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"On the Foundations of Ex Post Incentive Compatible Mechanisms"

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On the Foundations of Ex Post Incentive Compatible Mechanisms*

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Abstract

In private-value auction environments, Chung and Ely (2007) establish maxmin and Bayesian foundations for dominant-strategy mechanisms. We first show that similar foundation results for ex post mechanisms hold true even with interdependent values if the interdependence is only *cardinal*. This includes, for example, the one-dimensional environments of Dasgupta and Maskin (2000) and Bergemann and

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Morris (2009b). Conversely, if the environment exhibits *ordinal* interdependence, which is typically the case with multi-dimensional environments (e.g., a player's private information comprises a noisy signal of the common value of the auctioned good and an idiosyncratic private-value parameter), then in general, ex post mechanisms do not have foundation. That is, there exists a non-ex-post mechanism that achieves strictly higher expected revenue than the optimal ex post mechanism, regardless of the agents' high-order beliefs.

1 Introduction

The recent literature on mechanism design provides a series of studies on the robustness of mechanisms, motivated by the idea that a desirable mechanism should not rely too heavily on the agents' common knowledge structure.[1] One approach taken in the literature is to adopt stronger solution concepts that are insensitive to various common knowledge assumptions. For instance, in private-value environments, Segal (2003) studies dominant-strategy incentive compatible sales mechanisms. In interdependent-value environments, Dasgupta and Maskin (2000) study efficient auction rules that are independent of the details under the concept of ex post incentive compatibility.

However, a mechanism that achieves desired outcomes without the agents' common knowledge assumption does not immediately imply dominant-strategy or ex post incentive compatibility. In revenue maximization in private-value

[1] See, for example, Wilson (1985).

auction (under "regularity" conditions), Chung and Ely (2007) fill in this gap by establishing the *maxmin* and *Bayesian* foundation of the optimal dominant-strategy mechanism, in the following sense. Consider a situation where the seller in an auction (principal) only knows a joint distribution of the bidders' (agents) valuation profile for the auctioned object, which may be based on data about similar auctions in the past. On the other hand, he does not have reliable information about the bidders' beliefs about each other's value. For example, the bidders may have more or less information than the seller (e.g., through their information acquisition), or may simply have a "wrong" belief from the seller's point of view for various reasons. Thus, the seller's objective is to find a mechanism that achieves a good amount of revenue *regardless of the bidders' (high-order) beliefs.* Note that, in a dominant-strategy mechanism, it is always an equilibrium for each bidder to report his true value, and therefore, it always guarantees the same level of expected revenue. On the other hand, in non-dominant-strategy mechanisms, expected revenue may vary with the bidders' (high-order) beliefs. In the definition of Chung and Ely (2007), there is a maxmin foundation for a dominant-strategy mechanism if, for any non-dominant-strategy mechanism, there is a possible belief of the seller with which the dominant-strategy mechanism achieves (weakly) higher expected revenue than the non-dominant-strategy mechanism.[2]

[2]As a stronger concept, if the same belief can be found for any non-dominant-strategy mechanism with which a dominant-strategy mechanism achieves (weakly) higher expected revenue, then there is a *Bayesian* foundation, because, as long as the seller is Bayesian rational and has that particular belief, he finds it optimal to offer a dominant-strategy

In this paper, we examine the existence of such foundations for ex post incentive compatible mechanisms in interdependent-value environments. Our main observation is that the key property that guarantees such foundations is what we call the *cardinal* vs. *ordinal* interdependence. To explain these concepts, imagine an auction problem, where each bidder's willingness-to-pay depends both on his own type and the other bidders' types. If one type of each bidder always has a higher valuation for the good than another type *regardless of the other bidders' types* (even if each type's valuation itself may vary with the others' types), then we say that the environment exhibits only *cardinal* interdependence. Conversely, if the types cannot be ordered in such a uniform manner with respect to the others' types, then we say that the environment exhibits *ordinal* interdependence.[3]

We first show that, in the environments with only cardinal interdependence, (both maxmin and Bayesian) foundations exist for ex post mechanisms. This includes, for example, private-value environments (in this sense, our result is a generalization of Chung and Ely (2007)), and the one-dimensional environments of Dasgupta and Maskin (2000) and Bergemann and Morris (2009b).

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mechanism, even though he can also offer any other mechanism.

[3]These interdependence concepts are obviously related to the "size" of interdependence (e.g., private-value environments are special cases of cardinally interdependent cases). However, they are not necessarily corresponding to each other. For example, if a bidder's valuation in an auction is a sum of a function only of his own type and another function of the others' types, then however large is the second term, the environment never exhibits ordinal interdependence. In this sense, a more appropriate interpretation is that these interdependent concepts are related to the *diversity* of interdependence across types.

Conversely, if the environment exhibits *ordinal* interdependence, which is typically the case with multi-dimensional environments (e.g., a player's private information comprises a noisy signal of the common value of the auctioned good and an idiosyncratic private-value parameter), then in general, ex post mechanisms do not have foundation. That is, there exists a non-ex-post mechanism that achieves strictly higher expected revenue than the optimal ex post mechanism, regardless of the agents' high-order beliefs.

Regarding the foundation results, Chen and Li (2016) consider a general class of private-value environments where agents have multi-dimensional payoff types, and show that if the environment satisfies the *uniform-shortest-path-tree* property, then the maxmin (and Bayesian) foundation exists for dominant-strategy mechanisms. This property simply means that, for any allocation rule the principal desires to implement, the set of binding constraints is invariant. This holds true in the single-good auction environment of Chung and Ely (2007) with regularity, and in this sense, their result generalizes that of Chung and Ely (2007), keeping the private-value assumption. Our work is a complement to Chen and Li (2016) in that we consider interdependent-value environments. For our foundation result (Theorem 1), a similar property to their uniform-shortest-path-tree property holds, which suggests that some of their argument may be applicable even in interdependent-value environments.

Regarding the no-foundation results, there are several papers in the literature that provide examples or a restrictive class of environments in which (various versions of) foundations for dominant-strategy or ex post mechanisms do not exist. For example, for interdependent-value environments, Bergemann and Morris (2005) provide examples in the context of implementation of certain ("non-separable") social choice correspondences, and Jehiel, Meyer-ter Vehn, Moldovanu, and Zame (2006) provide an example for revenue maximization in sequential sales. Chen and Li (2016) also provide an instance of environment where, without their uniform-shortest-path-tree property, there might not exist a foundation for dominant-strategy mechanisms, even in private-value environments. Our work contributes to this line of research by providing a general class of environments with a no-foundation result (instead of providing examples), and the economic intuition based on the cardinal vs. ordinal interdependence.

Other closely related papers include Bergemann and Morris (2005) and Börgers (2013). In interdependent-value environments, Bergemann and Morris (2005) show that any *separable* social choice correspondence that is implementable given any (high-order) belief structure of the agents must satisfy ex post incentive compatibility. In this sense, they provide another sort of foundation for ex post incentive compatible mechanisms. Their separable social choice correspondence necessarily admits a unique non-monetary allocation for each payoff-type profile, and hence, in general, excludes revenue maximization as the principal's objective. Thus, our work is complementary to theirs in that we consider revenue maximization.

Börgers (2013) criticizes the foundation theorems by constructing a non-dominant-strategy (or more generally, a non-ex-post) mechanism that yields *weakly* higher expected revenue than the optimal dominant-strategy mechanism for any belief structure of the agents, while it yields *strictly* higher expected revenue for some belief structures. Our no-foundation result is stronger in that it provides a *strict* improvement in expected revenue for *any* (high-order) belief structure, though under stronger conditions on the environment.

2 Model

There exist $I \in \mathbb{N}$ agents. Agent i's privately-known *payoff type* is $\theta_i \in \Theta_i \subseteq \mathbb{R}^d$, where $|\Theta_i| < \infty$.[4] A payoff-type profile is written as $\theta = (\theta_1, \ldots, \theta_I) \in \Theta_1 \times \ldots \times \Theta_I = \Theta$. The principal's (subjective) prior belief for θ is given by $f \in \Delta(\Theta)$, where we assume $f(\theta) > 0$ for all $\theta \in \Theta$.

Each agent i's willingness-to-pay for $q_i \in Q_i \subseteq \mathbb{R}_+$ units of the good is denoted by $v_i(q_i, \theta)$. We assume that $0 \in Q_i$, $|Q_i| < \infty$,[5] $v_i(0, \theta) = 0$, and $v_i(\cdot, \theta)$ is increasing for all θ. Moreover, as a standard single-crossing condition, we assume that, for each $\theta_i \neq \theta_i'$, and θ_{-i}, we have either

$$v_i(q_i, \theta_i, \theta_{-i}) - v_i(q_i', \theta_i, \theta_{-i}) > v_i(q_i, \theta_i', \theta_{-i}) - v_i(q_i', \theta_i', \theta_{-i}), \quad \forall q_i > q_i';$$

[4]Potential extensions to cases with continuous payoff type spaces are discussed in Section 5.

[5]As it becomes clearer, the finiteness of Q_i is without loss of generality (though it simplifies the notation), given that Θ is finite and we only consider finite mechanisms (including ex post incentive compatible mechanisms).

or

$$v_i(q_i, \theta_i, \theta_{-i}) - v_i(q_i', \theta_i, \theta_{-i}) < v_i(q_i, \theta_i', \theta_{-i}) - v_i(q_i', \theta_i', \theta_{-i}), \quad \forall q_i > q_i'.$$

In the first (second) case, we denote $\theta_i \succ_i^{\theta_{-i}} \theta_i'$ ($\theta_i \prec_i^{\theta_{-i}} \theta_i'$, respectively). Our assumption throughout the paper is that $\prec_i^{\theta_{-i}}$ is a total ordering over Θ_i for any θ_{-i}, although $\prec_i^{\theta_{-i}}$ can be different from $\prec_i^{\theta_{-i}'}$. Let

$$\eta = \min_{i, \theta_i \neq \theta_i', \theta_{-i}, q_i \neq q_i'} |v_i(q_i, \theta_i, \theta_{-i}) + v_i(q_i', \theta_i', \theta_{-i}) - v_i(q_i', \theta_i, \theta_{-i}) - v_i(q_i, \theta_i', \theta_{-i})| \ (> 0).$$

In particular, this implies that, by setting $q_i > 0 = q_i'$,

$$|v_i(q_i, \theta_i, \theta_{-i}) - v_i(q_i, \theta_i', \theta_{-i})| \geq \eta$$

for all $\theta_i \neq \theta_i'$ and θ_{-i}.

Paying $p_i \in \mathbb{R}$ to the principal, agent i's final payoff is $v_i(q_i, \theta) - p_i$. The principal's objective is the total revenue, $\sum_i p_i$. The feasible set of $q = (q_1, \ldots, q_I)$ is denoted by $Q \subseteq \prod_i Q_i$, where the shape of Q depends on the specific environment of interest.

For example, auctions, trading, and public-goods environments are in this class, with (or without) interdependence. In terms of interdependence, our framework includes a typical "common + private" environment studied in the auction literature: Imagine that each agent i has a unit demand for the good, his payoff-type comprises $(c_i, d_i) \in \Theta_i \subseteq \mathbb{R}^2$, where c_i may be interpreted as a "common-value" component and d_i may be interpreted as an idiosyncratic "private-value" component, and his valuation for the good is $\pi_i(c_1, \ldots, c_N) + d_i$ for some function π_i.

