Exploring the Effects on the Electoral College of National and Regional Popular Vote Interstate Compact: An Electoral Engineering Perspective*

Olivier de Mouzon† Thibault Laurent‡ Michel Le Breton§
Dominique Lepelley¶

May 2018

Abstract

The main purpose of this paper is to explore the consequences of the formation of either a Regional Popular Vote Interstate compact or a National Popular Vote Interstate compact on the functioning of the Electoral College. The two versions of interstate Compact which are considered here differ in only one respect: in one case the interstate compact allocates its electoral votes to the regional popular winner while in the other case it allocates these votes to the national popular winner. They both differ from the ongoing National Popular Vote Interstate Compact as it is assumed that the agreement is effective as soon as the members sign it. The decisiveness and welfare analysis are conducted for a simplified symmetric theoretical version of the Electoral College where the malapportionment problems are absent. The three most popular probabilistic models

*The full source codes of our computer program *SimuElect*, dedicated to simulate the many voting scenarios studied in this paper, are available at http://www.thibault.laurent.free.fr/code/electoral. The authors thanks Arnaud Dellis for comments at an early stage of this research. They express their deepest gratitude to Nick Miller who has returned, as a non anonymous referee, a long and insightful report. We were lucky to benefit from his constructive criticisms concerning both the content and the exposition of our manuscript but he should not be held responsible for the imperfections, limitations and mistakes that remain. Michel Le Breton would like to express his immense respect and gratitude to Ken Arrow to whom this special issue is dedicated. He has always considered his insightful and deep work as a model to which research in social sciences should tend. Ken Arrow was interested in many topics (not to say all) and majority voting (the topic of the current paper) was among them. He is pleased to report that among the many occasions in which he had the great chance and opportunity to interact/talk with the master, he remembers, in particular, the meeting of the American Economic Association (AEA) in Boston, January 1994. He was invited by Amartya Sen to present a paper on Condorcet and majority voting in a session dedicated to social choice. Ken Arrow was discussant. Needless to say that his comments were, as always, generous but incisive. This intimidating but exceptional experience will remain a great moment of his intellectual life. He is pleased to pay his tribute to this giant through this new coauthored paper on majority voting.

†Toulouse School of Economics, INRA, University of Toulouse Capitole, Toulouse, France.

‡Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France.

§Institut Universitaire de France and Toulouse School of Economics, University of Toulouse Capitole, France.

¶CEMOI, Faculté de Droit et de Sciences Économiques et Politiques, Université de La Réunion, France.

are considered and the study is conducted either from the self-interest perspective of the initiators of the interstate compact or from a general interest perspective. The analysis combines analytical arguments and simulations.

Classification JEL : D71, D72.

Key Words : Electoral College, Voting Power.

1 Introduction

The President of the United States is elected, not by a direct national popular vote, but by an indirect Electoral College system[1] in which (in almost universal practice since the 1830s) separate state popular votes are aggregated by adding up state electoral votes awarded, on a winner-take-all basis, to the plurality winner in each state. Each state has electoral votes equal in number to its total representation in Congress and since 1964 the District of Columbia has three electoral votes. Therefore the U.S. Electoral College is a two-tier electoral system: individual voters cast votes in the first tier to choose between rival slates of 'Presidential electors' pledged to one or other Presidential candidate, and the winning elector slates then cast blocs of electoral votes for the candidate to whom they are pledged in the second tier. At the present time, there are 538 electoral votes, so 270 are required for election and a 269-269 electoral vote tie is possible. As is well-known, the Electoral College has produced a 'wrong winner' in the 2000 presidential election and in the recent 2016 election[2], and it has done so three times before[3].

What is the public support for the Electoral College? In December 2016 i.e. weeks after the election of Donald Trump[4], a poll from Gallup shows that Americans' support for keeping the Electoral College system for electing presidents has increased sharply. The country is now sharply divided on the issue: 47% of Americans say they want to keep the Electoral College, while 49% say they want to amend the Constitution to allow for a popular vote for president.

[1]Miller (2012b) contains an insightful presentation of the Electoral College that he qualifies as "a terrific boon for political science (and public choice) research (and teaching)".

[2]Trump is the fifth person in U.S. history to become president despite losing the nationwide popular vote (Clinton received about 2.9 million more votes nationwide, a margin of 2.1%).

[3]As pointed out by Neubauer et al. (2012): " (...) for the 56 presidential elections that have taken place since 1780, at least four unpopular elections—nearly 7%—have occurred. Further, the 1960 presidential election has been a matter of discussion among scholars. Some (for instance Edwards (2011)) argue that a fair accounting of the popular vote cast in Alabama would make Richard. Nixon -not John. F. Kennedy- the winner of the nationwide popular vote. If we include the 1960 election, fully 8.6% of all American presidential elections have been unpopular". But these authors also note that "estimates based on the historical record do not necessarily provide good predictions of the likelihood of an unpopular result in a future election because the sample size involved in these estimates is small, and because the political structure has changed dramatically many times in United States history. A more effective way to estimate the likelihood of a future unpopular election is to use the Monte Carlo method: simulate a large number of elections using a suitable randomization approach and then compute the percentage that result in unpopularly elected presidents, using as input data an appropriate set of recent presidential elections". They have used the principal components approach to generate 20,000 trials and found a 4.9% frequency of unpopular elections. Some other authors have reached similar conclusions. For instance, as a consequence of estimating a probabilistic-voting model of electoral competition, Strömberg (2008) obtains that the probability of an election inversion to be about 4% (see also chapter 13 in Morton (2006)). Ball and Leuthold (1988) as well as Merrill (1977) also estimate the likelihood of election inversions.

[4]At about the same time, a McClatchy-Marist poll revealed that 52 percent of registered voters think that the popular vote should be the deciding factor in future elections, and 45 percent think the Electoral College should remain in place. Three percent of those polled were unsure.

In the past, a clear majority favored amending the U.S. Constitution to replace the Electoral College with a popular vote system.

Such sentiment has clearly prevailed when Gallup asked this question twice in 2000 – after George W. Bush won the Electoral College while Al Gore won the popular vote – in 2004 and in 2011. In each instance, support for a constitutional amendment hovered around 60%.

From 1967 through 1980, Gallup asked a slightly different question that also found majority support for an amendment to base the winner on the popular vote. Support for an amendment peaked at 80% in 1968, after Richard Nixon almost lost the popular vote while winning the Electoral College. Ultimately, he wound up winning both by a narrow margin, but this issue demonstrated the possibility of a candidate becoming president without winning the popular vote. In the 1976 election, Jimmy Carter faced a similar situation, though he also won the popular vote and Electoral College. In a poll taken weeks after the election, 73% were in favor of an amendment doing away with the Electoral College.

This year, for the first time in the 49 years Gallup has asked about it, less than half of Americans want to replace the Electoral College with a popular vote system. The reason for this shift in opinion is clear: In the aftermath of this year's election, the percentage of Republicans wanting to replace the Electoral College with the popular vote has fallen significantly.

Currently, 19% of Republicans and Republican-leaning independents favor basing the winner on the popular vote, down from 49% in October 2004 and 54% in 2011. Democrats and Democratic-leaning independents already widely favored having the popular vote determine the winner and are slightly more likely to do so now than in the past. The opinion of U.S. citizens on the Electoral College has a lot to do with partisanship. If you feel like your camp is going to lose or gain by amending the Electoral College, then you oppose or favor the reform.

The Trump election has given rise to many petitions and initiatives to amend the constitution in order to replace the Electoral College by popular vote. The rules of the Electoral College are not set in stone. While Constitutional amendments are rare, they do happen. Over the history of the U.S., there have been at least 700 proposed amendments to modify or abolish the Electoral College - more than any other subject of Constitutional reform.

Among[5] the most recent reform attempts, let us mention:

"*1950: The Lodge-Gossett Amendment*, named for its co-sponsors Senator Henry Cabot Lodge (R-MA) and Rep. Ed Gossett (D-TX), was a classic example of a reform plan known as proportional allocation. The plan was introduced in the 81st Congress (1949-1950) as an amendment proposal that would abolish the Electoral College as it was known, replacing it

[5] The description below is borrowed from http://www.fairvote.org/past. Further historical information can be found in Wilmerding (1958), Peirce and Longley (1981, Chapter 6) and Schumaker and Loomis (2002). They discuss debates about changes in the Electoral College that go back to the early nineteenth century.

with a proportional electoral vote.

In this case, electors and the College would remain in place, but electoral votes would be allocated to presidential tickets in a manner directly proportional to the popular votes each ticket received in the states. The proposal was amended in the Senate to also require a 40% threshold of electoral votes for a ticket to be elected to the Presidency and Vice Presidency. If no one received such a threshold, the Senate and the House of Representatives, in a joint session, would then choose among the top two presidential candidates and their running mates.

The Lodge-Gossett Amendment passed the Senate with a super majority by a vote of 64-27, but died a bitter death in the House.

1956: Hubert Humphrey's (D-MN) S. J. 152 was a new, unique proposal of reform introduced in the 84th Congress. In this plan, the Electoral College would be abolished as known, but the then 531 electoral votes would still be put to use. Two electoral votes would be awarded to the candidate winning the overall popular vote in each of the then 48 states. The remaining 435 would then be divided nationally in proportion to the nationwide popular vote. The proposal passed the House of Representatives, but later died in the Senate.

1966: Delaware filed a lawsuit against New York, arguing that its "winner-take-all" system for awarding Electoral College votes effectively disenfranchised small states in the presidential election process. The Supreme Court, under whose original jurisdiction the case was filed, refused to hear it. However, Delaware's action generated support from several other states and 11 more joined in the lawsuit: Arkansas, Florida, Iowa, Kansas, Kentucky, North Dakota, Oklahoma, Pennsylvania, South Dakota, Utah, and Wyoming.

1969: This proposal came to be after the 1968 Presidential election, in which American Independent candidate George Wallace managed to obtain 46 electoral votes, generating concern over the possibilities of contingent elections and electoral vote-trading for political concessions. In the 91st Congress, Rep. Emanuel Celler (D-NY) introduced the proposal, which would abolish the Electoral College in favor of a direct popular election with a 40% threshold and a runoff if no threshold was achieved. The bill was wildly popular in the House, passing 338-70, yet failed to pass in the Senate due to a filibuster.

1979: After the close election between Jimmy Carter and Gerald Ford in 1976, Senator Birch Bayh (D-IN) introduced a proposal in the 96th Congress to abolish the Electoral College and replace it with direct election. The measure failed the Senate by a vote of 51-48 in 1979. Because of its failure in that chamber, the House decided not to vote on its version of the proposal.

2004: On November 2, 2004 the citizens of Colorado not only cast their vote for the president of the United States. They also decided how Colorado will be represented in the Electoral College. As most of the other states, Colorado has a winner-takes-all system, i.e. all electoral votes from Colorado are given to the candidate with the most votes within Colorado. Amendment 36 proposed to change this to proportional representation. If it had been accepted, the nine electoral votes of Colorado would have been divided between the different candidates in proportion of the votes they receive from the Coloradans. The amendment was defeated."

As we can see from that sample, some reforms suggest to abolish the Electoral College while some others suggest instead to keep the Electoral College and change the allocation rule of the electoral votes within the states. In this paper, we will be exclusively interested in those which keep the Electoral College in place. Among those reforms, let us mention:

Instant Runoff Voting to determine the state winner (instead of plurality). Once done, the winner gets all electoral votes. This reform matters as long as third candidates play a role in the election.

Proportional Allocation of Electoral Votes. It corresponds to the Lodge-Gossett Amendment discussed above.

Congressional District Method. This method has been used in Maine since 1972 and Nebraska since 1996[6]. They both use an alternative method of distributing their electoral votes, called the Congressional District Method. Currently, these two states are the only two in the union that diverge from the traditional winner-take-all method of electoral vote allocation.

With the district method, a state divides itself into a number of districts, allocating one of its electoral votes to each district. The winner of each district is awarded that district's electoral vote, and the winner of the state-wide vote is then awarded the state's remaining two electoral votes.

National Bonus Plan. This idea, proposed by historian Arthur Schlesinger Jr., retains the current Electoral College system, but also awards extra electoral votes as a bonus to the winner of the popular vote. The amount suggested by Schlesinger in his National Bonus Plan is 102 extra electoral votes (two for every state and two for Washington, DC). The extra boost of electoral votes would almost always be able to guarantee that the popular vote winner would also be the Electoral College winner. While technically maintaining the institution, this option compensates for the uneven power given to the states by the Electoral College.

Hereafter, we will focus on a specific initiative[7] called the *National Popular Vote Interstate Compact (NPVIC) or sometimes shortly NPV*[8]. The National Popular Vote bill would guar-

[6] Interestingly, since both states have adopted this modification, the statewide winners have consistently swept all of the state's districts as well. Consequently, neither state has ever split its electoral votes

[7] We refer our readers to the website: http://www.nationalpopularvote.com/ and chapter 6 in Koza et al (2013) for a more detailed exposition of that proposal.

[8] According to Elliot and Ali (1988) "In the United States of America, *an interstate compact is an agreement between two or more states which requires congressional ratification...* As stated in Article I, Section 10, Clause 3 of the United States Constitution "No State shall, without the Consent of Congress... enter into any Agreement or Compact with another State". Koza et al (2013, Chapter 5) contains a background on interstate compacts.

antee the Presidency to the candidate who receives the most popular votes in all 50 states and the District of Columbia. It has been enacted into law in 11 states (CA, DC, HI, IL, MA, MD, NJ, NY, RI, VT, WA) with 165 electoral votes (which is 30.7% of the total Electoral College vote and 61.1% of the votes needed for it to have legal force). It will take effect when enacted by states with 105 more electoral votes. It has passed at least one house in 12 additional states with 96 electoral votes (AR, AZ, CO, CT, DE, ME, MI, NC, NM, NV, OK, OR) and been approved unanimously by committee votes in two additional states with 27 electoral votes (GA, MO). Most recently, the bill was passed by a 40–16 vote in the Republican-controlled Arizona House, 28–18 in Republican-controlled Oklahoma Senate, 57–4 in Republican-controlled New York Senate, 37–21 in Democratic-controlled Oregon House, and 26-16 in the New Mexico Senate.

The compact would modify the way participating states implement Article II, Section 1, Clause 2 of the U.S. constitution which requires each state legislature to define a method to appoint its electors to vote in the Electoral College. The constitution does not mandate any particular legislative scheme for selecting electors, and instead vests state legislatures with the exclusive power to choose how to allocate its own electors. The agreement would go into effect among the participating states into the compact only after they collectively represent an absolute majority of votes in the Electoral College.

The first academic publication suggesting this plan is due to Bennett (2001), a law professor at Northwestern[9]. The academic plan uses two constitutional features: the presidential electors clause (which gives each state the power to determine the manner in which the electors are selected) and the compact clause (under which it creates an enforceable compact). It was noted that such plan could be enacted by the passage of laws in as few as eleven states. In 2006, Koza, a computer scientist, edited a book that makes a detailed case for his plan for an interstate compact. He formed with others a non-profit group to introduce the legislation. By the time of the group's opening news conference in February 2006, the proposed interstate compact had been introduced in the Illinois legislature. From that period, the NPVIC legislation was introduced in many state legislatures: 42 in 2007. Maryland became the first state to join the compact in 2007 and the last one was New York in April 2014. The plan has raised important debates between supporters and opponents including controversies about its legality.

On one hand, the NPV initiative has received a friendly welcome in some academic[10], media

[9] See also Amar and Amar (2001).

[10] For instance, Maskin and Sen (2017a), two outstanding social choice theorists, write "Currently, the most promising initiative to replace the Electoral College is the National Popular Vote Interstate Compact". While benevolent, they point out however that "This condition creates a coordinating mechanism—states would move to the new system together, not unilaterally. So far, ten states and the District of Columbia have joined the compact, amounting to 165 electoral votes. All of them are solidly Democratic—probably reflecting the election

and political circles[11]. According to Koza et al (2013) "The current system for electing the President and the Vice President of the United States has three major shortcomings: voters are effectively disenfranchised in four-fifths of the states in presidential elections, the current system does not reliably reflect the nationwide popular vote and not every vote is equal[12]". Interestingly, the possibility of election inversions is therefore advocated by the NPV proponents as one important reason to abandon the Electoral College (Sections 1.2.2 and 9.3 of Koza et al (2013)) contains a lot of developments on that issue. Beyond the counting of election inversions, they also emphasize that "there have been six presidential elections since World War II in which a shift of a relatively small number of votes on one or two states would have elected a presidential candidate who lost the popular vote nationwide"[13]. They also mention that "about half of American presidential elections have been landslides (that is, elections with a margin of 10% or more). If one consider only non-landslide elections, the four 'wrong winner' elections represent one in seven of the non-landslide elections (14%)". Finally, they assert that "given the relative closeness of all seven presidential elections between 1988 and 2012, and given today's closely divided political environment, problems with the operation of the state-by-state winner-take-all rule can be expected in the future"[14].

On the other hand, as pointed out in other academic, media and political circles[15] and papers, the NPV initiative raised a number of legal and practical problems. On the academic side, they have been discussed among others by Gaines (2012), Miller (2012, Section 10) and

---

of 2000, when Al Gore won the popular vote but lost in the Electoral College. In all likelihood, we will have to wait for an election in which the same thing happens to a Republican candidate before any red states sign on. More importantly, even if the compact succeeds (so that the Electoral College is in effect "replaced"), the election system will remain highly unsatisfactory unless plurality rule—election by less than a majority—is also replaced".

[11]The media coverage of the fact that Donald Trump, the winner of the 2016 U.S. presidential election, failed to win the popular vote has been very large. Even, if it had been already a matter of general discussion before (specially after the 2000 election), the 2016 election has resurrected the debate in the U.S. and beyond. About foreign coverage, let us mention for instance that French newspapers have dedicated a lot of space to that issue and other peculiarities of the U.S. Electoral College.

[12]For an early analysis of that third feature, see Owen(1975, 2011).

[13]Details are found in table 1.23 in Koza et al (2013).

[14]Koza et al (2013) refers to Abbott and Levine (1991) to predict that emerging political trends would lead to more frequent election inversions. They also report some analysis by insiders like for instance M. Dowd "How Obama could lose the popular vote and win the election", Huffington Post, June 6, 2012.

[15]The NPV plan is responsible for lively and partisan appraisals by politicians. Virgin (2017) reports some comments on the NPV plan December 7, 2011 from Mitch McConnell (R-KY), then-Senate Minority leader: a 'dangerous' and 'absurd' scheme Democratic lawmakers and activists were 'sneaking through' under cover of metaphorical darkness. "They are well-funded, unfortunately, as they are well-organized, and they are getting close to the finish line". "We need to kill in the cradle before it grows up". Similarly on November. 15, 2016, D. D. Wire reports in the Los Angeles Times the following statement by retiring Senator Barbara Boxer (D-Ca.): "In my lifetime, I have seen two elections where the winner of the general election did not win the popular vote. The Electoral College is an outdated, undemocratic system that does not reflect our modern society, and it needs to change immediately. Every American should be guaranteed that their vote counts."

De Witt and Schwartz (2016)[16]. De Witt and Schwartz (2016) asserts that "the result would be a procedural calamity in pursuit of quixotic goals". They assert that "the mischief that would create (especially procedural instability, non compliant electors, nation-wide recounts, vote manipulation and narrowed support), the compact's questionable Constitutionality, the weakness of its defense, and the availability of less calamitous alternatives are reasons to reject it". Miller (2012) states that "Apart from legal details and uncertainties pertaining to interstate compacts, NPV raises three sets of problems. The first includes all the issues that also pertain to a constitutional amendment that would abolish the Electoral College and replace with a national popular election. The second pertains to special problems that arise in trying to convert the Electoral College system into a national popular election through this particular device. The third pertains to the credibility of the commitments that states acceding to compact promise to uphold[17]. Some of the issues in the first set were noted earlier and some are alluded to below, but I won't otherwise address them here". For a social choice theorist, the second set of issues also discussed by DeWitt and Schwartz (2016)[18] is of primary importance. As pointed out forcefully by Miller "the most basic problem is that, at the present time, the 'national popular winner' is an unofficial designation bestowed only by the media and commentators. No official body designates such a winner and, in a very close election, the designation may be contested, and this is precisely the circumstance in which the NPVP would be mostly likely to make a difference. NPVP requires 'the chief election official of each member state' to make a determination of 'the national popular vote winner' and, in doing so, to treat as 'conclusive'

[16]Koza (2016) answers the 24 criticisms of NPV in De Witt and Schwartz (2016) while Koza et al (2013) develops "answers to 131 myths about the National Popular Vote Plan".

[17]To quote Miller: "The third set of issues pertains to the durability of the interstate compact itself, especially in the face of controversies such as those noted above. The compact provides a nice example of a 'social contract' in a cooperative game but also highlights the problem of 'credible commitment'—whether and how the terms of the 'social contract' among states could be enforced promptly and reliably in highly controversial cases. Clearly there would be strong incentives for some states to defect from the compact in precisely the circumstances in which the compact produces a winner different from the Electoral College winner, and such defections would tend to be legitimized in circumstances producing the kinds of controversies noted above".

[18]In response to the assertion the current system can reject popular-vote winners, they write "But what does that mean ? There are countless formulas for translating popular votes into 'winners'. What is the right one ? Proponents of the compact may think that they are enforcing rules by majorities, but what they have actually proposed is rule by pluralities: whoever wins the most votes wins the election, even if those votes are not a majority". This point is also raised by Maskin and Sen (2017a) who write: "More importantly, even if the compact succeeds (so that the Electoral College is in effect "replaced"), the election system will remain highly unsatisfactory unless plurality rule—election by less than a majority—is also replaced". Maskin and Sen (2017b) argue further that "There is a risk that the presence of additional major candidates might prevent any one of them from getting 270 votes in the Electoral College. This could be avoided by amending the Electoral College system so that the winner is the candidate who wins the nationwide vote under majority-rule voting. Such a change could be instituted, for example, by revising the National Popular Vote Interstate Compact initiative, in which a state pledges to award its electoral votes to the winner of the national popular vote as long as states totaling at least 270 electoral votes make the same pledge."

official statements concerning the popular vote for president from non-member (as well as member) states"[19].

The main feature of the NPV that our paper revisits is the stipulation that it takes effect only when agreed to by states controlling a majority of electoral votes. Instead, we examine the effect of implementing the compact among however how many states agree to it[20]. What could be the effects of implementing the compact as soon as its current members agree about it? This modified version of NPV has been proposed by Brams and Kilgour (2013). What they call the Modified Popular Vote Plan (MPV)[21] becomes effective immediately for any state that agrees to it[22]. Brams and Kilgour argue that this reform would have benefited the 41 states in the U.S. 2012 presidential election - including the three largest, California, Texas and New York that received no attention from the candidates because they were not considered battleground states, in which either candidate might win[23].

In this paper, we also examine two versions of the NPV which share with Brams and

[19]We refer to Miller (2012) for the details of his arguments. Along these lines, he also points out that "There is also the problem that states are not constitutionally required to hold direct popular votes for unified elector slates (so that non-member states might fail, even in principle, to produce statewide presidential popular vote counts), though of course at present all states (including Maine and Nebraska with their district systems) actually do this. The compact requires member states to hold such elections, but obviously it cannot require non-member states to do so." This echoes our early discussion of the 1960 U.S. presidential election and the controversies about counting popular votes in Alabama.

[20]They write: "*Unlike NPV, MPV does not require an interstate compact saying when it becomes law; it becomes effective immediately for any state that enacts it*".

