outside or at the boundary of the Laguerre measure. The novelty in this direction comes from the case of the asymptotic behaviour of Laguerre kernels and its partial derivatives within the oscillatory regime of classical Laguerre polynomials (Theorem 3.2.1) and this result will permit us to obtain asymptotics for Sobolev-type orthogonal polynomials.

- For the case of Sobolev-type polynomials orthogonal with respect to nontrivial probability measures with unbounded supports, we have studied the representative case of Laguerre-Sobolev type orthogonal polynomials. The novelty consists of considering for the first time a discrete Sobolev inner product with mass points inside the support, which is the oscillatory region for the corresponding polynomials. More precisely, we have studied the Outer Relative Asymptotics (Theorem 4.1.1) as well as the inner $L^2$-Asymptotics (Theorem 4.3.1). We have also obtained the asymptotic behavior of the coefficients in the five term recurrence relation associated to certain Sobolev-type polynomials (Theorem 4.2.1) .

- Finally, we get a Cohen type inequality for Fourier expansions in terms of the orthonormal polynomials associated with a Sobolev type inner product with a mass point located outside the support of the measure (Theorem 5.2.1), for which we have found a new family of test functions, different from those one can find in the literature. Then, as an immediate consequence, we deduce the divergence of Fourier expansions and Cesàro means of order $\delta$ in terms of this kind of Laguerre-Sobolev type polynomials (Corollary 5.2.1).

## 6.1 Future work

In view of the results of this thesis, we plan to follow several research lines in the next future.

- The properties of classical Laguerre polynomials are very well known. In particular, Hahn condition allows to determine the derivatives of Laguerre polynomials and using Perron or Féjer asymptotic formulas we can obtain the asymptotic behavior of these derivatives. It would be very useful to carry out a study about asymptotics of derivatives of $k$-iterated Laguerre polynomials.

- From a theoretical point of view, it would be interesting to study the pointwise convergence of the Jacobi-Fourier series associated with Sobolev type inner products when some mass points are located inside the support and other ones outside.

- For a denumerable set of mass points outside $[-1, 1]$ which satisfy the Blaschke's condition and for a measure $\mu$ such that its absolutely continuous part satisfies the Szegő condition, the outer strong asymptotics of standard orthogonal polynomials was given in [82]. In [16], Denisov proved the outer ratio asymptotics of standard polynomials with respect to a measure such that $\mu'(x) > 0$ a.e. $x \in [-1, 1]$ in the presence of a denumerable set of mass points off $[-1, 1]$ with the only condition that they accumulate at $\pm 1$. The idea will be to study the case of a denumerable set of mass points inside $[-1, 1]$, which remains an open problem.

- In Chapter 4 we have studied asymptotics for Laguerre-Sobolev type polynomials. It would be interesting to extend these results to another families of polynomials orthogonal with respect to measures supported on an unbounded set. Our technique is quite general and, basically, it requires to know strong asymptotics for the corresponding Sobolev polynomials.

- In Chapter 5 we have studied the convergence of some balanced Fourier-Sobolev expansions in some $p$-weighted space, when $p$ does not belong to the Pollard interval. The complementary study for the case when $p$ belongs to the Pollard interval remains an open problem.

- It would be interesting the study of Cohen type inequalities for a continuous Sobolev inner product associated with coherent pairs of measures when one of the measures is the Laguerre one. This case requires to work in weighted Sobolev spaces different from whose used since the first study of Markett.

# Bibliography

[1] M. Alfaro, F. Marcellán, M. L. Rezola, and A. Ronveaux, *On orthogonal polynomials of Sobolev type: algebraic properties and zeros*, SIAM J. Math. Anal. **23** (1992), 737–757.

[2] M. Alfaro, F. Marcellán, M. L. Rezola, and A. Ronveaux, *Sobolev-type orthogonal polynomials: the non-diagonal case*, J. Approx. Theory **83** (1995), 266–287.

[3] M. Alfaro, J. J. Moreno-Balcázar, A. Peña, and M. L. Rezola, *A new approach to the asymptotics of Sobolev type orthogonal polynomials*, J. Approx. Theory **163** (2011), 460–480.

[4] M. Alfaro, J. J. Moreno-Balcázar, and M. L. Rezola, *Laguerre-Sobolev orthogonal polynomials: asymptotics for coherent pairs of type II*, J. Approx. Theory **122** (2003), 79–96.

[5] R. Alvarez-Nodarse and J. J. Moreno-Balcázar, *Asymptotic properties of generalized Laguerre orthogonal polynomials*, Indag. Math. (N.S.) **15** (2004), 151–165.