2.1 Type space

The agents' private information includes their own payoff types, their (first-order) beliefs about their payoff types, and their arbitrarily higher-order beliefs. To model this, we introduce type spaces as in Bergemann and Morris (2005).

A ("known-own-payoff-type") type space, denoted by $\mathcal{T} = (T_i, \widehat{\theta}_i, \widehat{\pi}_i)_{i=1}^I$, is a collection of a measurable space of types T_i for each agent i, a measurable function $\widehat{\theta}_i : T_i \to \Theta_i$ that describes the agent's payoff type, and a measurable function $\widehat{\pi}_i : T_i \to \Delta(T_{-i})$ that describes his belief about the others' types. Let $\widehat{\beta}_i(t_i)$ denote the belief hierarchy associated with type t_i (i.e., it describes t_i's first-order belief about θ_{-i}, second-order belief, and so on, up to an arbitrary high order). We say that $\mathcal{T}$ has *no redundant types* if for each i, mapping $t_i \mapsto (\widehat{\theta}_i(t_i), \widehat{\beta}_i(t_i))$ is one-to-one.

In fact, there exists a (compact) *universal type space* $\mathcal{T}^* = (T_i^*, \widehat{\theta}_i^*, \widehat{\pi}_i^*)_{i=1}^I$, such that any type space without redundant types can be embedded into it, in the following sense.[6]

Lemma 1. Let $\mathcal{T}$ be a type space with no redundant types. Then, for each

[6]For constructions of universal type spaces, see Mertens and Zamir (1985) and Brandenburger and Dekel (1993).

i, there exist subsets $\widehat{T}_i \subset T_i^*$ and bijections $h_i : T_i \to \widehat{T}_i$ such that:

1. $\widehat{\theta}_i^*(h_i(t_i)) = \widehat{\theta}_i(t_i)$ for all $t_i \in T_i$; and

2. $\widehat{\pi}_i^*(h_i(t_i))[h_{-i}(t_{-i})] = \widehat{\pi}_i(t_i)[t_{-i}]$ for all $t_i \in T_i$ and $t_{-i} \in T_{-i}$,

where $h_{-i}(t_{-i}) = (h_1(t_1), \ldots, h_{i-1}(t_{i-1}), h_{i+1}(t_{i+1}), \ldots, h_I(t_I))$.

In what follows, we directly work with this universal type space.[7] Specifically, let $\mu \in \Delta(T^*)$ represent the principal's prior belief over T^* such that $\mu(\{t|\widehat{\theta}^*(t) = \theta\}) = f(\theta)$ for each θ, that is, the principal's (first-order) belief for θ is given by $f(\theta)$, as assumed above. The other information contained in μ captures the principal's belief over the agents' possible belief structures. Let $\mathcal{M} \subseteq \Delta(T^*)$ represent the set of all such μ.

In some contexts, it may be reasonable to assume that (the principal believes that) the agents do not have extreme (non-full-support) first-order beliefs. For example, instead of assuming that each agent's belief or knowledge is exogenous, one may be interested in a situation where each agent engages in his own information acquisition (through which his belief is updated), where the information acquisition cost is a linear function of relative entropy (Sims (2003)). Then, it is infinitely costly for each agent to know other agents' payoff types.

Let $\mathcal{M}^{\text{full}} \subset \mathcal{M}$ denote the set of μ such that every agent i has a full-support first-order belief about the other agents. More precisely, for each

[7] The results would not change even if we allow for type spaces with redundant types, but more notation would be involved.

agent i with type t_i, let $\widehat{\pi}_i^{*1}(t_i) \in \Delta(\Theta_{-i})$ denote his first-order belief, that is, $\widehat{\pi}_i^{*1}(t_i)[\theta_{-i}] = \int_{t_{-i}|\widehat{\theta}^*_{-i}(t_{-i})=\theta_{-i}} d\widehat{\pi}_i^*(t_i)[t_{-i}]$ for each θ_{-i}. Then, $\mathcal{M}^{\text{full}}$ is the set of all $\mu \in \mathcal{M}$ such that $\mu\left(\{t \mid \forall i, \theta_{-i},\ \widehat{\pi}_i^{*1}(t_i)[\theta_{-i}] > 0\}\right) = 1$.

2.2 Mechanism

The principal designs a mechanism, denoted by (M, q, p), where M_i represents a message set for each agent i, $M = M_1 \times \ldots \times M_I$, $q : M \to Q = [0,1]^I$ denotes an allocation rule, and $p : M \to \mathbb{R}^I$ denotes a payment function. Each agent i reports a message $m_i \in M_i$ simultaneously, and then he receives the good with probability $q_i(m)$ and pays $p_i(m)$ to the principal. We assume that M_i contains a non-participation message $\emptyset \in M_i$ such that $\big(q_i(\emptyset, m_{-i}), p_i(\emptyset, m_{-i})\big) = (0,0)$ for any $m_{-i} \in M_{-i}$.

We now introduce a class of mechanisms with ex post incentive compatibility (an EPIC mechanism for short).

Definition 1. An *EPIC mechanism* is a mechanism $\Gamma = (M, q, p)$ such that, for each i, (i) $M_i = \Theta_i$, and (ii) for each $\theta \in \Theta$ and $\theta_i \neq \theta_i' \in \Theta_i$:

$$
\begin{aligned}
v_i(q_i(\theta), \theta) - p_i(\theta) &\geq 0, \\
v_i(q_i(\theta), \theta) - p_i(\theta) &\geq v_i(q_i(\theta_i', \theta_{-i}), \theta) - p_i(\theta_i', \theta_{-i}).
\end{aligned}
$$

The expected revenue in the truth-telling (ex post) equilibrium in an

EPIC mechanism is given by:

$$R_f(\Gamma) = \sum_{\theta} \sum_{i} p_i(\theta) f(\theta).$$

Note that this does not depend on μ, and in this sense, $R_f(\Gamma)$ may be interpreted as a "robustly guaranteed" expected revenue with respect to the agents' beliefs and higher-order beliefs. Let R_f^{EP} denote the maximum expected revenue among all EPIC mechanisms.

Applying the standard argument, the optimal mechanism among all EPIC mechanisms is characterized by the corresponding virtual-value maximization. To explain this, let $F_i(\theta_i, \theta_{-i}) = \sum_{\tilde{\theta}_i \preceq_i^{\theta_{-i}} \theta_i} f(\tilde{\theta}_i, \theta_{-i})$ denote the cumulative distribution function of i's payoff types given the other agents' payoff-type profile θ_{-i}.

Agent i's *virtual valuation* at payoff-type profile θ is given by:

$$\gamma_i(q_i, \theta) = v_i(q_i, \theta) - \frac{1 - F_i(\theta)}{f(\theta)} \big(v_i^+(q_i, \theta) - v_i(q_i, \theta) \big),$$

where $v_i^+(\theta_i, \theta_{-i}) = \min_{\tilde{\theta}_i \succ_i^{\theta_{-i}} \theta_i} v_i(\tilde{\theta}_i, \theta_{-i})$ whenever the right-hand side is well-defined; otherwise $\gamma_i(q_i, \theta) = v_i(q_i, \theta)$.

The following result is standard, so we omit its proof.

Lemma 2.

$$\begin{aligned} R_f^{EP} \quad = \quad & \max_{q:\Theta\to Q} \quad \sum_\theta \sum_i \gamma_i(q_i(\theta),\theta) f(\theta) \\ & \text{sub. to} \quad \forall i, \theta_i, \theta_i', \theta_{-i}; \\ & \qquad \theta_i \succ_i^{\theta_{-i}} \theta_i' \;\Rightarrow\; q_i(\theta_i,\theta_{-i}) \geq q_i(\theta_i',\theta_{-i}). \qquad \text{(M)} \end{aligned}$$

We assume that the solution exists in this maximization problem, which we denote by $q^{EP} = (q_i^{EP}(\theta))_{i,\theta}$. The corresponding payment rule is denoted by $p^{EP} = (p_i^{EP}(\theta))_{i,\theta}$.[8]

As in Chung and Ely (2007), we further assume the following "regularity" condition throughout the paper.

Assumption 1. There exists $\varepsilon > 0$ such that, for any distribution over Θ, $\tilde{f}$, such that $\|\tilde{f} - f\| < \varepsilon$ (in a Euclidean distance), the monotonicity constraints (M) are not binding in the problem of $R_{\tilde{f}}^{EP}$. In particular, this implies

$$R_f^{EP} \quad = \quad \max_{q:\Theta\to Q} \quad \sum_\theta \sum_i \gamma_i(q_i(\theta),\theta) f(\theta).$$

Of course, the conditions on the environment that imply the above as-

[8] p^{EP} is given as follows. For each i, θ_i and θ_{-i}, (i) if there is no $\theta_i' \prec_i^{\theta_{-i}} \theta_i$, then

$$p_i^{EP}(\theta) = v_i(q_i^{EP}(\theta),\theta);$$

(ii) otherwise, letting $\theta_i' \prec_i^{\theta_{-i}} \theta_i$ be such that no θ_i'' satisfies $\theta_i' \prec_i^{\theta_{-i}} \theta_i'' \prec_i^{\theta_{-i}} \theta_i$,

$$p_i^{EP}(\theta) = v_i(q_i^{EP}(\theta),\theta) - v_i(q_i^{EP}(\theta_i',\theta_{-i}),\theta) + p_i^{EP}(\theta_i',\theta_{-i}).$$

sumption can vary with the environment. For example, in an auction environment with $Q = \{q \in \{0,1\}^N | \sum_i q_i \leq 1\}$, the regularity assumption is satisfied if, for each $i \neq j$, and θ,[9]

$$\gamma_i(\theta) \geq \gamma_j(\theta) \Rightarrow \forall \theta_i' > \theta_i,\ \gamma_i(\theta_i', \theta_{-i}) > \gamma_j(\theta_i', \theta_{-i}).$$

In a *digital-good* environment of Goldberg, Hartline, Karlin, Saks, and Wright (2006) with $Q = \{0,1\}^N$, the regularity assumption is satisfied under the strict monotone hazard rate condition, i.e., for each i and θ, $\frac{1-F_i(\theta)}{f(\theta)}$ is decreasing in θ_i. In a multi-unit sales environment as in Mussa and Rosen (1978), the regularity assumption is satisfied under the strict monotone hazard rate condition and concavity of each v_i with respect to q_i.

The following notation is extensively used in the subsequent analysis. For each i and $q_i > 0$, define

$$\Theta_i^*(q_i, \theta_{-i}) \quad = \quad \{\theta_i \in \Theta_i | q_i^{EP}(\theta_i, \theta_{-i}) \geq q_i\}$$

as the set of i's payoff types whose allocation given θ_{-i} is greater than or equal to q_i in the optimal EPIC mechanism. Note that, by monotonicity, if $\theta_i \in \Theta_i^*(q_i, \theta_{-i})$ and $\theta_i' \succ_i^{\theta_{-i}} \theta_i$, then $\theta_i' \in \Theta_i^*(q_i, \theta_{-i})$. Let $\theta_i^*(q_i, \theta_{-i})$ be the lowest element in $\Theta_i^*(q_i, \theta_{-i})$ with respect to $\prec_i^{\theta_{-i}}$, that is, for any $\theta_i \in$

[9] Chung and Ely (2007) call it the single-crossing condition. A stronger sufficient condition is the combination of the strict monotone hazard rate property (i.e., for each i and θ, $\frac{1-F_i(\theta)}{f(\theta)}$ is decreasing in θ_i), and affiliation in f (which includes independent f as a special case).