[21]Variants of Brams and Kilgour's MPV plan were discussed before the official NPV plan was formally proposed. Section 6.5 in Koza et al (2013) describes previous proposals for multi-state electoral legislation. They note on page 282 that "none of the earlier proposals contains a provision making the effective date of the system contingent on the enactment of identical laws in states that collectively possess a majority of the electoral votes". Section 6.5 develops arguments against these earlier proposals.

[22]Interestingly, Neubauer, Schilling and Zeitlin (2012) also consider that new plan when they ask: "What if states instead took action before they commanded a majority?"

[23]Their analysis is entirely conducted under the IC assumption. The first half of their contribution is dedicated to the IC examination of a numerical example with 3 states. In the second half of their analysis, they depart from IC by assuming that IC only applies to 9 states (called battleground states) and that the 41 states which are left are assumed to be biased either in the democrat or the republican direction. The all normative analysis is conducted by using the relative or normalized Banzhaf index of voting power. This is an unfortunate choice as these numerical values does not translate into payoff or utilities. Under IC, the Penrose formula states that there is a one to one affine relationship between the absolute Banzhaf power index and the utility of a voter but (to the best of our knowledge) no such relationship exists between the relative Banzhaf power and utility. Even worse, as soon as we depart from IC, the Penrose formula ceases to be true and no conclusion on utilities can be derived from information on power either absolute or relative. It is however immediate to see that for a single state (battleground or not) the MVP unilateral deviation is a dominated strategy. Indeed, if the deviation leads to a different outcome than the one that would have prevailed under the Electoral College, then it implies that the electoral votes of that state have been given to the minority winner in the states. This conclusion is at odds with the conclusion of Brams and Kilgour who conclude instead that the deviation is beneficial since the relative Banzhaf power index moves trivially from the value 0 to some positive value. But being powerful has no value per se; what truly matters is how your payoff is affected by the change.

Kilgour's MPV that the plan becomes effective as soon as its members agree on that[24]. A single member will never find beneficial to enact it on its own. This implies that a critical number of members are needed for such initiative to be considered. But critical does not mean a majority of states as we will show. The two versions of MPV that we will consider involve two different degrees of departure from political state sovereignty. The more extreme departure from state sovereignty called *National Modified Popular Vote Plan (NMPV)* would give all the electoral votes of the interstate compact to the nationwide popular winner as proposed by NPV and MPV.

A more moderate form, called hereafter *Regional Modified Popular Vote (RMPV)* in effect amalgamates, for purpose of Electoral College voting, the signatory states into a single 'mega-state' using the winner-take-all rule. In that second case, it is as if the coalition of states was forming a 'legislative caucus' as defined by Schattschneider (1942)[25] or a caucus as defined by Berg (1990)[26]. Variants of RMPV were also discussed before the NPV plan was proposed. In the 1990's, Charles Schumer proposed a bi-state NY-TX compact of this nature to create a competitive 'mega-state' (larger than California) that would attract the attention of the presidential candidates during presidential campaigns[27]. Koza et al (2013) describes as follows another 'bloc voting' variation due to Wilson (2006): "only the popular votes of only the enacting states would decide which candidate received the electoral votes of the enacting states"[28].

[24] As noted by our referee, both the NMPV and RNPV plans (but specially the former) present most of the practical problems presented by the NPV and discussed above.

[25] We are thankful to our referee for calling our attention on that classic book and in particular the relevance of chapter 3. Schattschneider writes "the practice of prior consultation in order to agree upon a united front is an old one usually described by the word caucus". In this chapter, Schattschneider pursues the strategic implications of caucuses in a legislative context. In our working paper version, we sketch some of the game-theoretical issues issues from the formation of a caucus. In particular, we raise the question of reaction other states which is an issue also addressed by Schattschneider under the heading 'rival caucus'.

[26] Berg refers himself to some earlier work. He defines the internal rule to decide within the caucus as being *caucus majority*. Note however that in our setting, since the caucus consists in a coalition of states whom are themselves populated by citizens, it is important to define what is meant by a caucus majority. Is it a majority of the state's representatives within the caucus (where each representative vote according to the majority opinion in his state) or a majority among the citizens living in the states forming the caucus? RMPV refers to the second of these two versions.

[27] We thank our referee for calling our attention on that proposal described in Koza et al (2013, section 6.5).

[28] Interestingly, the RMPV alternative also appears in discussion on public mailing lists. For instance, in 2014, on the https://department-lists.uci.edu/pipermail/law-election, Sean Parnell wrote: "Now here's my question: under this theory, NPV's inclusion of popular vote totals in non-compact states is basically a courtesy. If they wanted to, the NPV compact would be amended to simply say that member states would collectively award their electors to the candidate who receives the largest number of popular votes in the compact states, and simply ignore states that aren't members of compacts. Furthermore, while the compact currently says that any state may join the compact, I assume that could be amended to say that a majority of states already in the compact must vote to approve the membership of other states who want to join, or some other limiting feature could be devised."

Since both variants of the interstate compact abandon sovereignty, it is legitimate to ask why it could be beneficial to do so as, at first glance, it seems to go against the interest of the voters living in such states. And indeed, ex post, some states may and will end up supporting a candidate defeated in the state. To see that is not paradoxical at all, it is important to emphasize that the agreement is ex ante improving the welfare of all the members. The interstate compact acts exactly as a risk sharing agreement in economics. This is why we need at least two parties to make that agreement profitable.

This paper is theoretical and postulates some simplifications[29]. We consider only the case of two presidential candidates (thus avoiding spoiler effects within states), make no distinction between competitive ('battleground states') versus non competitive states (in contrast to Brams and Kilgour), and assume that all states have equal numbers of voters and cast equal (single) electoral votes.[30].

We explore in turn the implications of three different probabilistic assumptions. Like all the authors working in that area, we first consider the *a priori Banzhaf/IC* model. Then, we move to two other popular models known as the *a priori Shapley-Shubik/IAC* model and the *a priori May/IAC** model (it departs slightly from the IAC in that, while correlated within states, the votes remain independent across states). Notice that most of our theoretical analysis will focus on the IC and IAC^* models. While popular, the IAC model raises a number of questions. First, it introduces so much correlation across individual votes that the differences between the popular vote outcome, the Electoral College outcome and outcomes from variants tends to vanish[31] for large population of voters. Second, the correlation among votes attached to the IAC model introduces mathematical difficulties and it becomes more difficult to establish closed form formulas for the probabilities of the more relevant electoral events. As a consequence, we will mostly report simulation results on the IAC model.[32]

In the first part of the paper, we study the changes in power and utility of the voters in and out the Regional Modified Popular Vote Interstate Compact. We also study the changes of total power and total utility. For both IC and IAC^* models, we prove numerically *a square root result*. Precisely, this result asserts that the size of the interstate compact that maximizes

[29] In doing so, we follow an important part of the existing literature on the topic.

[30] Therefore, malapportionment issues are also absent. It is not conceptually problematic to consider models integrating all these effects. The probabilistic models describing these richer environments are more involved: the input is not binary, voters and candidates may be strategic and voters may be ex ante biased towards some candidates.

[31] The speed of convergence to zero of the probability of election inversion is studied in de Mouzon et al. (2017). They also show however that with a generalized version of IAC with lower (but not null) levels of correlation, the probability of election inversion while small does not vanishes to 0.

[32] Some further developments illustrating these points in the simplest conceivable versions of the Electoral College are offered in the 'long' version of this work. We also show that under IAC, we cannot make any difference between the popular vote and the Electoral College from the perspective of decisiveness!

the interest of the interstate compact voters is about $1.4\sqrt{K}$, where K denotes the number of states, i.e. around 10 when $K = 51$. There is always an incentive for an interstate compact of that type to form if the other states remain passive. But, we show that this gain is at the expense of the voters outside the compact and more critically at the expense of the society in general as the gains of the voters in the compact do not compensate for the losses of the voters outside the compact unless the compact contains 64% of the states.

In the second and shorter part of the paper, we study the National Modified Popular Vote Interstate Compact. *In such case, the externality exerted by the interstate on the other states is positive instead of being negative.* Therefore, the patterns describing the changes in utilities for voters inside and outside the interstate compact are totally different from the patterns exhibited in the regional version. We note that for voters outside the interstate compact, things work as if they had two votes instead of one: they vote in their state and influence the electoral outcome in the state and consequently the final outcome but they also participate to the nationwide popular vote and influence the final outcome through that channel too. As the mathematics of the problem become quite cumbersome, we use simulations to evaluate the impact of NMPV on the influence and utility of the different voters and the society. One interesting result that shows up through these simulations is that the maximal social gain with respect to the Electoral College (corresponding to the popular vote) is obtained for relatively small values of L, with L denoting the interstate size. Note that if the interstate compact contains a majority of states, then it implements the utility of the popular vote. Therefore, by a continuity argument there exists a smaller interstate compact for which the total utility is larger than the total utility of the Electoral College. The surprise is that the gain is already obtained for a low value of L. It has to be remembered however that payoffs are here ex ante payoffs. The gain which is evaluated is expected success; it has to be understood as the chance to be on the 'good side' on average i.e. over a long series of elections.

Related Literature

From a methodological perspective, our paper is fully aligned with the intellectual tradition pioneered by Banzhaf (1968), Shapley and Shubik (1954) and more recently[33] Barbera and Jackson (2006), Beisbart and Bovens (2007, 2008, 2013), Beisbart, Bovens and Hartmann (2005), Beisbart and Hartmann (2010), Edelman (2004), Felsenthal and Machover (1999), Miller (2009, 2012a) to cite the most important contributions in that area. Among those, we would like to call the attention of the reader on Beisbart and Bovens (2008) who provides a very insightful analysis of Amendment 36 in Colorado under IC and a data based probabilistic model and Miller (2009) who provides a remarkable comparative analysis of twelve potential reforms to

[33]The exceptional paper of Weber (1977) should not be forgotten.

amend the Electoral College. The common denominator of these papers also shared by our paper is to formulate the constitutional/electoral engineering problem as a mechanism design problem where the designer postulates that the inputs (utilities/preferences) are the random realizations of a stochastic model which can be either a priori i.e. totally unrelated to the preferences that could be inferred from the statistical data, or a posteriori i.e. a model which reflects the empirical likelihood of the profiles. Given these ingredients, any potential electoral reform is evaluated from an ex ante perspective through the use of social welfare function like for instance the Rawlsian or utilitarian ones. Among these authors, the vast majority use the (a priori) IC assumption which supposes that the votes are independent. Fewer authors also consider a priori models where the votes are correlated. Even fewer authors consider empirical models: the notable exceptions are Merrill (1977, 1978) and Beisbart and Bovens (2007) who estimate an empirical Gaussian model allowing biases/distortions among candidates and vote correlations within and across states.

Beisbart and Bovens (2008) consider the true Electoral College, denoted A^{51}, and examine if amendment 36 in Colorado, denoted $A^{50}P_C$, might have been beneficial. They also investigate what would happen if all states were to adopt proportional representation, a reform, denoted P^{51}. They conduct the comparison of the current Electoral College with these two reforms from two perspectives (from the self-interested perspective of a Coloradan and also from an impartial perspective[34]) and by using two alternative probabilistic models: IC and a second one denoted R which is a posteriori and builds upon data from past elections. They obtain very striking results which are mostly contained in tables 1, 2 and 3 of their paper. The results are very sensitive to the probabilistic model and to the perspective which is considered. From the Colorado perspective, $A^{50}P_C$ is the worst mechanism in both probabilistic models and the third one clearly dominates the second one in the IC model while the domination is slightly reversed in the R model. In contrast, from the impartial perspective, the ordering are more contrasted. For the IC model, P^{51} is clearly the worst while for the R model, it is clearly the best. This shows the extreme sensitivity of the electoral engineering exercise to the probabilistic model. The conclusion obtained under IC is surprising at first glance. The mechanics leading to that conclusion are however quite easy to understand. Under IC, it is almost likely that the election will be extremely tied in all states. It follows that with P^{51} the probability for a citizen in a state i with an even number of electoral votes is equal to 0. Under IC, proportional representation fails to do a good job because, due to a finite number of electoral votes, the

[34] They consider two measures to assess the mechanisms from an impartial perspective. One is the spread of the voting powers of citizens from different states evaluated through the standard deviation. The second one evaluate by how far the outcome reflects the preferences of the majority of voters: to do so they use the mean majority deficit.

allocation of the votes across the two parties is certain except in the case of an odd number where it amounts to allocate the reminder on the basis of majority. That last stage looks like the way it operates in Electoral College except for the fact that now the differences in state population sizes have been totally deleted in such process. Under R, things are back to the normal since in that case the votes received by the two parties may depart with a significant probability from the equal split. As a consequence, in any such event, the allocation of electoral votes across the two parties reflects in a much better way the strengths of the two groups of voters than does winner-takes all method. Simulations indicates that the expected majority deficit is reduced by a factor of 3.4.

Like us, Beisbart and Bovens (2013) consider a symmetric toy version of the Electoral College assuming equipopulated states. They investigate the implications of changing the number of states while maintaining the same or almost the same population per state. They consider IC and evaluation is done through the utilitarian criterion. They first show that when the total population is odd (dividing the total population into an odd number of equipopulated districts) the mean majority deficit is maximized when the district population is about the square root of the total population. Since in practice, the corresponding number of districts is too large, we may conclude that for any reasonably low number of districts, having less districts is better than having more. In contrast, when the total population is even, things are very peculiar. In particular, dividing the population into districts of size 2 or two districts containing each half of the population is the worst case. Interestingly, they derive some lessons for the analysis of the Electoral College. A first lesson is that the mean majority deficit is smaller for 51 states than it is for 435 districts. This is not the pure district plan according to which each state has as many equal-sized districts as it has electors, adding up to 538 districts. The mean majority deficit climbs to 899. Second, in the case of unequal-sized states and a move to a system with roughly equal-sized districts, they show through simulations (allowing random fluctuations of the population sizes within the units) that the mean majority deficit is about 1057. So the current EC is worse than either a partition into 51 equal-sized states or a partition into 435 equal-sized districts.

In a very rich paper, Miller (2009) evaluate from an utilitarian perspective and under IC twelve potential electoral reforms to amend the Electoral College. Some of them have already been discussed above including the pure district, modified district, national bonus, pure proportional and whole number proportional plans. He adds to this list various other reforms describing alternative ways to apportion the electoral votes across states: apportion electoral votes in whole numbers on the basis of population, apportion electoral votes fractionally, apportion electoral votes fractionally but then add back the two electoral votes based on senate representation, apportion electoral votes equally among states, redraw state boundaries so that the states have the same number of voters ans electoral votes (uniform apportionment) and apportion electoral votes using Penrose's Square Root rule either pure or in whole-number (Penrose apportionment). He is able to rank all the reforms along a single utilitarian dimension to obtain that unsurprisingly the direct popular vote (or the pure proportional system) is best followed closely by the national bonus plan (the bonus comprises 101 seats). The third position is occupied by the modified district plan. Then, there are several reforms which are more or less alike. This group includes the Penrose apportionment, the pure district plan, the uniform apportionment and fractional proportional apportionment plus two seats. The fourth position is occupied by the existing Electoral College. The fifth position contains three plans: the proportional apportionment, the equality apportionment and the whole number proportional apportionment plans. The bottom reform is the whole number proportional plan. Miller presents also some alternative social ranking of these reforms either based on equality principles or on the self-interested perspective of the states.

Finally, let us also point out the relevance of two papers respectively by Berg (1990) and Feix et al. (2007) which contain some insights on the problem tackled in our paper. Berg considers the formation of a coalition of voters (that he calls a caucus) in a society of voters which uses the direct popular mechanism. The main originality of his work lies in the generality of the probabilistic model that he considers. In fact, he considers a family of probabilistic models defined by a single parameter that he calls social cohesion and which contains the Banzhaf/IC model and the Shapley-Shubik/IAC model as special cases. His paper does not contain any general result but reports some few specific simulations on the effect of the caucus on the power and utilities of the voters inside and outside the caucus. His paper is also among the very few one to make a clear distinction between power and utility and to point out that computations differ as soon as we do not assume IC. In the second half of their paper, Feix et al. (2007) study under IC the design of the voting rule inside an alliance in order for the voters in the alliance to experiment an increase of their Banzhaf power. They show that, in the case of the direct popular mechanism (i.e. one voter per state), an alliance can be implemented if and only if some randomness is introduced in the decision making process within the alliance. In the two-tier case with a large number of voters per state, their difficulty disappears.

The paper is organized as follows. In section 2, we introduce the main notations and notions used in our paper. In section 3, we proceed to a detailed theoretical and numerical analysis of RMPV under IC. Then in section 4, we proceed to a similar exploration under IAC^*. In section 5, we offer a numerical analysis of RMPV under IAC. In section 6, we proceed to a detailed numerical analysis of NMPV under IC, IAC and IAC^*. The paper is completed by an appendix consisting of eight parts. A first part describes the simulator that we created to simulate IC, IAC and IAC^* elections for an arbitrary number K of states, an arbitrary size

L of the interstate and arbitrary number m of voters per state. The second part describes the working of IAC in simple versions of the Electoral College. The third and fourth parts describe an alternative way of computing the probability of election inversion for the IC and IAC^* models. The fifth appendix contains the analytics of individual and total utilities for the IAC^* model. The sixth illustrates the importance of conditioning in the case of NMPV. The seventh provides some theoretical insights explaining the asymptotic differences between IC, IAC and IAC^*, concerning the decomposition of the probability of being pivotal into its five components. Finally, the eighth illustrates the sensitivity of our conclusions to the existence of battle ground states.

2 Two Modified Popular Vote Interstate Compacts

2.1 Voting Mechanisms

We consider a society N of n voters which must chose among two alternatives/candidates: D versus R. Each member i of N is described by his/her preference P_i. There are two possible preferences: D or R. An electoral mechanism is a mapping F defined over $\{D, R\}^n$ with values in $\{D, R\}$: to each profile $P = (P_1, ..., P_n) \in \{D, R\}^n$, the mechanism attaches a decision $F(P) \in \{D, R\}$. Any such mechanism F is totally described by the list $F^{-1}(D)$ of coalitions $S \subseteq N$ such that $F(P) = D$. This list $\mathcal{W}_F = \mathcal{W}$ (if no ambiguity arises) of coalitions defines a *simple game* as soon as $\varnothing \notin \mathcal{W}$ and $N \in \mathcal{W}$[35] and $\mathcal{W}$ is monotonic ($S \in \mathcal{W}$ and $S \subseteq T$ $\Rightarrow T \in \mathcal{W}$).

The standard majority mechanism (popular vote) is defined as follows. Given a profile $P = (P_1, ..., P_n) \in \{D, R\}^n$, candidate D is elected iff the number of voters who prefer D to R is larger than $\frac{n}{2}$[36]. Hereafter, we will denote by $Maj(n)$ or simply Maj this electoral mechanism.

The definition of the U.S. Electoral College, and more generally, of any two-tier weighted majority voting electoral mechanism is based on a partition of the the population N of voters into K states : $N = \cup_{1 \leq k \leq K} N^k$. The n^k voters of state k are endowed with w^k electoral votes; to avoid notational complications with ties, we assume that n^k is an odd integer[37]. Given a profile $P \in \{D, R\}^n$, we declare D to be the winner of the election in state k if the number of voters preferring D to R in state k is larger than $\frac{n^k}{2}$. Let $D(P) = \{k = 1, .., K : \ D \text{ is the winner in state } k\}$.

[35]The mechanism is not constant.

[36]In case of a tie, the majority rule needs to be supplemented by a tie breaking rule. In our article, the choice of the tie breaking rule has no influence on the results. Therefore, the tie breaking rule will not be specified unless strictly necessary for the sake of clarity. Note that if n is odd, ties are impossible.

[37]Then, if K is an odd integer, then the popular majority vote does not display ties.

Given P, candidate D is declared to be the winner of the election if:

$$\sum_{k \in D(P)} w^k > \frac{\sum_{k=1}^{K} w^k}{2}.$$

Hereafter, we will denote by $EC(w)$ this electoral mechanism[38]; we emphasize the dependence upon the vector w of electoral votes (we omit the dependence upon the populations of states). We now explore the properties of two electoral mechanisms based on coalitional deviations from the two-tier weighted majority voting mechanism.

In both cases, the states are partitioned into two groups. The first group (hereafter, referred to as group 1) contains the states whose index k runs from 1 to L. The second group (hereafter, referred to as group 2) contains the states whose index k runs from $L+1$ to K.[39]

Modified Regional Popular Vote Interstate Compact (MRPV or shortly M1)

A coalition $S \subseteq N$ of voters is winning if the sum of $\widehat{n}^1$ and $\widehat{n}^2$ is larger than $\frac{\sum_{1 \leq k \leq K} w_k}{2}$ where:

(i) $\widehat{n}^1$ is either the total sum of the electoral votes of the states in group 1 iff the number of voters who prefer D to R in $S \cap \left(\cup_{1 \leq k \leq L} N^k\right)$ is larger than $\frac{\sum_{1 \leq j \leq L} n^j}{2}$ or 0 otherwise[40],

and

(ii) $\widehat{n}^2$ is the sum of the electoral votes of the states j in group 2 such that $\mid S \cap N^j \mid > \frac{n^j}{2}$.

MRPV describes the situation where the states in group 1 deviate from the current functioning of the Electoral College as follows. They commit ex ante to allocate the totality of the electoral votes of the states to the *regional popular winner* where the region consists of the union of the states which are part of the interstate compact. On one hand, it is deteriorating for these states to abandon their sovereignty: it is more difficult for a voter to be pivotal in group 1 than in his state because the group is larger than the state. On the other hand, it is beneficial as their unique large body of representatives will have a larger impact in the second tier. It is unclear to assert which effect dominates.

Modified National Popular Vote Interstate compact (MNPV or shortly M2))

A coalition $S \subseteq N$ of voters is winning if the sum of $\widetilde{n}^1$ and $\widetilde{n}^2$ is larger than $\frac{\sum_{1 \leq k \leq K} w_k}{2}$ where:

[38] In the terminology of simple games, the second tier is referred to as a weighted majority game.

[39] Strictly speaking, the range of L is $\{0, 1, ..., K\}$ where $L = 0$ means that no interstate forms. Note that in the case of MRPV, the cases $L = 0$ and $L = 1$ are both equivalent to the electoral college. While in the case of MNPV, there are not: $L = 0$ and $L = 1$ are two different mechanisms. Only $L = 0$ corresponds to the electoral college.

[40] If $\sum_{1 \leq j \leq L} n^j$ is an even integer (given our assumptions, this will happen when L is an even integer), the rule needs to be supplemented by a tie breaking rule. As already pointed out, this is inconsequential for our analysis. However, to run our simulations, we had to make a choice. To preserve neutrality among candidates, we decided to depart from our deterministic framework by breaking any given tie through a fair coin flip.

(i) $\widetilde{n}^1$ is either the total sum of the electoral votes of the states in group 1 iff the number of voters in N who prefer D to R is larger than $\frac{n}{2}$ or 0 otherwise,
and

(ii) $\widetilde{n}^2$ is the sum of the electoral votes of the states j in group 2 such that $\mid S \cap N^j \mid > \frac{n^j}{2}$, i.e. formally S is winning if:

$$\sum_{L+1 \leq j \leq K : |S \cap N^j| > \frac{n^j}{2}} w^j + \sum_{1 \leq j \leq L} w^j > \frac{\sum_{1 \leq k \leq K} w^k}{2} \quad \text{and } Maj(P) = D$$

or

$$\sum_{1 \leq j \leq L : |S \cap N^j| > \frac{n^j}{2}} w^j > \frac{\sum_{1 \leq k \leq K} w^k}{2} \quad \text{and } Maj(P) = R.$$

MNPV describes the situation where the states in group 1 deviate even further from the sovereignty of their voters as they commit ex ante to allocate the totality of the electoral votes of the states to the *nationwide popular winner*. MNPV displays the same tradeoffs as MRPV. Note however that the cost side is larger as the departure from sovereignty is more significant. Indeed, by allocating their electoral votes to the popular nationwide majority winner, they take into account the preferences of voters outside the group. Now, it may well be that the group votes against its majority will!