[6] P. Althammer, *Eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen und deren Anwendung auf die beste Approximation*, J. Reine Angew. Math. **211** (1962), 192–204.

[7] I. Alvarez Rocha and L. Salto, *Asymptotics of polynomials orthogonal with respect to a discrete-complex Sobolev inner product*, J. Comput. Appl. Math. **178** (2005), 1–19.

[8] R. Askey, *Orthogonal expansions with positive coefficients*, Proc. Amer. Math. Soc. **16** (1965), 1191–1194.

[9] R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708.

[10] L. Baratchart, A. Martínez-Finkelshtein, D. Jiménez, D. S. Lubinsky, H. N. Mhaskar, I. Pritsker, M. Putinar, M. Stylianopoulus, V. Totik, P. Varju, and Y.

Xu, *Open problems in Constructive Function Theory*, Electron. Trans. Numer. Anal. **25** (2006), 511–525.

[11] A. Branquinho, A. Foulquié, and F. Marcellán, *Asymptotic behavior of Sobolev type orthogonal polynomials on a rectifiable Jordan curve or arc*, Constr. Approx. **18** (2002), 161–182.

[12] A. Cachafeiro, F. Marcellán, and J. J. Moreno-Balcázar, *On asymptotic properties of Freud-Sobolev orthogonal polynomials*, J. Approx. Theory **125** (2003), 26–41.

[13] T. S. Chihara, *An Introduction to Orthogonal Polynomials*. Mathematics and its Applications Series, Gordon and Breach, New York (1978).

[14] P. J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** (1960), 191–212.

[15] A. Deaño, E. J. Huertas, and F. Marcellán, *Strong and ratio asymptotics for Laguerre polynomials revisited*, J. Math. Anal. Appl. **403** (2013), 477–486.

[16] S. A. Denisov, *On Rakhmanov's theorem for Jacobi matrices*, Proc. Amer. Math. Soc. **132** (2004), 847–852 .

[17] M. Derevyagin, J. C. García-Ardila, and F. Marcellán, *Multiple Geronimus transformation*, Linear Algebra Appl. **454** (2014) 158–183.

[18] H. Dueñas and F. Marcellán, *Asymptotic behaviour of Laguerre-Sobolev type orthogonal polynomials: a nondiagonal case*, J. Comput. Appl. Math. **235** (2010), 998–1007.

[19] H. Dueñas and F. Marcellán, *The Laguerre-Sobolev-type orthogonal polynomials. Holonomic equation and electrostatic interpretation*, Rocky Mount. J. Math. **41** (2011), 95–131.

[20] H. Dueñas, E. J. Huertas, and F. Marcellán, *Analytic properties of Laguerre-type orthogonal polynomials*, Integral Transforms Spec. Funct. **22** (2011), 107–122.

[21] H. Dueñas, E. J. Huertas, and F. Marcellán, *Asymptotic properties of Laguerre-Sobolev type Orthogonal Polynomials*, Numer. Algorithms **60** (2012), 51–73.

[22] B. Dreseler and P. M. Soardi, *A Cohen type inequality for ultraspherical series*, Arch. Math. (Basel) **38** (1982), 243–247.

[23] B. Dreseler and P. M. Soardi, *A Cohen type inequality for Jacobi expansions and divergence of Fourier series on compact symmetric spaces*, J. Approx. Theory **35** (1982), 214–221.

[24] A. J. Durán and W. Van Assche, *Orthogonal matrix polynomials and Higher-order recurrence relations*, Linear Algebra Appl. **219** (1995), 261–280.

[25] W. D. Evans, L. L. Littlejohn, F. Marcellán, C. Markett, and A. Ronveaux, *On recurrence relations for Sobolev orthogonal polynomials*, SIAM J. Math. Anal. **26** (1995), 446–467.

[26] J. Favard, *Sur les polynômes de Tchebicheff*, C. R. Acad. Sci. Paris **200** (1935), 2052–2053.

[27] B. Xh. Fejzullahu, *A Cohen type inequality for Fourier expansions with respect to non-discrete Laguerre-Sobolev inner product*, Numer. Func. Anal. Opt. **31**(12) (2010), 1330–1341.

[28] B. Xh. Fejzullahu, *Asymptotics for orthogonal polynomials with respect to the Laguerre measure modified by a rational factor*, Acta Sci. Math. (Szeged) **77** (2011), 73–85.