$\Theta_i^*(q_i, \theta_{-i})$, we have $\theta_i \succ_i^{\theta_{-i}} \theta_i^*(q_i, \theta_{-i})$. This $\theta_i^*(q_i, \theta_{-i})$ is called i's threshold type with respect to q_i given θ_{-i}. Finally, let

$$\Theta_{-i}^*(q_i, \theta_i) = \{\theta_{-i} \in \Theta_{-i} | \theta_i \in \Theta_i^*(q_i, \theta_{-i})\}$$

denote the set of θ_{-i} with which θ_i is allocated greater than or equal to q_i units in the optimal EPIC mechanism.

2.3 Foundations

For a non-EPIC mechanism, expected revenue may vary with the agents' belief structure, and the principal—who does not know the agents' belief structure—may not want to offer a mechanism if the expected revenue is low for some possible belief structures. Following Chung and Ely (2007), we say that there is a *maxmin foundation* for EPIC mechanisms if, for any non-EPIC mechanism $\Gamma = (M, q, p)$, there exists $\mu \in \mathcal{M}$ such that, for any Bayesian equilibrium σ^*, the expected revenue obtained in the equilibrium is less than R_f^{EP}, that is:

$$\int_{t \in T} \sum_i p_i(\sigma^*(t)) d\mu \leq R_f^{EP}.$$

If there exists a single $\mu \in \mathcal{M}$ that achieves the above inequality for all Γ, then we say that there is a *Bayesian foundation* for EPIC mechanisms.[10]

[10]These definitions are consistent with the verbal explanations of the corresponding definitions in Chung and Ely (2007). However, in fact, the mathematical definitions of

In the context where (the principal believes that) the agents have full-support first-order beliefs, we replace $\mathcal{M}$ by $\mathcal{M}^{\text{full}}$ in the above definitions, and we say that there is a *strong* maxmin / Bayesian foundation for EPIC mechanisms.

3 Without ordinal interdependence

First, we consider the case where, for each i, θ_{-i}, and θ'_{-i}, $\prec_i^{\theta_{-i}} = \prec_i^{\theta'_{-i}}$. This includes the private-value environment (as in Chung and Ely (2007)) as a special case, but also includes some interdependent-value environments. For example, assume that $\Theta_i \subseteq \mathbb{R}$ and $v_i(q_i, \theta_i, \theta_{-i})$ is an increasing function of θ_i for each given q_i, θ_{-i}. Because i's payoff is affected by θ_{-i}, the environment exhibits interdependence, but it is only *cardinal* interdependence in the sense that a higher value of θ_i corresponds to a higher type with respect to $\prec_i^{\theta_{-i}}$ for any θ_{-i}.

On the other hand, even if v_i is increasing in θ_i, if $\Theta_i \subseteq \mathbb{R}^d$ with $d > 1$, it

them in Chung and Ely (2007) are slightly different: for example, their mathematical definition of maxmin foundation says that, for any non-EPIC mechanism $\Gamma = (M, q, p)$,

$$\inf_{\mu \in \mathcal{M}} \left[\max_{\sigma^*:\text{Bayesian equilibrium}} \int_{t \in T} \sum_i p_i(\sigma^*(t)) d\mu \right] \leq R_f^{EP}.$$

To see the difference, let $R(\mu)$ denote the term inside the bracket on the left-hand side (i.e., the expected revenue given μ), and imagine a case where (i) $R(\mu) > R_f^{EP}$ for any μ, while (ii) for any $\varepsilon > 0$, there exists μ such that $R(\mu) - \varepsilon < R_f^{EP}$. That is, the non-EPIC mechanism Γ is a *strict* improvement over the optimal EPIC mechanism, while it is not a *uniform* improvement. The verbal definition of Chung and Ely (2007) (which we follow in this paper) suggests that there is no maxmin foundation, while their mathematical definition says there is. The difference is not innocuous, because the non-EPIC mechanism we propose is indeed such a mechanism.

is possible to have $\prec_i^{\theta_{-i}} \neq \prec_i^{\theta'_{-i}}$ for some θ_{-i} and θ'_{-i}. For example, consider an auction environment in which each agent i's payoff-type comprises $(c_i, d_i) \in \Theta_i \subseteq \mathbb{R}^2$, where c_i denotes a "common-value" component and d_i denotes an idiosyncratic "private-value" component, and his valuation for the good is $\pi_i(c_1, \ldots, c_N) + d_i$ for some function π_i strictly increasing in all the arguments. Then, for $(c_i, d_i), (c'_i, d'_i) \in \Theta_i$ such that $c_i < c'_i$ and $d_i > d'_i$, it is possible that, given some c_{-i}, (c_i, d_i) has a higher valuation for the good than (c'_i, d'_i) (i.e., $\pi_i(c_i, c_{-i}) + d_i > \pi_i(c'_i, c_{-i}) + d'_i$), while given another c'_{-i}, (c_i, d_i) has a lower valuation than (c'_i, d'_i).

Such environments with *ordinal* interdependence are studied in the next section.

Definition 2. We have *ordinal interdependence* if there exists i, θ_{-i}, and θ'_{-i} such that $\prec_i^{\theta_{-i}} \neq \prec_i^{\theta'_{-i}}$.

Generalizing Chung and Ely (2007) (for private-value auction environments), we show that no ordinal interdependence implies the strong maxmin / Bayesian foundations for EPIC mechanisms.

Theorem 1. With Assumption 1 and no ordinal interdependence, EPIC mechanisms have the strong Bayesian (and hence strong maxmin) foundation.

Our proof for Theorem 1 is a direct extension of Chung and Ely (2007) in the private-value setting to the interdependent-value environment. We provide a sketch of the proof here, and the formal proof in the Appendix.

First, we impose the *non-singularity* condition on the payoff-type distribution f, which says that f satisfies certain full-rank conditions, and consider the Bayesian mechanism design problem with a simple type space having a particular belief structure. We show that under such a belief structure, it is without loss of generality to treat all participation constraints and all "adjacent downward" incentive constraints with equality, and ignore all the other constraints. Then we show that the total expected revenue in this Bayesian problem is maximized by the optimal EPIC mechanism.

The next step is to relax the non-singularity assumption by choosing a sequence of non-singular distributions which converge to the given payoff-type distribution. Since the optimal EPIC mechanisms achieve the highest expected revenue over the sequence of simple type spaces with the particular belief structure, by taking the limit, we show that the Bayesian foundation also exists for any arbitrary payoff-type distribution, as long as Assumption 1 is satisfied.[11]

4 With ordinal interdependence

In this section, we consider the environment that further satisfies the following conditions.

Assumption 2 ("Highest Payoff Type")**.** For each i, there exists $\bar{\theta}_i \in \Theta_i$ such that, for each $\theta_i \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$, we have $\bar{\theta}_i \succ_i^{\theta_{-i}} \theta_i$.

[11] In their paper, they show by example that, without the condition corresponding to Assumption 1, there may not exist a Bayesian foundation.

Assumption 3 ("Richness"). For each i, q_i, θ_i, θ'_i and θ_{-i} such that $v_i(q_i, \theta_i, \theta_{-i}) > v_i(q_i, \theta'_i, \theta_{-i})$, there exists θ'_{-i} such that $\theta_i \in \Theta^*_i(q_i, \theta'_{-i})$ and $\theta'_i \notin \Theta^*_i(q_i, \theta'_{-i})$.

The highest-payoff-type assumption is satisfied if Θ is a complete sub-lattice in $\mathbb{R}^d$, $v_i(q_i, \theta)$ is increasing in θ. The richness assumption connects the difference among i's different payoff types and the difference among their allocations. For example, consider an auction environment where each i's payoff type is $(c_i, d_i) \in C_i \times D_i (= \Theta_i)$ where $C_i, D_i \subseteq \mathbb{R}$, and his valuation is $\pi_i(c) + d_i$. The richness assumption would be easily satisfied if each D_j is rich enough so that, if $\pi_i(c_i, c_{-i}) + d_i > \pi_i(c'_i, c_{-i}) + d'_i$ for some $c_i, c'_i \in C_i$, $d_i, d'_i \in D_i$ and $c_{-i} \in C_{-i}$, then we can find some d_{-i} such that agent i's virtual value is the highest given (c_i, d_i) (and (c_{-i}, d_{-i})) but not given (c'_i, d'_i) (and (c_{-i}, d_{-i})).

In this environment, EPIC mechanisms have the strong (maxmin and Bayesian) foundation if and only if we do not have ordinal interdependence.

Theorem 2. Under Assumptions 1-3, EPIC mechanisms have the strong foundation if and only if we do not have ordinal interdependence.

Before the formal proof of the theorem, we provide the basic intuition through the following two examples.

Example 1. Assume $I = 2$, $\Theta_1 = \Theta_2 = \{1, 2\}$, and $Q = \{0, 1\}^2$. Table 1 collects payoff-type distribution f, agent 1's valuation and virtual value at each payoff type profile, and the corresponding optimal EPIC allocation for agent 1. For agent 2, assume that $v_2(\theta) = \theta_2 + 1$ for all θ so that the optimal EPIC allocation for him is $(q_2^{EP}(\theta), p_2^{EP}(\theta)) = (1, 2)$ for all θ.

Table 1: Auction environment of Example 1.

$f, v_1, \gamma_1, (q_1^{EP}, p_1^{EP})$	$\theta_2 = 1$				$\theta_2 = 2$			
$\theta_1 = 1$	$\frac{1}{6}$,	2,	2,	$(1, 1)$	$\frac{1}{6}$,	1,	-1,	$(0, 0)$
$\theta_1 = 2$	$\frac{1}{3}$,	1,	$\frac{1}{2}$,	$(1, 1)$	$\frac{1}{3}$,	2,	2,	$(1, 2)$

We have $\Theta_1^*(q_1, \theta_2) = \{1, 2\}$ if $(q_1, \theta_2) = (1, 1)$ and $\Theta_1^*(q_1, \theta_2) = \{2\}$ if $(q_1, \theta_2) = (1, 2)$. Hence, the threshold payoff type of agent 1 given $\theta_2 = 1$ (i.e., $\theta_1 = 2$) is assigned the goods given $\theta_2 = 2$, but the non-threshold winning payoff type of agent 1 given $\theta_2 = 1$ (i.e., $\theta_1 = 1$) is unassigned given $\theta_2 = 2$. This reversal of the order over agent 1's payoff types is crucial for the no-foundation result.

Now we consider a modification of the optimal EPIC mechanism, which asks agent 1's first-order belief. More specifically, agent 1 is asked to report his payoff type θ_1 and his belief for $\theta_2 = 1$, that is:

$$y(t_1) = \int_{t_2 | \widehat{\theta}_2(t_2) = 1} d\widehat{\pi}_1(t_1)[t_2].$$

If he reports $\theta_1 = 1$ and first-order belief $y \in [0, 1]$, agent 1 obtains the goods by paying $(2 - \cos\alpha)$ under $\theta_2 = 1$, but fails to get the goods and still needs to pay $(1 - \sin\alpha)$ under $\theta_2 = 2$, where $\alpha = \arctan \frac{1-y}{y}$. We keep the optimal EPIC allocations for both agents in the other cases. It is easy to verify that the new mechanism is Bayesian incentive compatible over the universal type space.