Note that when $L = 1$, MRPV corresponds to $EC(w)$ and when $L = K$ it corresponds to Maj. When $\sum_{1 \leq j \leq L} w^j \geq \frac{\sum_{1 \leq k \leq K} w^k}{2}$, MNPV corresponds to Maj.

MNPV is far more complicated to analyze than MRPV. The origin of the difficulties lies in the existence of correlations which prevent the use of a multiplicative formula to evaluate the influence of a voter.

2.2 Probabilistic Models

To evaluate the power and the utility of any voter, we need to introduce a probabilistic model[41] π on the set of profiles $\{D, R\}^n : \pi(P)$ denotes the probability of profile P. To evaluate how often voter i is influential, we consider the frequency of profiles P such that $F(D, P_{-i}) \neq F(R, P_{-i})$ or equivalently of coalitions $S = S(P)$, such that $S \in \mathcal{W}$ and $S \setminus \{i\} \notin \mathcal{W}$ or $S \notin \mathcal{W}$ and $S \cup \{i\} \in \mathcal{W}$. The probability $Piv_i(\mathcal{W}, \pi)$ of such an event is:

$$\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} \pi(T) + \sum_{T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}} \pi(T).$$

[41] On probabilistic models in general, see Gehrlein (2006) and Gehrlein and Lepelley (2011).

This formula makes clear that the evaluation depends upon the probability π which is considered. Two popular specifications have attracted most of the attention and dominate the literature. The first (known under the heading *Impartial Culture (IC)*) leads to the Banzhaf's index. It corresponds to the setting where all the preferences P_i proceed from independent Bernoulli draws with parameter $\frac{1}{2}$. In this case, $\pi_{-i}(T) = \frac{1}{2^{n-1}}$ for all $T \subseteq N\backslash\{i\}$. The Banzhaf power $B_i(\mathcal{W}) \equiv Piv_i(\mathcal{W}, IC)$ of voter i is equal to:

$$\frac{\eta_i(\mathcal{W})}{2^{n-1}}.$$

where $\eta_i(\mathcal{W})$ denotes the number of coalitions $T \subseteq N\backslash\{i\}$ such that $T \notin \mathcal{W}$ and $T \cup \{i\} \in \mathcal{W}$ (in the literature, any such coalition T is referred to as a "swing" for voter i). The second model, known under the heading *Impartial Anonymous Culture (IAC) Assumption* (Chamberlain and Rothschild (1981), Fishburn and Gehrlein (1976), Good and Mayer (1975), Kuga and Nagatani (1974)) leads to the Shapley-Shubik's index. It is defined as follows. Conditionally to a draw of the parameter p in the interval $[0, 1]$, according to the uniform distribution, the preferences P_i proceed from independent Bernoulli draws with parameter p. In such a case, we obtain: $\pi_{-i}(T) = p^t(1-p)^{n-1-t}$ where $t \equiv \#T$. The Shapley-Shubik power $Sh_i(\mathcal{W}) = Piv_i(\mathcal{W}, IAC)$ of voter i is equal to:

$$\int_0^1 \left(\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} p^t(1-p)^{n-1-t} \right) dp.$$

Hereafter we will mostly focus on the two models and a third one that we call IAC^* and which is intermediate between IC and IAC. It was first introduced by May (1948) in his analysis of election inversions and studied more recently in full generality by Le Breton et al. (2014, 2016). This model is defined as follows.

Conditionally on K independent and identically distributed draws $p_1, ..., p_K$ in the interval $[0, 1]$, according to the uniform distribution, the preferences in state N_k proceed from independent Bernoulli draws with parameter p_k. In such a case, the probability of profile P is equal to:

$$\prod_{k=1}^{K} \binom{n_k}{l_k} \frac{1}{(n_k+1)}.$$

where l_k denotes the number of voters voting D in state N_k.

Regarding utilities, we denote by $U_i(x)$ the utility derived from alternative x and we will assume that each voter derives a utility of 1 for his best alternative and a utility of 0 for his worst alternative. Therefore, the expected utility of a player is equal to the probability of success. We denote by $U_i(\mathcal{W},\pi)$ the expected utility of voter i when the voting mechanism is $\mathcal{W}$ and the probabilistic model is π. It is well known (see for instance Felsenthal and Machover (1998)) that under IC, for any electoral mechanism $\mathcal{W}$ and any voter $i \in N$, there is an affine relationship between expected utility $U_i(\mathcal{W}, IC)$ and Banzhaf power $B_i(\mathcal{W})$. This link known as the Penrose's formula states that for any electoral mechanism $\mathcal{W}$:

$$U_i(\mathcal{W}, IC) = \frac{1}{2} + \frac{1}{2}B_i(\mathcal{W}).$$

Unfortunately, in general there is no such simple link between $U_i(\mathcal{W},\pi)$ and $Piv_i(\mathcal{W},\pi)$ when π is different from IC (see for instance, Laruelle et al. (2006) and Le Breton and Van Der Straeten (2015)). To evaluate ex ante an electoral mechanism, we need to aggregate the expected utilities of the voters. In this paper, we will consider[42] the utilitarian criterion:

$$U(\mathcal{W},\pi) \equiv \sum_{i\in N} U_i(\mathcal{W},\pi).$$

It is well know that the optimal utilitarian mechanism is the mechanism Maj. In fact, for the utilitarian criterion, the electoral mechanism Maj is ex post optimal[43]; for any profile, it is always optimal to chose the majority preference.

A consequence of the Penrose's formula is that under IC we can compare mechanisms either in terms of total expected utility or, equivalently, in terms of total power:

$$S(\mathcal{W},\pi) = \sum_{i\in N} Piv_i(\mathcal{W},\pi) \equiv \sum_{i\in N} B_i(\mathcal{W}).$$

This sum called *sensitivity* by Felsenthal and Machover (1998, 1999) is itself equivalent up to an affine negative transformation to the *mean majority deficit* $MD(\mathcal{W},\pi)$

$$\sum_{P\in\{D,R\}^n} \pi(P)\left(\sum_{i\in N} U_i(Maj(P)) - \sum_{i\in N} U_i(F(P))\right).$$

Given the optimality of Maj, the quantity $\sum_{i\in N} U_i(Maj(P) - \sum_{i\in N} U_i(F(P)))$ is always non negative: it simply counts (each time the mechanism F picks a minority decision) how many voters are missing with respect to the majority benchmark. Felsenthal and Machover (1998) have shown that:

[42] This is equivalent to Weber (1977) *effectiveness* of an electoral mechanism defined as

$$\frac{U(\mathcal{W},\pi) - \frac{n}{2}}{U(Maj,\pi) - \frac{n}{2}}.$$

.

[43] Therefore it is ex ante optimal for any π.

$$MD(\mathcal{W}, IC) = \frac{S(Maj, IC) - S(\mathcal{W}, IC)}{2}.$$

Therefore, under IC the three social criteria are equivalent. In the case of IAC and IAC^*, we will compute and examine both total utility and sensitivity. We will also refer to an additional popular measure of the suboptimality of an electoral mechanism. We will say that the voting mechanism F displays *election inversion* at profile P if $F(P) \neq Maj(P)$. The probability of election inversion is the probability of the event $\{\{D, R\}^n : F(P) \neq Maj(P)\}$. Note that in contrast to the mean majority deficit, the election inversion criteria does not keep track of the magnitude of the majority deficit whenever there is a majority deficit i.e. an election inversion.

Let us conclude this section by an important methodological remark. In a limited number of cases, we can derive an analytical closed form expression of some of these quantities but most of the time such closed forms are out of reach. The numerical computations reported in this paper could be done in several ways. In some cases, in particular for mechanism $M1$ and under IC and IAC^*, we can program the numerical computation of the expression we are interested in. This is what we have done for pivotality and sensitivity for $M1$ under IC and IAC^* and for large values of r.However in general, this approach turns to be complicated if not impossible to implement in full generality when we deal with $M2$ and/or IAC. This is why we have conceived a program, called *SimuElect* which simulates the electoral outcomes of mechanisms $M1$ and $M2$ for IC, IAC^* and IAC. The program has so far been conceived for the case of equipopulated districts but the integers K and r can be arbitrary, as well as the number of simulations. In what follows, we will mostly limit our numerical presentation to the case $K = 51$ and[44] $r = 10^6$ but occasionally we will report extra evidence for others values of these two parameters. Besides, all results in the paper are based on 10^9 simulations. The algorithm and the codes corresponding to this program are described at length in appendix 1. In a nutshell, let us just mention that the basic idea of this program is to emulate independent random draws of electorates for each of the three probabilistic models and to derive from that procedure estimates of the probabilities of the events of main interest: election inversions, individual and total utilities and pivotalities.

3 Analysis of a MRPV under IC in a Toy Symmetric Version of the Electoral College

In this section and the next one, following the pioneering works of Beisbart and Bovens (2013), Feix et al. (2004, 2011) and May (1948), we first consider a toy symmetric version of the U.S.

[44] We have similar figures for $r = 10^k$ for $0 \leq k \leq 5$. For $k \geq 7$, computing is time consuming.

Electoral College. Miller (2012b) makes a distinction between apportionment and distribution effects. He notes that "apportionment effects encompass whatever may cause deviations from perfect apportionment. The Electoral College system is imperfectly apportioned for the following reasons: (i) electoral votes are apportioned in small whole numbers, and therefore cannot be precisely proportional to anything; (ii) electoral vote apportionments are anywhere from two to ten years out-of-date at the time of a presidential election; (iii) the apportionment of electoral votes is skewed in favor of smaller states; (iv) electoral votes are apportioned to states on the basis of their total population and not on the basis of their voting age population, or voting eligible population (excluding non-citizens, etc.), or number of registered voters, or number of actual voters in a given election". In contrast, distribution effects result from winner-take all at the state (or district) level, which can make one candidate's popular vote support be more 'efficiently' depending upon the geography of the votes. To eliminate apportionment effects, we will assume that the states are equipopulated and have one electoral vote each i.e. with our earlier notations: $n^k = 2r + 1 \equiv m$ and $w^k = 1$ for all $k = 1, ..., K$ where r is an integer.

The main objective of this section is to analyze the implications of a MRPV on the utilities of the citizens. Given our symmetry assumptions, they are only two utility levels to be calculated: the utility of the citizens from a state belonging to the interstate compact and the utility level of the citizens from a state outside the interstate compact.

In what follows, we first compute numerically the utility of a citizen inside the interstate compact as a function of the size L of the interstate compact. Then, we compute the utility of a citizen outside the interstate compact and the total utility. Finally, we compute the probability of an election inversion.

3.1 Evaluation of the Power of a Citizen Living inside the Interstate Compact: A New Square Root Rule

The Banzhaf power of any citizen in a state in the interstate compact of L states is the Banzhaf power of the citizen in the coalition multiplied by the Banzhaf power of the representative of the interstate compact in the upper tier. The upper tier is a weighted majority game with $K - L + 1$ representatives. The representative of the coalition has a weight of L while the others have a weight of 1. The majority quota is $Q = \frac{K+1}{2}$.

It is straightforward to show that, whenever $L \leq Q$, the Banzhaf power of the representative in the upper-tier is equal to:

$$\sum_{i=Q-L}^{Q-1} \frac{H!}{(i!)\,((H-i)!)} \frac{1}{2^H}$$

where $H \equiv K - L$. Further, since the winner in the interstate compact is the majority outcome in the coalition, the Banzhaf power of any citizen in the interstate compact is[45]:

$$\frac{(L(2r+1)-1)!}{\left(\frac{L(2r+1)-1}{2}\right)!^2}\frac{1}{2^{L(2r+1)-1}}$$

In what follows, we will concentrate on the case where r is a large number. From Stirling's formula, we deduce that the above expression is approximately equal to:

$$\frac{1}{\sqrt{\pi r L}}$$

When r is large, the Banzhaf power of a citizen in the interstate compact is therefore approximately:

$$\frac{1}{\sqrt{\pi r L}}\sum_{i=Q-L}^{Q-1}\frac{H!}{(i!)\left((H-i)!\right)}\frac{1}{2^H}$$

Up to the value of $\frac{1}{\sqrt{r}}$ which enters multiplicatively into that formula, we obtain that the Banzhaf power of any such citizen if equal to:

$$\Phi_{IC}(K,L) = \frac{1}{\sqrt{\pi L}}\sum_{i=Q-L}^{Q-1}\frac{H!}{(i!)\left((H-i)!\right)}\frac{1}{2^H}$$

Note that the function $\Phi_{IC}(K,L)$ is defined for all $1 \leq L \leq Q$. When $L \geq Q$, the Banzhaf power of the representative of the interstate compact is equal to 1 and the Banzhaf power of a citizen of the interstate compact is then:

$$\Phi_{IC}(K,L) = \frac{1}{\sqrt{\pi L}}$$

We derive the first easy conclusion that there is certainly nothing to gain from the perspective of the interstate compact to have an extra state as soon as this interstate compact contains a majority of states as only the negative force is at work. When the size L is smaller than Q, things are more subtle as, on one hand (as already noted) $\frac{1}{\sqrt{\pi L}}$ decreases with L, but on the other hand, $\sum_{i=Q-L}^{Q-1}\frac{H!}{(i!)((H-i)!)}\frac{1}{2^H}$ increases with L. How the graph of Φ_{IC} looks like and in particular what is the value of L for which $\Phi_{IC}(K,L)$ is maximal?

The numerical evidence resulting from programming[46] the function Φ_{IC} shows that it is single peaked with a peak at (up to integer rounding) $c\sqrt{L}$ where the constant c is about 1.4.

[45] When L is odd. When L is even, it is slightly different. We obtain something similar when we break the ties by either choosing one candidate or by flipping a coin.

[46] As already pointed out, here, the numerical analysis of pivotality uses directly the mathematical expressions which have been derived. Unsurprisingly, we obtained exactly the same results when using *SimuElect*.

We have not being able to prove analytically that this was true. On figure 1 below, we have plotted the graph of $\Phi_{IC}(51, L)$: $\Phi_{IC}(51, L)$ corresponds to the curve called inside.

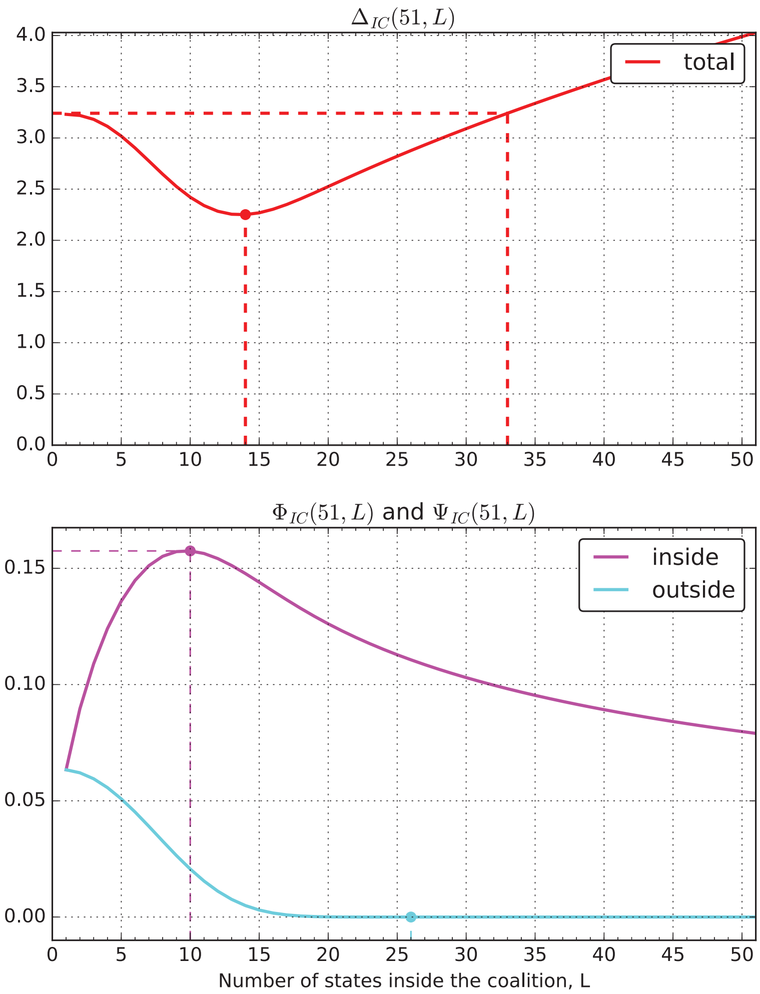

Figure 1: Graph of $\Phi_{IC}(51, L)$.

Starting from the initial Electoral College, the utility of a citizen in the interstate compact increases until $\lfloor 1.4\sqrt{51} \rfloor = \lfloor 9.998\,0 \rfloor = 10$ and decreases then continuously until $L = 51$. Since the popular vote is superior to the Electoral College, we have also: $\Phi_{IC}(51, 51) > \Phi_{IC}(51, 1)$.

We were able[47] to show more generally that $\Phi_{IC}(K, L)$ was maximized at a unique value $L^*(K)$ for all $K \leq 51019531$ and that $\frac{L^*(K)}{\sqrt{K}}$ converges to a a constant that is close to 1.4.

[47] We have similar figures for $K = 101, 1001$ and 10001.

3.2 Evaluation of the Power of a Citizen Living in a State outside the Interstate Compact

It is straightforward to show that, whenever $L \leq Q$, the Banzhaf power in the upper-tier of the representative of a state outside the interstate compact is equal to:

$$\frac{\binom{H-1}{Q-1}}{2^{H-1}}.$$

Further, the Banzhaf power of any citizen in his state is:

$$\frac{(2r)!}{((r)!)^2}\frac{1}{2^{2r}}.$$

Thus, when r is large, the Banzhaf power of a citizen living in a state outside the interstate compact is approximately equal to:

$$\frac{1}{\sqrt{\pi r}}\frac{\frac{(H-1)!}{((Q-1)!)((H-Q)!)}}{2^{H-1}}$$

i.e. up to the multiplicative factor $\frac{1}{\sqrt{r}}$:

$$\Psi_{IC}(K,L) \equiv \frac{1}{\sqrt{\pi}}\frac{\frac{(H-1)!}{((Q-1)!)((H-Q)!)}}{2^{H-1}}.$$

This function is a decreasing function of L over the whole range of values. Indeed, since:

$$\Psi_{IC}(K,L+1) = 2\frac{H-Q}{H-1}\Psi_{IC}(K,L)$$

this follows from the fact that $2\frac{H-Q}{H-1} \leq 1$ when $L \geq 0$. Further, since from Felsenthal and Machover (1998, Theorem 3.2.18):

$$\sqrt{L+1}\Phi_{IC}(K,L+1) = \sqrt{L}\Phi_{IC}(K,L) + \Psi_{IC}(K,L)$$

we deduce by induction[48] that: $\Psi_{IC}(K,L+1) \leq \Phi_{IC}(K,L+1)$. Indeed, if $\Psi_{IC}(K,L) \leq \Phi_{IC}(K,L)$, then from the above equality, we deduce that:

$$\Phi_{IC}(K,L+1) \geq \frac{\sqrt{L}+1}{\sqrt{L+1}}\Psi_{IC}(K,L) > \Psi_{IC}(K,L) > \Psi_{IC}(K,L+1)$$

i.e. that it is preferable to be in rather than out of the interstate compact. Note also that $\Psi_{IC}(K,L)$ is equal to 0 when L reaches the value Q. This means that the interstate compact exerts a negative externality on the citizens outside the interstate compact. When L remains

[48] Note that $\Psi_{IC}(K,1) = \Phi_{IC}(K,1)$.

smaller than $L^*(K)$, increasing L benefits to those already in the interstate compact and to the newcomers while it hurts those who remain outside. For values of L larger than $L^*(K)$, the utility of insiders goes down as well as the utility of remaining outsiders. The happy few are those entering in the interstate compact. This means that if the interstate compact could enact barriers to entry, then it will find profitable to do so when the size $L^*(K)$ is reached.

In the case where $K = 51$, we obtain the graph describing the evolution of the decisiveness of the citizens outside the coalition as a function of L; this graph appears on figure 1 above under the label outside. We observe that it is indeed much below the decisiveness of the citizens inside the coalition and continuously decreasing.

3.3 Evaluation of the Sensitivity

When L changes, not only the per capita utilities of citizens in and out of the interstate compact but the numbers of citizens in the two groups also change through the process. The net effect on total utility is therefore unclear. Sensitivity is here equal to

$$(K-L)(2r+1)\left[\frac{\frac{(H-1)!}{((Q-1)!)((H-Q)!)}}{2^{H-1}}\times\frac{1}{\sqrt{\pi}}\right]+L(2r+1)\times\frac{1}{\sqrt{\pi L}}\sum_{i=\frac{K+1}{2}-L}^{\frac{K-1}{2}}\frac{(K-L)!}{(i!)\,((K-L-i)!)}\frac{1}{2^{K-L}}$$

i.e. up to the multiplicative constant $\frac{2r+1}{\sqrt{\pi r}}$

$$\Delta_{IC}(K,L)\equiv\frac{1}{2^{H-1}}(K-L)\left[\frac{(H-1)!}{((Q-1)!)\,((H-Q)!)}\right]+\frac{1}{2^{H-1}}\times\frac{\sqrt{L}}{2}\sum_{i=\frac{K+1}{2}-L}^{\frac{K-1}{2}}\frac{(K-L)!}{(i!)\,((K-L-i)!)}$$

when $L\leq Q$, and

$$\Delta_{IC}(K,L)\equiv\sqrt{L}$$

when $L\geq Q$.

Let us first examine for which values of L the sensitivity of the mechanism for $L \geq Q$ is at least equal to the sensitivity $\Delta_{IC}(K,1)$ of the initial Electoral College. Using Stirling's formula we obtain that the sensitivity of the Electoral College is equal to:

$$K(2r+1)\times\frac{1}{\sqrt{\pi r}}\times\frac{(K-1)!}{\frac{(K-1)}{2}!^2}\times\frac{1}{2^{K-1}}\simeq K(2r+1)\times\frac{1}{\sqrt{\pi r}}\times\sqrt{\frac{2}{\pi K}}\text{ if } K \text{ is large enough.}$$

When $L\geq Q$, $\Delta_{IC}(K,L)=\sqrt{L}$ is larger than the sensitivity of the Electoral College when:

$$\sqrt{L} \geq \sqrt{\frac{2K}{\pi}} \text{ i.e. } \frac{L}{K} \geq \frac{2}{\pi} \simeq 63.66\%$$

i.e. when about two thirds of the states are in the interstate compact. Besides this specific observation, the complete graph of $\Delta_{IC}(K, L)$ is not obvious to derive analytically. In the case where $K = 51$, we obtain the graph describing the evolution of the sensitivity of the citizens as a function of L which appears on figure 1 above under the label total. We observe that it is single-dipped with a minimum around $L = 14$ and that we need to reach $L = 33$ to recover the level of the Electoral College. If sensitivity is taken as a measure of the performance of the voting mechanism, this computation reveals that the worst situation is experimented for this value of L. We note that this value is slightly to the right of the value of L for which the decisiveness of voters inside the coalition is maximal. So the lesson here is that, if a single interstate forms, the total benefit of the insiders is much below than the total cost that they inflict to the outsiders. Unless L is very large, we end up in a situation which is worst than the initial Electoral College. We do not have a precise conjecture on how the value of L minimizing $\Delta_{IC}(K, L)$ changes with K but a square root principle seems at work in that case too[49].