[29] B. Xh. Fejzullahu and F. Marcellán, *A Cohen type inequality for Laguerre-Sobolev expansions*, J. Math. Anal. Appl. **352** (2009), 880–889.

[30] B. Xh. Fejzullahu, F. Marcellán, and J. J. Moreno-Balcázar, *Jacobi-Sobolev orthogonal polynomials: asymptotics and a Cohen type inequality*, J. Approx. Theory **170** (2013), 78–93.

[31] B. Xh. Fejzullahu and R. Xh. Zejnullahu, *Orthogonal polynomials with respect to the Laguerre measure perturbed by the canonical transformations*, Integral Transforms Spec. Funct. **21** (2010), 569–580.

[32] A. Foulquié, F. Marcellán, and K. Pan, *Asymptotic behavior of Sobolev-type orthogonal polynomials on the unit circle*, J. Approx. Theory **100** (1999), 345–363.

[33] G. Freud, *Orthogonal Polynomials*, Pergamon Press, Oxford-New York (1971).

[34] J. S. Geronimo, D. S. Lubinsky, and F. Marcellán, *Asymptotics for Sobolev orthogonal polynomials for exponential weights*. Constr. Approx. **22** (2005), 309–346.

[35] S. Giulini, P. M. Soardi, and G. Travaglini, *A Cohen type inequality for compact Lie groups*, Proc. Amer. Math. Soc. **77** (1979), 359–364.

[36] G. H. Hardy and J. E. Littlewood, *A new proof of a theorem on rearrangements*, J. London Math. Soc. **23** (1948), 163–168.

[37] E. J. Huertas, *Analytic properties of Krall-type and Sobolev-type Orthogonal Polynomials*. Doctoral Dissertation, Universidad Carlos III de Madrid. 2012.

[38] E. J. Huertas, F. Marcellán, M. F. Pérez-Valero, and Y. Quintana, *Asymptotics for Laguerre-Sobolev type orthogonal polynomials modified within their oscillatory regime*. Appl. Math. Comput. **236** (2014), 260–272.

[39] E. J. Huertas, F. Marcellán, M. F. Pérez-Valero, and Y. Quintana, *A Cohen type inequality for Laguerre-Sobolev expansions with a mass point outside their oscillatory regime*. Turkish J. Math. **38** (2014), 994–1006.

[40] A. Iserles, P. E. Koch, S. P. Nørsett, and J. M. Sanz-Serna, *Orthogonality and approximation in a Sobolev space*, in Algorithms for Approximation (J. C. Mason and M. G. Cox, Eds) Chapman & Hall, London, 1990, 117–124.

[41] M. E. H. Ismail, *Classical and quantum orthogonal polynomial in one variable*, Encyclopedia of Mathematics and its Applications. Vol. **98**, Cambridge University Press, Cambridge (2005).

[42] G. Kerkyacharian, P. Petrushev, D. Picard, and Y. Xu, *Decomposition of Triebel-Lizorkin and Besov spaces in the context of Laguerre expansions*, J. Funct. Anal. **256** (2009), 1137–1188.

[43] G. Klambauer, *Aspects of Calculus*, Springer-Verlag, New York 1986.

[44] R. Koekoek and H. G. Meijer, *A generalization of Laguerre Polynomials*, SIAM J. Math. Anal. **24** (1993), 768–782.

[45] G. López Lagomasino, *Relative asymptotics of orthogonal polynomials on the real axis*, Mat. Sb. **137** (179), (1988), 266–287 . (English translation: Math. USSR-Sb. **65** (1990), 505–527).

[46] G. López Lagomasino, *Asymptotics of polynomials orthogonal with respect to varying measures*, Constr. Approx. **5** (1989), 199–219.

[47] G. López Lagomasino, F. Marcellán, and W. Van Assche, *Relative asymptotics for polynomials orthogonal with respect to a discrete Sobolev inner product*, Constr. Approx. **11** (1995), 107–137.

[48] G. López Lagomasino, I. Pérez Izquierdo, and H. Pijeira, *Asymptotic of extremal polynomials in the complex plane*, J. Approx. Theory **137** (2005), 226–237.

[49] G. López Lagomasino and H. Pijeira, *Zero location and n-th root asymptotics of Sobolev orthogonal polynomials*, J. Approx. Theory **99** (1999), 30–43.

[50] D. C. Lewis, *Polynomial least square approximations*, Amer. J. Math. **69** (1947), 273–278.

[51] D. S. Lubinsky, *Universality limits in the bulk for arbitrary measures on compact sets*, J. Anal. Math. **106** (2008), 373–394.

