WORKING PAPERS

N° TSE –1013

May 2019

"Estimates for the SVD of the Truncated Fourier Transform on $L^2(\cosh(b.))$ and Stable Analytic Continuation"

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ESTIMATES FOR THE SVD OF THE TRUNCATED FOURIER TRANSFORM ON $L^2(\cosh(b\cdot))$ AND STABLE ANALYTIC CONTINUATION

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Abstract. The Fourier transform truncated on $[-c,c]$ is usually analyzed when acting on $L^2(-1/b,1/b)$ and its right-singular vectors are the prolate spheroidal wave functions. This paper considers the operator acting on the larger space $L^2(\cosh(b\cdot))$ on which it remains injective. We give nonasymptotic upper and lower bounds on the singular values with similar qualitative behavior in m (the index), b, and c. The lower bounds are used to obtain rates of convergence for stable analytic continuation of possibly nonbandlimited functions which Fourier transform belongs to $L^2(\cosh(b\cdot))$. We also derive bounds on the sup-norm of the singular functions. Finally, we provide a numerical method to compute the SVD and apply it to stable analytic continuation when the function is observed with error on an interval.

1. Introduction

Extrapolating an analytic square integrable function f from its observation with error on $[-c,c]$ to $\mathbb{R}$ and generalisation of this idea has a wide range of applications, for example in imaging and signal processing [25], in geostatistics and with big data [17], and finance [24]. A researcher may want to estimate a density from censored observations outside a rectangle and to recover the density on a larger rectangle (see, *e.g.*, [6] and an application to clinical trials). Estimation of an analytic density under random censorship has been the topic of

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Date: This version: May 16, 2019.

Keywords: Analytic continuation, Nonbandlimited functions, Heavy tails, Uniform estimates, Extrapolation, Singular value decomposition, Truncated Fourier transform, Singular Sturm Liouville Equations, Super-resolution.

AMS 2010 Subject Classification: Primary 42C10, 45Q05; secondary 41A60, 45C05, 65L15.

Eric Gautier acknowledges financial support from the grant ERC POEMH 337665. The authors are grateful to Aline Bonami for useful discussions and sending us the work [11].

an important literature (see, *e.g.*, [5] for optimal rates and references therein). When the function is a Fourier transform, this is a type of super-resolution in image restauration [7, 8, 22] which can be done under auxiliary information such as information on the support of the object. The related problem of out-of-band extrapolation (see, *e.g.*, [2, 7] and the references therein) consists in recovering a function from partial observation of its Fourier transform. It is customary to rely on analytic functions and use Hilbert space techniques which are numerically more appealing than local methods. For the problem of extrapolation, one can restrict attention to bandlimited functions which are square integrable functions which Fourier transforms have support in $[-1/b, 1/b]$. For the second problem, one can work with functions which support is a subset of $[-1/b, 1/b]$ in which case their Fourier transform is analytic by the Paley-Wiener theorem. Prolate spheroidal wave functions (henceforth PSWF, see [36, 40]) are the right-singular functions of the truncated Fourier transform restricted to functions with support in $[-1/b, 1/b]$. They form an orthonormal basis of $L^2(-c, c)$, are restrictions of square integrable orthogonal analytic functions on $\mathbb{R}$, and are a complete system of the bandlimited functions with bandlimits $[-1/b, 1/b]$. Hence, a bandlimited function on the whole line is simply the series expansion on the PSWF basis, sometimes called Slepian series, which coefficients only depend on the function on $(-c, c)$, almost everywhere on $\mathbb{R}$. This makes sense if we understand the PSWF functions as their extension to $\mathbb{R}$. In this framework, analytic continuation is an inverse problem in the sense that the solution does not depend continuously on the data, more specifically severely ill-posed (see, *e.g.*, [20, 39, 47]), and many methods have been proposed (see, *e.g.*, [4, 9, 16, 17, 19, 28, 30, 42]). To obtain precise error bounds, it is useful to obtain nonasymptotic upper and lower bounds on the singular values rather than asymptotic estimates. In several applications, uniform estimates on singular functions are useful as well. This occurs for example to show that certain nonparametric statistical procedures involving series are so-called adaptive (see, *e.g.*, [15]). This means that an estimator with a data-driven smoothing parameter reaches the optimal minimax rate of convergence. Such a program providing nonasymptotic bounds on the SVD has been carried recently in relation to the PSWF functions in [11, 12, 13]. A second important aspect is the access to efficient methods to obtain the SVD. While numerical solutions to the inverse problems have for a long time relied on the Tikhonov or iterative methods such as the Landweber method (Gerchberg method for out-of band extrapolation, see [7]) to avoid using the SVD, recent developments have made it possible to approximate efficiently the PSWF and the SVD.

Assuming that the function observed on an interval is the restriction of a bandlimited function can be questionable. Moreover, one would require an upper bound on $1/b$ which might not be available in practice (see [41]). For this reason, this paper considers the larger class of functions which Fourier transforms belong to $L^2(\cosh(b\cdot))$. It means that we replace the compact support by a weaker integrability property. This is an alternative assumption under which super-resolution is possible. Though with a different scalar product, we have $L^2\left(\cosh(b\cdot)\right) = L^2\left(e^{b|\cdot|}\right)$ and (see Theorem IX.13 in [38]), for $a > 0$, $\left\{f \in L^2(\mathbb{R}) :\ \forall b < a,\ f \in L^2\left(e^{b|\cdot|}\right)\right\}$ is the set of square-integrable functions which Fourier transform have an analytic continuation on $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < a/2\}$. Hence, when the function is the Fourier transform, the space $L^2\left(\cosh(b\cdot)\right)$ is in some sense the largest space that we can consider for the Fourier transform of a square-integrable function that we can extrapolate with Hilbert space techniques. When, we rather observe the Fourier transform on a domain, or an approximation of it, like in the random coefficients linear model (see [21]), the above shows that assuming the probability density of the random coefficients belongs to $L^2(\cosh(b\cdot))$ means that its Laplace transform is finite near 0 or equivalently that it does not have heavy-tails. The broader class $L^2(\cosh(b\cdot))$ has rarely been used and unlike the PSWF much fewer results are available, with the notable exception of [31, 44]. It is considered in [5] in the case of random censoring and in [21] for the problem of estimation the density of random coefficients in a random coefficients model. In both problems, it is sometimes not desirable to assume that the density has a compact support and that the researcher knows a superset which contains the support. Rather it can be more meaningful to assume the probability distribution does not have heavy-tails. [21] makes use of results on the singular value decomposition (henceforth SVD) of the truncated Fourier operator on $L^2(\cosh(b\cdot))$ to obtain optimal rates of convergence and a numerical algorithm to compute the SVD. These are the results of this paper which are much more broadly applicable.

This paper provides nonasymptotic upper and lower bounds on the singular values, with similar qualitative behavior, and applies the lower bounds to error bounds for stable analytic continuation using the spectral cut-off method. We also analyze a differential operator which commutes with $\mathcal{Q}_c$. The corresponding eigenvalue problem involves singular Sturm-Liouville equations. This allows to prove uniform estimates on the right-singular functions. Solving numerically singular differential equations allows to obtain these functions, hence all the SVD. Working with the differential equations is useful because the eigenvalues increase quadratically while those of $\mathcal{Q}_c$ decrease faster than exponentially. Finally, we illustrate numerically the proposed method for stable analytic continuation by an adaptive spectral cut-off method and compare it to one that relies on the PSWF.

2. Preliminaries

We use *a.e.* for almost everywhere and $f(\cdot)$ for a function f of some generic argument, denote by $L^2(-1,1)$ and $L^2(\mathbb{R})$ the usual L^2 spaces of complex-valued functions equipped with the Hermitian product, for example $\langle f,g\rangle_{L^2(-1,1)} = \int_{-1}^1 f(x)\overline{g}(x)dx$, by $L^2(W)$ for a positive function W on $\mathbb{R}$ the weighted L^2 spaces equipped with $\langle f,g\rangle_{L^2(W)} = \int_{\mathbb{R}} f(x)\overline{g}(x)W(x)dx$, and by $S^\perp$ the orthogonal of the set S in a Hilbert space. We denote by $\|f\|_{L^\infty([a,b])}$ the sup-norm of the function f on $[a,b]$. For two differentiable functions f and g, their Wronskian is $W(f,g) = fg' - gf'$. Our analysis uses

$$\begin{array}{rccc} \mathcal{C}_c: & L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ & f & \to & cf(c\cdot) \end{array}, \tag{1}$$

$\mathcal{E}$ the operator which extends a function in $L^2(-1,1)$ to $L^2(\mathbb{R})$ by assigning the value 0 outside $[-1,1]$, and $\Pi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that $\Pi f(\cdot) = f(-\cdot)$. The inverse of a mapping f, when it exists, is denoted by f^I. We denote, for $b,c>0$, by

$$\begin{array}{rccc} \mathcal{F}_{b,c}: & L^2(\cosh(b\cdot)) & \to & L^2(-1,1) \\ & f & \to & \mathcal{F}[f](c\cdot) \end{array}, \tag{2}$$

by $\mathcal{F}[f](\cdot) = \int_{\mathbb{R}} e^{ix\cdot} f(x)dx$ the Fourier transform of f in $L^1(\mathbb{R})$ and also use the notation $\mathcal{F}[f]$ for the Fourier transform in $L^2(\mathbb{R})$. $\mathcal{F}_{b,c}^*$ denotes the Hermitian adjoint of $\mathcal{F}_{b,c}$. Recovering f such that for b small enough $f \in L^2(\cosh(b\cdot))$ based on its Fourier transform on $[-c,c]$ amounts to inverting $\mathcal{F}_{b,c}$. This can be achieved using the SVD. Define the finite convolution operator

$$\begin{array}{rccc} Q_c: & L^2(-1,1) & \to & L^2(-1,1) \\ & h & \mapsto & \int_{-1}^1 \pi c \,\mathrm{sech}\left(\frac{\pi c}{2}(\cdot - y)\right) h(y)dy. \end{array}$$

It is compact, symmetric, and positive on spaces of real and complex valued functions. Denote by $(\rho_m^c)_{m\in\mathbb{N}_0}$ its positive real eigenvalues in decreasing order and repeated according to multiplicity and by $(g_m^c)_{m\in\mathbb{N}_0}$ its eigenfunctions which can be taken to be real valued.

Proposition 1. For $b>0$, $c\in\mathbb{R}$, we have $c\mathcal{F}_{b,c}\mathcal{F}_{b,c}^* = Q_{c/b}$.