Because we are interested in the strong foundation, assume that (the principal believes that) agent 1 always has a full-support first-order belief, that is, $y \in (0,1)$ with (μ-)probability one. Then, agent 1 with $\theta_1 = 1$ always pays strictly more than under the optimal EPIC mechanism regardless of his (full-support) first-order belief and agent 2's true payoff type: if $\theta_2 = 1$, agent 1 pays $2 - \cos\alpha$ for some $\alpha \in (0, \frac{\pi}{2})$, which is strictly greater than 1; if $\theta_2 = 2$, agent 1 pays $1 - \sin\alpha$ for some $\alpha \in (0, \frac{\pi}{2})$, which is strictly greater than 0.

Therefore, this new mechanism raises strictly higher expected revenue than the optimal EPIC mechanism, as long as agent 1 has a full-support first-order belief.

Example 2. Assume $I = 2$, $\Theta_1 = \{1, 2, 3\}$, $\Theta_2 = \{1, 2\}$ and $Q = \{0, 1\}^2$. Table 2 collects payoff-type distribution f, agent 1's valuation and virtual value at each payoff type profile, and the corresponding optimal EPIC allocation for agent 1. Clearly, agent 1's preference exhibits ordinal interdependence.

Table 2: Auction environment of Example 2.

$f, v_1, \gamma_1, (q_1^{EP}, p_1^{EP})$	$\theta_2 = 1$	$\theta_2 = 2$
$\theta_1 = 1$	$\frac{1}{6}$, 3, 3, $(1,2)$	$\frac{1}{6}$, 3, 3, $(1,2)$
$\theta_1 = 2$	$\frac{1}{6}$, 2, 1, $(1,2)$	$\frac{1}{6}$, 1, -1, $(0,0)$
$\theta_1 = 3$	$\frac{1}{6}$, 1, -1, $(0,0)$	$\frac{1}{6}$, 2, 1, $(1,2)$

We have $\Theta_1^*(q_1, \theta_2) = \{1, 2\}$ if $(q_1, \theta_2) = (1, 1)$ and $\Theta_1^*(q_1, \theta_2) = \{1, 3\}$ if $(q_1, \theta_2) = (1, 2)$. Hence, neither of these two sets is the subset of the other one, which never happens when we have only cardinal interdependence. Now we construct a new detail-free mechanism as follows. When agent 1 reports $\theta_1 = 1$ and first-order belief y, agent 1 obtains the goods by paying $(3-\cos\alpha)$ under $\theta_2 = 1$ and obtains the goods by paying $(3-\sin\alpha)$ under $\theta_2 = 2$, where $\alpha = \arg\tan\frac{1-y}{y}$. We keep the optimal EPIC mechanism for both agents in the other cases. It is easy to check that the new mechanism is Bayesian incentive compatible over the universal type space. Since we assume full-support beliefs, that is, $y \in (0,1)$, then the payment from agent 1 is always strictly greater than 2, the optimal EPIC payment rule, given any θ_2. Thus, the new mechanism raises strictly higher expected revenue than the optimal EPIC mechanism regardless of the agents' (high-order) beliefs, resulting in no foundation for the EPIC mechanisms.

The two examples above identify some cases where revenue improvement is possible. Motivated by them, we define the concept of *improvability* as follows.

Definition 3 ("Improvability")**.** Revenue from i is improvable with respect to $(\theta_i, \theta_{-i}, \theta'_{-i})$ if there exists q_i and q'_i such that at least one of the following holds:

(i) $\theta_i \in \Theta^*_i(q'_i, \theta'_{-i}) \cap \Theta^*_i(q_i, \theta_{-i})$, and $\theta^*_i(q_i, \theta_{-i}) \notin \Theta^*_i(q'_i, \theta'_{-i})$, and $\theta^*_i(q'_i, \theta'_{-i}) \notin \Theta^*_i(q_i, \theta_{-i})$;

(ii) $\theta_i \in \Theta^*_i(q'_i, \theta'_{-i}) \setminus \Theta^*_i(q_i, \theta_{-i})$, and $\theta^*_i(q'_i, \theta'_{-i}) \in \Theta^*_i(q_i, \theta_{-i})$;

(iii) $\theta_i \in \Theta^*_i(q_i, \theta_{-i}) \setminus \Theta^*_i(q'_i, \theta'_{-i})$, and $\theta^*_i(q_i, \theta_{-i}) \in \Theta^*_i(q'_i, \theta'_{-i})$.

The examples essentially show that, given the optimal EPIC mechanism, if the revenue from some agent i is improvable with respect to some $(\theta_i, \theta_{-i}, \theta'_{-i})$, then the strong foundation does not exist. Thus, we complete the proof of Theorem 2 by showing that the ordinal interdependence necessarily implies the improvability. See the appendix for the formal proof.

Next, we study if EPIC mechanisms have the (not necessarily strong) foundation. The following example suggests that the same mechanism as above does not generally work, if the agents have non-full-support first-order beliefs.

Example 3. In the new mechanism proposed in Example 1, if we allow for non-full-support beliefs, there exists a situation where agent 1 always correctly predicts agent 2's payoff types. Formally, let $C = \{t \in T | \widehat{\theta}(t) = (1,1), \widehat{\pi}_1(t_1)[1] = 1\}$, $C' = \{t \in T | \widehat{\theta}(t) = (1,2), \widehat{\pi}_1(t_1)[2] = 1\}$, and consider μ such that $\mu(C) = f(1,1)$ and $\mu(C') = f(1,2)$. Because the optimal choice for agent 1 is $y = 1$ (or reporting $y = 1$ as his belief for $\theta_2 = 1$) if $t \in C$, and $y = 0$ if $t \in C'$, the equilibrium payments in the new mechanism coincide with those in the optimal EPIC mechanism. Thus, without the full-support belief assumption, the new mechanism in Example 1 only *weakly* improves the expected revenue.

Now we further modify the mechanism as follows. Unless agent 1 reports $\theta_1 = 1$ and $y = 0$, the allocation is the same as the previous mechanism proposed in Example 1. If agent 1 reports $\theta_1 = 1$ and $y = 0$, then the following events happen: agent 1 does not buy the good for any θ_2, he pays

$M(>3)$ if $\theta_2 = 1$ (i.e., when his belief turns out to be "wrong"), and the principal offers price 3 for agent 2 (so that agent 2 buys only if $\theta_2 = 2$, i.e., when agent 1's belief turns out to be "right"), instead of price 2. It is easy to verify that the new mechanism is Bayesian incentive compatible on the universal type space $\mathcal{T}^*$.

This new mechanism achieves a weakly higher expected revenue than in the optimal EPIC mechanism. First, this weak improvement is obvious unless $\theta_1 = 1$ and $y = 0$. If $\theta_1 = 1$ and $y = 0$, the principal earns $M > 3$ from agent 1 if $\theta_2 = 1$ (while the optimal EPIC mechanism yields total revenue 3), and earns 3 from agent 2 if $\theta_2 = 2$ (while the optimal EPIC mechanism yields total revenue 2).

To show a strict improvement in expected revenue for any $\mu \in \mathcal{M}$, consider the case where $\theta_1 = 1$ and $\theta_2 = 2$. Because $f(1,2) > 0$, it suffices to show that, for any $y \in [0,1]$ reported by agent 1, the new mechanism achieves a strictly higher revenue than 2, the revenue in the optimal EPIC mechanism. First, as we see above, if $y = 0$ is reported, then the new mechanism yields 3 (from agent 2), and hence there is a strict improvement. If $y > 0$, then agent 2 pays 2, and agent 1 pays $1 - \sin(\arctan \frac{1-y}{y}) > 0$, and hence, there is again a strict improvement.

Notice that the key for strict improvement is to use agent 1's belief to modify the price for agent 2. If agent 1 is correct, such modification is profitable for the principal. Otherwise, the principal collects a "fine" from agent 1, which is also profitable.

As suggested in the example, it seems impossible to raise any additional revenue from an agent if he always correctly predicts the other agents' payoff types. Instead, in such a case, a natural alternative idea is to use this agent's prediction to raise additional revenue from the other agents (and to fine him if his prediction turns out to be wrong in order for the principal to "hedge", as in the example above). Because this means that we need to be able to change an agent's allocation without changing the others', we assume that the feasible allocation set Q is a product set, $Q = \prod_i Q_i$, in what follows.

In addition, even if an agent correctly predicts the occurrence of some θ_{-i} (or its non-occurrence), such information does not necessarily make the principal earn strictly more revenue from the other agents (for example, in auction, imagine that any $j(\neq i)$'s virtual valuation is negative given θ_{-i}). Thus, we need a stronger version of the improvability.

Definition 4. We have the *strong improvability* if there exist $i, \theta_i, \theta_j, q_j, \theta_{-ij}$ such that $\theta_j \in \Theta_j^*(q_j, \theta_i, \theta_{-ij})$, and that revenue from i is improvable with respect to $\big(\theta_i, (\theta_j, \theta_{-ij}), (\theta_j^*(q_j, \theta_i, \theta_{-ij}), \theta_{-ij})\big)$.

Roughly, the strong improvability implies that, if agent i with θ_i correctly predicts that $-i$'s payoff types are not θ'_{-i}, then (given θ_{-ij}) the principal can know that j's type is not a threshold type for some q_j. Such information enables the principal to earn higher expected revenue from j.

Proposition 1. Under Assumptions 2-3 and $Q = \prod_i Q_i$, strong improvability implies no foundation of EPIC mechanisms.

A natural question is, under which additional conditions on the environment, the ordinal interdependence implies the strong improvability, so that EPIC mechanisms do not have the foundation if and only if we do not have the ordinal interdependence. A sufficient condition is the following richness condition on Q.

Assumption 4. For each i, θ_i, and θ_{-i}, we have $q_i^{EP}(\theta_i, \theta_{-i}) > 0$, and for each $\theta_i' \neq \theta_i$, we have $q_i^{EP}(\theta_i, \theta_{-i}) \neq q_i^{EP}(\theta_i', \theta_{-i})$.

A representative example is a monopoly problem with multiple buyers and multiple units of trading.[12] Note that this excludes some situations where the lowest payoff type of an agent (given the other agents' payoff types) is "excluded" from trading.

Theorem 3. Under Assumption 4 and $Q = \prod_i Q_i$, EPIC mechanisms have the foundation if and only if we do not have ordinal interdependence.

5 Discussion: Continuous payoff-type space

In the previous finite payoff-type setup, given the others' payoff-type profiles, the difference in valuations between any two payoff types is strictly positive and bounded away from 0, which enables us to exploit the "gaps" in valuations and increase the payments. However, if we have infinitely many payoff

[12] See Mussa and Rosen (1978) and Segal (2003) (or their straightforward generalizations) for such environments, although they focus on private-value environments.

types, such "gaps" may not exist, since η defined in Section 2 could be equal to 0, and then the previous construction no longer works.

In this sense, the continuous payoff-type space case is more complicated than the finite case, and hence the analysis of the continuous case is beyond the scope of the current paper.[13] Nevertheless, it may be interesting to note how our approach may be useful (with appropriate modifications), even in the continuous case. The following example explains this.

We assume $I = 2$, $\Theta_1 = \{0, 1\} \times [0, 2](\ni (c_1, d_1))$, $\Theta_2 = \{0, 1\}(\ni c_2)$, and $Q = \{0, 1\}^2$. Agent 1's valuation for $q_1 = 1$ is $v_1(c_1, d_1, c_2) = c_1 c_2 + d_1$, and agent 2's valuation for $q_2 = 1$ is $v_2(c_2) = 1 + \frac{8}{7} c_2$.[14] One may interpret c_i as a (binary) common-value component, and d_1 as a private-value component for agent 1. Essentially, the only difference from the previous sections is that agent 1 now has a continuous payoff-type space. The other specifications are for simplicity.