From the Penrose formula, we know that the evolution of individual utilities on one hand and the total utility on the other hand is the same as the evolution of individual pivot probabilities on one hand and sensitivity on the other hand. Unsurprisingly, this is exactly what is reflected on figure 2 below.

Our analysis does not say that a partition of the K states into an interstate compact of $L^*(K)$ states on one side and $K - L^*(K)$ isolated states on the other side is stable. We have just shown that this scenario maximizes the utility of the members of the interstate compact. We also shown that it does so at the expense of the general interest. Likely, the states outside the interstate compact would react to such a move.

This raises the following general question. For any partition $\Lambda = \left\{S^1, ..., S^M\right\}$ of the states into coalitions (where the representatives of each coalition give their votes to the popular majority winner in the coalition), we can compute the expected utility $U_i(\Lambda)$ of any citizen i. These utilities are always the same for citizens living in the same state and will be denoted $U^k(\Lambda)$ where:

$$U^k(\Lambda) = \frac{1}{\sqrt{|S^{m(k)}|}} B_{m(k)}(\Lambda)$$

[49]We can show that for $K = 101$ and $K = 1001$, the picture remains qualitatively the same for a larger number of states i.e. $\Delta_{IC}(K, L)$ is single-dipped with a minimum at $L = L^{**}(K)$. We speculate that $L^{**}(K)$ is also a small number as compared to K but we do not have any specific mathematical conjecture to report on its asymptotic behavior.

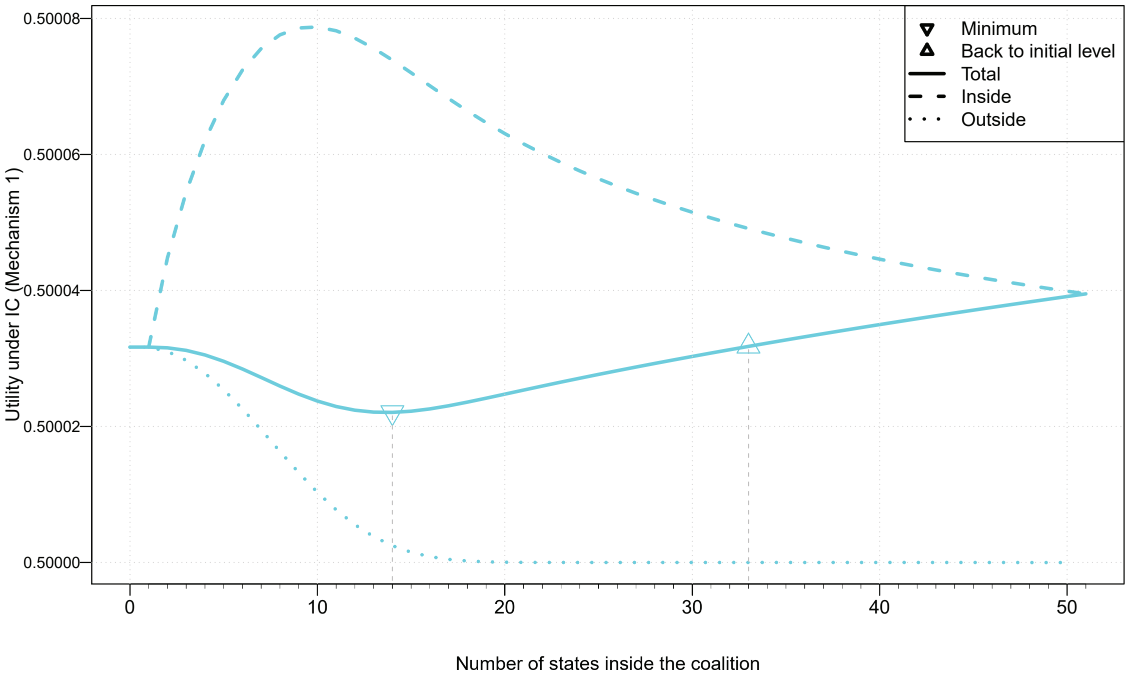

Figure 2: Graphs of individual and total utilities.

with $m(k)$ denoting the index of the coalition to which state k belongs and $B_m(\Lambda)$ denotes the Banzhaf power of the representative of coalition m in the upper tier when the upper tier consists of a weighted majority game with M representatives where representative m has a voting weight equal to $|S^m|$. This raises a side question question of independent interest. For an arbitrary weighted majority game $\left[Q; w^1, w^2, ..., w^K\right]$ with K voters, how the Banzhaf index of the voters changes when some voters act together by adding their weights. For instance, how $B_1\left[Q; w^1, w^2, w^3, ..., w^K\right]$ and $B_2\left[Q; w^1, w^2, w^3, ..., w^K\right]$ compare to $B_{12}\left[Q; w^1 + w^2, w^3, ..., w^K\right]$. It increases but it important to know by how much. Felsenthal and Machover (1998) report an intriguing result[50]: For any electoral mechanism and for any pair of players, the sum of their Banzhaf powers is equal to their Banzhaf power if they form a coalition. This additivity result does not extend of course to coalitions of size 3 or more.

The above ingredients define a *game in partition form* (Bloch (1996), Hart and Kurz (1983)). They exist several notions of stability in the vein of the core concept. For instance, a partition $\Lambda = \left\{S^1, ..., S^M\right\}$ is said to be $\delta-$stable if there does not exist a coalition T such that:

$$U^k(\Lambda_T) > U^k(\Lambda) \text{ for all } k \in T$$

where Λ_T denotes the partition $\left\{T, S^1 \backslash T, ..., S^M \backslash T\right\}$. It is said to be $\gamma-$stable if there does

[50]Theorem 3.2.18 that we have already used.

not exist a coalition T such that:

$$U^k\left(\widetilde{\Lambda}_T\right) > U^k(\Lambda) \text{ for all } k \in T$$

where $\widetilde{\Lambda}_T$ denotes the partition $\{T, T^1, ..., T^R\}$ where the R coalitions T^R are formed as follows: either $S^m \cap T \neq \emptyset$ and $\neq T$ and then the coalition $S^m \backslash T$ decomposes itself into singletons or $S^m \cap T = \emptyset$ and the coalition S^m remains as such. Given the fact that externalities are negative, the γ scenario eases coalitional deviations.

Our square root result sheds light on the incentives of coalitions in this game but more work is needed to characterize stable partitions if any. For sure, there is no $\gamma-$coalition structure Λ with one coalition T and $\{1, ..., K\} \backslash T$ decomposed into singletons. In such a case the graph of $U^k(\Lambda)$ for $k \in T$ is single-peaked with a maximum in $\left\lfloor 1.4\sqrt{K} \right\rfloor$. So $\gamma-$stability imposes $|T| = \left\lfloor 1.4\sqrt{K} \right\rfloor$. But if $|T| = \left\lfloor 1.4\sqrt{K} \right\rfloor$, we do not get $\gamma-$stability either as any coalition $S \subseteq \{1, ..., K\} \backslash T$ with $|S| = \left\lfloor 1.4\sqrt{K} \right\rfloor$ can profitably deviates. Note in particular that the grand coalition is not δ or γ stable because it is always profitable to leave one state out.

3.4 Election Inversions

In this section, we evaluate the probability $\Xi(K, L)$ of election inversions when an interstate compact of L states forms. We can derive an analytic form of the probability inversions if r is large[51] but here we only report results obtained through the use of SimuElect. When $K = 51$ and $r = 10^6$, our program delivers the following graph:

The function is single peaked with a peak around $L = 13$. As expected, the function is the same as the one depicted on figure 3 for the range $[26, 51]$. The value of $\Xi_{IC}(51, L)$ jumps from a theoretical value slightly above 20% to a value larger than 30% when $L = 13$. For a large range of values of L, it stays above the original value of 20%. In fact, like for sensitivity and total utility, the interstate compact is beneficial from the perspective of minimizing election inversions when L is above 34.

4 Analysis of a MRPV under IAC* in a Toy Symmetric Version of the Electoral College

This section replicates the analysis of the preceding section when preferences are drawn according to IAC^*. Like IC, this a priori probabilistic model is neutral but now the votes are

[51] These developments are reported in appendix 3. They suggest an alternative program to perform numerical simulations.

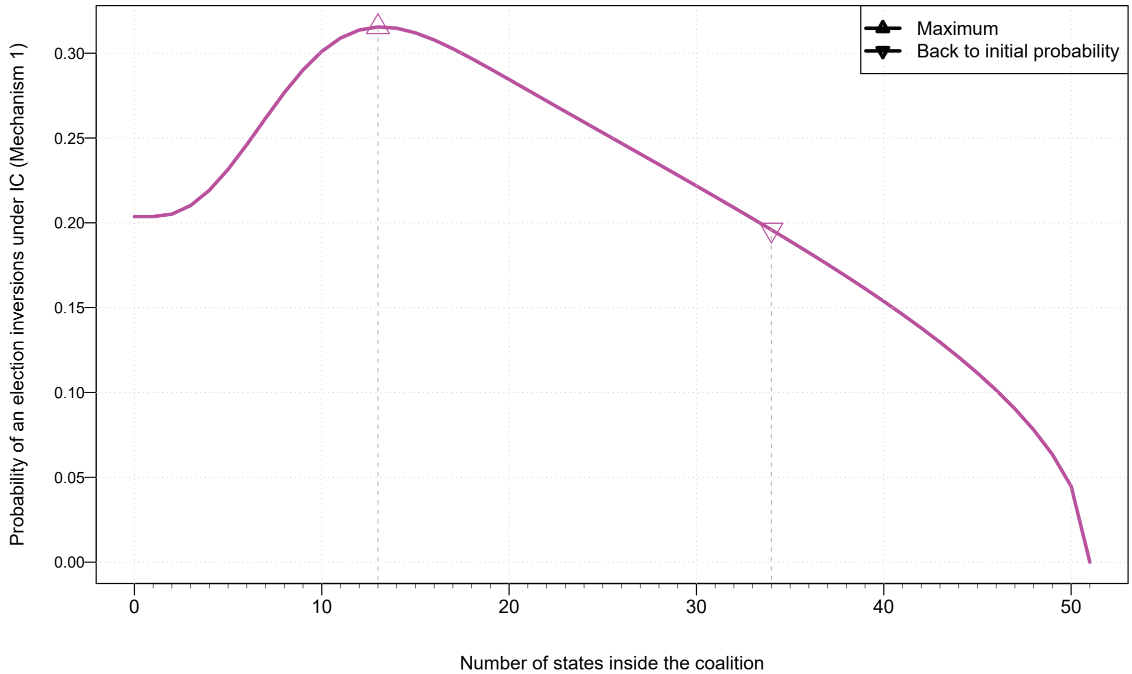

Figure 3: Graph of $\Xi_{IC}(51, L)$.

correlated within each state. The main objective of this section is to show that the conclusions derived under IC remain valid. As before, we first compute numerically the power of a citizen inside the interstate compact as a function of the size L of the interstate compact. Then, we compute the power of a citizen outside the interstate compact and the total power (sensitivity). We also compute the probability of an election inversion. Finally, we compute the utilities of citizens inside and outside the interstate compact. In contrast to IC, this new subsection is needed since the Penrose's formula does not hold for IAC^*.

4.1 Evaluation of the IAC* Power of a Citizen Living inside (or outside) the Interstate Compact

The IAC^* power of any citizen in a state in the interstate compact of L states is the IAC^* power of the citizen in the interstate compact multiplied by the Banzhaf power of the representative of the interstate compact in the upper tier. The upper tier is a weighted majority game with $K - L + 1$ representatives. The representative of the interstate compact has a weight of L while the others have a weight of 1. The majority quota is $Q = \frac{K+1}{2}$.

Since the winner in the coalition is the majority outcome in the interstate compact, we deduce from Le Breton et al. (2016) that when r is large, the IAC^* power of any citizen in the interstate compact is approximately:

$$\frac{c_L}{L(2r+1)}$$

where the constant c_L is reported in the following tables for the first[52]values of L.

L	5	6	7	8	9	10	11	12	13	14	15	16
c_L	2.995	3.3	3.577	3.835	4.076	4.304	4.521	4.727	4.925	5.115	5.298	5.476
L	17	18	19	20	21	22	23	24	25	26	27	28
c_L	5.647	5.814	5.976	6.134	6.288	6.438	6.584	6.728	6.870	7.006	7.143	7.273
L	29	30	31	32	33	34	35	36	37	38	39	40
c_L	7.408	7.531	7.657	7.781	7.903	8.023	8.141	8.257	8.372	8.485	8.597	8.708
L	41	42	43	44	45	46	47	48	49	50	51	
c_L	8.817	8.924	9.031	9.136	9.240	9.342	9.444	9.545	9.644	9.743	9.840	

Table 1 : Exact values of c_L for the first values of L

When r is large, the IAC^* power of a citizen in the interstate compact is therefore:

$$\frac{c_L}{L(2r+1)} \sum_{i=Q-L}^{Q-1} \frac{H!}{(i!)\,((L-i)!)} \frac{1}{2^H}$$

Up to the value of $\frac{1}{2r+1}$ which enters multiplicatively into that formula, we obtain that the Banzhaf power of any such citizen if equal to:

$$\Phi_{IAC*}(K,L) = \frac{c_L}{L} \sum_{i=Q-L}^{Q-1} \frac{H!}{(i!)\,((H-i)!)} \frac{1}{2^H}$$

Note that the function $\Phi_{IAC*}(K,L)$ is defined for all $1 \leq L \leq Q$. When $L \geq Q$, the Banzhaf power of the representative of the interstate compact is equal to 1 and the Banzhaf power of a citizen of the interstate compact is then:

$$\Phi_{IAC*}(K,L) = \frac{c_L}{L}$$

Like for IC, there is nothing to gain from the perspective of an interstate compact to welcome an extra state as soon as this interstate contains a majority of states. When the size L is smaller than Q, we face a tradeoff similar to the tradeoff identified in the IC case. On one hand[53] $\frac{c_L}{L}$ decreases with L but on the other hand $\sum_{i=Q-L}^{Q-1} \frac{H!}{(i!)((H-i)!)} \frac{1}{2^H}$ increases with L. How the graph of $\Phi_{IAC*}(K,L)$ looks like and in particular what is the value of L for which the power is maximal?

[52]It is conjectured in Le Breton et al. (2016) that $c_L \simeq \sqrt{\frac{6L}{\pi}}$ when L is large enough.

[53]We do not have a mathematical proof of this assertion and conjecture that it holds true.

We also observe that the IAC^* power of a citizen living in a state outside the interstate compact is:

$$\Psi_{IAC^*}(K,L) = \frac{\frac{(H-1)!}{((Q-1)!)((H-Q)!)}}{2^{H-1}} \times \frac{1}{2r+1}$$

i.e. it is the same as in the IC case up to the multiplicative factor $\frac{1}{2r+1}$.

Our direct numerical leads to the function $\Phi_{IAC^*}(K,L)$ is also single peaked with a peak very close to the the peak of $\Phi(K,L)$. On figure 4 below, we have plotted the graph of $\Phi_{IAC^*}(51,L)$ together with the graph of $\Psi_{IAC^*}(51,L)$. They appear respectively under the headings inside and outside.

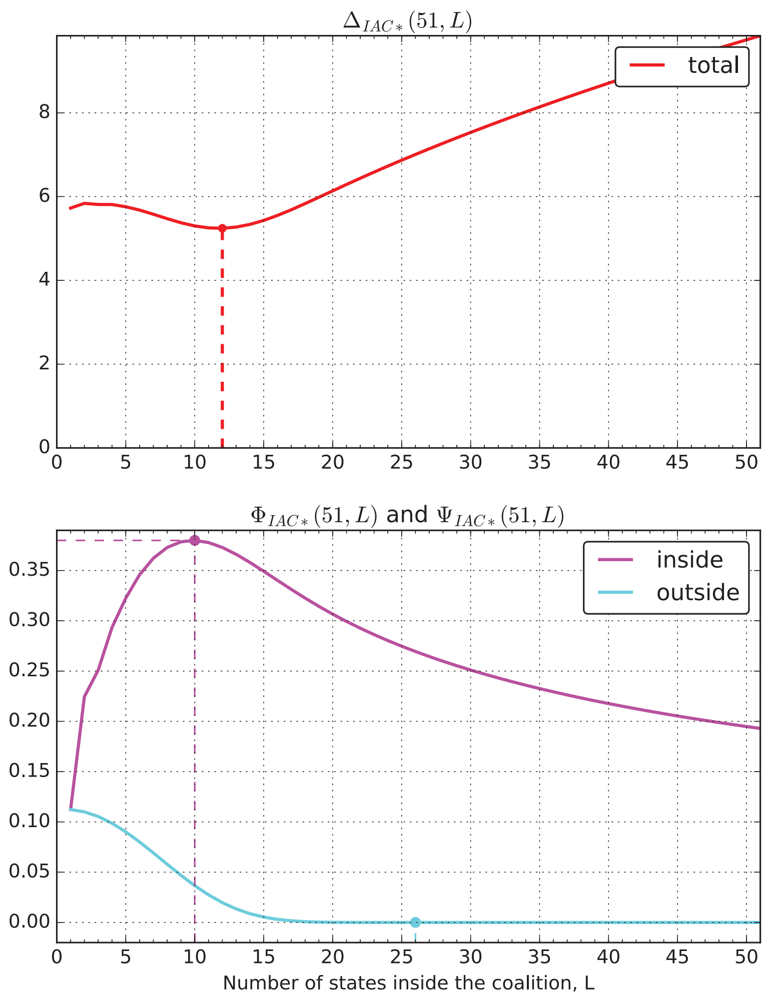

Figure 4: Graph of $\Phi_{IAC^*}(51,L)$ and $\Psi_{IAC^*}(51,L)$.

Starting from the initial Electoral College, the utility of a citizen in the coalition increases until 10 and decreases then continuously until $L = 51$[54].

On the basis of this numerical analysis, we cannot make any difference between the IC and IAC^* settings. They share the property that there are no interstate correlations between the votes but they differ strongly from the perspective of intrastate correlations between the votes. We do not have any intuitive explanation to report on the reasons explaining this intriguing coincidence.

4.2 Evaluation of the IAC* Sensitivity

Summing up the IAC^* powers of citizens inside and outside the coalition of states leads to the following expression of the IAC^*sensitivity:

$$\Delta_{IAC^*}(K,L) = (K-L)(2r+1)\left[\frac{\frac{(H-1)!}{((Q-1)!)((H-Q)!)}}{2^{H-1}} \times \frac{1}{2r+1}\right]$$

$$+L(2r+1) \times \frac{c_L}{L(2r+1)} \sum_{i=\frac{K+1}{2}-L}^{\frac{K-1}{2}} \frac{(K-L)!}{(i!)\left((K-L-i)!\right)} \frac{1}{2^{K-L}}$$

i.e.:

$$\Delta_{IAC^*}(K,L) = \frac{1}{2^{H-1}}(K-L)\left[\frac{(H-1)!}{((Q-1)!)\left((H-Q)!\right)}\right] + \frac{1}{2^{H-1}} \times \frac{c_L}{2} \sum_{i=\frac{K+1}{2}-L}^{\frac{K-1}{2}} \frac{(K-L)!}{(i!)\left((K-L-i)!\right)}$$

when $L \leq Q$. And:

$$\Delta_{IAC^*}(K,L) \equiv c_L$$

when $L \geq Q$.

The graph[55] of $\Delta_{IAC^*}(51, L)$ appears on figure 4 above under the heading total.

We can hardly make any difference between $\Delta_{IAC^*}(K,L)$ and $\Delta(K,L)$. They are both single-dipped with a minimum for values of L which are almost the same. There is however a small difference between the two when we zoom on the area located near the origin. Indeed instead of decreasing straight from $L = 0$ as in the IC setting, $\Delta_{IAC^*}(K,L)$ increases over a small interval to reach a local maximum around a small value of L when $K = 51$.

Besides this peculiarity, the IC and IAC^* settings display the same features.

[54]The same shape shows up for all values of K that we have explored including $K = 101, 1001$ and 10001.

[55]The same shape shows up for $K = 101$ and 1001.

4.3 Election Inversions under IAC*

Like for the IC case, we can derive an analytic form of the probability inversions if r is large[56] but here we only report results obtained through the use of SimuElect. The graph of $\Xi_{IAC^*}(51, L)$ appears on figure 5 below.

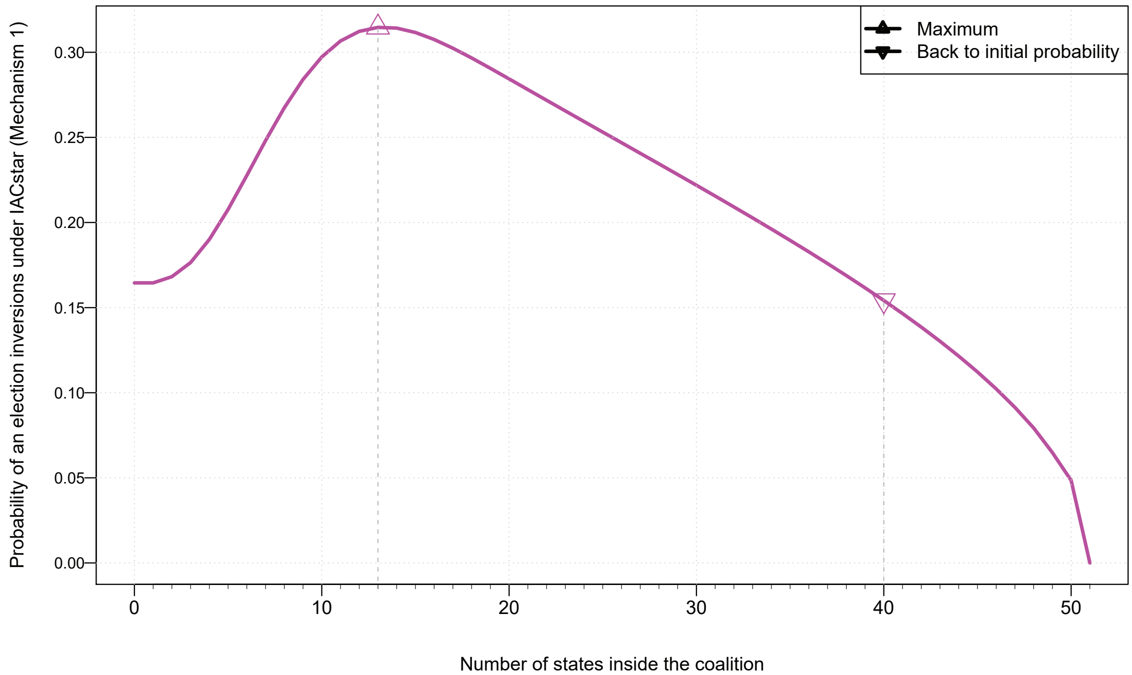

Figure 5: Graph of $\Xi_{IAC^*}(51, L)$.

We cannot make any qualitative difference between the probabilities in the IC and IAC^* cases. Again the function $\Xi_{IAC^*}(51, L)$ is single peaked. It jumps from a value close to 17% to a value above 30% when L is around 14. Further, we need now to reach the value $L = 40$ to recover the value of the probability for the Electoral College.

4.4 Total Utility under IAC*

In appendix 5, we present an analytical form for individual and total utilities. In this section, we report only results obtained through the use of SimuElect. On figure 6 below, we have reported the graphs of the individual expected utilities (inside and outside) and total expected utility as a function of L when $K = 51$.

Interestingly, the graphs display the same features as those obtained in the IC case. and as long as we interested in total utility, it echoes exactly the comparisons drawn from either

[56]These developments are reported in appendix 4. Like for the IC case, they suggest an alternative program to perform numerical simulations.