Proof. Because $\mathcal{F}_{b,c} = \mathcal{F}\mathcal{C}_{c^{-1}} = c^{-1}\mathcal{C}_c\mathcal{F}$, $\Pi\mathcal{F}_{b,c} = \mathcal{F}_{b,c}\Pi$,

$$\mathcal{F}_{b,c}^* = \mathrm{sech}(b\cdot)\Pi\mathcal{F}_{b,c}\mathcal{E}, \tag{3}$$

and $\operatorname{sech}(b\cdot)$ is even, we obtain $\mathcal{F}_{b,c}^* = \Pi\left(\operatorname{sech}(b\cdot)\mathcal{F}_{b,c}\mathcal{E}\right)$ and

$$\begin{aligned} c\mathcal{F}_{b,c}\mathcal{F}_{b,c}^* &= c\Pi\mathcal{F}_{b,c}\left(\operatorname{sech}(b\cdot)\mathcal{F}_{b,c}\mathcal{E}\right) \\ &= 2\pi\mathcal{F}^I\left(\mathcal{C}_{c^{-1}}\left(\operatorname{sech}(b\cdot)\mathcal{C}_c\mathcal{F}\mathcal{E}\right)\right) \\ &= 2\pi c\mathcal{F}^I\left(\mathcal{C}_{c^{-1}}\left(\operatorname{sech}(b\cdot)\right)\mathcal{F}\mathcal{E}\right), \end{aligned}$$

where, for $a.e.$ $x \in \mathbb{R}$,

$$2\pi c\mathcal{F}^I\left(\mathcal{C}_{c^{-1}}\left(\operatorname{sech}(b\cdot)\right)\right)(x) = \int_{\mathbb{R}} e^{-itx}\operatorname{sech}\left(\frac{bt}{c}\right)dt = \frac{\pi c}{b}\operatorname{sech}\left(\frac{\pi c}{2b}x\right).$$

As a result, we have, for $f \in L^2(-1,1)$,

$$c\mathcal{F}_{b,c}\mathcal{F}_{b,c}^*[f] = \mathcal{C}_{c/b}\left[\pi\operatorname{sech}\left(\frac{\pi}{2}\cdot\right)\right] * \mathcal{E}[f] = Q_{c/b}[f].$$

□

Proposition 2. For all $b, c > 0$, $\mathcal{F}_{b,c}$ is injective and $\left(\varphi_m^{b,c}\right)_{m\in\mathbb{N}_0}$ is a basis of $L^2(\cosh(b\cdot))$.

Proof. We use that, for every $h \in L^2(\cosh(b\cdot))$, if we do not restrict the argument in the definition of $\mathcal{F}_{b,c}[h]$ to $[-1,1]$, $\mathcal{F}_{b,c}[h]$ can be defined as a function in $L^2(\mathbb{R})$. In what follows, for simplicity, we use $\mathcal{F}_{b,c}[h]$ for both the function in $L^2(-1,1)$ and in $L^2(\mathbb{R})$.
Let us now show that $\mathcal{F}_{b,c}$ defined in (2) is injective. Denote by $M_k = \left(\int_{\mathbb{R}} t^{2k}\operatorname{sech}(bt)dt\right)^{1/2}$, we have

$$\begin{aligned} M_k^{1/k} &\le \left(\frac{4\Gamma(2k+1)}{b^{2k+1}}\right)^{1/(2k)} \\ &\le \left(\frac{4(2k+1)^{2k+1/2}}{e^{2k}b^{2k+1}}\right)^{1/(2k)} \qquad \text{(by (1.3) in [32])} \\ &\le \frac{4^{1/(2k)}(2k+1)^{1+1/(4k)}}{eb^{1+1/(2k)}} \end{aligned}$$

hence $\sum_{k\in\mathbb{N}} M_k^{-1/k} = \infty$. $\mathcal{F}_{b,c}[h]$ belongs to $C^\infty(\mathbb{R})$ by the Lebesgue dominated convergence theorem because, for all $(k,u) \in \mathbb{N}_0 \times \mathbb{R}$,

$$\int_{\mathbb{R}} \left|c^k t^k e^{ictu} h(t)\right| dt \le c^k \|h\|_{L^2(\cosh(b\cdot))} M_k.$$

We obtain, for all $(k,u) \in \mathbb{N}_0 \times \mathbb{R}$,

$$\left|\mathcal{F}_{b,c}[h]^{(k)}(u)\right| \le c^k \|h\|_{L^2(\cosh(b\cdot))} M_k.$$

Theorem B.1 in [18] and the fact that, by the Cauchy-Schwarz inequality, for all $k \in \mathbb{N}_0$, $M_k \le M_{k-1}M_{k+1}$ yield that $\mathcal{F}_{b,c}[h]$ is zero on $\mathbb{R}$. Thus, $\mathcal{F}[h]$ and h are zero *a.e.* on $\mathbb{R}$.
The second part of Proposition 2 holds by Theorem 15.16 in [27] and the injectivity of $\mathcal{F}_{b,c}$. $\square$

The SVD of $\mathcal{F}_{b,c}$, denoted by $\left(\sigma_m^{b,c}, \varphi_m^{b,c}, g_m^{c/b}\right)_{m\in\mathbb{N}_0}$, is such that, for $m \in \mathbb{N}_0$,

$$\sigma_m^{b,c} = \sqrt{\rho_m^{c/b}/c} \tag{4}$$

and $\varphi_m^{b,c} = \mathcal{F}_{b,c}^* g_m^{c/b}/\sigma_m^{b,c}$. It yields, for all $f \in L^2(\cosh(b\cdot))$,

$$f = \sum_{m\in\mathbb{N}_0} \frac{1}{\sigma_m^{b,c}} \left\langle \mathcal{F}_{b,c}[f], g_m^{c/b} \right\rangle_{L^2(-1,1)} \varphi_m^{b,c}. \tag{5}$$

(5) is a core element to obtain constructive methods to approximate a function from partial observations of its Fourier transform when the signal f does not have compact support.

Because $\mathcal{F}[\mathrm{sech}(\pi c\cdot/2)](\star) = (2/c)\mathrm{sech}(\star/c)$, Theorem II in [44] provides the equivalent

$$\log(\rho_m^c) \underset{m\to\infty}{\sim} -\pi m \frac{K(\mathrm{sech}(\pi c))}{K(\tanh(\pi c))},$$

where $K(r) = \int_0^{\pi/2}\left(1 - r^2\sin(x)^2\right)^{-1/2}dx$ is the complete elliptic integral of the first kind. This paper provides nonasymptotic upper and lower bounds on the eigenvalues and upper bounds on the sup-norm of the functions $(g_m^c)_{m\in\mathbb{N}_0}$. A similar analysis is carried in [11, 13] for

$$\begin{array}{llll} \mathcal{F}_c^{W_{[-1,1]}}: & L^2\left(W_{[-1,1]}\right) & \to & L^2(-1,1), \\ & f & \to & \mathcal{F}[f](c\cdot) \end{array}$$

where we denote, for a subset $\mathcal{S}$, by $W_{\mathcal{S}} = \mathbb{1}\{\mathcal{S}\} + \infty\,\mathbb{1}\{\mathcal{S}^c\}$.

3. Lower bounds on the eigenvalues of Q_c and application

3.1. Lower bounds on the eigenvalues of Q_c.

Lemma 1. For all $m \in \mathbb{N}_0$, $c \in (0,\infty) \mapsto \rho_m^c$ is nondecreasing.

Proof. Take $m \in \mathbb{N}_0$. Using the maximin principle (see Theorem 5 page 212 in [10]), the $m+1$-th eigenvalue ρ_m^c satisfies

$$\rho_m^c = \max_{V\in S_{m+1}} \min_{f\in V\setminus\{0\}} \frac{\langle Q_c f, f\rangle_{L^2(-1,1)}}{\|f\|^2_{L^2(-1,1)}},$$

where S_{m+1} is the set of $m+1$-dimensional vector subspaces of $L^2(-1,1)$. Using (3) and Proposition 1, we obtain

$$
\begin{aligned}
\langle Q_c f, f\rangle_{L^2(-1,1)} &= c\left\langle \mathcal{F}_{1,c}\mathcal{F}_{1,c}^*[f], f\right\rangle_{L^2(-1,1)} \\
&= c\left\langle \mathcal{F}_{1,c}^*[f], \mathcal{F}_{1,c}^*[f]\right\rangle_{L^2(\cosh)} \\
&= c\left\|\mathrm{sech}\times\mathcal{F}_{1,c}\left[\mathcal{E}\left[f\right]\right]\right\|_{L^2(\cosh)}^2 \\
&= c\int_{\mathbb{R}} \mathrm{sech}\left(x\right)\left|\int_{\mathbb{R}} e^{ictx}\mathcal{E}\left[f\right](t)dt\right|^2 dx \\
&= \int_{\mathbb{R}} \mathrm{sech}\left(\frac{x}{c}\right)\left|\mathcal{F}\left[\mathcal{E}\left[f\right]\right](x)\right|^2 dx
\end{aligned}
\tag{6}
$$

hence

$$
\rho_m^c = \max_{V\in S_{m+1}} \min_{f\in V\setminus\{0\}} \frac{2\pi\int_{\mathbb{R}} \mathrm{sech}\left(x/c\right)\left|\mathcal{F}\left[\mathcal{E}\left[f\right]\right](x)\right|^2 dx}{\left\|\mathcal{F}\left[\mathcal{E}\left[f\right]\right]\right\|_{L^2(\mathbb{R})}^2}. \tag{7}
$$

Then, using that $t\mapsto\cosh(t)$ is even, nondecreasing, and positive, we obtain that, for all $0<c_1\le c_2$ and $x\in\mathbb{R}$, $\mathrm{sech}\left(x/c_2\right)\ge\mathrm{sech}\left(x/c_1\right)$ hence that $\rho_m^{c_1}\le\rho_m^{c_2}$. □

Theorem 1. For all $c>0$, we have $\rho_0^c\ge 2\pi$. For all $m\in\mathbb{N}$, we have

$$
\forall\ 0<c<\pi/2,\ \rho_m^c \ge \frac{2\pi\sin(2c)^2}{(2ec)^2}\exp\left(-2\log\left(\frac{e^2}{2c}\right)m\right) \tag{8}
$$

$$
\forall c>0,\ \rho_m^c\ \ge \pi\exp\left(-\frac{\pi(m+1)}{2c}\right). \tag{9}
$$

Proof. Let $m\in\mathbb{N}_0$, $c>0$, and $M=(m+1)/(2c)$. For $R>0$, we denote by $PW(R)$ the Paley-Wiener space of functions which Fourier transform has compact support in $[-R,R]$ and by $S_{m+1}(R)$ the set of $m+1$-dimensional subspaces of $PW(R)$. Using (7), we have

$$
\rho_m^c = \max_{V\in S_{m+1}(1)} \min_{g\in V\setminus\{0\}} \frac{2\pi\int_{\mathbb{R}} \mathrm{sech}\left(x/c\right)\left|g(x)\right|^2 dx}{\|g\|_{L^2(\mathbb{R})}^2}.
$$