For the principal's prior for θ, assume that each c_i takes 0 or 1 equally likely, independently from c_{-i}. Independently from c_2, the density of d_1 given

[13] Generalizing Theorem 1 to the continuous case (even in the private-value environment as in Chung and Ely (2007)) may also be non-trivial.

[14] We omit q_1, q_2 in the arguments of v_1, v_2 for brevity.

c_1 is:

$$f(d_1|c_1=0) = \begin{cases} \frac{3}{4} & \text{if} \quad d_1 \in [0,1], \\ \frac{1}{4} & \text{if} \quad d_1 \in [1,2], \end{cases}$$
$$f(d_1|c_1=1) = \begin{cases} \frac{1}{4} & \text{if} \quad d_1 \in [0,1], \\ \frac{3}{4} & \text{if} \quad d_1 \in [1,2]. \end{cases}$$

We can show that the optimal EPIC mechanism $(q_i^*, p_i^*)_{i=1,2}$ is given as follows:

$$q_1^*(c_1,d_1,c_2) = \begin{cases} 1 & \text{if} \quad c_1c_2 + d_1 \geq \frac{3}{4}c_2 + 1, \\ 0 & \text{otherwise}, \end{cases}$$
$$p_1^*(c_1,d_1,c_2) = (\frac{3}{4}c_2+1)q_1^*(c_1,d_1,c_2),$$
$$q_2^*(c_1,d_1,c_2) = \begin{cases} 1 & \text{if} \quad c_2 = 1, \\ 0 & \text{if} \quad c_2 = 0, \end{cases}$$
$$p_2^*(c_1,d_1,c_2) = \frac{15}{7}q_2^*(c_1,d_1,c_2).$$

This mechanism can be interpreted as a posted-price mechanism, where the price for agent 2 is always $\frac{15}{7}$ (so that only high-value type of agent 2 buys), and the price for agent 1 is $\frac{3}{4}c_2+1$, varying with c_2.

Our basic idea for improvement is very similar to the finite case in the previous section. However, to explain the basic incentive issues, we first consider the following "bundling" interpretation. Imagine that the seller is selling to agent 1 a right to obtain the good when $c_2 = 0$, and another right to obtain the good when $c_2 = 1$. To buy the bundle, agent 1 pays 1 when $c_2 = 0$ and $\frac{7}{4}$ when $c_2 = 1$, as in the optimal EPIC mechanism. Similarly, to buy only when $c_2 = 1$ (but not when $c_2 = 0$), agent 1 pays $\frac{7}{4}$ when $c_2 = 1$. However, to buy only when $c_2 = 0$ (but not when $c_2 = 1$), agent 1 pays $1 - \varepsilon$ for a small $\varepsilon > 0$.

If agent 1's purchase behavior is the same as in the optimal EPIC mechanism (in particular, if agent 1 buys both when $c_2 = 0$ and $c_2 = 1$ as long as $d_1 \geq 1$ and $c_1 + d_1 \geq \frac{7}{4}$), then this new mechanism achieves a strictly higher expected revenue. However, such a behavior may not be incentive compatible. For example, if agent 1 believes that $c_2 = 0$ with probability very close to 1, then no payoff type of agent 1 would buy the bundle: even for the highest payoff-type (i.e., $(c_1, d_1) = (1, 2)$), it would be better to buy the good only when $c_2 = 0$ (with price $1 - \varepsilon$) than to buy the good for both $c_2 \in \{0, 1\}$. Such a deviation makes the expected revenue much smaller than under the optimal EPIC mechanism.

To avoid this, we introduce the following side bet: if (and only if) agent 1 buys the bundle, he can further buy a lottery that yields to him ε if $c_2 = 0$, and $-b\varepsilon$ if $c_2 = 1$ (for some $b \in (\frac{8}{7}, \frac{9}{7})$). Furthermore, if agent 1 buys this lottery, then with probability ε, the principal offers price 1 instead of $\frac{15}{7}$ to agent 2 (and the principal continues to offer price $\frac{15}{7}$ to agent 2 with the other probability $1 - \varepsilon$).

Then, as long as $d_1 \geq 1$ and $c_1 + d_1 \geq \frac{7}{4} + b\varepsilon$, agent 1 prefers to "buying the bundle and the lottery" to "buying only when $c_2 = 0$", regardless of his belief. Whether or not he actually buys the lottery depends on his belief. However, observe that regardless of the true state, the expected revenue of the principal from the lottery is always non-negative: if $c_2 = 0$, the principal pays ε to agent 1 while he earns additional revenue 1 from agent 2 with probability ε (hence, the expected revenue gain is non-negative); if $c_2 = 1$, the principal receives $b\varepsilon$ from agent 1 while he loses revenue $\frac{8}{7}$ from agent 2 with probability ε (hence, the expected revenue gain is non-negative for $b > \frac{8}{7}$). Therefore, the worst-case scenario for the principal is that agent 1 *never* buys the lottery.

On the other hand, if $d_1 > 1$ and $c_1 + d_1 \in (\frac{7}{4}, \frac{7}{4} + b\varepsilon)$, the worst-case scenario is that agent 1 buys only when $c_2 = 0$, even though he buys both for $c_2 \in \{0, 1\}$ in the optimal EPIC mechanism. This revenue loss occurs regardless of agent 1's belief, and this is one of the fundamental differences from the (generic) finite case where only the gain exists as long as the agents have full-support first-order beliefs.

Nevertheless, the overall expected revenue change is strictly positive, at least for sufficiently small ε, which is approximately:

$$
\begin{aligned}
& (1-\varepsilon)\Pr(d_1 \in (1-\varepsilon,1),\ c_1+d_1<\frac{7}{4})\Pr(c_2=0) \\
& -\varepsilon\Pr(d_1>1,\ c_1+d_1<\frac{7}{4})\Pr(c_2=0) \\
& -\Pr(d_1>1, c_1+d_1\in(\frac{7}{4},\frac{7}{4}+b\varepsilon))\Pr(c_2=1) \\
\simeq\ & \frac{3\varepsilon}{16}-\frac{3\varepsilon}{64}-\frac{7b\varepsilon}{64},
\end{aligned}
$$

which is positive if $b<\frac{9}{7}$. The first term (on the left-hand side) is because agent 1 whose payoff type satisfies $d_1 \in (1-\varepsilon,1)$ and $c_1+d_1<\frac{7}{4}$ does not buy in any state in the optimal EPIC mechanism, while he buys when $c_2=0$ in the modified mechanism. The second term is because, for agent 1 whose payoff type satisfies $d_1>1$ and $c_1+d_1<\frac{7}{4}$, the price he pays in the modified mechanism (when $c_2=0$) is smaller by ε. The third term is because agent 1 whose payoff type satisfies $d_1>1$ and $c_1+d_1\in(\frac{7}{4},\frac{7}{4}+b\varepsilon)$ buys both for $c_2\in\{0,1\}$ in the optimal EPIC mechanism, while he buys only when $c_2=0$ in the modified mechanism.[15]

This is, of course, just one example, and whether a similar approach works more generally is left to be determined. However, we believe that, as demonstrated in this example, our basic idea of modifying mechanisms carries over even to some continuous environments.

[15] There are other changes in agent 1's behavior, but their effects on the expected revenue are $o(\varepsilon)$, and hence omitted.

6 Conclusion

If the environment exhibits only cardinal interdependence (and certain regularity conditions), then there exist the maxmin and Bayesian foundations for EPIC mechanisms, in the sense of Chung and Ely (2007). If the environment exhibits ordinal interdependence, (and certain additional conditions), then such a foundation may not exist.

In interdependent-value environments, Yamashita (2015) provides an alternative solution concept (that is, *incentive compatibility in value revelation*), which is also robust to the agents' belief structure in a related sense and useful in the implementation of social choice correspondences in undominated strategies. It may be interesting to investigate similar sorts of foundation results for this alternative solution concept.

A Proof of Theorem 1

Because $\succ_i^{\theta_{-i}} = \succ_i^{\theta'_{-i}}$ for all i, θ_{-i}, and θ'_{-i}, we denote this ordering by $\succ_i$ with no superscript. Also, let $\Theta_i = \{\theta_i^1, \ldots, \theta_i^N\}$ (where $N = |\Theta_i|$) so that $\theta_i^n \prec_i \theta_i^{n+1}$ for all $n = 1, \ldots, N-1$.

Consider the simple type space $\widehat{\mathcal{T}}^f = (T_i, \widehat{\theta}_i, \widehat{\pi}_i)_{i=1}^I$ with $T_i = \Theta_i$ and the agents' beliefs defined by $\widehat{\pi}_i(\theta_i^n)[\theta_{-i}] = \big(\sum_{\theta'_{-i} \in \Theta_{-i}} G_i(\theta_i^n, \theta'_{-i})\big)^{-1} G_i(\theta_i^n, \theta_{-i})$ for all $\theta_{-i} \in \Theta_{-i}$, where $G_i(\theta_i^n, \theta_{-i}) = \sum_{k=n}^N f(\theta_i^k, \theta_{-i})$. By convention, $G_i(\theta_i^{N+1}, \theta_{-i}) = 0$.

The optimal Bayesian mechanism given this simple type space achieves:

$$
\begin{aligned}
V(f) = & \max_{(q,p):\Theta \to Q \times \mathbb{R}^I} \sum_{\theta \in \Theta} f(\theta) \sum_{i \in I} p_i(\theta) \\
& s.t. \quad \forall i \in I, \forall n, l \in \{1, \ldots, N\}, \forall \theta \in \Theta : \\
& \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}] \big(v_i(q_i(\theta_i^n, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^n, \theta_{-i}) \big) \geq 0, \qquad (BIR_i^n) \\
& \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}] \big(v_i(q_i(\theta_i^n, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^n, \theta_{-i}) \big) \\
\geq & \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}] \big(v_i(q_i(\theta_i^l, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^l, \theta_{-i}) \big). \qquad (BIC_i^{n \to l})
\end{aligned}
$$

Because the identity function $\widehat{\theta}_i$ is one-to-one, by Lemma 1, $\widehat{\mathcal{T}}^f$ can be embedded in the universal type space $\mathcal{T}^*$ through a bijection h such that $t_i^n = h_i(\theta_i^n)$. Thus, $V(f)$ provides an upper bound for the best expected revenue given the universal type space $\mathcal{T}^*$ (and the principal's belief $\mu^* \in \mathcal{M}$

such that $\mu^*(h(\widehat{\theta}^{-1}(\theta))) = f(\theta))$. Therefore, in order to show the Bayesian foundation for EPIC mechanisms given f, it suffices to show that $V(f) \leq R_f^{EP}$.

We first prove the claim by imposing the non-singularity condition on f, which assumes that $\Omega_i = (f(\theta_i^1, \cdot), \ldots, f(\theta_i^N, \cdot))^\intercal$ has rank N for each i, where $f(\theta_i^n, \cdot) = (f(\theta_i^1, \theta_{-i}))_{\theta_{-i} \in \Theta_{-i}}$ is a $(I-1)N$-dimensional vector.