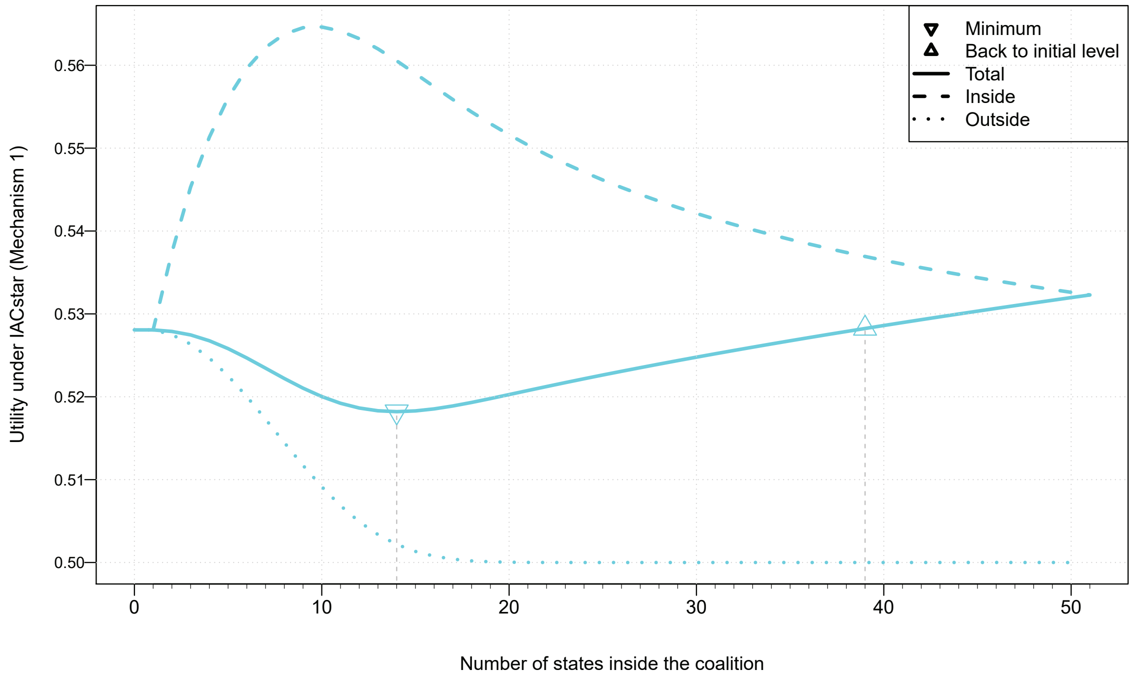

Figure 6: Graphs of individual and total utilities.

sensitivity or the probability of election inversions. Since the Penrose formula does not hold in the IAC^*, we could not anticipate such perfect correlation between the three objectives.

5 Analysis of a MRPV under IAC in a Toy Symmetric Version of the Electoral College

This section is entirely dedicated to the analysis of MRPV under IAC. Besides the limit case of the popular vote, it is indeed extremely difficult to derive closed form expression of the different probabilities and utilities of interest. The strong correlation among *all* the votes complicates in a very severe way the analysis of the upper tier of the Electoral College. To see why, consider for instance the case where $L = 1$. The probability that a voter from state 1 is pivotal is now the probability that he is pivotal in state 1 multiplied by the probability that electoral vote of state 1 is pivotal in the second tier conditional upon the information that the electoral popular margin was equal to 1 in state 1. Conditioning matters. The complexities that arise even in the simplest cases $K = 2$ and $K = 3$ are presented in Appendix 2. In fact, even the analysis of the Electoral College under IAC remains largely unexplored besides our recent paper (de Mouzon et al. (2017)). Under IAC, the high correlation among the votes implies that the probability of pivot events is much lower than under IC or IAC^*. Further, the differences between the different voting mechanisms including the popular vote and the Electoral College are also much lower. In particular, as demonstrated by de Mouzon et al. (2017), the probability of election inversions, which always remains distant from zero under IC or IAC^*, converges to 0 as the inverse of the square root of the population when the population of voters gets larger and larger. Some more peculiar implications of IAC on the relationships between decisiveness and utility and electoral geography are also presented in Appendix 2.

Due to these mathematical complexities, the results presented in this shorter section are derived from SimuElect. Figure 7 presents the graph of the probability of election inversions when $K = 51$ and $r = 10^6$.

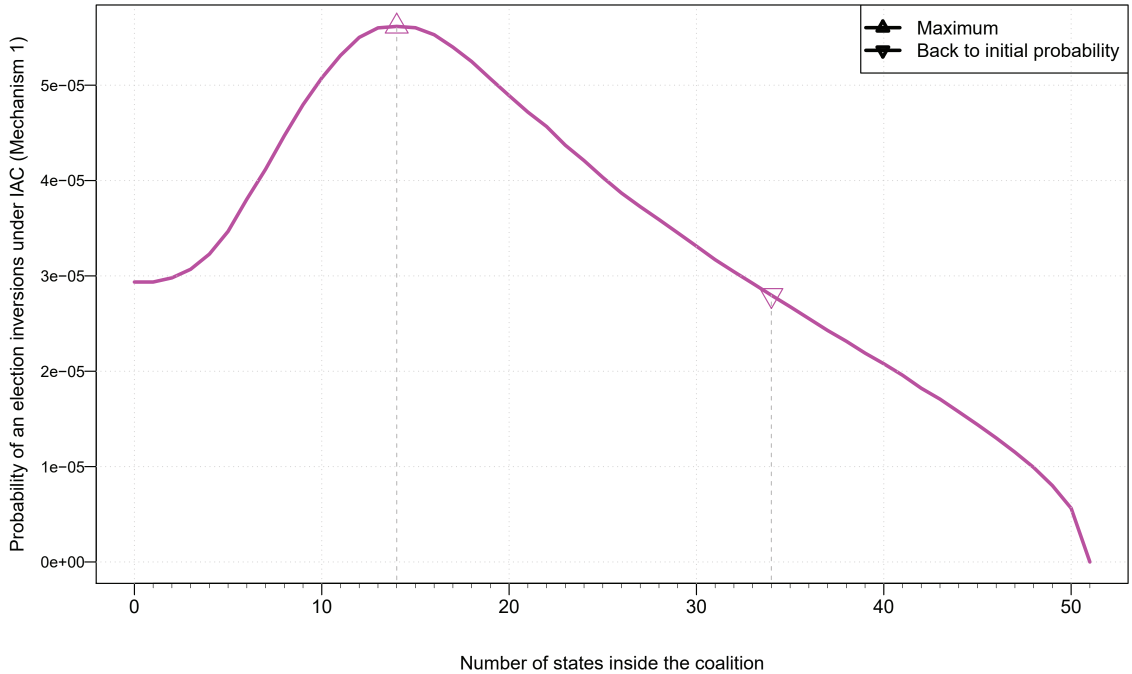

Figure 7: Graph of the IAC probability of election inversions when $K = 51$ and $r = 10^6$.

As already pointed out above, the numbers which appear on the vertical scale are very small as compared to IC or IAC^*. Here the order of magnitude is 10^{-5} while it is 10^{-4} as demonstrated by de Mouzon et al. (2017) when $K = 3$ and $r = 10^6$. The surprise however is that within the margins of this scale, the graph is again single peaked with a maximum obtained when $L = 14$ like in the IC and IAC^* cases.

Figure 8 below reports the graph of the different probabilities of being pivotal.

The curves are highly chaotic. With $r = 10^6$, the events are very rare (the probabilities are of order 10^{-8}) and even a very large number of simulations does not annihilate the effects of randomness. This is why we have reported on figure 9 the graph when $K = 51$ and $r = 10$.

Now, the probabilities are of the order 10^{-3}. The first thing that we observe is that the

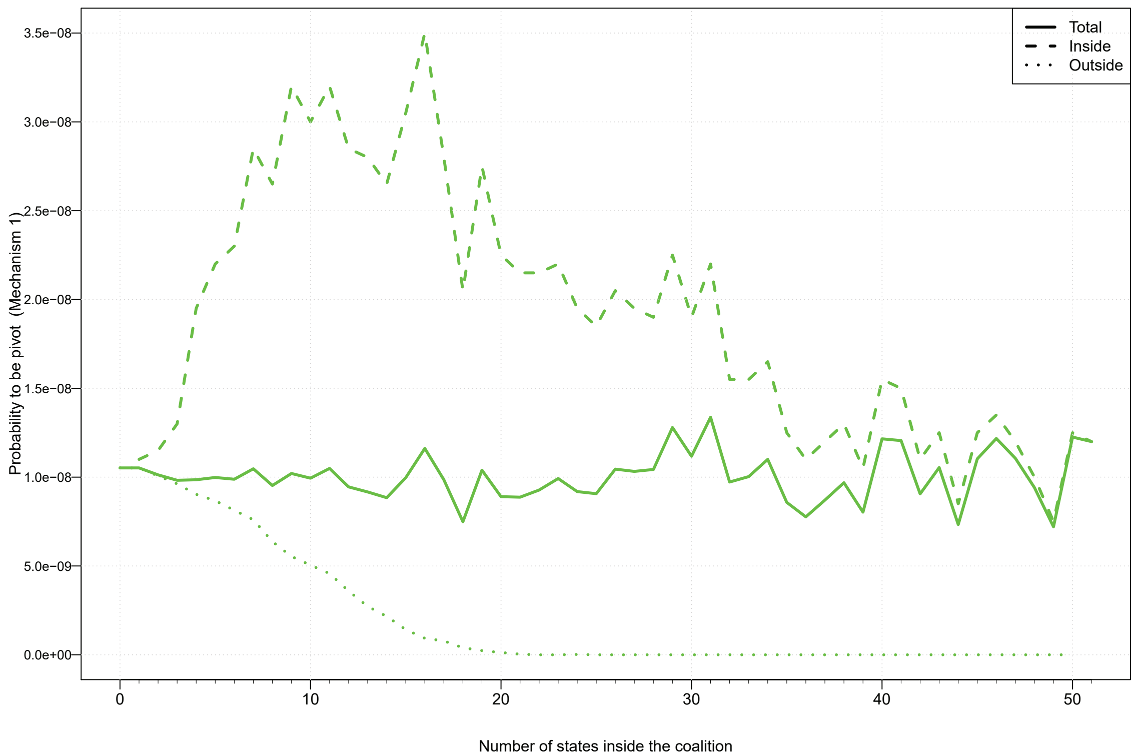

Figure 8: Graph of the IAC probability of being pivotal when $K = 51$ and $r = 10^6$.

average probability is constant over the whole range[57]. Further, the probability of being pivotal of the insiders is single peaked with a peak around $L = 14$ while the probability of the voters outside the coalition decreases and tends rapidly to very low values.

We conclude this section by presenting on figure 10 the graphs describing the evolution of individual and total utilities in the case where $r = 10$, since again high values of r leads to some chaos in the curves making more difficult the interpretation.

We can see that in contrast to the average decisiveness curve, the average utility curve is not constant. There is a minimum around $L = 15$ and the value in $L = 0$ is not equal to the value in $L = 51$. On the other hand, the value oscillates in an interval of an order equal to 10^{-3}. Further, the graph of the curve describing the evolution of the utility of the insiders is single peaked with a peak around $L = 10$ while the curve describing the evolution of the utility of the outsiders is single dipped with a minimum around $L = 17$. This last feature does not appear under IC and IAC^*.

In spite of some differences between IAC and IC and IAC^*, it remains that many properties of the mechanism MRPV show up in the three models. In particular, from an utilitarian

[57] This is not surprising. Indeed, the IAC average decisiveness curve is always constant since for any simple game, the sum of the IAC individual probabilities of being pivotal is equal to 1.

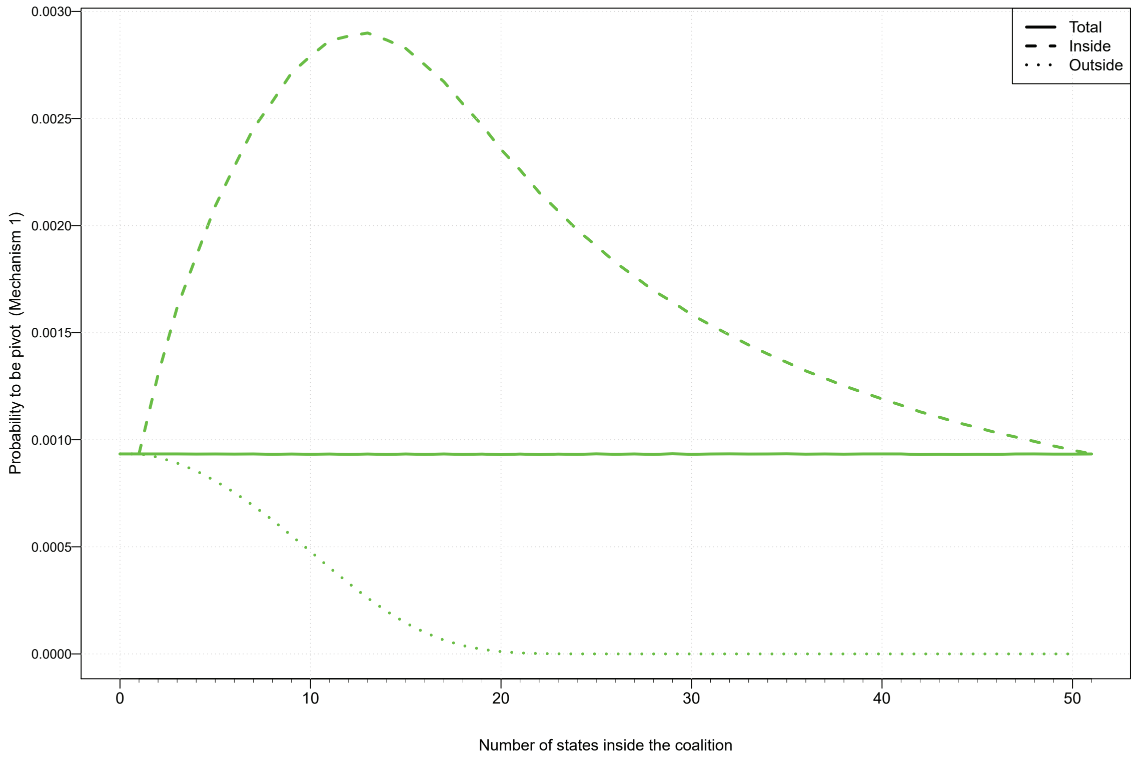

Figure 9: Graph of the IAC probability of being pivotal when $K = 51$ and $r = 10$.

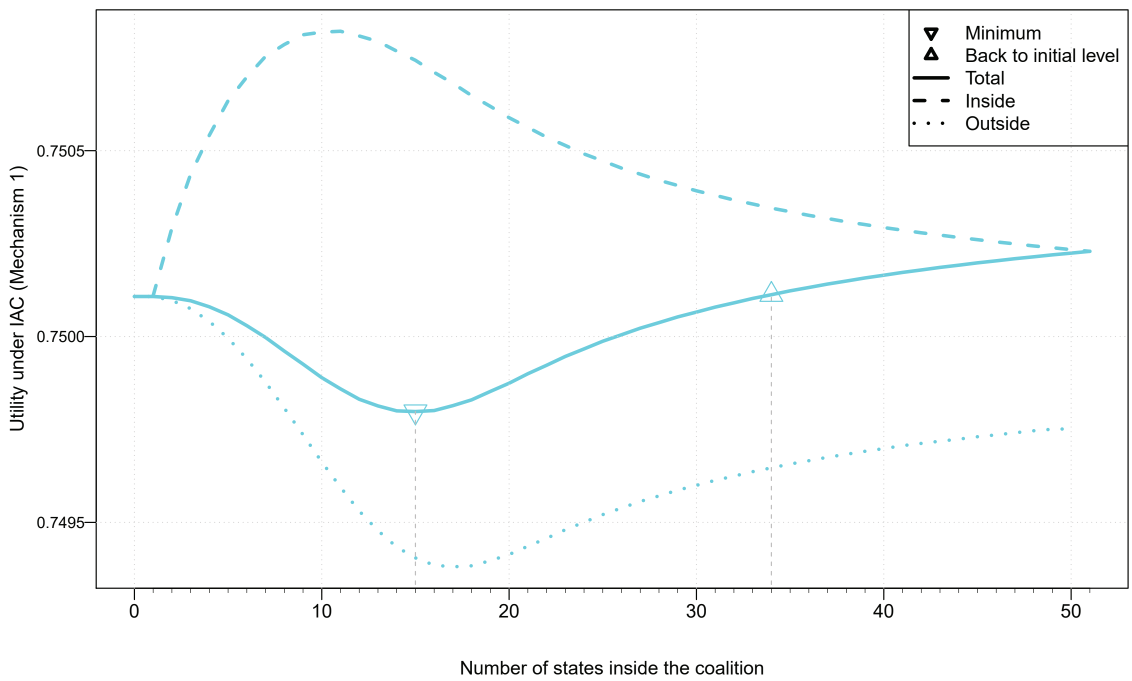

Figure 10: Graph of the IAC utilities when $K = 51$ and $r = 10$.

perspective we observe that this mechanism is worse than the Electoral College and that the interstate compact has to be quite large to recover that utility level. They also share the conclusion that the worse situation is obtained from a quite small size of the interstate compact and that there is a strong conflict of interest between insiders and outsiders, with an "optimal" size from the insiders self-interested perspective also quite small.

6 Analysis of MNPV in a Toy Symmetric Version of the Electoral College

In this last section, we turn our attention towards the analysis of the second version of the interstate compact i.e. we assume now that the interstate compact allocates the totality of its electoral votes to the national popular winner. Clearly, the subsequent departure from sovereignty is much more significant than in the regional counterpart as the voters living in states from the interstate compact can vote collectively against their preference: at the extreme, it could be the case that the population of voters from these states prefers unanimously one candidate and still allocates its votes to the other one.

What is the rationale for a state to be part of such arrangement? If we remember that the Electoral College is suboptimal with respect to the popular vote, then we may expect that if the size of the interstate compact is big enough[58], the probability of being pivotal in the second tier will be large enough to compensate the small probability of being pivotal in the national popular vote. *However, in contrast to the regional version, even under IC and IAC^*, we cannot use a simple multiplicative formula to describe this tradeoff.* The multiplicative formula is now conditional[59]. Let i be an arbitrary voter from a state within the interstate compact. The probability of being pivotal denoted $Piv(i)$ is the product of two terms: $Piv(Maj(Km))$ the probability of being pivotal in the popular majority mechanism and $Piv(d$ |the country vote without i is divided equally between D and $R)$ the probability that the block representative from the interstate compact is decisive in the upper tier conditional upon the fact that voter i is pivotal in the country vote. The root of the difficulty is that we cannot ignore conditioning since the votes of the states and the national vote are obviously correlated. The voting outcomes of states outside the interstate are, under IC and IAC^*, independent random variables but they are correlated with the vote of the block of representatives from the interstate. By using the orthant tools presented in the preceding sections, we could determine the probability law (and

[58]Since this is true when the coalition contains a majority of states, it is true when the size of the coalition is close to that size. Note also that this argument does not hold when the interstate contains a single state since the first effect is totally absent.

[59]In appendix 6, we report some analytical arguments to illustrate the importance of conditioning.

therefore the correlations) of the vector of votes of representatives in the second tier.

Given the mathematical complexities that we have just described, we use again SimuElect to evaluate as a function of L (the size of the interstate) of the probability of being pivotal for inside and outside voters, the sensitivity, the utilities of inside and outside voters, the total utility and the probability of election inversions.

6.1 Election Inversions

We will present here the outcome of our simulations in the case where $r = 10^6$ i.e. the total population of the country is around 10^8. Under IC, figure 11 below shows clearly that the probability of an election inversion decreases rapidly from about 20% when $L = 0$ to a value close to 0 when L is equal or larger than 10.

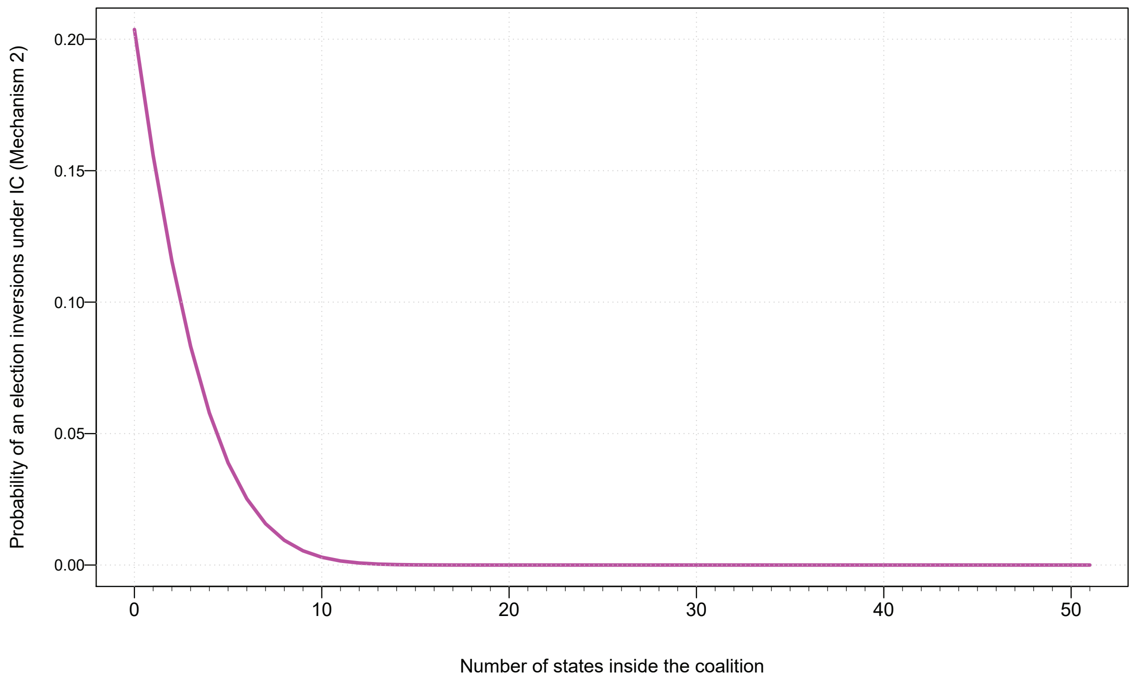

Figure 11: Graph of the IC probability of election inversion when $K = 51$.

Under IAC^*, the situation is similar (except that the value is about 16% when $L = 0$) as illustrated on figure 12 below

This result is striking and to some extend largely unexpected. It shows that the convergence to the popular vote is obtained for relatively low sizes of the coalition. There is no need to reach the critical value of $L = 26$ to get it.