Then, for $g\in PW(1)$, the function $g_{Mc}:\ x\in\mathbb{R}\mapsto(Mc)^{1/2}g(Mcx)$ satisfies $\|g\|_{L^2(\mathbb{R})}^2 = \|g_{Mc}\|_{L^2(\mathbb{R})}^2$ and belongs to $PW(Mc)$ because, for *a.e.* $t\in\mathbb{R}$,

$$
\begin{aligned}
\mathcal{F}\left[g_{Mc}\right](t) &= \int_{\mathbb{R}}(Mc)^{1/2}e^{itx}g(Mcx)dx \\
&=(Mc)^{-1/2}\mathcal{F}\left[g\right]\left(\frac{t}{Mc}\right),
\end{aligned}
$$

and the support of $t \mapsto \mathcal{F}[g](t/(Mc))$ is in $[-1,1]$. Using

$$\int_{\mathbb{R}} \operatorname{sech}\left(\frac{x}{c}\right)|g(x)|^2 dx = \int_{\mathbb{R}} \operatorname{sech}(Mx)|g_{Mc}(x)|^2 dx,$$

we have, for $V \in S_{m+1}(Mc)$,

$$\rho_m^c \geq \min_{g \in V \setminus \{0\}} \frac{2\pi \int_{\mathbb{R}} \operatorname{sech}(Mx)|g(x)|^2 dx}{\|g\|^2_{L^2(\mathbb{R})}}. \tag{10}$$

Let us now choose a convenient such space V defined, for $\varphi : t \in \mathbb{R} \mapsto \sin(t/2)/(\pi t)$, as

$$V = \left\{ \sum_{k=0}^{m} P_k e^{i(k-m/2)\cdot} \varphi(\cdot), \quad (P_k)_{k=0}^m \in \mathbb{C}^{m+1} \right\}.$$

The Fourier transform of an element of V is of the form $\sum_{k=0}^m P_k \mathcal{F}[\varphi](\cdot - k + m/2)$ and, because $\mathcal{F}[\varphi](\cdot) = \mathbb{1}\{|\cdot| \leq 1/2\}$, it has support in $[-1/2 - m/2, 1/2 + m/2] = [-Mc, Mc]$. This guarantees that $V \in S_{m+1}(Mc)$.

We now obtain a lower bound on the right-hand side of (10). Let $g \in V$, defined via the coefficients $(P_k)_{k=0}^m$, and P the polynomial function with these coefficients. Let $0 < x_0 < \pi$. We have, using $\forall x \in [0, 2x_0)$, $\sin(x/2)/x \geq \sin(x_0)/(2x_0)$ for the last display,

$$\begin{aligned}
\int_{\mathbb{R}} \operatorname{sech}(Mx)|g(x)|^2 dx &\geq \int_{-2x_0}^{2x_0} \operatorname{sech}(Mx) \left| \sum_{k=0}^m P_k e^{ikx} \right|^2 |\varphi(x)|^2 dx \\
&\geq \frac{1}{\cosh(2Mx_0)} \min_{x \in [-2x_0, 2x_0]} |\varphi(x)|^2 \int_{-2x_0}^{2x_0} \left| \sum_{k=0}^m P_k e^{ikx} \right|^2 dx \\
&\geq \frac{\sin(x_0)^2}{(2\pi x_0)^2} e^{-2Mx_0} \left\| P\left(e^{i\cdot}\right) \right\|^2_{L^2(-2x_0, 2x_0)}.
\end{aligned}$$

Now, using that, for $k \in \mathbb{N}_0$, $t \mapsto \mathcal{F}[\varphi](t - k + m/2)$ have disjoint supports, we obtain

$$\begin{aligned}
\|g\|^2_{L^2(\mathbb{R})} &= \frac{1}{2\pi} \|\mathcal{F}[g]\|^2_{L^2(\mathbb{R})} \\
&= \frac{1}{2\pi} \sum_{k=0}^m |P_k|^2 \|\mathcal{F}[\varphi]\|^2_{L^2(\mathbb{R})} \\
&= \frac{1}{(2\pi)^2} \left\| P\left(e^{i\cdot}\right) \right\|^2_{L^2(-\pi,\pi)} \|\mathcal{F}[\varphi]\|^2_{L^2(\mathbb{R})} \\
&= \frac{1}{(2\pi)^2} \left\| P\left(e^{i\cdot}\right) \right\|^2_{L^2(-\pi,\pi)},
\end{aligned}$$

hence, by the Turan-Nazarov inequality (see [33] page 240),

$$\begin{aligned}\rho_m^c &\geq \frac{2\pi \sin(x_0)^2}{x_0^2} e^{-2Mx_0} \frac{\left\|P\left(e^{i\cdot}\right)\right\|^2_{L^2(-2x_0,2x_0)}}{\left\|P\left(e^{i\cdot}\right)\right\|^2_{L^2(-\pi,\pi)}} \\ &\geq \frac{2\pi \sin(x_0)^2}{x_0^2} e^{-2Mx_0} \left(\frac{e}{x_0}\right)^{-2m}.\end{aligned}$$

We obtain, for $0 < x_0 < \pi$ and $m \in \mathbb{N}_0$,

$$\rho_m^c \geq \frac{2\pi \sin(x_0)^2}{x_0^2} \exp\left(-\frac{x_0}{c}(m+1) - 2\log\left(\frac{e}{x_0}\right) m\right).$$

Thus, for all $c > 0$, we have $\rho_0^c \geq 2\pi$ (by letting x_0 tend to 0) and, for all $m \in \mathbb{N}$,

$$\rho_m^c \geq 2\pi e^{-2m} \sup_{x_0 \in (0,\pi)} \left(\frac{\sin(x_0)}{x_0}\right)^2 e^{-x_0/c} \exp\left(-\left(\frac{x_0}{c} - 2\log\left(x_0\right)\right) m\right). \tag{11}$$

Using that if $2c < \pi$, $x_0 \mapsto x_0/c - 2\log(x_0)$ admits a minimum at $x_0 = 2c$, we obtain, for all $0 < c < \pi/2$,

$$\rho_m^c \geq \frac{2\pi \sin(2c)^2}{(2ec)^2} \exp\left(-2\log\left(\frac{e^2}{2c}\right) m\right).$$

We now prove the second bound on ρ_m^c. Let $m \in \mathbb{N}_0$ and $c_m = \pi(m+1)/2$. For all $x \in \mathbb{R}$, we have $\operatorname{sech}(x/c) \geq \operatorname{sech}(c_m/c)\, \mathbb{1}\{|x| \leq c_m\}$, hence, by (7), we have

$$\begin{aligned}\rho_m^c &= \max_{V \in S_{m+1}} \min_{f \in V\setminus\{0\}} \int_{\mathbb{R}} \operatorname{sech}\left(\frac{x}{c}\right) \left|\mathcal{F}\left[\mathcal{E}\left[f\right]\right](x)\right|^2 dx \frac{1}{\|f\|^2_{L^2(-1,1)}} \\ &\geq \operatorname{sech}\left(\frac{c_m}{c}\right) \max_{V \in S_{m+1}} \min_{f \in V\setminus\{0\}} \int_{\mathbb{R}} \mathbb{1}\{|x| \leq c_m\} \left|\mathcal{F}\left[\mathcal{E}\left[f\right]\right](x)\right|^2 dx \frac{1}{\|f\|^2_{L^2(-1,1)}} \\ &\geq \operatorname{sech}\left(\frac{c_m}{c}\right) \rho_m^{W_{[-1,1]},c_m},\end{aligned} \tag{12}$$

where $\rho_m^{W_{[-1,1]},c_m}$ is the m^{th} eigenvalue of $\mathcal{Q}_c^{W_{[-1,1]}} = c\mathcal{F}_c^{W_{[-1,1]}}\left(\mathcal{F}_c^{W_{[-1,1]}}\right)^*$. Using that $0 \leq m \leq \lfloor 2c_m/\pi \rfloor$ and the first inequality in Lemma A.2 in [21] (with a difference by a factor $1/(2\pi)$ in the normalisation of $\mathcal{Q}_c^{W_{[-1,1]}}$), we have $\rho_m^{W_{[-1,1]},c_m} \geq \pi$ hence, for all $m \in \mathbb{N}_0$,

$$\begin{aligned}\rho_m^c &\geq \exp\left(-\frac{c_m}{c}\right) \rho_m^{W_{[-1,1]},c_m} \quad \text{(by (12))} \\ &\geq \pi \exp\left(-\frac{\pi(m+1)}{2c}\right).\end{aligned}$$

□

Some proof techniques are from the proof of Proposition 2.1 in [11] for the PSWF. The best of the two lower bounds in terms of the factor in the exponential is (8) for $c \le c_0$, where $c_0 = 0.32179$ (a numerical approximation), and (9) for larger c. This yields

$$\forall m \in \mathbb{N}_0,\ \forall c > 0,\ \rho_m^c \ge \theta(c,m) e^{-2\beta(c)m}, \tag{13}$$

where

$$\beta : c \mapsto \log\left(\frac{e^2}{2c}\right) \mathbb{1}\{c \le c_0\} + \frac{\pi}{4c}\mathbb{1}\{c > c_0\},$$

$$\theta : (c,m) \mapsto 2\pi \mathbb{1}\{m = 0\} + \frac{2\pi \sin(2c)^2}{(2ec)^2}\mathbb{1}\{c \le c_0, m \ne 0\} + \frac{\pi}{e^{\pi/(2c)}}\mathbb{1}\{c > c_0, m \ne 0\}.$$

Clearly, because $c_0 \le \pi/4$ and $x \mapsto \sin(x)/x$ is decreasing on $(0, \pi/2)$, the lower bound holds when we replace θ by

$$\widetilde{\theta} :\ (c,m) \mapsto 2\pi \mathbb{1}\{m = 0\} + \frac{2\pi \sin(2c_0)^2}{(2ec_0)^2}\mathbb{1}\{c \le c_0, m \ne 0\} + \frac{\pi}{e^{\pi/(2c)}}\mathbb{1}\{c > c_0, m \ne 0\}.$$

3.2. **Application: Error bounds for stable analytic continuation of functions which Fourier transform belongs to $L^2(\cosh(b\cdot))$.** In this section, we consider the problem where we observe the function f with error on $(x_0 - c, x_0 + c)$, for $c > 0$ and $x_0 \in \mathbb{R}$,

$$f_\delta(cx + x_0) = f(cx + x_0) + \delta\xi(x), \quad \text{for } a.e.\ x \in (-1,1), \quad \mathcal{F}[f] \in L^2(\cosh(b\cdot)), \tag{14}$$

where $\xi \in L^2(x_0 - c, x_0 + c)$, $\|\xi\|_{L^2(x_0-c,x_0+c)} \le 1$, and $\delta > 0$. We consider the problem of approximating $f^0 = f$ on $L^2(\mathbb{R})$ from f^δ on $(x_0 - c, x_0 + c)$. Noting that, for a.e. $x \in (-1, 1)$,

$$\frac{1}{2\pi}\mathcal{F}_{b,c}\left[\mathcal{F}\left[f(x_0 - \cdot)\right]\right](x) = f(cx + x_0) \tag{15}$$

suggests the two steps regularising procedure:

(1) approximate $\mathcal{F}\left[f(x_0 - \cdot)\right]/(2\pi) \in L^2(\cosh(b\cdot))$ by the spectral cut-off regularization,

$$F_\delta^N = \sum_{m \le N} \frac{1}{\sigma_m^{b,c}} \left\langle f_\delta(c \cdot + x_0), g_m^{c/b}(\cdot)\right\rangle_{L^2(-1,1)} \varphi_m^{b,c}, \tag{16}$$

(2) take the inverse Fourier transform and define

$$f_\delta^N(\cdot) = 2\pi \mathcal{F}^I\left[F_\delta^N\right](x_0 - \cdot). \tag{17}$$

The lower bounds on the eigenvalues of $\mathcal{Q}_{c/b}$ of Theorem 1 are useful to obtain rates of convergence when $\mathcal{F}[f]$ satisfies a source condition. This means that, for a given sequence $(\omega_m)_{m\in\mathbb{N}_0}$, f belongs to

$$\mathcal{H}_{\omega,x_0}(M) = \left\{ f: \quad \sum_{m\in\mathbb{N}_0} \omega_m^2 \left| \left\langle \mathcal{F}[f(x_0-\cdot)], \varphi_m^{b,c}(\cdot) \right\rangle_{L^2(\cosh(b\cdot))} \right|^2 \leq M^2 \right\}. \tag{18}$$

Theorem 2. Take $M>0$ and define β as in (13), then we have

(1) for $(\omega_m)_{m\in\mathbb{N}_0} = (m^\sigma)_{m\in\mathbb{N}_0}$, $\sigma > 1/2$, $N = \lfloor \overline{N} \rfloor$, and $\overline{N} = \ln(1/\delta)/(2\beta(c/b))$,

$$\sup_{f\in\mathcal{H}_{\omega,x_0}(M), \|\xi\|_{L^2(-1,1)}\leq 1} \left\| f_\delta^N - f \right\|_{L^2(\mathbb{R})} \underset{\delta\to 0}{=} O((-\log(\delta))^{-\sigma}), \tag{19}$$

(2) for $(\omega_m)_{m\in\mathbb{N}_0} = (e^{\kappa m})_{m\in\mathbb{N}_0}$, $\kappa > 0$, $N = \lfloor \overline{N} \rfloor$, and $\overline{N} = \ln(1/\delta)/(\kappa + \beta(c/b))$,

$$\sup_{f\in\mathcal{H}_{\omega,x_0}(M), \|\xi\|_{L^2(-1,1)}\leq 1} \left\| f_\delta^N - f \right\|_{L^2(\mathbb{R})} \underset{\delta\to 0}{=} O\left(\delta^{\kappa/(\kappa+\beta(c/b))} \right). \tag{20}$$

Proof. We have, using the Plancherel equality for the first equality,

$$\begin{aligned}
\left\| f_\delta^N - f \right\|_{L^2(\mathbb{R})}^2 =& \frac{1}{2\pi} \left\| \mathcal{F}\left[f_\delta^N\right] - \mathcal{F}\left[f\right] \right\|_{L^2(\mathbb{R})}^2 \\
=& \frac{1}{2\pi} \left\| \mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right] \right\|_{L^2(\mathbb{R})}^2 \\
\leq& \frac{1}{2\pi} \left\| \mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right] \right\|_{L^2(\cosh(b\cdot))}^2 \\
\leq& \frac{1}{\pi} \left\| \mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f_0^N(x_0-\cdot)\right] \right\|_{L^2(\cosh(b\cdot))}^2 \\
&+ \frac{1}{\pi} \left\| \mathcal{F}\left[f_0^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right] \right\|_{L^2(\cosh(b\cdot))}^2 .
\end{aligned} \tag{21}$$

Using (16) for the first equality, the Cauchy-Schwarz inequality and (4) for the first inequality, and (13) for the second inequality, we obtain

$$\begin{aligned}
&\left\| \mathcal{F}\left[f_\delta^N(x_0-\cdot)\right] - \mathcal{F}\left[f_0^N(x_0-\cdot)\right] \right\|_{L^2(\cosh(b\cdot))}^2 \\
&= \left\| \sum_{m\leq N} \frac{2\pi}{\sigma_m^{b,c}} \left\langle (f_\delta - f)(c\cdot + x_0), g_m^{c/b}(\cdot) \right\rangle_{L^2(-1,1)} \varphi_m^{b,c}(\cdot) \right\|_{L^2(\cosh(b\cdot))}^2 \\
&= \sum_{m\leq N} \left(\frac{2\pi}{\sigma_m^{b,c}} \right)^2 \left| \left\langle (f_\delta - f)(c\cdot + x_0), g_m^{c/b}(\cdot) \right\rangle_{L^2(-1,1)} \right|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq (2\pi)^2 \left\|(f_\delta - f)(c\cdot + x_0)\right\|^2_{L^2(-1,1)} \sum_{m\leq N} \frac{c}{\rho_m^{c/b}} \\
&\leq \frac{(2\pi)^2 c\delta^2}{\theta(c,1)} \|\xi\|^2_{L^2(x_0-c,x_0+c)} \sum_{m\leq N} e^{2\beta(c/b)m} \\
&\leq \frac{(2\pi)^2 c\delta^2}{\theta(c,1)(1-e^{-2\beta(c/b)})} e^{2\beta(c/b)N}.
\end{aligned}
$$

Using (17), we have

$$
\begin{aligned}
\mathcal{F}\left[f_0^N(x_0-\cdot)\right](\star) &= \sum_{m\leq N} \frac{2\pi}{\sigma_m^{b,c}} \left\langle f(c\cdot + x_0), g_m^{c/b}(\cdot)\right\rangle_{L^2(-1,1)} \varphi_m^{b,c}(\star) \\
&= \sum_{m\leq N} \frac{2\pi}{\sigma_m^{b,c}} \left\langle \mathcal{F}_{b,c}\left[\frac{1}{2\pi}\mathcal{F}\left[f(x_0-\cdot)\right]\right], g_m^{c/b}\right\rangle_{L^2(-1,1)} \varphi_m^{b,c}(\star) \\
&= \sum_{m\leq N} \frac{1}{\sigma_m^{b,c}} \left\langle \mathcal{F}\left[f(x_0-\cdot)\right], \mathcal{F}_{b,c}^*\left[g_m^{c/b}\right]\right\rangle_{L^2(\cosh(b\cdot))} \varphi_m^{b,c}(\star) \\
&= \sum_{m\leq N} \left\langle \mathcal{F}\left[f(x_0-\cdot)\right], \varphi_m^{c/b}\right\rangle_{L^2(\cosh(b\cdot))} \varphi_m^{b,c}(\star).
\end{aligned}
$$

Thus, using Proposition 2 and Pythagoras' theorem, we obtain

$$
\begin{aligned}
\left\|\mathcal{F}\left[f_0^N(x_0-\cdot)\right] - \mathcal{F}\left[f(x_0-\cdot)\right]\right\|^2_{L^2(\cosh(b\cdot))} &= \sum_{m>N} \left|\left\langle \mathcal{F}[f(x_0-\cdot)], \varphi_m^{b,c}(\cdot)\right\rangle_{L^2(\cosh(b\cdot))}\right|^2 \\
&\leq \sum_{m\in\mathbb{N}_0} \left(\frac{\omega_m}{\omega_N}\right)^2 \left|\left\langle \mathcal{F}[f(x_0-\cdot)], \varphi_m^{b,c}(\cdot)\right\rangle_{L^2(\cosh(b\cdot))}\right|^2 \\
&\leq \frac{M^2}{\omega_N^2} \quad (\text{using } f\in\mathcal{H}_{\omega,x_0}(M)).
\end{aligned}
\tag{22}
$$

Finally, using (21)-(22) yields

$$
\left\|f_\delta^N - f\right\|^2_{L^2(\mathbb{R})} \leq \frac{1}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c,1)\left(1-e^{-2\beta(c/b)}\right)}\delta^2 e^{2\beta(c/b)N} + \frac{M^2}{\omega_N^2}\right). \tag{23}
$$

Consider case (1). Take δ small enough so that $\overline{N}\geq 2$ and $\log\left(\delta\log(1/\delta)^{2\sigma}\right)\leq 0$. By (23) and the definition of $(\omega_N)_{n\in\mathbb{N}_0}$ in the first display below, $\overline{N}-1\leq N\leq \overline{N}$ in the second display, and $\overline{N}\geq 2$ in the third display, we obtain

$$
\left\|f_\delta^N - f\right\|^2_{L^2(\mathbb{R})} \leq \frac{N^{-2\sigma}}{\pi}\left(\frac{(2\pi)^2 c}{\theta(c,1)(1-e^{-2\beta(c/b)})}\delta^2 e^{2\beta(c/b)N} N^{2\sigma} + M^2\right)
$$

$$\leq \frac{\overline{N}^{-2\sigma}\left(1-1/\overline{N}\right)^{-2\sigma}}{\pi}\left(\frac{(2\pi)^2c}{\theta(c,1)(1-e^{-2\beta(c/b)})}\delta^2 e^{2\beta(c/b)\overline{N}}\overline{N}^{2\sigma}+M^2\right)$$
$$\leq \frac{\overline{N}^{-2\sigma}2^{2\sigma}}{\pi}\left(\frac{(2\pi)^2c}{\theta(c,1)(1-e^{-2\beta(c/b)})}\delta^2 e^{2\beta(c/b)\overline{N}}\overline{N}^{2\sigma}+M^2\right).$$

Using that

$$\delta^2\exp\left(2\beta\left(\frac{c}{b}\right)\overline{N}\right)\overline{N}^{2\sigma}=\exp\left(2\sigma\log\left(\frac{1}{2\beta(c/b)}\right)+\log\left(\delta\log\left(\frac{1}{\delta}\right)^{2\sigma}\right)\right)\leq\left(\frac{1}{\beta(c/b)}\right)^{2\sigma},$$

yields

(24)
$$\left\|f_\delta^N-f\right\|^2_{L^2(\mathbb{R})}\leq\frac{1}{\pi}\left(4\beta\left(\frac{c}{b}\right)\right)^{2\sigma}\left(\frac{(2\pi)^2c}{\theta(c,1)(1-e^{-2\beta(c/b)})}\left(\frac{1}{\beta(c/b)}\frac{\sigma}{e}\right)^{2\sigma}+M^2\right)(-\log(\delta))^{-2\sigma},$$

hence the result.