[52] D. S. Lubinsky, *A new approach to universality limits involving orthogonal polynomials*, Ann. of Math. **170** (2009), 915–939.

[53] F. Marcellán, M. Alfaro, and M. L. Rezola, *Orthogonal polynomials on Sobolev spaces: Old and new directions*, J. Comput. Appl. Math. **48** (1993), 113–131.

[54] F. Marcellán, A. Branquinho, and J. C. Petronilho, *Classical orthogonal polynomials: a functional approach*, Acta Appl. Math. **34** (1994), 283–303.

[55] F. Marcellán, H. G. Meijer, T. E. Pérez, and M. A. Piñar, *An asymptotic result for Laguerre-Sobolev orthogonal polynomials*, J. Comput. Appl. Math. **87** (1997), 87–94.

[56] F. Marcellán and J. J. Moreno-Balcázar, *Asymptotics and zeros of Sobolev orthogonal polynomials on unbounded supports*, Acta Appl. Math. **94** (2006), 163–192.

[57] F. Marcellán, B. P. Osilenker, and I. A. Rocha, *On Fourier Series of Jacobi-Sobolev orthogonal polynomials*, J. Inequal. Appl. **7** (2002), 673–699.

[58] F. Marcellán, B. P. Osilenker, and I. A. Rocha, *On Fourier Series of a discrete Jacobi-Sobolev inner product*, J. Approx. Theory **117** (2002), 1–22.

[59] F. Marcellán, M. F. Pérez-Valero, Y. Quintana, and A. Urieles, *Recurrence relations and outer relative asymptotics of orthogonal polynomials with respect to a discrete Sobolev type inner product*, Bull. Math. Sci. **4** (1) (2014), 83–97.

[60] F. Marcellán, M. F. Pérez-Valero, and Y. Quintana, *Asymptotic behavior of derivatives of Laguerre kernels and some applications*, J. Math. Anal. Appl. **421** (2015), 314–328.

[61] F. Marcellán, Y. Quintana, and A. Urieles, *On $W^{1,p}$-convergence of Fourier-Sobolev expansions*, J. Math. Anal. Appl. **398** (2013), 594–599.

[62] F. Marcellán, Y. Quintana, and A. Urieles, *On the Pollard decomposition method applied to some Jacobi-Sobolev expansions*, Turk. J. Math. **37** (2013), 934–948.

[63] F. Marcellán and F. R. Rafaeli, *Monotonicity and asymptotics of zeros of Laguerre-Sobolev type orthogonal polynomials of higher order derivatives*, Proc. Amer. Math. Soc. **139** (2011), 3929–3936.

[64] F. Marcellán and A. Ronveaux, *On a class of polynomials orthogonal with respect to a discrete Sobolev inner product*, Indag. Math. (N.S.) **1** (1990), 451–464.

[65] F. Marcellán and A. Ronveaux, *A bibliography of Sobolev orthogonal polynomials*, Internal Report Universidad Carlos III de Madrid, (2014).

[66] F. Marcellán and W. Van Assche, *Relative asymptotics for orthogonal polynomials with a Sobolev inner product*, J. Approx. Theory **72** (1993), 193–209.

[67] F. Marcellán and Y. Xu, *On Sobolev orthogonal polynomials*, Expo. Math. In press.

[68] F. Marcellán, R. Xh. Zejnullahu, B. Xh. Fejzullahu, and E. J. Huertas, *On orthogonal polynomials with respect to certain discrete Sobolev inner product*, Pacific J. Math. **257** (2012), 167–188.

[69] C. Markett, *Mean Césaro summability of Laguerre expansions and norm estimates with shifted parameter*, Anal. Math. **8** (1982), 19–37.

[70] C. Markett, *Cohen type inequalities for Jacobi, Laguerre and Hermite expansions*, SIAM J. Math. Anal. **14** (1983), 819–833.

[71] A. Martínez-Finkelshtein, *Asymptotic properties of Sobolev orthogonal polynomials*, J. Comput. Appl. Math. **99** (1998), 491–510.

[72] A. Martínez-Finkelshtein, *Analytic aspects of Sobolev orthogonal polynomials revisited*, J. Comput. Appl. Math. **127** (2001), 255–266.

[73] H. G. Meijer, *A short history of orthogonal polynomials in a Sobolev space I. The non-discrete case*, Niew Arch. Wisk. **14** (1996), 93–112.