Like always, as illustrated by figure 13 below, under IAC the differences are relatively tiny as compared to the two previous probabilistic models. But the overall picture is the same. The

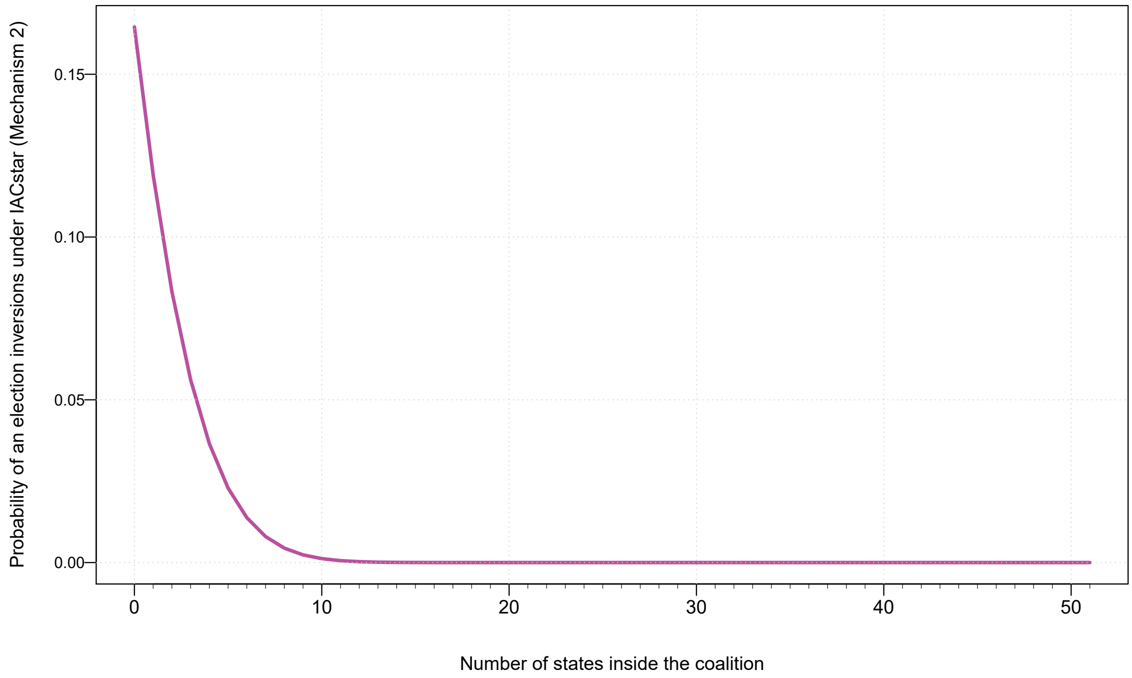

Figure 12: Graph of the IAC^* probability of election inversion when $K = 51$.

probability of election inversion is about 3×10^{-5} when $L = 0$, 5×10^{-6} when $L = 5$ and closer to 0 when L is larger than 10.

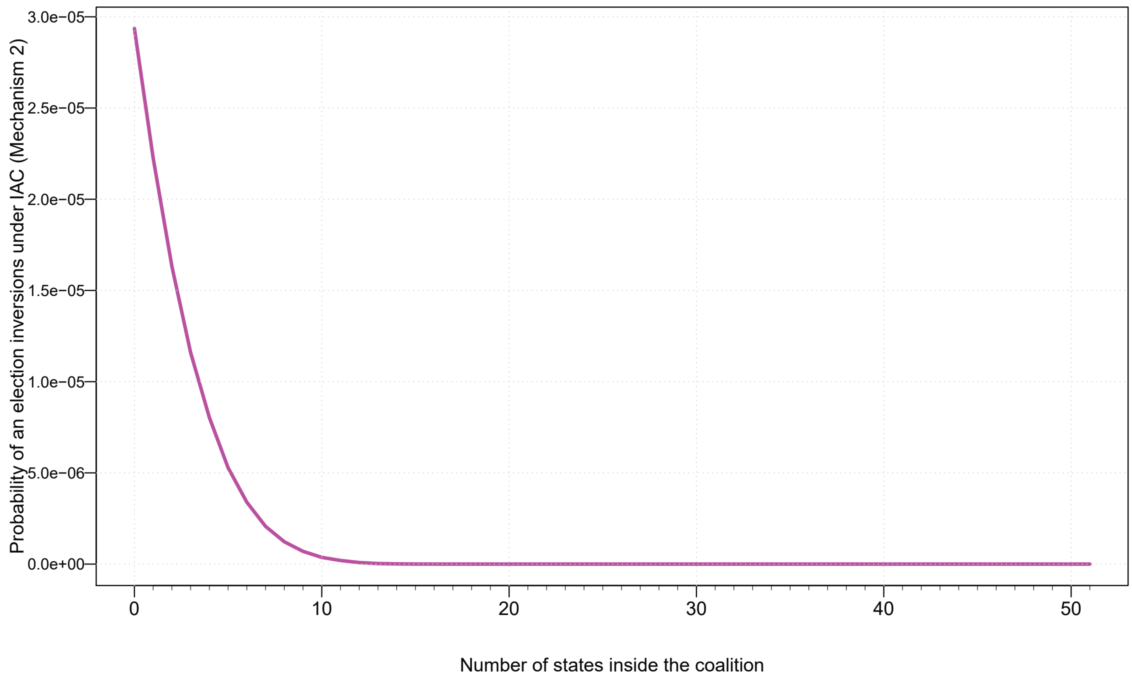

Figure 13: Graph of the IAC probability of election inversion when $K = 51$.

For the three models, the qualitative message is the same. There is no need to wait for $L = 26$ to get the benefit of the popular vote.

6.2 Evaluation of Individual and Total Utilities

Like in the case of the regional version, while election inversion is a excellent indicator of the utilitarian quality of a voting mechanism, it is not a perfect measure. Further it does not tell us how the situation of the voters is impacted depending upon the fact that they are inside or outside the coalition. The graphs of total utility, utility of insiders and utility of outsiders for the three models are displayed on figures 14, 15 and 16. Like in Section 5, for the IAC model, the simulations were performed with $r = 10$ to avoid chaotic behavior.

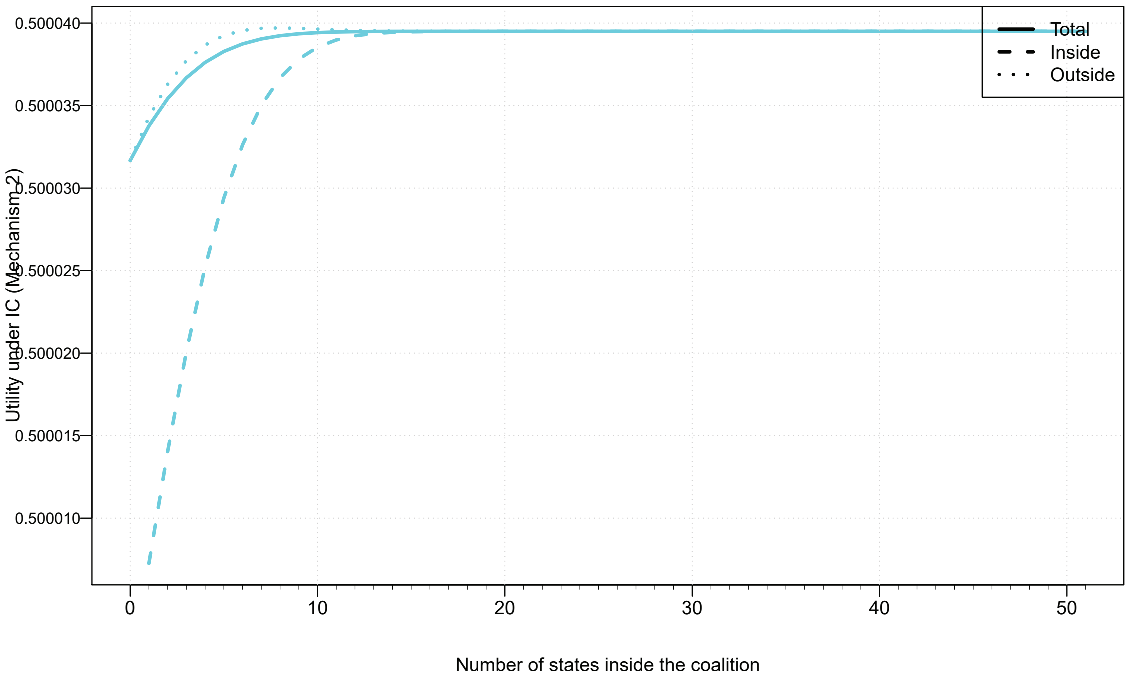

Figure 14: Graph of the IC individual and total utilities when $K = 51$.

If we look at total utility, we have confirmation of what we discovered from the analysis of election inversions. It jumps from 0.50003[60] when $L = 0$ to 0.50004[61] (the expected utility under the popular vote) when L is equal or larger than 10. But the three figures contain also very important information. If we look at the evolution of the utilities of the insiders and the outsiders with respect to L, we see that becoming part of the interstate compact hurts (as compared to the initial Electoral College status quo situation) if the size of the interstate

[60]Precisely, $0.5 + (0.5) \times \frac{2}{10^4 \pi}$.

[61]Precisely $0.5 + (0.5) \times \frac{\sqrt{2}}{10^4 \sqrt{\pi}}$.

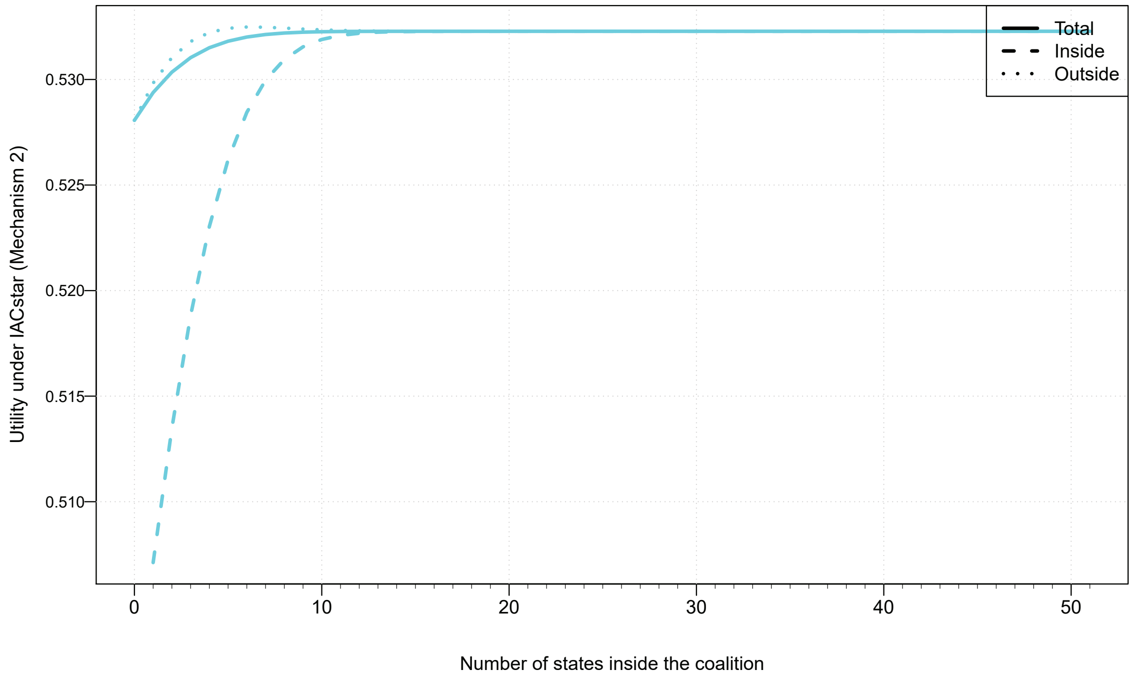

Figure 15: Graph of the IAC^* individual and total utilities when $K = 51$.

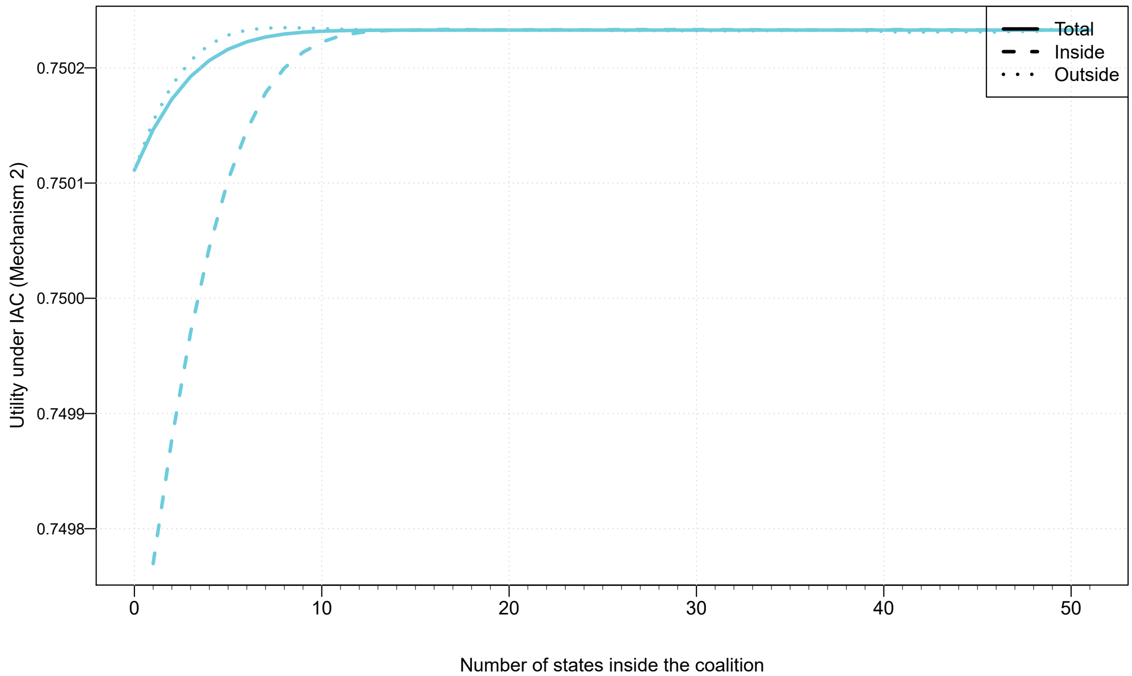

Figure 16: Graph of the IAC individual and total utilities when $K = 51$.

compact is too small. Unsurprisingly, it is certainly not a good idea for a single state to do it as it would abandon its sovereignty without receiving anything in return. This is transparent on the figures. What the numerical calculations also shows is that we need about five states to recover the initial utility level. Then, rapidly, around $L = 10$, the utility of the insiders converges to their utility in the popular vote mechanism. On the other hand we see, as expected, that the utility of the insiders is always strictly above the utility of the outsiders when L is small and that they are about the same when L reaches the value 10. These calculations temperate the overoptimistic conclusion drawn from the analysis of the election inversions. Outsiders free ride on insiders and there is a critical size of the interstate that is needed for the insiders to get something out of this operation. This raises a coordination problem.

The same comments apply to IAC^* and IAC. Unfortunately, the graph for IAC displays some chaotic behavior due to the large population size that we have considered.

6.3 Evaluation of the Power of Citizens Living inside and outside the Interstate Compact

In this last section, we analyze the simulations concerning the average probability of being decisive and the probability of being decisive for the two categories of citizens. We know that there is an exact relationship between utility and pivotality under IC but that we have to be more careful when dealing with $IAC^*$and IAC. The problem with rare events is that with large populations of voters (like the number 10^8 considered in the previous section) we are confronted to some instability. The curves cease to be smooth and display some chaotic features, in particular when the number of simulations is not large enough. Therefore, in this section we will consider the case where $r = 10^3$. The results are reproduced below on figures 17, 18 and 19.

For IC and IAC^* the graphs confirm the analysis of the previous section. In spite of the fact that there is not always a direct relationship between the two quantities, the evolution of pivotality with respect to L parallels the evolution of utility for IAC^*. For IAC, things are less clear. For IAC, things are different since as already pointed out the average theoretical pivotality curve is flat.

It is interesting to know the relative importance of each channel of decisiveness of outside voters. Take an arbitrary voter i from a state k outside the coalition i.e. $k \geq L + 1$. We have partitioned the event "voter i is pivotal" into five subevents. The first one, denoted DO is: "Voter i is pivotal in state k but is not pivotal countrywise and the delegate of state k is pivotal in the college". The second one, denoted IO is: "Voter i is not pivotal in state k but is pivotal countrywise and the block of delegates from the interstate is pivotal in the college". The third one, denoted BD is: "Voter i is pivotal in state k and is pivotal countrywise and the delegate of state k is pivotal in the college". The fourth one, denoted BI is: "Voter i is pivotal in state

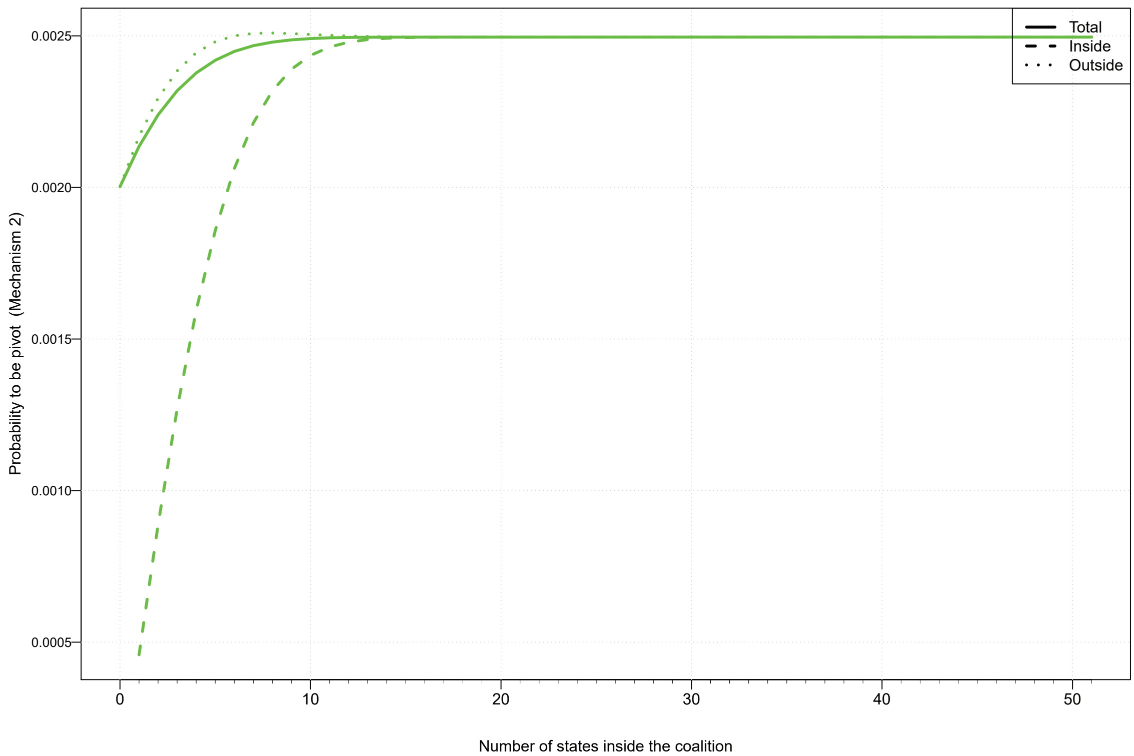

Figure 17: Graph of the IC probabilities of being pivotal when $K = 51$ and $r = 10^3$.

k and is pivotal countrywise, the delegate of state k is not pivotal in the college but the block of delegates from the interstate is pivotal in the college". The fifth one, denoted B is: "Voter i is pivotal in state k and is pivotal countrywise, the block of delegates from the interstate is not pivotal in the college but the block of delegates together with the delegate of state k is pivotal in the college".

The simulations deliver interesting results concerning the probabilities of these five events. On figures 20, 21 and 22 below, we have reported the *relative* probabilities of the five events as a function of L and for each of the three probabilistic models in the case where $r = 10^3$.

Without much surprise, the "pure" popular vote channel denoted IO is increasingly dominant when L gets larger and larger for the three probabilistic models[62]. Except for some small differences, the three graphs display the same features. The share of IO is very large and constant once L has reached a critical value between 10 and 15. Under IC we can derive the explicit values of the IO and BI shares(the other shares are trivially equal to 0) when $L \geq \frac{K+1}{2}$. They do not depend upon L. We can also prove that when m is large, the BI share behaves approximatively as $\sqrt{\frac{2}{\pi m}}$: it tends towards 0 as the inverse of the square rule of m.

[62]This feature is amplified when r gets larger and larger.

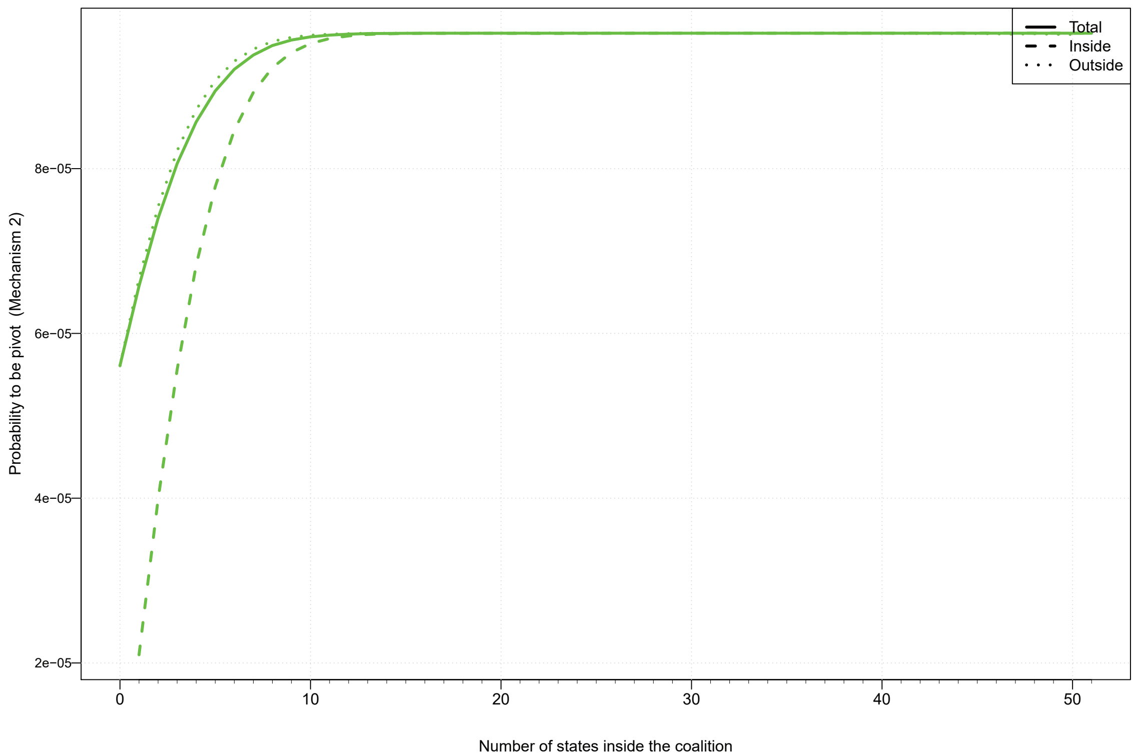

Figure 18: Graph of the IAC^* probabilities of being pivotal when $K = 51$ and $r = 10^3$.

This asymptotic disappearance of BI is also shared by IAC and IAC^*[63]. On the other hand, unsurprisingly the importance of DO decreases with L. What is more surprising is that out of the five components , only two or three, namely IO, BI and DO show up for small values of L. For much smaller values of m, the figures are different. For instance[64], when $m = 3$, the three components are still there but now the share of the two other ones is not negligible when L is small.

7 Conclusion

In this paper, we have evaluated two possible versions of a "Popular Vote Interstate Compact"from an engineering electoral perspective. We have shown that the two versions display opposite properties. A major difference between the ongoing NPV initiative and the MNPV version considered in our paper is the fact that the mechanism is effective as soon as the members of the interstate compact have signed the agreement. This means that if the interstate

[63]The differences to which we alluded before can be made more precise when m is large. We can prove that, up to a multiplicative constant, the BI share behaves approximately as $\frac{1}{\sqrt{m}}$ under IAC and as $\frac{1}{m}$ under IAC^*. The analytical arguments are provided in appendix 7 of the working paper version.