Consider now case (2). Using $\overline{N}-1\leq N\leq\overline{N}$ in the first display and $\delta^2\exp\left(2\left(\beta\left(\frac{c}{b}\right)+\kappa\right)\overline{N}\right)=1$ and the definition of $\overline{N}$ in the second display, yields

$$\left\|f_\delta^N-f\right\|^2_{L^2(\mathbb{R})}\leq\frac{e^{-2\kappa(\overline{N}-1)}}{\pi}\left(\frac{(2\pi)^2c}{\theta(c,1)(1-e^{-2\beta(c/b)})}\delta^2 e^{2(\beta(c/b)+\kappa)\overline{N}}+M^2\right)$$
$$\leq\frac{e^{2\kappa}}{\pi}\left(\frac{(2\pi)^2c}{\theta(c,1)(1-e^{-2\beta(c/b)})}+M^2\right)\delta^{2\kappa/(\kappa+\beta(c/b))},$$

hence the result. □

Unlike (20), the rate in (19) depends on c only through the constant. The rate in (20) deteriorates for small values of c. The result (20) is related to those obtained for the so-called "2exp-severely ill-posed problems" (see [14] for a survey and [43] which obtains similar polynomial rates). We give a numerical illustration in Section 8.

4. Upper bounds on the eigenvalues of Q_c

Theorem 3. For $m\in\mathbb{N}_0$ and $0<c<2$, we have

$$\rho_m^c\leq\frac{4ec(c/2)^{2m}}{\sqrt{2m+1}(1-(c/2)^2)}.$$

Some of the proof techniques below are from the proof of Theorem 3.1 in [12] for the PSWF.

Proof. Using the minimax principle (see Theorem 4 page 212 in [10]), the $m+1$-th eigenvalue ρ_m^c satisfies

$$\rho_m^c = \min_{S_m} \max_{f \in S_m^\perp} \frac{\langle Q_c f, f\rangle_{L^2(-1,1)}}{\|f\|^2_{L^2(-1,1)}},$$

where S_m is the set of m-dimensional vector subspaces of $L^2(-1,1)$. We use (6), which yields

$$\rho_m^c = \min_{S_m} \max_{f \in S_m^\perp} \frac{c\left\langle \mathcal{F}^*_{1,c}[f], \mathcal{F}^*_{1,c}[f]\right\rangle_{L^2(\cosh)}}{\|f\|^2_{L^2(-1,1)}}.$$

We denote by $(P_m)_{m\in\mathbb{N}_0}$ the Legendre polynomials with the normalization $P_m(1)=1$. They are such that $\left(\sqrt{m+1/2}P_m\right)_{m\in\mathbb{N}_0}$ is an orthonormal basis of $L^2(-1,1)$. Let S_m be the vector space spanned by $P_0,\dots,P_{m-1}$. Take $f\in S_m^\perp$ of norm 1. It is of the form $f=\sum_{k=m}^\infty a_k\sqrt{k+1/2}P_k$, where $\sum_{k=m}^\infty |a_k|^2=1$. The Cauchy-Schwarz inequality yields, for *a.e.* $x\in\mathbb{R}$,

$$\left|\mathcal{F}^*_{1,c}f(x)\right|^2 \le \left(\sum_{k=m}^\infty |a_k|^2\right)\left(\sum_{k=m}^\infty \left(k+\frac{1}{2}\right)\left|\mathcal{F}^*_{1,c}P_k(x)\right|^2\right)$$

and after integration

$$\left\|\mathcal{F}^*_{1,c}f\right\|^2_{L^2(\cosh)} \le \sum_{k=m}^\infty \left(k+\frac{1}{2}\right)\left\|\mathcal{F}^*_{1,c}P_k\right\|^2_{L^2(\cosh)}.$$

Thus, we have

$$\rho_m^c \le c\sum_{k=m}^\infty \left(k+\frac{1}{2}\right)\left\|\mathcal{F}^*_{1,c}P_k\right\|^2_{L^2(\cosh)}. \tag{25}$$

Then, using (18.17.19) in [35], we obtain, for *a.e.* x and $c>0$,

$$\begin{aligned}\mathcal{F}^*_{1,c}\left[P_k\right](x) &= \operatorname{sech}(x)\,\mathcal{F}_{1,c}\left[\mathcal{E}\left[P_k\right]\right](-x)\\ &= \operatorname{sech}(x)\, i^{-k}\sqrt{\frac{2\pi}{c\,|x|}}J_{k+1/2}(c\,|x|),\end{aligned}$$

where $J_{k+1/2}$ is the Bessel function of order $k+1/2$. Using (9.1.62) in [1] in the first display, $\Gamma(k+3/2)=(k+1/2)\Gamma(k+1/2)$ and $\Gamma(k+1/2)\ge\sqrt{2\pi}e^{-k-1/2}(k+1/2)^k$ (see (1.4) in [32]) in the second display, we obtain

$$\begin{aligned}\left|\mathcal{F}^*_{1,c}\left[P_k\right](x)\right| &\le \operatorname{sech}(x)\sqrt{\pi}\frac{|cx/2|^k}{\Gamma(k+3/2)}.\\ &\le \operatorname{sech}(x)\sqrt{\frac{e}{2}}\frac{1}{k+1/2}\left(\frac{ec}{2(k+1/2)}\right)^k|x|^k.\end{aligned}$$

Thus, we have

$$\begin{aligned}\left\|\mathcal{F}^*_{1,c}\left[P_k\right]\right\|^2_{L^2(\cosh)} &\le \frac{e}{2(k+1/2)^2}\left(\frac{ec}{2(k+1/2)}\right)^{2k}\int_{\mathbb{R}} x^{2k}\mathrm{sech}^2\left(x\right)dx\\ &\le \frac{2e}{(k+1/2)^2}\left(\frac{ec}{2(k+1/2)}\right)^{2k}\int_0^\infty x^{2k}e^{-2x}dx\\ &\le \frac{2e}{(k+1/2)^2}\left(\frac{ec}{4(k+1/2)}\right)^{2k}\Gamma(2k+1).\end{aligned}$$

Then, by (25), we have

$$\begin{aligned}\rho^c_m &\le 2ec\sum_{k=m}^\infty \frac{1}{k+1/2}\left(\frac{ec}{4(k+1/2)}\right)^{2k}\Gamma(2k+1)\\ &\le 2ec\sum_{k=m}^\infty \frac{1}{k+1/2}\left(\frac{ec}{4(k+1/2)}\right)^{2k}(2k+1)^{2k+1/2}e^{-2k}\quad \text{(using (1.3) in [32])}\\ &= 4ec\sum_{k=m}^\infty \frac{1}{\sqrt{2k+1}}\left(\frac{c}{2}\right)^{2k}\\ &\le \frac{4ec}{\sqrt{2m+1}}\sum_{k=m}^\infty \left(\frac{c}{2}\right)^{2k}\end{aligned}$$

hence, for $0 < c < 2$, this yields the result. □

Theorem 3 holds for a limited range of values of c but this range is enough to construct the so-called test functions to prove the minimax lower bounds in [21]. Note also that, by (8) and Theorem 3, we have, for all $0 < c \le c_0 < \pi/2$ and $m \in \mathbb{N}$,

$$\frac{2\pi\sin(2c_0)^2}{(2ec_0)^2}\exp\left(-2\left(\log\left(\frac{2}{c}\right)+2-\log(4)\right)m\right) \le \rho^c_m \le \frac{4ec_0}{\sqrt{2m+1}(1-c_0^2)}\exp\left(-2\log\left(\frac{2}{c}\right)m\right).$$

These upper and lower bounds do not exactly match but have a similar behavior as c approaches 0 and else differ by a small exponential factor (indeed $2(2-\log(4)) = 1.2274$).

5. Properties of a differential operator which commutes with $\mathcal{Q}_c$

In this section, we consider differential operators $\mathcal{L}[\psi] = -\left(p\psi'\right)' + q\psi$ on $L^2(-1,1)$, with (1) $p(x) = \cosh(4c) - \cosh(4cx)$ and $q(x) = 3c^2\cosh(4cx)$, (2) $p(x) = 1 - x^2$ and $q(x) = q^c(x)$ where, for $Y(x) = \sin\left(X(x)\right)$, $X(x) = (\pi/U(c))\int_0^x p(\xi)^{-1/2}d\xi$, and $U(c) = \int_{-1}^1 p(\xi)^{-1/2}d\xi$,

$$q_c(Y(x)) = \frac{1}{2} + \frac{1}{4}\tan\left(X(x)\right)^2 - \left(\frac{U(c)c}{\pi}\right)^2\left(\cosh(4cx) + \frac{\sinh^2(4cx)}{p(x)}\right), \tag{26}$$

and (3) $p(x) = 1 - x^2$ and $q(x) = 0$. It is stated in [44] that in the unpublished work [31] it is proved that the eigenfunctions of $\mathcal{Q}_c$ are those of the differential operator in case (1) with domain $\mathcal{D} \subset \mathcal{D}_{\max} = \left\{\psi \in L^2(-1,1) : \mathcal{L}[\psi] \in L^2(-1,1)\right\}$ with boundary conditions of continuity at ± 1. This is an important property for the asymptotic analysis in [44] and to obtain bounds on the sup-norm of these functions in Section 6 and numerical approximations of them in Section 7. To study $\mathcal{L}$ in case (1), [44] uses the changes of variable and function, for all $x \in (-1,1)$ and $\psi \in \mathcal{D}_{\max}$,

$$y = Y(x), \tag{27}$$

$$\forall y \in (-1,1),\ \Gamma(y) = F(y)\psi\left(Y^{-1}(y)\right),\ F(y) = \left(\frac{p(Y^{-1}(y))}{1-y^2}\right)^{1/4}, \tag{28}$$

where Y is a $C^\infty-$diffeomorphism on $(-1,1)$. This relates an eigenvalue problem for (1) to an eigenvalue problem for (2) and it is useful to view the operator in case (2) as a perturbation of the operator in case (3). In the three cases, $1/p$ and q are holomorphic on a simply connected open set $(-1,1) \subseteq E \subseteq \mathbb{C}$. The spectral analysis involves the solutions to (H_λ): $-(p\psi')' + (q-\lambda)\psi = 0$ with $\lambda \in \mathbb{C}$ which are holomorphic on E and span a vector space of dimension 2 (see Sections IV 1 and 10 in [26]). So they are infinitely differentiable on $(-1,1)$, have isolated zeros in $(-1,1)$, and the condition of continuity (or boundedness) at ± 1 makes sense.

We now present a few useful estimates.