Lemma 3. *In the solution of $V(f)$, $(BIC_i^{n \to n-1})$ holds with equality for all i and $n \neq 1$, and (BIR_i^n) holds with equality for all i and n.*

The lemma implies that, for all i and n:

$$\begin{aligned}
&\sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^n, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^n, \theta_{-i})\big) \\
= &\sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^{n-1}, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^{n-1}, \theta_{-i})\big) = 0,
\end{aligned}$$

or equivalently:

$$\sum_{\theta_{-i} \in \Theta_{-i}} \Big(\sum_{\theta'_{-i} \in \Theta_{-i}} G_i(\theta_i^n, \theta'_{-i})\Big)^{-1} G_i(\theta_i^n, \theta_{-i})\big(v_i(q_i(\theta_i^n, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^n, \theta_{-i})\big) = 0,$$

$$\sum_{\theta_{-i} \in \Theta_{-i}} \Big(\sum_{\theta'_{-i} \in \Theta_{-i}} G_i(\theta_i^n, \theta'_{-i})\Big)^{-1} G_i(\theta_i^n, \theta_{-i})\big(v_i(q_i(\theta_i^{n-1}, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^{n-1}, \theta_{-i})\big) = 0.$$

This implies:

$$\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i})v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) = \sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i})p_i(\theta_i^n,\theta_{-i}),$$
$$\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i})v_i(q_i(\theta_i^{n-1},\theta_{-i}),\theta_i^n,\theta_{-i}) = \sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i})p_i(\theta_i^{n-1},\theta_{-i})\big),$$

and therefore, the objective becomes:

$$\begin{aligned}
&\sum_{i\in I}\sum_{n=1}^{N_i}\sum_{\theta_{-i}\in\Theta_{-i}} f(\theta_i^n,\theta_{-i})p_i(\theta_i^n,\theta_{-i})\\
=&\sum_{i\in I}\sum_{n=1}^{N_i}\sum_{\theta_{-i}\in\Theta_{-i}} \big(G_i(\theta_i^n,\theta_{-i})-G_i(\theta_i^{n+1},\theta_{-i})\big)p_i(\theta_i^n,\theta_{-i})\\
=&\sum_{i\in I}\sum_{n=1}^{N_i}\left(\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i})p_i(\theta_i^n,\theta_{-i}) - \sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^{n+1},\theta_{-i})p_i(\theta_i^n,\theta_{-i})\right)\\
=&\sum_{i\in I}\sum_{n=1}^{N_i}\sum_{\theta_{-i}\in\Theta_{-i}} \big(G_i(\theta_i^n,\theta_{-i})v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) - G_i(\theta_i^{n+1},\theta_{-i})v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^{n+1},\theta_{-i})\big)\\
=&\sum_{i\in I}\sum_{\theta\in\Theta} f(\theta)\gamma_i(q_i,\theta).
\end{aligned}$$

Therefore, under Assumption 1, we have $V(f) = R_f^{EP}$.

Proof. (of the lemma)

We first show that each upward incentive constraint, $(BIC_i^{n\to l})$ with $n < l$, can be ignored without loss. Let $\Pi_i = \big(\widehat{\pi}_i(\theta_i^1),\ldots,\widehat{\pi}_i(\theta_i^N)\big)^\intercal$ denote the matrix of agent i's beliefs, where each $\widehat{\pi}_i(\theta_i^n)$ is a $(I-1)N$-dimensional vector.

Then:

$$\Pi_i = \begin{pmatrix} \kappa_i^1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_i^N \end{pmatrix}_{N\times N} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{N\times N} \Omega,$$

where $\kappa_i^n = \left(\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n, \theta_{-i})\right)^{-1}$, and hence Π_i has a rank N. Thus, there exists $\lambda \in \mathbb{R}^{(I-1)N}$ such that:

$$\Pi_i \lambda = (1, \ldots, 1, \underbrace{0}_{l\text{-th element}}, \ldots, 0)^\intercal.$$

If we add λ to $p_i(\theta_i^l, \cdot)$, each $BIC_i^{n\to l}$ with $n < l$ is relaxed, while no other (BIC) and (BIR) constraints are affected. Moreover, from $\widehat{\pi}_i(\theta_i^l) \cdot \lambda = 0$ and $\widehat{\pi}_i(\theta_i^{l+1}) \cdot \lambda = 0$, we obtain:

$$\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^l, \theta_{-i})\lambda(\theta_{-i}) = 0, \quad \sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^{l+1}, \theta_{-i})\lambda(\theta_{-i}) = 0,$$

which implies that $\sum_{\theta_{-i}\in\Theta_{-i}} f(\theta_i^l, \theta_{-i})\lambda(\theta_{-i}) = 0$, that is, the principal's expected revenue is also unaffected.

Next, we show that for any mechanism (q, p) satisfying the remaining constraints, there exists a mechanism (q', p') which satisfies not only the remaining constraints, but also (BIR_i^n) for $n = 1, \ldots, N$ and $(BIC_i^{n\to n-1})$ for $n = 2, \ldots, N$ with equality, and raises at least as high expected revenue as (q, p).

Given any such mechanism (q, p), if $(BIC_i^{n\to n-1})$ is satisfied with strict inequality for some i and n, then let $\beta_i^{n\to n-1}$ be the amount of the slackness of this constraint $(BIC_i^{n\to n-1})$. Let Π_i' be the matrix generated by substituting the n-th row of Π_i with the vector $f(\theta^{n-1}, \cdot)$. That is:

$$\Pi_i' = \begin{pmatrix} \kappa_i^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \kappa_i^{n-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \kappa_i^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \kappa_i^N \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \Omega,$$

and hence, Π_i' has a rank N. Thus, there exists $\lambda \in \mathbb{R}^{(I-1)N}$ such that:

$$\Pi_i'\lambda = (0, \ldots, 0, \underbrace{1}_{n\text{-th element}}, 0, \ldots, 0)^\intercal.$$

Because $\widehat{\pi}_i(\theta_i^{n-1}) \cdot \lambda = 0$ and $f(\theta^{n-1}, \cdot) \cdot \lambda = 1$, we have:

$$\widehat{\pi}_i(\theta_i^n) \cdot \lambda = \frac{\kappa_i^n}{\kappa_i^{n-1}} \widehat{\pi}_i(\theta_i^{n-1}) \cdot \lambda - \kappa_i^n f(\theta^{n-1}, \cdot) \cdot \lambda < 0,$$

and thus, $\varepsilon = -\beta_i^{n\to n-1}/(\widehat{\pi}_i(\theta_i^n) \cdot \lambda) > 0$. If we add $\varepsilon\lambda$ to $p_i(\theta_i^{n-1}, \cdot)$, then all the constraints for types θ_i^l with $l \neq n$ are unaffected because $\widehat{\pi}_i(\theta_i^l) \cdot \lambda = 0$ for all $l \neq n$, and for type θ_i^n only constraint $(BIC_i^{n\to n-1})$ is changed, which holds with equality under the new payment rule. Because $f(\theta^{n-1}, \cdot) \cdot (\varepsilon\lambda) = \varepsilon > 0$, the expected revenue increases under the new payment rule.

Similarly, if (BIR_i^n) is satisfied with strict inequality for some i and n, then let β_i^n be the amount of the slackness of this constraint (BIR_i^n). Because Π_i has a rank N, there exists $\lambda \in \mathbb{R}^{(I-1)N}$ such that:

$$\Pi_i \lambda = (\beta_i^1, \ldots, \beta_i^N)^\intercal \geq 0.$$

Adding λ to each $p_i(\theta_i^n, \cdot)$ does not affect any (BIC) constraint, while all the participation constraints are satisfied with equality in the new mechanism. The change in the total expected revenue is:

$$\begin{aligned}
\sum_{n=1}^{N} \sum_{\theta_{-i} \in \Theta_{-i}} f(\theta_i^n, \theta_{-i}) \lambda(\theta_{-i}) &= \sum_{\theta_{-i} \in \Theta_{-i}} \lambda(\theta_{-i}) \sum_{n=1}^{N} f(\theta_i^n, \theta_{-i}) \\
&= \sum_{\theta_{-i} \in \Theta_{-i}} \lambda(\theta_{-i}) G_i(\theta_i^1, \theta_{-i}) \\
&= \frac{1}{\kappa_i^1} \sum_{\theta_{-i} \in \Theta_{-i}} \lambda(\theta_{-i}) \widehat{\pi}_i(\theta_i^1)[\theta_{-i}] \\
&= \beta_i^1,
\end{aligned}$$

which is non-negative. □

Next, we consider the case where f is singular, that is, for some i, Ω_i has a rank strictly less than N. Consider a sequence of distributions over Θ, $\{f^r\}_{r=1}^{\infty}$, such that each f^r is full-support and $f_r \to f$ (in the standard

Euclidean distance).[16] By Assumption 1, without loss of generality, we assume that the monotonicity constraints (M) are not binding in the problem of $R^{EP}_{f_r}$.

We prove the following continuity lemma.

Lemma 4. For each $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{N}$ such that, for any $r \geq r_\varepsilon$, $R^{EP}_{f_r} \leq R^{EP}_f + \varepsilon$ and $V(f_r) \geq V(f) - \varepsilon$.

Proof. (of the lemma)

For the first inequality, recall that $R^{EP}_f = \sum_i \sum_\theta \max\{\gamma_i(\theta), 0\} f(\theta)$, which is obviously continuous in f.

For the second inequality, let (q, p) be a solution to the problem of $V(f)$.

In the following, for each r, we construct another mechanism (q, p^r) (note that we keep the same q), so that it satisfies all the constraints of the problem of $V(f_r)$, namely:

$$\sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}^r_i(\theta^n_i)[\theta_{-i}] \big(v_i(q_i(\theta^n_i, \theta_{-i}), \theta^n_i, \theta_{-i}) - p^r_i(\theta^n_i, \theta_{-i}) \big) \geq 0, \qquad (BIR^n_i(r))$$

$$\begin{aligned} & \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}^r_i(\theta^n_i)[\theta_{-i}] \big(v_i(q_i(\theta^n_i, \theta_{-i}), \theta^n_i, \theta_{-i}) - p^r_i(\theta^n_i, \theta_{-i}) \big) \\ \geq & \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}^r_i(\theta^n_i)[\theta_{-i}] \big(v_i(q_i(\theta^l_i, \theta_{-i}), \theta^n_i, \theta_{-i}) - p^r_i(\theta^l_i, \theta_{-i}) \big). \qquad (BIC^{n \to l}_i(r)) \end{aligned}$$

[16] We can always find such a sequence because the set of all non-singular distributions is a dense subset of the set of all distributions over Θ.