[64]These figures are available from the authors upon request.

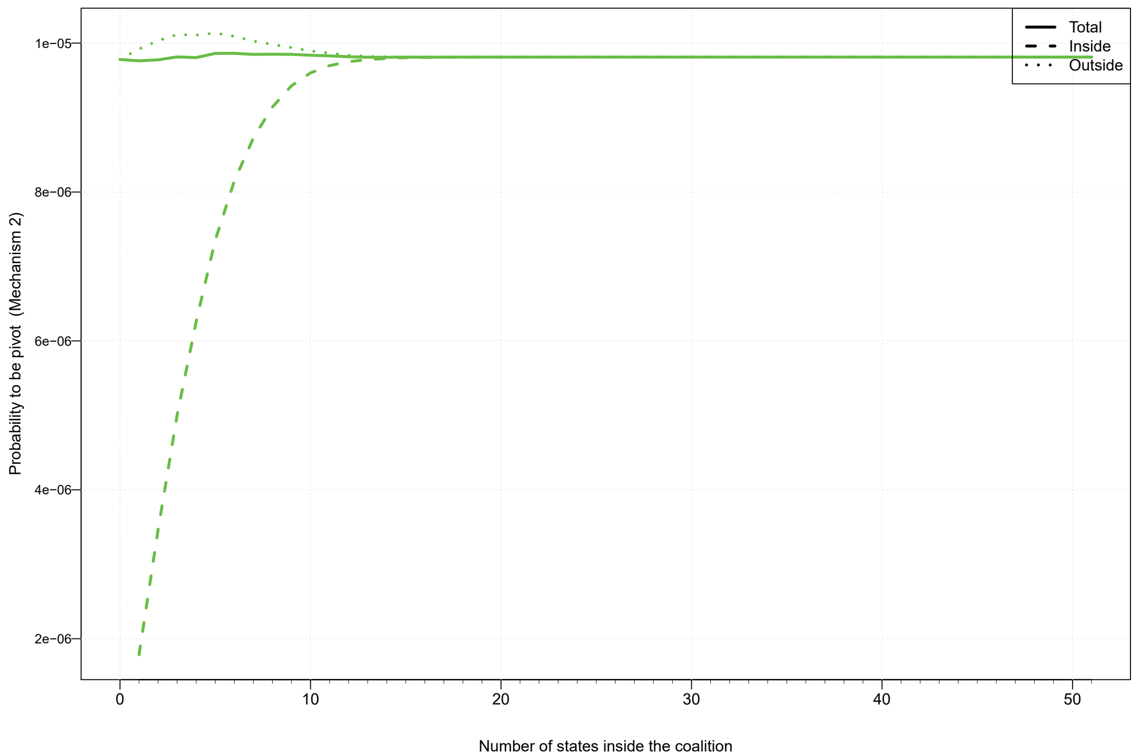

Figure 19: Graph of the IAC probabilities of being pivotal when $K = 51$ and $r = 10^3$.

compact does not control a majority of the electoral votes, then the winner of the election does not need to be a popular winner. Indeed, from an ex ante perspective and with an unbiased and symmetric probabilistic model (like the three popular ones considered in this paper), when we ask citizens to compare the Electoral College and the popular vote, they should unanimously vote in favor of the popular vote. But of course, ex post some states may regret the move from one system to the other.

One of the most striking conclusions obtained in our analysis of MNPV is that for the three a priori models which are considered here, the gains of the popular vote are already there for a relatively small size of the interstate compact. The minimal size is critical however in the sense that any smaller size hurts (when compared to the Electoral College) the participants to the interstate compact.

This conclusion has been derived however under strong assumptions. First, the states have been assumed to be equipopulated. Second, the probabilistic models have been assumed to be a priori. Third, probabilistic models have been assumed to be unbiased towards either of the candidates.

Concerning the first qualification, the extension of our framework to an asymmetric setting does not raise any particular conceptual problems. In particular, in the case of the U.S. Electoral

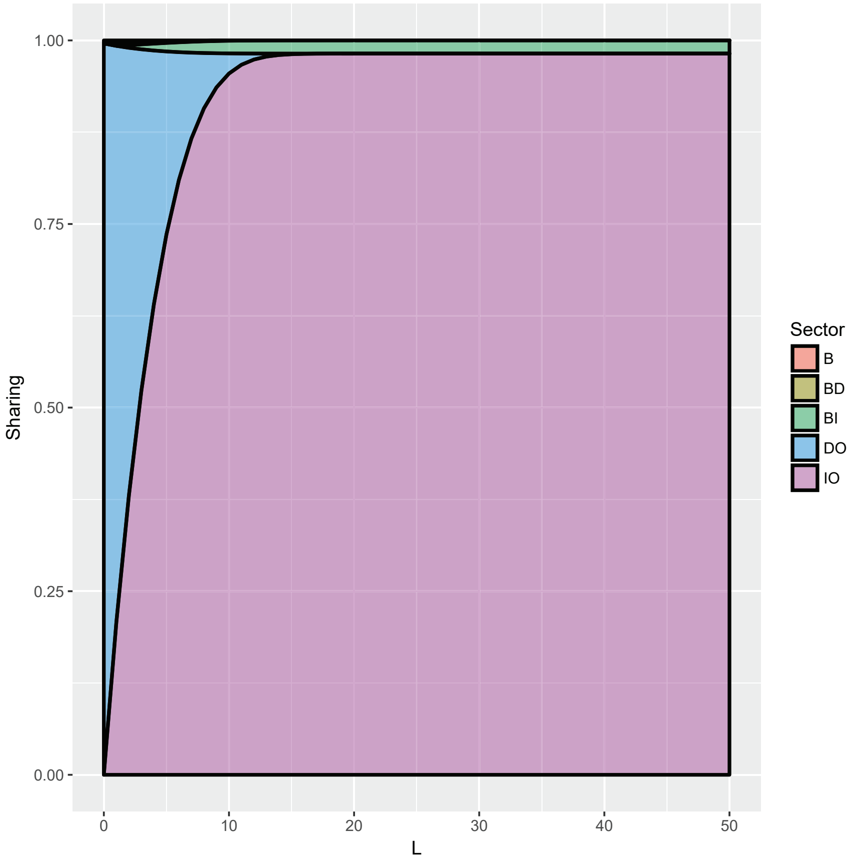

Figure 20: Decomposition of the outsider IC probability of being pivotal when $K = 51$ and $r = 10^3$.

College i.e. on the basis of the current populations and electoral votes per state, it would be possible but computationally demanding to identify the implications of all 2^{51} conceivable interstate compact. Under IC, analysis of a large set of potential reforms of the real Electoral College (i.e. taking into account the demographic parameters and the electoral votes) are already performed in Beisbart and Bovens (2008) and Miller (2009).

The second qualification raises some fundamental issues. The use of these a priori probabilistic models can be disputed on the grounds of realism by those who have in mind a *theoretical or empirical positive analysis of the interstate compact*. Such scholars would rather prefer the use of an a posteriori probabilistic model estimated through short term historical voting data. Such model will likely exhibits biases and asymmetries which are not captured by the a priori model. But our electoral engineering perspective is normative and should not be based on a vision of preferences that would be excessively historical. This issue is forcefully discussed by Beisbart and Bovens (2008)[65]: "Whether an a priori model or an a posteriori model is a

[65]As also noted by Miller (2009) in his analysis of the Electoral College and some of its reforms: "A measure

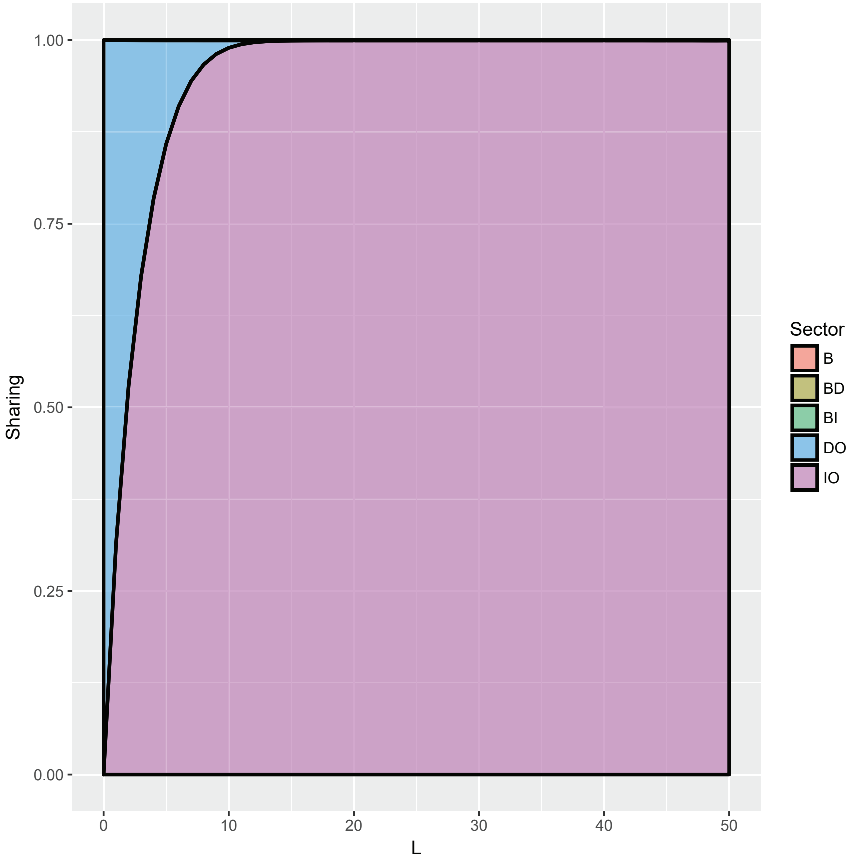

Figure 21: Decomposition of the outsider IAC^* probability of being pivotal when $K = 51$ and $r = 10^3$.

better estimate for the future, depends on what exactly our task is. If our task is to design immutable institutions that will perform well for the U.S. in the long run, we cannot rely on past preferences because preferences are subject to change (see, e.g., Felsenthal and Machover 1998, p. 13). Thus, an a priori model is more fitting... On the other hand, if our task is to design institutions that will do well for the U.S. in the predictable future and that can be updated when there is a substantial change in voting patterns, an a posteriori model is more fitting... Felsenthal and Machover (1998) argue that an impartial assessment of a voting scheme requires one to put oneself behind the Rawlsian veil of ignorance and that the veil of ignorance excludes knowledge of preferences... Nonetheless, Felsenthal and Machover may be correct in their resistance to import empirical information about interest profiles. There does seem to be

---

of a priori voting power takes account of the fundamentals of a voting rule but nothing else. Thus the following analysis takes account only of the 2000 population of each state and the District of Columbia, the apportionment of electoral votes based on that population profile, and the requirement that a Presidential candidate receive 270 electoral votes to be elected. It does not take account of other demographic factors, historical voting patterns, differing turnout rates, relative party strength, survey or polling data, etc. This indicates the sense in which a priori voting power analysis is conducted behind a 'veil of ignorance' and is blind to empirical contingencies."

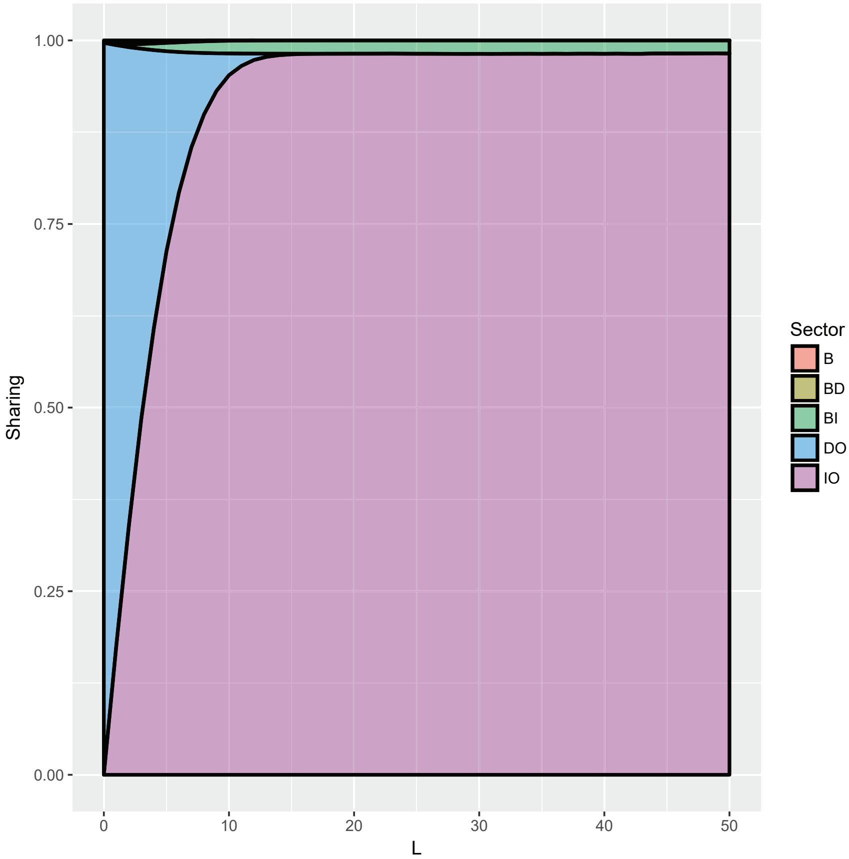

Figure 22: Decomposition of the outsider IAC probability of being pivotal when $K = 51$ and $r = 10^3$.

something untoward in tuning a voting scheme to the preferences of the voters."

We leave it as an open question what kind of information one should take into account when assessing whether one voting scheme is more beneficial than another from an impartial point of view. It is however important to point out that the methods developed in our paper can be used for a positive analysis of the different versions of the interstate compact. It would start from an arbitrary probabilistic model π on the space of preferences $\{D, R\}^n$, possibly constructed from a statistical analysis as for instance the second model of Beisbart and Bovens (2008) or the model of Strömberg (2008). Given any such π, we can repeat the analysis conducted in this paper for any electoral mechanism i.e. computation of the individual and total probabilities of being pivotal, computation of the individual and total utilities and computation of the probability of an election inversion[66].

The third qualification cannot be separated from the second one and needs a particular

[66] In particular in the case of the U.S. Electoral College i.e. on the basis of the current populations and electoral votes per state, Neubauer, Schilling and Zeitlin (2012) estimate the consequences of different reforms including the NMPV plan.

attention. If instead of an a priori electoral ex ante mechanism design perspective, we opt for a positive analysis of the game describing the interstate compact formation, we need an a priori or empirical probabilistic model with competitive and non-competitive states like the a priori IC one considered by Brams and Kilgour (2013). From a theoretical perspective, we can construct probabilistic models with bias that produce very different payoffs and incentives depending upon the bias. In contrast to the symmetric unbiased version, for which all the states (all the citizens) would benefit from changing the Electoral College into the popular vote, we can now have situations where some states still prefer to reform the Electoral College while others prefer the status quo[67]. The positive analysis echoes many of the discussions about NPV among opponents and supporters about the benefits and costs of such a reform depending whether you are on the Democrat or the Republican side[68].

To conclude, let us also mention two extra issues which have been discussed marginally in the paper.

For many authors and commentators, the main objective of a reform of the Electoral College like NPV is to modify completely the incentives of the presidential candidates during their campaigns. In the current system, candidates do not campaign in states in which they are comfortably ahead or hopelessly behind[69]. We may wonder why states should worry about receiving the visit of the candidates and why candidates should pay attention only to swing states if their platform's on one hand and citizen's preferences on the other hand are exogenous. Our current model is too rudimentary[70] to describe precisely the strategic campaigning decisions of the candidates and the responses of the voters to visits and promises[71]. Note however that

[67] This could be illustrated in a stylistic situation with a biased state (all citizens vote for sure for one candidate) and two swing states (described by the IC model). Evaluations of the states are ex ante evaluations like an insurance contract between two parties. If the classroom situation where preferences are such that in half plus one of the states, one candidate wins for sure with a minimal majority and do not get any votes in the other half minus one of the states, uncertainty has disappeared. With such extreme ideological bias, ex ante gains do not exist. We are left with a straightforward ex post dispute. The candidate winning a majority of the states will always prefer the Electoral College while the other will always prefer the popular vote. As noted by Hinich et al. (1975), in such case election inversion is certain.

[68] For instance, Section 9.31 in Koza et al (2013) entitled "Myth that a Nationwide Vote for President Would favor One Political Party Over the Other" examines 14 issues listed under that heading. In his econometric study, Virgin (2017) finds evidence that loyalties to the state ad to the party may be competing. Although NPV advances furthest when Democrats control state lawmaking, a state's status a swing- but not as an over-represented- state weakens the relationship to the point where even Democrats are unlikely to aid NPV.

[69] This asymmetric treatment of citizens is listed as one of the three shortcomings of the current system by Koza et al (2013).

[70] See for instance, among many, Brams and Davis (1974) for an ealy model addressing that question and Strömberg (2008) for a recent analysis of how US presidential candidates should allocate resources across states to maximize the probability of winning the election, based on the estimation of a probabilistic-voting model of political competition under the Electoral College system.

[71] Scholar works on campaigning games and redistributive politics coincide with common wisdom to conclude that indeed swing states receive more attention at equilibrium.

in the case of IC[72], there is no contradiction between the conventional objective of maximizing utility and the objective of maximizing pivotality if pivotality is measured by the Banzhaf absolute index.

Finally, one important extension of the current framework and assumptions concerns the number of candidates. We have postulated that their number was exogenous and assumed to be equal to two. As already pointed out, he NPV plan is considered by some of its opponents to be imprecise about the definition of the popular winner[73]. This is not a secondary issue since we know, from social choice theory, that the notion of popular winner is not uncontroversial as soon as we have at least three candidates[74]. This issue is also practically[75] important as the number of candidates is not exogenous and may depend upon the electoral system itself[76].

8 References

Abbott, D.W. and J.P. Levine (1991) *Wrong Winner: The Coming Debacle in the Electoral College*, Westport, C.T. Praeger.

Amar, A.R. and V.D. Amar (2001) "How to Achieve Direct National Election of the president without Amending the Constitution: Part Three of a Three-Part Series on the 2000 Election and the Electoral College", *FindLaw* (December 28) http://writ.lp.findlaw.com/amar/20011228.html

Ball, W. J. and D. A. Leuthold. (1991) "Estimating the likelihood of an unpopular verdict in the Electoral College", *Public Choice*, 70, pp.215-224.

Banzhaf, J.F. (1968) "One man, 3.312 votes: a mathematical analysis of the Electoral College", *Villanova Law Review*, 13, 304-332.

Barberà, S. and M.O. Jackson (2006) "On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union", *Journal of Political Economy*, 114, 317-339.

Beisbart, C. and L. Bovens (2007) "Welfarist Evaluations of Decision Rules for Board of Representatives", *Social Choice and Welfare*, 29, 581-608.

Beisbart, C. and L. Bovens (2008) "A Power Measure Analysis of Amendment 36 in Colorado", *Public Choice*, 124, 231-246.

[72]Under IC, IAC and IAC*, the probability of a tied election is very small in the case of the popular vote if the population of voters is large. If asymmetries among the candidates are introduced, tied elections this probability is much more smaller (see for instance, Chamberlain and Rothschild (1981)).

[73]This is illustrated in Appendix 8.

[74]See however sections 9.7 and 9.8 in Koza et al. for discussions of some of these issues.

[75]The spoiler effect is an important issue. Section 3 in Hinich et al. (1975) contains an interesting analysis of the consequences of having national or regional third parties.

[76]In the current version of the Electoral College, the winner takes all of the electoral votes in a state according to plurality. But even without altering the winner-take-all principle, we could conceive alternative rules: Borda, instant runoff, plurality with runoff, Condorcet, ... to cite a few.

Beisbart, C. and L. Bovens (2013) “Minimizing the Threat of a Positive Majority Deficit in Two-Tier Voting Systems with Equipopulous Units”, *Public Choice*, 145, 75-94.

Beisbart, C., L. Bovens and S. Hartmann (2005) “Welfarist Foundations of Alternative Decision Rules in the Council of Ministers”, *European Union Politics*, 6, 395-419.

Beisbart, C. and S. Hartmann (2010) “Welfarist Evaluations of Decision Rules under Interstate Utility Dependencies”, *Social Choice and Welfare*, 34, 315-344.

Bennett, R.W. (2001) “State Coordination in Popular Election of the president without a Constitutional Amendment”, *Green Bag 4*, 2, 241-245.

Berg, S. (1990) “The Probability of casting a Decisive Vote: The Effects of a Caucus”, *Public Choice*, 64, 73-92.

Brams, S.J. and M. D. Davis. (1974) ”The 3/2’s Rule in Presidential Campaigning”, *American Political Science Review*, 68, 113-134.

Brams, S.J. and D.M. Kilgour (2013) “Voting Power under a Modified Popular Vote Plan in US Presidential Elections”, NYU, Mimeo.

Chamberlain, G. and M. Rothschild (1981) “A Note on the Probability of Casting a Decisive Vote”, *Journal of Economic Theory*, 25,152-162.

Choosing a President: The Electoral College and Beyond, Schumaker, P.D. and B.A. Loomis (Eds), Seven Bridges Press, 2002.

de Mouzon, Laurent, Le Breton and Lepelley (2017) “The Theoretical Shapley-Shubik Probability of an Election Inversion in a Toy Symmetric Version of the U.S. presidential Electoral System”, Toulouse School of Economics, Mimeo.

De Witt, D. and T. Schwartz (2016) “A Calamitous Compact”, *PS: Political Science and Politics*, 49, 791-796.

Edelman, P.H. (2004) “Voting Power and At-Large Representation”, *Mathematical Social Sciences*, 47, 219-232.

Edwards, G.C. III (2011) *Why the Electoral College is Bad for America*, Yale University press, Second Edition, New Haven.

Electoral College Reform: Challenges and Possibilities, Bugh. G.E. (Ed), V.T. Ashgate, Burlington, 2010.

Elliot, J.M. and S.R. Ali (2007) *The State and Local Government Political Dictionary*, Wildside Press.

Feix, M., Lepelley, D., Merlin, V. and J.L. Rouet (2004) “The probability of conflicts in a U.S. presidential type election”, *Economic Theory,* 23, 227-257.

Felsenthal, D.S. and M. Machover (1998) *The Measurement of Voting Power*, Edward Edgar, Northampton.

Felsenthal, D. S. and Machover, M. (1999) “Minimizing the mean majority deficit: the second square-root rule”, *Mathematical Social Sciences,* 37, 25-37.

Fishburn, P.C. and Gehrlein, W.V. (1976) “Borda’s Rule, Positional Voting, and Condorcet’s Simple Majority Principle”, *Public Choice*, 28, 79-88.

Gains, B.J. (2010) “Compact Risk: Some Downsides to Establishing national Plurality Presidential Elections by Contingent Legislation”, *Electoral College Reform: Challenges and Possibilities*, Bugh, G.E. (Ed), V.T. Ashgate, Burlington, 2010.

Gehrlein, W. (2006) *Condorcet’s Paradox*, Springer.

Gehrlein, W. and D. Lepelley (2011) *Voting Paradox and Group Coherence: The Condorcet Efficiency of Voting Rules*, Springer.

Good, I.J. and L.S. Mayer (1975) “Estimating the Efficacy of a Vote”, *Behavioral Science*, 20, 25 -33.

Gupta, S.S. (1963) “Bibliography on the Multivariate Normal Integrals and Related Topics, *Annals of Mathematical Statistics*, 34, 829-838.