[74] H. G. Meijer, T. E. Pérez, and M. A. Piñar, *Asymptotics of Sobolev Orthogonal Polynomials for Coherent Pairs of Laguerre Type*, J. Math. Anal. Appl. **245** (2000), 528–546.

[75] B. Muckenhoupt, *Mean convergence of Hermite and Laguerre series II*, Trans. Amer. Math. Soc. **147** (1970), 433–460.

[76] T. Muir, *The theory of determinants in the historical order of development,* 4 vols., Macmillan, London, (1906–1923).

[77] P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. Vol. 213, Amer. Math. Soc. Providence, RI (1979).

[78] P. Nevai, *Géza Freud, orthogonal polynomials and Christoffel functions. A case study,* J. Approx. Theory **48** (1986), 1–167.

[79] A. F. Nikiforov and V. B. Uvarov, *Special functions of Mathematical Physics: an unified approach*. Birkhauser Verlag. Basel (1988).

[80] B. P. Osilenker, *Fourier series in orthogonal polynomials*, World Scientific, Singapore (1999).

[81] K. Pan, *On Sobolev Orthogonal Polynomials with Coherent Pairs: The Laguerre case, Type I*, J. Math. Anal. Appl. **223** (1998), 319–334.

[82] F. Peherstorfer and P. Yuditskii, *Asymptotics of orthonormal polynomials in the presence of a denumerable set of mass points,* Proc. Amer. Math. Soc. **129** (2001), 3213–3220 .

[83] A. Peña and M. L. Rezola, *Discrete Laguerre-Sobolev expansions: A Cohen type inequality*, J. Math. Anal. Appl. **385** (2012), 254–263.

[84] A. Portilla, J. M. Rodríguez, and E. Tourís, *The multiplication operator, zero location and asymptotic for non-diagonal Sobolev norms*, Acta Appl. Math. **111** (2010), 205–218.

[85] A. Portilla, Y. Quintana, J. M. Rodríguez, and E. Tourís, *Zero location and asymptotic behavior for extremal polynomials with non-diagonal Sobolev norms*, J. Approx. Theory **162** (2010), 2225–2242.

[86] A. Portilla, Y. Quintana, J. M. Rodríguez, and E. Tourís, *Concerning asymptotic behavior for extremal polynomials associated to non-diagonal Sobolev norms*, J. Funct. Spaces Appl. **2013**, article ID 628031 (2013), 1–11.

[87] E. A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials*, Mat. Sb. **103** (145) (1977), 271–291. (English translation (1977): Math. USSR-Sb. **32**, 199–213).

[88] E. A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials II*, Mat. Sb. **118** (160) (1982), 271–291. (English translation (1983): Math. USSR-Sb. **46**, 105–117).

[89] E. A. Rakhmanov, *On the asymptotics of polynomials orthogonal on the circle with weights not satisfying Szegő's condition*, Mat. Sb. **118** (172) (1986), 151–169. (English translation (1987): Math. USSR-Sb. **58**, 149–167).

[90] I. A. Rocha, F. Marcellán, and L. Salto, *Relative asymptotics and Fourier Series of orthogonal polynomials with a discrete Sobolev inner product*, J. Approx. Theory **121** (2003), 336–356.

[91] I. A. Rocha and L. Salto, *Asymptotics of polynomials orthogonal with respect to a discrete-complex Sobolev inner product*, J. Comput. Appl. Math. **178** (2005), 1–19.

[92] F. W. Schäfke, *Zu den Orthogonalpolynomen von Althammer*, J. Reine Angew. Math. **252** (1972), 195–199.

[93] B. Simon, *The Christoffel-Darboux kernel*. In *Perspectives in Partial Differential Equations, Harmonic Analysis and Applications*, 295–335, Proc. Sympos. Pure Math. **79**, Amer. Math. Soc., Providence, RI (2008), 295–335.

[94] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Cambridge University Press, Cambridge (1992).

[95] G. Szegő, *Orthogonal Polynomials*, Colloq. Publ. Amer. Math. Soc. Vol. **23**, (4th ed.), Amer. Math. Soc. Providence, RI (1975).

[96] S. Thangavelu, *Transplantation, summability and multipliers for multiple Laguerre expansions*, Tôhoku Math. J. **44** (1992), 279–298.

[97] V. B. Uvarov, *The connection between systems of polynomials that are orthogonal with respect to different distributions*, USSR Comput. Math. and Math. Phys. **9** (1969), 25–36.

[98] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials*, J. Comput. Appl. Math. **85** (1997), 67–86.