Let:

$$S_i^n(r) = \max\left\{0, \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(p_i(\theta_i^n,\theta_{-i}) - v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i})\big)\right\},$$

denote the size of violation of $(BIR_i^n(r))$ by p. If we consider a modified payment rule p' so that $p_i'(\theta_i^n,\cdot) = p_i(\theta_i^n,\cdot) - S_i^n(r)\mathbf{1}$, then this new payment rule satisfies the participation constraints, but may not satisfy the incentive compatibility constraints. Thus, let:

$$\begin{aligned} L_i^{n\to l}(r) = \max\Big\{ \quad 0, & \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^l,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i'(\theta_i^l,\theta_{-i})\big) \\ & - \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i'(\theta_i^n,\theta_{-i})\big)\Big\}, \end{aligned}$$

denote the size of violation of $(BIC_i^{n\to l}(r))$ by p'. As in the first part of the proof, the matrix of agent i's belief in the simple type space $\widehat{\mathcal{T}}^{f_r}$, $\Pi_i^r = \big(\widehat{\pi}_i^r(\theta_i^1),\ldots,\widehat{\pi}_i^r(\theta_i^N)\big)^\intercal$, has a rank N, and hence, there exists $\lambda_i^1(r),\ldots,\lambda_i^N(r) \in \mathbb{R}^{(I-1)N}$ such that:

$$\Pi_i^r\left(\lambda_i^1(r),\ldots,\lambda_i^N(r)\right) = \left(L_i^{n\to l}(r)\right)_{N\times N},$$

which we denote by $\mathbf{L}_r$. Or equivalently:

$$\mathbf{L}_r = \underbrace{\begin{pmatrix} \kappa_i^1(r) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_i^N(r) \end{pmatrix}_{N\times N}}_{\triangleq \mathbf{K}_r} \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{N\times N}}_{\triangleq \mathbf{A}} \Omega_r\left(\lambda_i^1(r), \ldots, \lambda_i^N(r)\right).$$

Define $p_i^r(\theta_i^n, \cdot) = p_i(\theta_i^n, \cdot) - S_i^n(r)\mathbf{1} + \lambda_i^n(r)$. Then, together with q, it satisfies all the constraints of the problem of $V(f_r)$.

We complete the proof by showing that $\sum_\theta \sum_i (p_i^r(\theta) - p_i(\theta)) f_r(\theta) \to 0$ as $r \to \infty$. Because it is obvious that $S_i^n(r) \to 0$, it suffices to show that:

$$\sum_{n=1}^{N} f_r(\theta_i^n, \cdot) \cdot \lambda_i^n(r) \to 0.$$

Indeed:

$$\sum_{n=1}^{N} f_r(\theta_i^n, \cdot) \cdot \lambda_i^n(r) = tr\big(\mathbf{A}^{-1}\mathbf{K}_r^{-1}\mathbf{L}_r\big) \to 0,$$

as $r \to \infty$, because $\mathbf{L}_r \to 0$. □

Finally, contrarily to the original claim, suppose that $V(f) > R_f^{EP}$, and let $\varepsilon \in (0, \frac{V(f)-R_f^{EP}}{2})$. Then, there exists r_ε such that:

$$V(f_r) - R_{f_r}^{EP} \geq V(f) - R_f^{EP} - 2\varepsilon > 0,$$

which contradicts the first part of this proof.

B Proof of Theorem 2

The previous theorem already states that EPIC mechanisms have the foundation if we do not have ordinal interdependence. Therefore, we only prove its converse in this proof.

We first observe an implication of ordinal interdependence under Assumptions 3.

Lemma 5. Under Assumptions 3, ordinal interdependence implies at least one of the following: (i) there exists i, q_i, q_i', θ_{-i} and θ'_{-i} such that $\theta_i^*(q_i, \theta_{-i}) \notin \Theta_i^*(q_i', \theta'_{-i})$ and $\theta_i^*(q_i', \theta'_{-i}) \notin \Theta_i^*(q_i, \theta_{-i})$; or (ii) there exists i, θ_i, q_i, q_i', θ_{-i} and θ'_{-i} such that $\theta_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i', \theta'_{-i})$ and $\theta_i^*(q_i, \theta_{-i}) \in \Theta_i^*(q_i', \theta'_{-i})$.

Proof. By definition of ordinal interdependence, there exists i, $\tilde{\theta}_{-i}$ and $\tilde{\theta}'_{-i}$ such that $\prec_i^{\tilde{\theta}_{-i}} \neq \prec_i^{\tilde{\theta}'_{-i}}$. Single-crossing condition implies that, for any $q_i > 0$, any θ_{-i}, and any distinct pair $\theta_i \neq \theta_i'$, we have $v_i(q_i, \theta_i', \theta_{-i}) < v_i(q_i, \theta_i, \theta_{-i})$ if and only if $\theta_i' \prec_i^{\theta_{-i}} \theta_i$. Thus, there exists θ_i and θ_i' such that $v_i(q_i, \theta_i', \tilde{\theta}_{-i}) < v_i(q_i, \theta_i, \tilde{\theta}_{-i})$ and $v_i(q_i, \theta_i', \tilde{\theta}'_{-i}) > v_i(q_i, \theta_i, \tilde{\theta}'_{-i})$ hold for any $q_i > 0$. Fixed any $q_i > 0$, by Assumption 3, there exists θ_{-i} and θ'_{-i} such that

$$\begin{cases} \theta_i \in \Theta_i^*(q_i, \theta_{-i}), & \theta_i' \notin \Theta_i^*(q_i, \theta_{-i}); \\ \theta_i \notin \Theta_i^*(q_i, \theta'_{-i}), & \theta_i' \in \Theta_i^*(q_i, \theta'_{-i}). \end{cases}$$

Next, we show that if (i) is violated, then we must have (ii). Without loss of generality, we can assume that $\theta_i^*(q_i, \theta_{-i}) \in \Theta_i^*(q_i, \theta'_{-i})$. Since $\theta_i \in \Theta_i^*(q_i, \theta_{-i})$

and $\theta_i \notin \Theta_i^*(q_i, \theta'_{-i})$, we have $\theta_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i, \theta'_{-i})$, which means (ii) holds. Therefore, we must have either (i) or (ii) is satisfied.

□

We show that, for each of these cases, there exists a mechanism that yields a strictly higher expected revenue than the optimal EPIC mechanism.

Case (i): $\theta_1^*(q_1, \theta_{-1}) \notin \Theta_1^*(q'_1, \theta'_{-1})$ and $\theta_1^*(q'_1, \theta'_{-1}) \notin \Theta_1^*(q_1, \theta_{-1})$.

Consider a new mechanism (M, q^*, p^*) such that $M_1 = \Theta_1 \times [0, 1]$, $M_j = \Theta_j$ for $j \neq 1$, and for each $((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) \in M$,

$$
\begin{aligned}
q^*((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) &= q^{EP}(\tilde{\theta}), \\
p_j^*((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) &= p_j^{EP}(\tilde{\theta}), \ \forall j \neq 1,
\end{aligned}
$$

and for p_1^*, we set $p_1^*((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) = p_1^{EP}(\tilde{\theta})$ unless $\tilde{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \cap \Theta_1^*(q'_1, \theta'_{-1})$ and $\tilde{\theta}_{-1} \in \{\theta_{-1}, \theta'_{-1}\}$; and for each $\tilde{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \cap \Theta_1^*(q'_1, \theta'_{-1})$, we set

$$
\begin{aligned}
p_1^*((\tilde{\theta}_1, x), \theta_{-1}) &= p_1^{EP}(\tilde{\theta}_1, \theta_{-1}) + \eta(1 - x), \\
p_1^*((\tilde{\theta}_1, x), \theta'_{-1}) &= p_1^{EP}(\tilde{\theta}_1, \theta'_{-1}) + \eta\psi(x),
\end{aligned}
$$

where $\psi(x) = 1 - \sqrt{1 - x^2}$.

Intuitively, $x \in [0, 1]$ is related to agent 1's first-order belief over θ_{-i} and θ'_{-i} (more precisely, their likelihood ratio). Indeed, if agent 1 reports his payoff type θ_1 truthfully, his optimal choice of x is given by $x^*(\beta, \beta') = \sqrt{\frac{(\beta/\beta')^2}{1+(\beta/\beta')^2}}$, where β is 1's first-order belief for θ_{-1} and β' is 1's first-order belief for θ'_{-1}. Note that, given any $\mu \in \mathring{\mathcal{M}}$, agent 1 chooses $x \in (0, 1)$ with probability one.

It is then obvious that, if the agents report their payoff types truthfully (and agent 1 chooses x optimally), then this new mechanism yields a strictly higher expected revenue than the optimal EPIC mechanism.

For any agent $j \neq 1$, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR.

We show the incentive compatibility of agent 1 with $\tilde{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \cap \Theta_1^*(q'_1, \theta'_{-1})$ (for the other payoff types, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR). First, obviously, any deviation to $\hat{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \cap \Theta_1^*(q'_1, \theta'_{-1})$ is not profitable. Second, any deviation to $\hat{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \setminus \Theta_1^*(q'_1, \theta'_{-1})$ is not profitable either, because, letting β and β' be his first-order beliefs for θ_{-1} and θ'_{-1} respectively, the expected gain by deviation is at most

$$\beta[\eta(1 - x^*(\beta, \beta'))] + \beta'[-\eta + \eta\psi(x^*(\beta, \beta'))] \leq 0.$$

Similarly, we can show that any deviation to $\hat{\theta}_1 \in \Theta_1^*(q'_1, \theta'_{-1}) \setminus \Theta_1^*(q_1, \theta_{-1})$ and $\hat{\theta}_1 \notin \Theta_1^*(q'_1, \theta'_{-1}) \cup \Theta_1^*(q_1, \theta_{-1})$ is not profitable either.

Case (ii): $\Theta_1^*(q_1, \theta_{-1}) \setminus \Theta_1^*(q'_1, \theta'_{-1}) \neq \emptyset$ and $\theta_1^*(q_1, \theta_{-1}) \in \Theta_1^*(q'_1, \theta'_{-1})$.

Consider a new mechanism (M, q^*, p^*) such that $M_1 = \Theta_1 \times [0,1]$, $M_j = \Theta_j$ for $j \neq 1$, and for each $((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) \in M$,

$$\begin{aligned} q^*((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) &= q^{EP}(\tilde{\theta}), \\ p_j^*((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) &= p_j^{EP}(\tilde{\theta}), \ \forall j \neq 1, \end{aligned}$$

and for p_1^*, we set $p_1^*((\tilde{\theta}_1, x), \tilde{\theta}_{-1}) = p_1^{EP}(\tilde{\theta})$ unless $\tilde{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \backslash \Theta_1^*(q_1', \theta_{-1}')$ and $\tilde{\theta}_{-1} \in \{\theta_{-i}, \theta_{-i}'\}$; and for each $\tilde{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \setminus \Theta_1^*(q_1', \theta_{-1}')$, we set

$$\begin{aligned} p_1^*((\tilde{\theta}_1, x), \theta_{-1}) &= p_1^{EP}(\tilde{\theta}_1, \theta_{-1}) + \eta(1 - x), \\ p_1^*((\tilde{\theta}_1, x), \theta_{-1}') &= p_1^{EP}(\tilde{\theta}_1, \theta_{-1}') + \eta\psi(x), \end{aligned}$$

where $\psi(x) = 1 - \sqrt{1 - x^2}$.

Again, $x \in [0,1]$ is related to agent 1's first-order belief over θ_{-i} and θ_{-i}'. Indeed, if agent 1 reports his payoff type θ_1 truthfully, his optimal choice of x is given by $x^*(\beta, \beta') = \sqrt{\frac{(\beta/\beta')^2}{1+(\beta/\beta')^2}}$, where β is 1's first-order belief for θ_{-1} and β' is 1's first-order belief for θ_{-1}'. Note that, given any $\mu \in \mathring{\mathcal{M}}$, agent 1 chooses $x \in (0,1)$ with probability one.

It is obvious that, if the agents report their payoff types truthfully (and agent 1 chooses x optimally), then this new mechanism yields a strictly higher expected revenue than the optimal EPIC mechanism.

For any agent $j \neq 1$, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR.