Lemma 2. We have, for all $c > 0$,

$$\frac{e^{2c}}{\sinh(2c)(1+\cosh(4c))^{1/2}} < U(c) < \frac{\pi e^{2c}}{\sinh(2c)(1+\cosh(4c))^{1/2}}.$$

Proof. By the second equation page 229 of [44]

$$U(c) = \frac{1}{c(1+\cosh(4c))^{1/2}} K\left(\frac{e^{4c}-1}{e^{4c}+1}\right),$$

and the result follows from the fact that, by Corollary 3.3 in [3],

$$\frac{ce^{2c}}{\sinh(2c)} < K\left(\frac{e^{4c}-1}{e^{4c}+1}\right) < \frac{\pi ce^{2c}}{\sinh(2c)}.$$

□

We make use of the identity, for all $x \in [-1,1]$,

$$p(x) = 4c\sinh(4c)(1-x)(1+u(x)), \tag{29}$$

$$u(x) = \int_x^1 \frac{4c\cosh(4ct)}{\sinh(4c)(1-x)}(x-t)dt, \tag{30}$$

which is obtained by Taylor's theorem with remainder in integral form

$$\cosh(4cx) = \cosh(4c) + (x-1)4c\sinh(4c) + \int_1^x 16c^2\cosh(4ct)(x-t)dt.$$

Also, u is increasing on $[-1,1]$ because, for all $x \in [-1,1]$,

$$u'(x) = \frac{4c}{\sinh(4c)(1-x)^2}\int_x^1 \cosh(4ct)(1-t)dt > 0 \tag{31}$$

and, for all $x \in [0,1]$,

$$-1 + \frac{1}{4c\sinh(4c)}(\cosh(4c)-1) \leq u(x) \leq 0. \tag{32}$$

Lemma 3. We have, for all $c > 0$ and $x \in [0,1]$,

$$\frac{c\sinh(4c)}{1-x} - \frac{8c^3\sinh(4c)\cosh(4c)}{3(\cosh(4c)-1)} \leq \left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^{-2} \leq \frac{c\sinh(4c)}{1-x}.$$

Proof. We have

$$\begin{aligned}
\left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^{-2} &= \left(\frac{1}{(4c\sinh(4c))^{1/2}}\left(2(1-x)^{1/2} - \int_x^1\int_1^\xi \frac{u'(t)}{2\sqrt{1-\xi}(1+u(t))^{3/2}}d\xi dt\right)\right)^{-2} \\
&= \frac{c\sinh(4c)}{1-x}\frac{1}{(1+\widetilde{u}(x))^2} \qquad (33) \\
&= \frac{c\sinh(4c)}{1-x} - \frac{c\sinh(4c)(2+\widetilde{u}(x))\widetilde{u}(x)}{(1-x)(1+\widetilde{u}(x))^2}, \qquad (34)
\end{aligned}$$

where

$$\widetilde{u}(x) = \int_x^1 \frac{1}{4\sqrt{(1-\xi)(1-x)}}\left(\int_\xi^1 \frac{u'(t)}{(1+u(t))^{3/2}}dt\right)d\xi. \tag{35}$$

The upper bound in Lemma 3 uses that, for all $x \in [0,1]$, $\widetilde{u}(x) \geq 0$. We now consider the lower bound. By (31), u is a C^1-diffeomorphism and

$$\int_x^1 \frac{u'(t)dt}{(1+u(t))^2}dt = -\frac{u(x)}{1+u(x)}. \tag{36}$$

Now, by (30), we have, for all $x \in [0,1]$,

$$\begin{aligned}
-u(x) &\leq \frac{4c\cosh(4c)}{\sinh(4c)(1-x)}\int_x^1 (t-x)dt \\
&= \frac{2c\cosh(4c)}{\sinh(4c)}(1-x),
\end{aligned}$$

and, by (32),

$$\int_x^1 \frac{u'(t)dt}{(1+u(t))^2}dt \le 8c^2\frac{\cosh(4c)}{\cosh(4c)-1}(1-x). \tag{37}$$

Now, using that $\widetilde{u}(x)\ge 0$ and that $g:\ t\mapsto (2+t)/(1+t)^2$ is decreasing on $[0,\infty)$ hence $g(t)t\le 2t$ for $t\ge 0$, we have

$$\begin{aligned}
\frac{(2+\widetilde{u}(x))\widetilde{u}(x)}{(1+\widetilde{u}(x))^2} &\le \int_x^1 \frac{1}{2\sqrt{(1-\xi)(1-x)}}\left(\int_\xi^1 \frac{u'(t)}{(1+u(t))^2}dt\right)d\xi \quad \text{(by (35))}\\
&\le \frac{4c^2\cosh(4c)}{\cosh(4c)-1}\int_x^1\sqrt{\frac{1-\xi}{1-x}}d\xi \quad \text{(by (37))}\\
&\le \frac{8c^2\cosh(4c)}{3\left(\cosh(4c)-1\right)}(1-x). \qquad (38)
\end{aligned}$$

□

Proposition 3. *F is such that*

$$\|F\|^4_{L^\infty([-1,1])} \le \frac{\pi^2 e^{4c}c^2\sinh(4c)^2}{\sinh(2c)^2(1+\cosh(4c))} \tag{39}$$

$$\|1/F\|^4_{L^\infty([-1,1])} \le \pi^2 e^{-4c}\frac{\sinh(2c)^2(1+\cosh(4c))(1+4c^2/3)^2}{2c\sinh(4c)\left(\cosh(4c)-1\right)}. \tag{40}$$

For all $c>0$ and $\lambda\in\mathbb{C}$, the change of variable and function (27)-(28) maps a solution of $\left(H_{U(c)^2\lambda/\pi^2}\right)$ in case (2) to a solution of (H_λ) in case (1) and reciprocally the inverse transformation maps a solution of (H_λ) in case (1) to a solution of $\left(H_{U(c)^2\lambda/\pi^2}\right)$ in case (2) and is a bijection of $\mathcal{D}$. Also, q^c can be extended by continuity to $[-1,1]$ and, for all $y\in[-1,1]$,

$$\frac{1}{2}-\left(\frac{U(c)c}{\pi}\right)^2 - R(c) \le q^c(y) \le \frac{1}{2}-\left(\frac{U(c)c}{\pi}\right)^2, \tag{41}$$

where

$$R(c)=\frac{2}{\pi^2}+\left(\frac{U(c)c}{\pi}\right)^2\left(\cosh(4c)\left(1+\frac{2c\sinh(4c)}{3\left(\cosh(4c)-1\right)}\right)-1\right).$$

Proof. To prove (39) and (40) it is sufficient, by parity, to consider $x\in[0,1]$.
(39) is a obtained by the following sequence of inequalities

$$\frac{p(x)}{1-Y(x)^2} = \frac{p(x)}{\sin^2\left(\pi\int_x^1 p(\xi)^{-1/2}d\xi/U(c)\right)}$$

$$
\begin{aligned}
&\leq \left(\frac{U(c)}{2}\right)^2 \frac{p(x)}{\left(\int_x^1 p(\xi)^{-1/2} d\xi\right)^2} \quad \text{(because } \sin(x) \geq 2x/\pi) \\
&\leq \left(\frac{U(c)}{2}\right)^2 p(x) \frac{c \sinh(4c)}{1-x} \quad \text{(by Lemma 3)} \\
&\leq \left(\frac{U(c)}{2}\right)^2 4c^2 \sinh(4c)^2 (1+u(x)) \quad \text{(by (29))} \\
&\leq \frac{\pi^2 e^{4c} c^2 \sinh(4c)^2}{\sinh(2c)^2 (1+\cosh(4c))} \quad \text{(by Lemma 2 and (32)).}
\end{aligned}
$$

We obtain (40) by the inequalities below. Using for the first display that, for $x \in [0, \pi/2]$, $\sin(x) \leq x$, (29) and (33) for the second display, (32) and (38) for the third, and Lemma 2 for the fourth, we obtain, for all $x \in [0,1)$,

$$
\begin{aligned}
\frac{1-Y(x)^2}{p(x)} &\leq \left(\frac{\pi}{U(c)}\right)^2 \frac{\left(\int_x^1 p(\xi)^{-1/2} d\xi\right)^2}{p(x)} \\
&\leq \left(\frac{\pi}{U(c)}\right)^2 \frac{2\,(1+\widetilde{u}(x))^2}{(4c \sinh(4c))^2} \frac{1}{1+u(x)} \\
&\leq \left(\frac{\pi}{U(c)}\right)^2 \frac{(1+4c^2/3)^2}{2c \sinh(4c)^2} \frac{\sinh(4c)}{\cosh(4c)-1} \\
&\leq \frac{\pi^2 e^{-4c}}{2c} \frac{\sinh(2c)^2 (1+\cosh(4c))(1+4c^2/3)^2}{\sinh(4c)\,(\cosh(4c)-1)}.
\end{aligned}
$$

Let Γ and ψ related via (28). Buy the above if one function is in $\mathcal{D}$ the other is s well. Moreover, by (28), we have

$$
\begin{aligned}
F'(y) &= \frac{F(y)}{4} \left(\frac{p'}{pY'} \left(Y^{-1}(y)\right) + \frac{2y}{1-y^2} \right) \\
(1-y^2)\Gamma'(y) &= \frac{F(y)}{4} \left((1-y^2) \left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'} \right) \left(Y^{-1}(y)\right) + 2y\psi\left(Y^{-1}(y)\right) \right)
\end{aligned}
$$

so differentiating a second time and injecting the above inequality, yields

$$
\begin{aligned}
\left((1-y^2)\Gamma'\right)'(y) = \frac{F(y)}{4} \Big(&\frac{1}{4} \left(\frac{p'}{pY'} \left(Y^{-1}(y)\right) + \frac{2y}{1-y^2} \right) \left((1-y^2) \left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'} \right) \left(Y^{-1}(y)\right) + 2y\psi\left(Y^{-1}(y)\right) \right) \\
&- 2y \left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'} \right) \left(Y^{-1}(y)\right) + (1-y^2) \left[\frac{1}{Y'} \left(\frac{p'\psi}{pY'} + 4\frac{\psi'}{Y'} \right)' \right] \left(Y^{-1}(y)\right) \\
&+ 2 \left(\psi\left(Y^{-1}(y)\right) + y\frac{\psi'}{Y'}\left(Y^{-1}(y)\right) \right) \Big).
\end{aligned}
$$

Dividing by $F(y)/4$ and using (27), Γ is solution of $\left(H_{U(c)^2\lambda/\pi^2}\right)$ iff ψ is solution on $(-1,1)$ of

$$\begin{aligned}&\frac{1}{4p(x)}\left(p'(x)+\frac{2YY'p}{1-Y^2}(x)\right)\left(\frac{1-Y^2}{(Y')^2p}(x)\left(p'\psi+4p\psi'\right)(x)+2\frac{Y}{Y'}(x)\psi(x)\right)\\&-2\frac{Y}{Y'}(x)\left(\frac{p'\psi}{p}+4\psi'\right)(x)+\frac{1-Y^2}{Y'}(x)\left(\frac{p'\psi}{pY'}+4\frac{\psi'}{Y'}\right)'(x)+2\left(\psi(x)+\frac{Y}{Y'}(x)\psi'(x)\right)\\&=4\left(q_c(Y(x))-\frac{U(c)^2\lambda}{\pi^2}\right)\psi(x).\end{aligned}$$

We now use, for all $x\in(-1,1)$,

$$Y'(x)=\frac{\pi}{U(c)p(x)^{1/2}}\cos(X(x)), \tag{42}$$

which yields the equality between C^∞ functions: $(1-Y^2)/\left((Y')^2p\right)=(U(c)/\pi)^2$ and

$$\begin{aligned}&\left(1+2\frac{Y}{p'Y'}\left(\frac{\pi}{U(c)}\right)^2\right)\left(\left(\frac{U(c)}{\pi}\right)^2\left(\frac{(p')^2}{4p}\psi+p'\psi'\right)+\frac{Y}{2pp'Y'}\left((p')^2\psi\right)\right)\\&-2\frac{Y}{p'Y'}\left(\frac{(p')^2}{p}\psi+4p'\psi'\right)+\left(\frac{U(c)}{\pi}\right)^2pY'\left(\frac{p'\psi}{pY'}+4\frac{\psi'}{Y'}\right)'+2\frac{Y}{p'Y'}p'\psi'\\&=4\left(q_c(Y)-\frac{1}{2}-\frac{U(c)^2\lambda}{\pi^2}\right)\psi.\end{aligned}$$

The term in factor of ψ on the left-hand side of the above equality is

$$\left(\frac{U(c)}{\pi}\right)^2\left(1+2\frac{Y}{p'Y'}\left(\frac{\pi}{U(c)}\right)^2\right)^2\frac{(p')^2}{4p}-2\frac{Y}{p'Y'}\frac{(p')^2}{p}+\left(\frac{U(c)}{\pi}\right)^2\frac{pp''Y'-(p')^2Y'-pp'Y''}{pY'}$$

Using $-2pY''=p'Y'+2\left(\pi/U(c)\right)^2Y$ which is obtained from (42), this becomes

$$\begin{aligned}&\left(\frac{U(c)}{\pi}\right)^2\left(1+2\frac{Y}{p'Y'}\left(\frac{\pi}{U(c)}\right)^2\right)^2\frac{(p')^2}{4p}-\frac{Y}{Y'}\frac{p'}{p}+\left(\frac{U(c)}{\pi}\right)^2\left(p''-\frac{(p')^2}{2p}\right)\\&=\left(\frac{Y}{Y'}\right)^2\left(\frac{\pi}{U(c)}\right)^2\frac{1}{p}+\left(\frac{U(c)}{\pi}\right)^2\left(p''-\frac{(p')^2}{4p}\right)\\&=\tan(X(x))^2+\left(\frac{U(c)}{\pi}\right)^2\left(p''-\frac{(p')^2}{4p}\right).\end{aligned}$$

hence

$$4\left(\frac{U(c)}{\pi}\right)^2(p\psi')'=4\left(q_c(Y)-\frac{1}{2}-\frac{1}{4}\tan(X(x))^2+\frac{1}{4}\left(\frac{U(c)}{\pi}\right)^2\left(\frac{(p')^2}{4p}-p''\right)-\left(\frac{U(c)}{\pi}\right)^2\lambda\right)\psi$$

and ψ is solution of (H_λ) in case (1).

We now obtain upper and lower bounds on the even function $q^c(Y(x))$, for $x \in [0,1]$, and start with the lower bound. To bound $\tan(X)^2$ in (26), we use

$$\tan\left(\frac{\pi}{U(c)}\int_0^x p(\xi)^{-1/2}d\xi\right)^2 = \left(\tan\left(\frac{\pi}{U(c)}\int_x^1 p(\xi)^{-1/2}d\xi\right)\right)^{-2}, \tag{43}$$

and (96) in [45] in the first display and Lemma 3 and the fact that $(a-b)^2 \geq a^2 - 2ab$ for $a, b > 0$ in the second display. We obtain

$$\begin{aligned}\tan\left(X(x)\right)^2 &\geq \left(\frac{U(c)}{\pi}\left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^{-1} - \frac{4}{\pi U(c)}\int_x^1 p(\xi)^{-1/2}d\xi\right)^2 \\ &\geq \left(\frac{U(c)c}{\pi}\right)^2\left(\frac{\sinh(4c)}{c(1-x)} - \frac{8c\sinh(4c)\cosh(4c)}{3\left(\cosh(4c)-1\right)}\right) - \frac{8}{\pi^2}.\end{aligned}$$

To bound the second term in the bracket in (26) we proceed as follows. We have

$$\begin{aligned}\frac{4c\sinh(4cx)^2}{p(x)} &= \frac{\sinh(4c)}{1-x}\frac{1}{1+u(x)}\frac{\sinh(4cx)^2}{\sinh(4c)^2} \quad \text{(by (29))} \\ &= \frac{\sinh(4c)}{1-x}\left(1 + \int_x^1 \frac{u'(t)dt}{(1+u(t))^2}\right) \quad \text{(by (36))} \\ &\leq \frac{\sinh(4c)}{1-x}\left(1 + 8c^2(1-x)\right) \quad \text{(by (37))},\end{aligned} \tag{44}$$

hence

$$\begin{aligned}q^c(Y(x)) \geq& \frac{1}{2} - \frac{2}{\pi^2} - \left(\frac{U(c)c}{\pi}\right)^2\cosh(4c)\left(1 + \frac{2c\sinh(4c)}{3\left(\cosh(4c)-1\right)}\right) \\ \geq& \frac{1}{2} - \left(\frac{U(c)c}{\pi}\right)^2 - R(c).\end{aligned}$$

Consider the upper bound on q^c. For $x \in [0,1]$, by (43) and $0 < z \leq \tan(z)$ on $(0, \pi/2]$, we have

$$q^c(Y(x)) \leq \frac{1}{2} + \left(\frac{U(c)c}{\pi}\right)^2\left(\frac{1}{4c^2\left(\int_x^1 p(\xi)^{-1/2}d\xi\right)^2} - \frac{\sinh(4cx)^2}{p(x)} - \cosh(4cx)\right).$$

Using Lemma 3, (44), and (31), we have

$$q^c(Y(x)) \leq \frac{1}{2} - \left(\frac{U(c)c}{\pi}\right)^2. \tag{45}$$

□

The operator $(\mathcal{L},\mathcal{D})$ in case (3) is self-adjoint. Indeed, it is shown page 571 of [34] that $\mathcal{D}$ is the domain of the self-adjoint Friedrichs extension of the minimal operator corresponding to the differential operator on $L^2(-1,1)$ on the domain $\mathcal{D}_{\min}$ (the subset of $\mathcal{D}_{\max}$ of functions with support in $(-1,1)$, see page 173 in [46], we removed one condition on $\mathcal{D}_{\max}$ which is automatically satisfied). By Proposition 3, the multiplication defined, for $\psi\in\mathcal{D}_{\max}$, by $\psi\to q^c\psi$ is bounded and symmetric on $L^2(-1,1)$. Thus, by the Kato-Rellich theorem (see, *e.g.*, [37]), the operator $(\mathcal{L},\mathcal{D})$ in case (2) is self-adjoint. Denote by $\left((U(c)/\pi)^2\chi_m^c\right)_{m\in\mathbb{N}_0}$ the eigenvalues of $(\mathcal{L},\mathcal{D})$ in case (2) arranged in increasing order and repeated according to multiplicity. They are real and, because the operator is bounded below, they are bounded below by the same constant. Moreover, Proposition 3 yields that $(\chi_m^c)_{m\in\mathbb{N}_0}$ are the eigenvalues of $(\mathcal{L},\mathcal{D})$ in case (1). The following result gives exact constants and a behavior uniform over m which is coherent with the asymptotic result on page 14 of [44].

Theorem 4. We have, for all $m\in\mathbb{N}_0$ and $c>0$,

$$\left(\frac{\pi}{U(c)}\right)^2\left(m(m+1)+\frac{1}{2}-R(c)\right)+c^2\le\chi_m^c\le\left(\frac{\pi}{U(c)}\right)^2\left(m(m+1)+\frac{1}{2}\right)-c^2.$$

Proof. This follows from the min-max theorem and (41). □

6. Uniform estimates on the singular functions g_m^c

Theorem 5. We have, for all $m\in\mathbb{N}_0$ and $c>0$,

$$\|g_m^c\|_{L^\infty([-1,1])}\le\frac{\pi\sinh(2c)\sqrt{(1+\cosh(4c))(1+4c^2/3)}}{\left(2c\sinh(4c)\left(\cosh(4c)-1\right)\right)^{1/4}e^{2c}}\left(\frac{2R(c)}{m+1/2}+\left(1+\sqrt{\frac{2}{3}}\frac{R(c)}{m+1/2}\right)\sqrt{m+\frac{1}{2}}\right).$$

Proof. Using in the first display the change of variable (27) and the change of function (28) with $\psi=g_m^c$, and denoting by $\Gamma_m^c(\cdot)=F(\cdot)g_m^c\left(Y^{-1}(\cdot)\right)$ and $\widetilde{\Gamma}_m^c=\Gamma_m^c\sqrt{U(c)/\pi}$, which is real valued, and (42) and (28) in the second display, we obtain

$$\begin{aligned}\int_{-1}^1\left|\widetilde{\Gamma}_m^c(y)\right|^2dy&=\frac{U(c)}{\pi}\int_{-1}^1Y'(x)\left|F(Y(x))\right|^2\left|g_m^c(x)\right|^2dx\\&=\int_{-1}^1\frac{\cos(X(x))}{\sqrt{1-\sin(X(x))^2}}\left|g_m^c(x)\right|^2dx=1.\end{aligned}$$

Also, by Proposition 3, for all $y\in(-1,1)$,

$$\left((1-y^2)\left(\widetilde{\Gamma}_m^c\right)'\right)'(y)+m(m+1)\widetilde{\Gamma}_m^c(y)=\left(m(m+1)-\left(\frac{U(c)}{\pi}\right)^2\chi_m^c+q^c(y)\right)\widetilde{\Gamma}_m^c(y).\quad(46)$$