We show the incentive compatibility of agent 1 with $\tilde{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \setminus \Theta_1^*(q_1', \theta_{-1}')$ (for the other payoff types, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR). First, obviously, any deviation to $\hat{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \setminus \Theta_1^*(q_1', \theta_{-1}')$ is not profitable. Second, any deviation to $\hat{\theta}_1 \in \Theta_1^*(q_1, \theta_{-1}) \cap \Theta_1^*(q_1', \theta_{-1}')$ is not profitable either, because, letting β and β' be his first-order beliefs for θ_{-1} and θ_{-1}' respectively, the expected gain by deviation is at most

$$\beta[\eta(1 - x^*(\beta, \beta'))] + \beta'[-\eta + \eta\psi(x^*(\beta, \beta'))] \leq 0.$$

Similarly, we can show that any deviation to $\hat{\theta}_1 \in \Theta_1^*(q_1', \theta_{-1}') \setminus \Theta_1^*(q_1, \theta_{-1})$ and $\hat{\theta}_1 \notin \Theta_1^*(q_1', \theta_{-1}') \cup \Theta_1^*(q_1, \theta_{-1})$ is not profitable either.

In conclusion, EPIC mechanisms do not have the strong foundation.

C Proof of Proposition 1

Assume that $(i, \theta_i, \theta_j, q_j, \tilde{\theta}_{-ij})$ satisfies the definition of strong improvability. We use the same mechanism as above, except that the allocation for agent i changes in case he reports θ_i and $x = 1$. Recall that, given his truthfully reporting θ_i, agent 1's optimal choice of x is $\sqrt{\frac{(\beta/\beta')^2}{1+(\beta/\beta')^2}}$ where β, β' are his first-order beliefs for θ_{-i}, θ_{-i}', respectively, with $\theta_{-i} = (\theta_j, \tilde{\theta}_{-ij})$ and $\theta_{-i}' = (\theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij}), \tilde{\theta}_{-ij})$; $x = 1$ means that he predicts that j does not have a threshold type for q_j given $\tilde{\theta}_{-ij}$. The allocations from agents i and j are then modified as follows (and all the other parts of the mechanism are the same as before):

$$\begin{aligned}
q_j^{**}((\theta_i,1),\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij}),\tilde{\theta}_{-ij}) &= q_j^*((\theta_i,1),\hat{\theta}_j,\tilde{\theta}_{-ij}), \\
p_j^{**}((\theta_i,1),\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij}),\tilde{\theta}_{-ij}) &= p_j^*((\theta_i,1),\hat{\theta}_j,\tilde{\theta}_{-ij}), \\
p_j^{**}((\theta_i,1),\theta_j,\tilde{\theta}_{-ij}) &= p_j^*(\theta_i,\theta_j,\tilde{\theta}_{-ij})+\eta, \qquad \forall \theta_j \succ_j^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij}) \\
p_i^{**}((\theta_i,1),\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij}),\tilde{\theta}_{-ij}) &= M,
\end{aligned}$$

where $\hat{\theta}_j$ is j's payoff type that is just below $\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$ with respect to $\prec_j^{\theta_i,\tilde{\theta}_{-ij}}$, and $M>0$ is sufficiently large.

Observe that the modified mechanism satisfies all the constraints. First, except for agents i and j, the allocations are the same as in the previous mechanism. For agent i, large fine M is irrelevant unless he assigns zero probability for θ_{-i} (because $x=1$ is not optimal for him); on the other hand, if he assigns zero probability for θ_{-i}, then this large fine is payoff-irrelevant for him. Finally, for agent j, we only need to check his incentive if i reports $(\theta_i,1)$ and $-ij$ report $\tilde{\theta}_{ij}$: in such a case, j with payoff type $\tilde{\theta}_j \precsim_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$ has no incentive of misreporting, because their on-path payoffs would be the same as in the original mechanism, while the other types' payments are higher than in the original mechanism. For $\tilde{\theta}_j \succ_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$, his payoff by deviation is at most

$$
\begin{aligned}
& v_j(q_j^*((\theta_i, 1), \hat{\theta}_j, \tilde{\theta}_{-ij}), \theta_i, \tilde{\theta}_j, \tilde{\theta}_{-ij}) - p_j^*((\theta_i, 1), \hat{\theta}_j, \tilde{\theta}_{-ij}) \\
\leq \; & v_j(q_j^*((\theta_i, 1), \tilde{\theta}_j, \tilde{\theta}_{-ij}), \theta_i, \tilde{\theta}_j, \tilde{\theta}_{-ij}) - p_j^*((\theta_i, 1), \tilde{\theta}_j, \tilde{\theta}_{-ij}) - \eta,
\end{aligned}
$$

but the right-hand side is precisely his on-path payoff. The individual rationality constraints can be checked similarly.

Finally, we show that this modified mechanism achieves a strictly higher expected revenue than the original mechanism. First, observe that it does not yield a lower payoff given any payoff-type profile. It is obvious except when a payoff type $(\theta_i, \theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij}), \tilde{\theta}_{-ij})$ and agent i chooses $x = 1$; if this is the realized payoff-type profile, and agent i reports $x = 1$, agent i pays a large fine M. Therefore, the principal would be better off by setting M large enough.

Moreover, consider a payoff-type profile $(\theta_i, \tilde{\theta}_j, \tilde{\theta}_{-ij})$ such that $\tilde{\theta}_j \succ_i^{\theta_i, \tilde{\theta}_{-ij}} \theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij})$. If agent i chooses $x < 1$ (at least with a positive probability), then i pays $\eta(1 - x)(> 0)$ more than in the original mechanism, and hence, strict improvement is achieved. If agent i chooses $x = 1$ (with probability one), then the principal increases j's payment by η as explained above, and thus, again strict improvement is achieved.

D Proof of Theorem 3

It suffices to show that ordinal interdependence implies strong improvability.

By ordinal interdependence, there exist $i, \theta_{-i}, \theta'_{-i}$ such that $\prec_i^{\theta_{-i}} \neq \prec_i^{\theta'_{-i}}$. We first observe the following lemma.

Lemma 6. Ordinal interdependence implies that there exist $j \neq i$, θ_j, θ'_j, and $\tilde{\theta}_{-ij}$ such that $\prec_i^{\theta_j, \tilde{\theta}_{-ij}} \neq \prec_i^{\theta'_j, \tilde{\theta}_{-ij}}$.

Proof. Let $i = 1$ without loss of generality, and for each $n = 1, \ldots, I$, let $\theta_{-1}^n = ((\theta'_j)_{j=2}^n, (\theta_j)_{j=n+1}^I)$. Note that $\theta_{-1}^1 = \theta_{-1}$ and $\theta_{-1}^I = \theta'_{-1}$.

If $\prec_1^{\theta_{-1}^{n-1}} = \prec_1^{\theta_{-1}^n}$ for all $n = 2, \ldots, I$, then we have $\theta_{-1}^1 = \theta_{-1}^I$, contradicting that $\prec_1^{\theta_{-1}} \neq \prec_1^{\theta'_{-1}}$. Therefore, there exists $n \in \{2, \ldots, I\}$ such that $\prec_1^{\theta_{-1}^{n-1}} \neq \prec_1^{\theta_{-1}^n}$. We complete the proof of the lemma by setting $j = n$ and $\tilde{\theta}_{-1j} = ((\theta'_k)_{k=2}^{n-1}, (\theta_k)_{k=n+1}^I)$. □

By the lemma, there exists θ_i, θ'_i such that $\theta_i \succ_i^{(\theta_j, \tilde{\theta}_{-ij})} \theta'_i$ and $\theta'_i \succ_i^{(\theta'_j, \tilde{\theta}_{-ij})} \theta_i$. Letting $q_i = q_i^{EP}(\theta'_i, \theta_j, \tilde{\theta}_{-ij})$ and $q'_i = q_i^{EP}(\theta'_i, \theta'_j, \tilde{\theta}_{-ij})$, by Assumption 4, we have $\theta'_i = \theta_i^*(q'_i, \theta'_j, \tilde{\theta}_{-ij}) = \theta_i^*(q_i, \theta_j, \tilde{\theta}_{-ij})$. It follows that $\theta_i \in \Theta_i^*(q_i, \theta_i, \tilde{\theta}_{-ij}) \setminus \Theta_i^*(q'_i, \theta'_j, \tilde{\theta}_{-ij})$ and $\theta_i^*(q_i, \theta_j, \tilde{\theta}_{-ij}) \in \Theta_i^*(q'_i, \theta'_j, \tilde{\theta}_{-ij})$. Then, revenue from agent i is improvable with respect to $\big(\theta_i, (\theta_j, \tilde{\theta}_{-ij}), (\theta'_j, \tilde{\theta}_{-ij})\big)$.

Without loss of generality, we assume that $\theta'_j \prec_j^{\theta_i, \tilde{\theta}_{-ij}} \theta_j$. Letting $q_j = q_j^{EP}(\theta'_j, \theta_i, \tilde{\theta}_{-ij})$, by Assumption 4, we have $\theta'_j = \theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij})$ and $\theta_j \in \Theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij}) \setminus \{\theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij})\}$. Thus, revenue from i is improvable with respect to $\big(\theta_i, (\theta_j, \tilde{\theta}_{-ij}), (\theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij}), \tilde{\theta}_{-ij})\big)$, where $\theta_j \in \Theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij})$, which establishes the strong improvability.

References

Bergemann, D., and S. Morris (2005): "Robust mechanism design," *Econometrica*, 73(6), 1771–1813.

——— (2009b): "Robust implementation in direct mechanisms," *The Review of Economic Studies*, 76(4), 1175–1204.

Börgers, T. (2013): "(No) Foundations of Dominant-Strategy Mechanisms: A Comment on Chung and Ely (2007)," mimeo, University of Michigan.

Brandenburger, A., and E. Dekel (1993): "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, 59, 189–198.

Chen, Y.-C., and J. Li (2016): "Revisiting the Foundations of Dominant-Strategy Mechanisms," mimeo, National University of Singapore.

Chung, K.-S., and J. C. Ely (2007): "Foundations of dominant-strategy mechanisms," *The Review of Economic Studies*, 74(2), 447–476.

Dasgupta, P., and E. Maskin (2000): "Efficient auctions," *Quarterly Journal of Economics*, 115(2), 341–388.

Goldberg, A. V., J. D. Hartline, A. R. Karlin, M. Saks, and A. Wright (2006): "Competitive auctions," *Games and Economic Behavior*, 55(2), 242–269.

Jehiel, P., M. Meyer-ter Vehn, B. Moldovanu, and W. R. Zame (2006): "The limits of ex post implementation," *Econometrica*, 74(3), 585–610.

Mertens, J.-F., and S. Zamir (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14, 1–29.

Mussa, M., and S. Rosen (1978): "Monopoly and Product Quality," *Journal of Economic Theory*, 18(2), 301–317.

Segal, I. (2003): "Optimal pricing mechanisms with unknown demand," *The American economic review*, 93(3), 509–529.

Sims, C. A. (2003): "Implications of rational inattention," *Journal of Monetary Economics*, 50(3), 665–690.

Wilson, R. (1985): "Incentive efficiency of double auctions," *Econometrica: Journal of the Econometric Society*, pp. 1101–1115.

Yamashita, T. (2015): "Implementation in Weakly Undominated Strategies: Optimality of Second-Price Auction and Posted-Price Mechanism," *Review of Economic Studies*, 82(3), 1223–1246.