We proceed like in the proof of Proposition 5 in [13] and obtain, by the method of variation of constants and knowledge of the solutions to the homogenous equation corresponding to the left-hand side of (46), that there exist $A, B \in \mathbb{R}$ such that, for $y \in (-1,1)$,

$$\widetilde{\Gamma}^c_m(y) = A\overline{P}_m(y) + BQ_m(y) + \frac{1}{m+1/2}\int_y^1 L_m(y,z)\sqrt{1-z^2}G_c(z)\widetilde{\Gamma}^c_m(z)dz, \tag{47}$$

where $\overline{P}_m$ is the Legendre polynomial of degree m and norm 1 in $L^2(-1,1)$, Q_m is the Legendre function of the second kind, $G_c(y) = m(m+1) - (U(c)/\pi)^2\chi^c_m + q^c(y)$, and $L_m(y,z) = \sqrt{1-z^2}\left(\overline{P}_m(y)Q_m(z) - \overline{P}_m(z)Q_m(y)\right)$. By Theorem 4 and Proposition 3, we have $\|G_c\|_{L^\infty([-1,1])} \leq R(c)$. Because $\Gamma^c_m(1)$ is finite, $\overline{P}_m$ is bounded but $\lim_{y\to 1} Q_m(y) = \infty$, we know that $B = 0$. By the result after Lemma 9 in [13], for all $0 \leq y \leq z \leq 1$, $|L_m(y,z)| \leq 1$. Hence, by the Cauchy-Schwarz inequality, we have, for all $y \in (1,1)$,

$$\begin{aligned}\left|\widetilde{\Gamma}^c_m(y) - A\overline{P}_m(y)\right| &\leq \frac{1}{m+1/2}\left(\int_y^1 (L_m(y,z))^2(1-z^2)dz\right)^{1/2}\left(\int_y^1 G_c(z)^2\widetilde{\Gamma}^c_m(z)^2dz\right)^{1/2}, \\ &\leq \frac{R(c)}{m+1/2}(1-y) \end{aligned} \tag{48}$$

so

$$\int_{-1}^1 \left|\widetilde{\Gamma}^c_m(y) - A\overline{P}_m(y)\right|^2 dy \leq \frac{2R(c)^2}{3(m+1/2)^2}$$

and, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{-1}^1 \left|\widetilde{\Gamma}^c_m(y) - A\overline{P}_m(y)\right|^2 dy \geq& 1 + A^2 - 2|A|\int_{-1}^1 \left|\widetilde{\Gamma}^c_m(y)\right|^2 dy \int_{-1}^1 \left|\overline{P}_m(y)\right|^2 dy \\ \geq& (1-|A|)^2,\end{aligned}$$

hence

$$|A| \leq 1 + \sqrt{\frac{2}{3}}\frac{R(c)}{m+1/2}. \tag{49}$$

Also, by (40) and Lemma 2, we have

$$\|1/F\|_{L^\infty([-1,1])}\sqrt{\frac{\pi}{U(c)}} \leq \left(1+\frac{4c^2}{3}\right)^{1/2}\frac{\pi\sinh(2c)(1+\cosh(4c))^{1/2}}{\left(2c\sinh(4c)\left(\cosh(4c)-1\right)\right)^{1/4}e^{2c}},$$

and we obtain the result by (48), (49), and $\left\|\overline{P}_m\right\|_{L^\infty([-1,1])} \leq \sqrt{m+1/2}$. □

The following bound is an easy consequence of the above result: , for all $m \in \mathbb{N}_0$ and $c > 0$,

$$\|g_m^c\|_{L^\infty([-1,1])} \leq H(c)\sqrt{m+\frac{1}{2}}, \tag{50}$$

where

$$H(c) = \left(1+\frac{4c^2}{3}\right)^{1/2} \frac{\pi \sinh(2c)(1+\cosh(4c))^{1/2}}{\left(2c\sinh(4c)\left(\cosh(4c)-1\right)\right)^{1/4} e^{2c}} \left(1+2\sqrt{2}R(c)\left(2+\frac{1}{\sqrt{3}}\right)\right).$$

7. Numerical method to obtain the SVD of $\mathcal{F}_{b,c}$

We have used in Section 5 that because $\mathcal{Q}_c$ commutes with $\mathcal{L}_c$, $(g_m^c)_{m\in\mathbb{N}_0}$ are the eigenfunctions of $\mathcal{L}_c$. To obtain a numerical approximation of these functions, we use $\mathcal{L}_c$, which eigenvalues are of the order of m^2 (see Lemma (2)), rather than $\mathcal{Q}_c$, which eigenvalues decay to zero faster than exponentially. This is achieved by solving numerically for the eigenfunctions of a singular Sturm-Liouville operator. We approximate the values of the eigenfunctions on a grid on $[-1, 1]$ using the MATLAB package MATSLISE 2 (it implements constant perturbation methods for limit point nonoscillatory singular problems, see [29] chapters 6 and 7). By Proposition A.1 in [21], we have $\varphi_m^{b,c}(\cdot) = \varphi_m^{1,c/b}(b\cdot)\sqrt{b}$ for all $m \in \mathbb{N}_0$. Finally, we use $\mathcal{F}_{1,c/b}^*\left[g_m^{c/b}\right] = \sigma_m^{1,c/b}\varphi_m^{1,c/b}$ and that $\varphi_m^{1,c/b}$ has norm 1 to obtain the remaining of the SVD. $\mathcal{F}_{1,c/b}^*\left[g_m^{c/b}\right]$ is computed using the fast Fourier transform (FFT). The availability of such a method to obtain the SVD mirrors recent developments on numerical approximations of the PSWF which allows to go beyond the usual toolbox based on the Tikhonov or iterative methods such as the Landweber method (Gerchberg method for out-of band extrapolation, see [7]).

8. Illustration: application to analytic continuation

We solve for f in (14) of Case (a) $f(\cdot) = 0.5/\cosh(2\cdot)$, which is not bandlimited, and Case (b) $f(\cdot) = \mathrm{sinc}(2\cdot)/6$ which is bandlimited, when $c = 0.5$, $x_0 = 0$, and $\xi(\cdot) = \cos(50\cdot)$. We use approximation f_δ^N described in Section 3.2 with $b = 1$ for (Case (a)), $b = 1/6.5$ for (Case (b)), and the selection rule for the parameter N based on a type of Goldenshluger-Lepski method (see [23] and the references therein):

$$\widehat{N} \in \underset{N'\in\{0,\dots,N_{\max}\}}{\operatorname{argmin}} B(N) + \Sigma(N),$$

$$B(N) = \sup_{N\leq N'\leq N_{\max}} \left(\left\|F_\delta^{N'\vee N} - F_\delta^N\right\|^2_{L^2(\cosh(b\cdot))} + \Sigma(N')\right)_+, \ \Sigma(N) = \frac{(2\pi)^2 c\delta^2}{\theta(c,1)\left(1-e^{-2\beta(c/b)}\right)} e^{2\beta(c/b)N},$$

and $N_{\max} = \lfloor \log(1/\delta) \rfloor$. By analogy with the statistical problem, we use the terminology estimator rather than approximation. Results are presented in figures 1 and 2. We compare f_δ^N to a similar estimator based on (16) but with the PSWF instead of g_m^c. The first coefficients of the decomposition of the PSWF on the Legendre polynomials can be obtained by solving for the eigenvectors of two tridiagonal symmetric Toeplitz matrices (for even and odd values of m, see Section 2.6 in [36]). Also computing their image by $\mathcal{F}_c^*$ is easy because $\mathcal{F}_c^* = \mathbb{1}\{[-1,1]\}\mathcal{F}_{-c}\mathcal{E}xt$ applied to the Legendre polynomials has a closed form involving the Bessel functions of the first kind (see (18.17.19) in [35]). This approach can only be used to perform analytic continuation of bandlimited functions (Case (b)). In the bandlimited case, using the estimator based on $g_m^{c/b}$ allows to perform analytic continuation without the knowledge of an interval containing the support of the Fourier transform of the function.

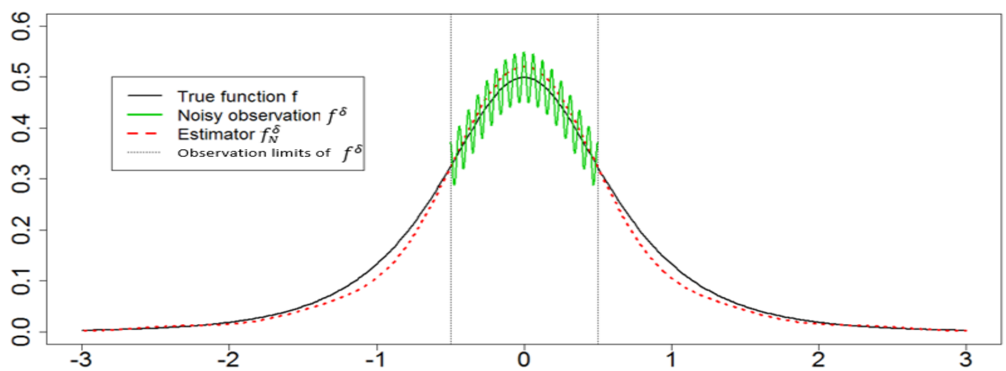

FIGURE 1. Case (a) with noise ($\delta = 0.05$), where F_δ^N in (16) uses $g_m^{c/b}$.

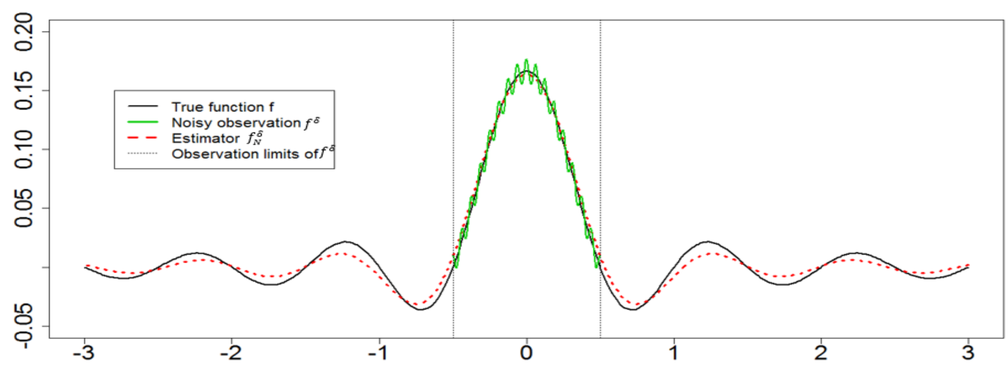

FIGURE 2. Case (b) with noise ($\delta = 0.01$), where F_δ^N in (16) uses $g_m^{c/b}$.

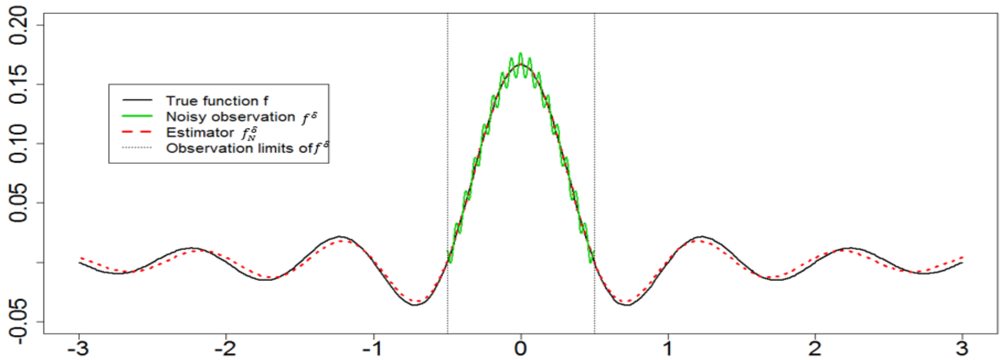

FIGURE 3. Case (b) with noise ($\delta = 0.01$), where F_δ^N in (16) uses the PSWF.

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