Hinich, M.J. Mickelsen, R. and P.C. Ordeshook (1972) “The Electoral College Versus a Direct Vote: Policy Bias, Reversals, and Indeterminate Outcomes”, *Journal of Mathematical Sociology*, 4, 3-35.

Koza, J.R. et al. (2013) *Every Vote Equal: A State-Based Plan for Electing the president by National Popular Vote*, Los Altos, C.A: National Popular Vote Press, Fourth Edition.

Koza, J.R. (2016) A Not-So-Calamitous Compact: A Response to De Witt and Schwartz”, *PS: Political Science and Politics*, 49, 797-804..

Kuga, K. and H. Nagatani (1974) “Voter Antagonism and the Paradox of Voting”, *Econometrica*, 42, 1045-1067.

Laruelle, A., R. Martínez and F. Valenciano (2006) “Success Versus Decisiveness: Conceptual Discussion and Case Study”, *Journal of Theoretical Politics*, 18, 185-205.

Le Breton, M. and D. Lepelley (2014) “Une Analyse de la Loi Electorale du 29 Juin 1820”, *Revue Économique*, 65, 469–518.

Le Breton, M., Lepelley, D. and H. Smaoui (2016) “Correlation, Partitioning and the Probability of Casting a Decisive Vote under the Majority Rule”, *Journal of Mathematical Economics*, 64,11–22.

Le Breton, M. and K. Van Der Straeten (2015) “Influence versus utility in the evaluation of voting rules: a new look at the Penrose formula”, *Public Choice*, 165, 2015, 103-122.

Lepelley, D., Merlin, V. and J.L. Rouet (2011) “Three Ways to Compute Accurately the Probability of the Referendum Paradox”, *Mathematical Social Sciences*, 62, 28-33.

May, K. (1948) “Probability of Certain Election Results”, *American Mathematical Monthly*, 55, 203-209.

Maskin, E. and A.K. Sen (2017a) "The Rules of the Game: A New Electoral System", *The New York Review of Books*, January 19 Issue.

Maskin, E. and A.K. Sen (2017b) "A Better Way to Chose Presidents", *The New York Review of Books*, January 19 Issue.

May, K. (1958) "The Frequency of Election Victories without Pluralities", *American Philosophical Society Yearbook*, 342-344.

Merrill, S. (1977) "Empirical estimates for the Likelihood of a Divided Verdict in a Presidential election", *Public Choice*, 33, 127-133.

Merrill, S. (1978) "Citizen Voting Power Under the Electoral College: A Stochastic Model Based on State Voting Patterns", *Siam Journal on Applied Mathematics*, 34, 376-390.

Neubauer, M.G., Schilling, M. et J. Zeitlin (2012) "Exploring Unpopular Presidential Elections", California State University, Mimeo.

Miller, N. R. (2009). "A priori voting power and the U.S. Electoral College", *Homo Oeconomicus*, 26, 341–380.

Miller, N. R. (2012a) Election inversions by the U.S. Electoral College, in D. S. Felsenthal & M. Machover (Eds.), *Electoral Systems: Paradoxes, Assumptions, and Procedures*, Berlin: Springer.

Miller, N.R. (2012b) "Why the Electoral College is good for political science (and public choice), *Public Choice*, 150, 1–25.

Morton, R. (2006) *Analyzing Elections*, W.W. Norton, New York.

Owen, G. (1975) "Evaluation of a Presidential Election Game", *American Political Science Review*, 69, 947-953.

Owen, G. (2001) *Game Theory*, Third Edition, Academic Press, New York..

Niemi, R.G. and H.F. Weisberg (Eds) *Probability Models of Collective Decision Making*, Charles. E Merrill Company, Columbus, 1972.

Penrose, L.S. (1946) "The Elementary Statistics of Majority Voting", *Journal of the Royal Statistical Society*, 109, 53-57.

Penrose, L.S. (1952) *On the Objective Study of Crowd Behavior*, H.K. Lewis and Co, London.

Peirce, N.R. and L.D. Longley (1981) *The People's president: The Electoral College in American History and the Direct Vote Alternative*, Yale University Pess, New Haven.

Schattschneider, E.E. (1942) *Party Government*, Holt, Rinehart and Winston, New York.

Schumaker, P.D. (2002) "Analyzing the Electoral College and Its Alternatives", Chapter 2 in *Choosing a President: The Electoral College and Beyond*, Schumaker, P.D. and B.A. Loomis (Eds), Chatam House Publishers, New York.

Shapley, L.S. and M. Shubik (1954) "A Method for Evaluating the Distribution of Power in a Committee System", *American Political Science Review*, 48, 787-792.

Stanley, R.P. (1997) *Enumerative Combinatorics*, Vol 1, Cambridge University Press.

Straffin, P.D. (1978) “Probability Models for Power Indices”, *Game Theory and Political Science*, Ordeshook (Ed), New York.

Straffin, P.D., Davis, M.D. and S.J. Brams (1981) “Power and Satisfaction in an Ideologically Divided Voting Body”, in *Power, Voting and Voting Power*, M.J. Holler (Ed) Physica-Verlag, 239-256.

Strömberg, D. (2008) “How the Electoral College Influences Campaigns and Policy: The Probability of being Florida”, *American Economic Review*, 98, 769-807.

Virgin, S.G. (2017) “Competing Loyalties in Electoral reforms: An Analysis of the U.S. Electoral College”, *Electoral Studies*, 49, 38-48.

Weber, R.J. (1978) “Comparison of Public Choice Systems”, Cowles Foundation Discussion Paper N 498, Yale University.

Wilderming, L.Jr (1958) *The Electoral College*, Rutgers University Press, New Brunswick.

Wilson, J.J. (2006) “Bloc Voting in the Electoral College: How the Ignored States could Become Relevant and Implement Popular Election along the Way”, *Election Law Journal*, 5, 384-409.

9 Appendices

The Appendix is divided into six parts parts. Appendix 1 contains a description of the program *SimuElect* which is used in the paper to simulate elections for the three probabilistic models which are considered. Appendix 2 contains some few insights on the IAC model. Finally, Appendices 3 and 4 contain respectively analytical formulas for the probability of election inversions in MRPV under IC and $IAC^*$when the number of voters per state is large. Appendix 5 contains analytical formulas for total utility in MRPV under IAC^*. Appendix 6 illustrates the importance of conditioning in the case of NMPV. Appendix 7 provides some theoretical insights explaining the asymptotic differences between IC, IAC and IAC^*, concerning the decomposition of the probability of being pivotal into its five components. Finally, appendix 8 illustrates the sensitivity of our conclusions to the existence of battle ground states.

9.1 Appendix 1: Description of the Program *SimuElect*

The current version of the program, *SimuElect*, allows to compute the probability of election inversions, the utility of the voters inside and outside the interstate compact and the probability of being decisive of the voters inside and outside the interstate compact for any value of the parameters r and K, for the IC, IAC^* and IAC probability models, and for the two versions, MRPV and MNPV, of the interstate compact arrangements.

For readability purpose, the algorithm is divided in several parts. The main steps of our algorithm are reported in Algorithm 1.

Algorithm 1: Main steps of our algorithm

Initialization of program constants

K = number of states (odd integer)

B = number of simulations

n = number of electors in each state (odd integer)

Initialization of result variables

Inversion, UtilityIn, UtilityOut, PivotIn, PivotOut = Matrix of size $(2, K+1)$ filled with $0s$

PivotOutM2 = Matrix of size $(5, K+1)$ filled with zeros

Computation

for *b in 1 to B* **do**

  Create a vector *vote* of size K which gives the number of electors voting democrat in each state (details in Algorithm 2)

  National popular winner computation ($1 \leftrightarrow Democrat; 0 \leftrightarrow Republican$):

  $PopuWinner = ((\sum vote) \geq (K \times n + 1)/2)$

  Compute the results for each coalition size and each mechanism:

  for *L in 0 to K* **do** `// L is the size of the coalition`

    for *M in 1(MRPV) or 2(MNPV)* **do** `/* Coalition States give their Electoral College votes to the regional (coalition) popular winner in the case of MRPV and to the national popular winner in the case of MNPV */`

      Computation of the Coalition and Electoral College vote winners: $CoalWinner$ and $ElColWinner$ (details in Algorithm 3)

      Adding inversion cases:

      $Inversion(M, L) = Inversion(M, L) + (ElColWinner \neq PopuWinner)$

      Adding number of happy voters inside and outside the coalition: $UtilityIn(M, L)$ and $UtilityOut(M, L)$ (details in Algorithm 4)

      Adding number of pivotal voters inside and outside the coalition: $PivotIn(M, L)$ and $PivotOut(M, L)$ plus, when M is 2, $PivotOutM2(1:5, L)$ (details in Algorithm 5)

Final computation and presentation of the results for each coalition size and each mechanism:

for *L in 0 to K* **do**

  for *M in 1(MRPV) or 2(MNPV)* **do**

    Percentage of election inversion:

    $Inversion(M, L) = Inversion(M, L)/B$

    Percentage of happy voters overall, inside, and outside the coalition:

    $Utility(M, L) = (UtilityIn(M, L) + UtilityOut(M, L))/(B \times K \times n)$

    $UtilityIn(M, L) = UtilityIn(M, L)/(B \times L \times n)$

    $UtilityOut(M, L) = UtilityOut(M, L)/(B \times (K - L) \times n)$

    Percentage of pivotal voters overall, inside, and outside the coalition:

    $Pivot(M, L) = (PivotIn(M, L) + PivotOut(M, L))/(B \times K \times n)$

    $PivotIn(M, L) = PivotIn(M, L)/(B \times L \times n)$

    $PivotOut(M, L) = PivotOut(M, L)/(B \times (K - L) \times n)$

  Distribution of pivotal voters by type in the case of MNPV ($M2$):

  $PivotOutM2(1:5, L) = PivotOutM2(1:5, L)/PivotOut(2, L)$

Algorithm 2: Detailed steps to obtain votes simulations

Create a **vector** $\boldsymbol{p} = (p_1, ..., p_K)$ of size K which gives the probability of voting democrat in each state:

switch *Probabilistic model* **do**
- **case** IC **do**
  - $p = (0.5, ..., 0.5)$
- **case** IAC **do**
  - $u = \mathcal{U}(0,1)$
  - $p = (u, ..., u)$
- **case** IAC^* **do**
  - $p = (\mathcal{U}(0,1), ..., \mathcal{U}(0,1))$

Create a **vector** ***vote*** of size K which gives the number of electors voting democrat in each state:

$vote = (\mathcal{B}(n, p_1), ..., \mathcal{B}(n, p_K))$

Algorithm 3: Detailed steps to obtain Coalition and Electoral College vote winners

Computation of the **Electoral College vote inside the coalition**, $\boldsymbol{CoalWinner}$ $(1 \leftrightarrow Democrat; 0 \leftrightarrow Republican)$:

if *M is 2(MNPV)* **then**
- $CoalWinner = PopuWinner$

else if $(\sum_{k=1}^{L} vote_k) = L \times n/2$ **then** `/* M is 1(MRPV) but there is a tie (may happen only when L is even) */`
- $CoalWinner$ is randomly chosen 0 or 1

else `// M is 1(MRPV) and there is a clear winner`
- $CoalWinner = ((\sum_{k=1}^{L} vote_k) \geq (L \times n + 1)/2)$

Computation of the number of **Democrat votes at Electoral College** level:

$\boldsymbol{ElColVote} = CoalWinner \times L + \sum_{k=L+1}^{K} (vote_k \geq (n+1)/2)$

Computation of the **Electoral College vote winner** $(1 \leftrightarrow Democrat; 0 \leftrightarrow Republican)$:

$\boldsymbol{ElColWinner} = (ElColVote \geq (K+1)/2)$

Algorithm 4: Detailed steps to obtain number of happy voters inside and outside the coalition

if *ElColWinner=1* **then** `// Democrat wins`
- $UtilityIn(M,L) = UtilityIn(M,L) + (\sum_{k=1}^{L} vote_k)$
- $UtilityOut(M,L) = UtilityOut(M,L) + (\sum_{k=L+1}^{K} vote_k)$

else `// Republican wins`
- $UtilityIn(M,L) = UtilityIn(M,L) + L \times n - (\sum_{k=1}^{L} vote_k)$
- $UtilityOut(M,L) = UtilityOut(M,L) + (K-L) \times n - (\sum_{k=L+1}^{K} vote_k)$

Algorithm 5: Detailed steps to obtain number of pivotal voters inside and outside the coalition

if *M is 1(MRPV)* **then**

Adding number of pivotal voters inside and outside the coalition (under MRPV): $PivotIn(M, L)$ and $PivotOut(M, L)$ (details in Algorithm 6)

else `// M is 2(MNPV)`

Adding number of pivotal voters inside and outside the coalition (under MNPV): $PivotIn(M, L)$, $PivotOut(M, L)$ and its decomposition by type of pivotal voters: $PivotOutM2(1:5, L)$ (details in Algorithm 7)

There are five types T of pivotal voters (each type has its counter, $PivotOutM2(T, L)$):

$T = 1$: $PivotOutM2(1, L)$ is also named $PivotOutM2_{DO}(L)$, which counts the number of voters (outside the coalition) which are pivotal through Direct mechanism Only (i.e. changing the vote of their state, without changing the coalition vote).

$T = 2$: $PivotOutM2(2, L)$ is also named $PivotOutM2_{BD}(L)$, which counts the number of voters (outside the coalition) which are pivotal through Both mechanisms, although Direct one is sufficient (i.e. changing the vote of their state is sufficient to change election outcome, but they also change the coalition vote).

$T = 3$: $PivotOutM2(3, L)$ is also named $PivotOutM2_{IO}(L)$, which counts the number of voters (outside the coalition) which are pivotal through Indirect mechanism Only (i.e. changing the vote of the coalition but not the one of their state).

$T = 4$: $PivotOutM2(4, L)$ is also named $PivotOutM2_{BI}(L)$, which counts the number of voters (outside the coalition) which are pivotal through Both mechanisms, when Indirect one only is necessary (i.e. changing the vote of the coalition is necessary to change election outcome, but they also change the vote of their state).

$T = 5$: $PivotOutM2(5, L)$ is also named $PivotOutM2_{B}(L)$, which counts the number of voters (outside the coalition) for which Both (direct and indirect) mechanisms are necessary to be pivotal.

Algorithm 6: Detailed steps to obtain number of pivotal voters inside and outside the coalition in the case of MRPV

```
Outside:            /* Remark: ElColVote gives the number of Democrat votes at
  Electoral College level (details in Algorithm 3) */
if ElColVote = (K + 1)/2 then     /* Democrat wins by one Electoral College vote
  only */
    PivotOut(M, L) =
      PivotOut(M, L) + (n + 1)/2 × (Σ_{k=L+1}^{K} (vote_k = (n + 1)/2))
else if ElColVote = (K − 1)/2 then   /* Republican wins by one Electoral College
  vote only */
    PivotOut(M, L) =
      PivotOut(M, L) + (n + 1)/2 × (Σ_{k=L+1}^{K} (vote_k = (n − 1)/2))
Inside:
if abs((Σ_{k=L+1}^{K} (vote_k ≥ (n + 1)/2)) − (K − L)/2) < L then      /* Coalition vote at
  Electoral College is decisive */
    CoalitionPopularVoteDistanceToMiddle = abs((Σ_{k=1}^{L} vote_k) − L × n/2)
    if CoalitionPopularVoteDistanceToMiddle = 0.5 then   /* Coalition winner wins
      by one vote (and L and n are odd) */
        PivotIn(M, L) = PivotIn(M, L) + (L × n + 1)/2
    else if CoalitionPopularVoteDistanceToMiddle = 0 then  /* Tie in the coalition
      (needs L even) */
        PivotIn(M, L) = PivotIn(M, L) + L × n/2    /* As the winner is here random,
          half of the voters (the happy ones) are pivotal.  */
    else if CoalitionPopularVoteDistanceToMiddle = 1 then      /* Coalition winner
      wins by two votes (needs L even).  So if one happy voter changes its
      vote, there is a tie and the outcome of the vote is randomly Democrat or
      Republican */
        if Coalition winner changes (which is randomly true or false) then      // This is
          true half of the times.
            PivotIn(M, L) = PivotIn(M, L) + (L × n/2 + 1) ;
        /* Else coalition winner does not change (half of the times) and there are no pivotal
          voters in that case. */
```

Algorithm 7: Detailed steps to obtain number of pivotal voters inside and outside the coalition in the case of MNPV

$PopularVoteDistanceToMiddle = (\sum vote) - K \times n/2$

Inside:

if $abs((\sum_{k=L+1}^{K}(vote_k \geq (n+1)/2)) - (K-L)/2) < L$ **then** `/* Coalition vote at Electoral College is decisive */`

  if $PopularVoteDistanceToMiddle = 0.5$ **then** `/* Coalition winner wins by one vote and is Democrat */`

    $PivotIn(M, L) = PivotIn(M, L) + \sum_{k=1}^{L}(vote_k)$

  else if $PopularVoteDistanceToMiddle = -0.5$ **then** `/* Coalition winner wins by one vote and is Republican */`

    $PivotIn(M, L) = PivotIn(M, L) + L \times n - \sum_{k=1}^{L}(vote_k)$

Outside: `/* Remark:` $ElColVote$ `gives the number of Democrat votes at Electoral College level (details in Algorithm 3) */`

Computation of $DVIHSBO$ (standing for $DemocratVotesInHappyStateByOne$), which gives the number of Democrat votes in a state where the winner wins by one vote only and corresponds to the Electoral College winner too:

if $ElColWinner$ **then** `/* Democrat wins the Electoral College */`

  $DVIHSBO = (n+1)/2$

else `/* Republican wins the Electoral College */`

  $DVIHSBO = (n-1)/2$

if $abs(PopularVoteDistanceToMiddle) = 0.5$ *and* $L \geq 1$ **then** `/* Coalition Electoral College vote outcome may be changed by any happy voter */`

  Computation of $PivotOutM2_B(L)$, $PivotOutM2_{IO}(L)$, $PivotOutM2_{BD}(L)$ and $PivotOutM2_{BI}(L)$ (details in Algorithm 8)

else if $abs(ElColVote - K/2) = 0.5$ **then** `/* Coalition Electoral College vote cannot be changed, but a State Electoral College vote is sufficient to change the election outcome. */`

  $PivotOutM2_{DO}(L) =$ `/* Number of voters outside the coalition that are pivotal through Direct mechanism Only (i.e. changing the vote of their state) */`

   $PivotOutM2_{DO}(L) + (n+1)/2 \times (\sum_{k=L+1}^{K}(vote_k = DVIHSBO))$

Algorithm 8: Detailed steps to obtain number of pivotal voters outside the coalition in the case of MNPV and when Coalition Electoral College vote outcome may be changed by any happy voter

Computation of $PivotOutM2_B(L)$, $PivotOutM2_{IO}(L)$, $PivotOutM2_{BD}(L)$ and $PivotOutM2_{BI}(L)$: `/* Remarks:` $ElColVote$ `gives the number of Democrat votes at Electoral College level (details in Algorithm 3) and` $DVIHSBO$ `gives the number of Democrat votes in a state where the winner wins by one vote only and corresponds to the Electoral College winner (details in Algorithm 7). */`

if $abs(ElColVote - K/2) = (L+1)/2$ **then** `/* Both direct and indirect contributions are necessary to change Electoral College vote outcome */`

- $PivotOutM2_B(L) =$ `/* Number of voters outside the coalition that are pivotal through Both (direct and indirect) mechanisms, when both are necessary */`
  $PivotOutM2_B(L) + (n+1)/2 \times (\sum_{k=L+1}^{K}(vote_k = DVIHSBO))$

else if $abs(ElColVote - K/2) \leq L/2$ **then** `/* Indirect contribution alone is sufficient */`

- $PivotOutM2_{IO}(L) =$ `/* Number of voters outside the coalition that are pivotal through Indirect mechanism Only (i.e. changing the vote of the coalition but not the one of their state) */`
  $PivotOutM2_{IO}(L) + (\sum_{k=L+1}^{K}(vote_k * \mathbb{1}_{vote_k <> DVIHSBO}))$
- **if** $abs(ElColVote - K/2) = 1/2$ **then** `/* One electoral college vote is sufficient to change the election */`
  - $PivotOutM2_{BD}(L) =$ `/* Number of voters outside the coalition that are pivotal through Both mechanisms, although Direct one is sufficient (i.e. changing the vote of their state) */`
    $PivotOutM2_{BD}(L) + (n+1)/2 \times (\sum_{k=L+1}^{K}(vote_k = DVIHSBO))$
- **else** `//` $1/2 < abs(ElColVote - K/2) \leq L/2$
  - $PivotOutM2_{BI}(L) =$ `/* Number of voters outside the coalition that are pivotal through Both mechanisms, when Indirect one only is necessary (i.e. changing the vote of the coalition) */`
    $PivotOutM2_{BI}(L) + (\sum_{k=L+1}^{K}(vote_k * \mathbb{1}_{vote_k = DVIHSBO}))$

9.2 Appendix 2: The IAC Model

The main purpose of this Appendix is to call the attention of the reader upon the mathematical complexities arising from the use of the IAC model. It becomes much more difficult to derive the exact formulas of the probabilities of the important relevant events[77]. Even the asymptotic

[77] As already pointed out several times, the symmetric setting postulated in this paper rules out malapportionment problems. Owen (1975, 2001) found a very intriguing result on the U.S. Electoral College using data from the sixties and the seventies. He computed the Banzhaf/IC and Shapley-Shubik/IAC power indices of U.S. citizens as a function of the state. While the two 51−dimensional vectors differ in order of magnitude, he shows

behavior of these probabilities is not straightforward to determine. This explains the appeal to simulations as a substitute. Below we illustrate some of these difficulties in two of the the simplest versions of the Electoral College, respectively the cases where the country is divided in either two or three equipopulated states. Among the important messages that we want to convey is that the geography of the votes resulting from the IAC model is rather intricate. Further, the magnitude of the correlation among individual votes is so important that some of the peculiarities of the Electoral College and its variants tend to vanish as soon as the population is large.

9.2.1 Two Equipopulated States: The Bivariate Distribution, Election Inversions and the Seats Votes Curve

We assume first $K = 2$ i.e. that the country is divided into two equipopulated states. We further assume that when the representatives of the two districts disagree, then a fair coin is flipped[78]. To describe the geography of the votes and the correlation among states, we introduce the discrete bivariate random variable $(\widetilde{k}, \widetilde{r})$. A basic event will be a pair of integers (k, r) where k denotes the number of voters voting D in the first district and r is the number of voters voting D in the second district. The IAC probability $\lambda_{IAC}(k, r)$ of the event (k, r) is as follows[79]:

$$\int_0^1 \binom{m}{k}\binom{m}{r} p^k (1-p)^{m-k} p^r (1-p)^{m-r} dp =$$

$$\frac{m!}{k!(m-k)!} \times \frac{m!}{r!(m-r)!} \times \frac{(k+r)!(2m-k-r)!}{(2m+1)!}$$

This bivariate distribution displays some interesting features. If we consider its continuous extension, we obtain:

$$\frac{\Gamma(m+1)}{\Gamma(x+1)\Gamma(m-x+1)} \times \frac{\Gamma(m+1)}{\Gamma(y+1)\Gamma(m-y+1)} \times \frac{\Gamma(x+y+1)\Gamma(2m-x-y+1)}{\Gamma(2m+2)}$$

For the sake of illustration consider the case where $m = 11$. We obtain:

that the two 50−dimensional vectors of ratios (where District of Columbia is normalized to 1) are about the same. For instance, the normalized power of a Californian citizen is about 3.2 times larger than the power of a voter of the district of Columbia for both indices. This intriguing equivalence for which we have no explanation does not hold for IAC^*.

[78]We depart slightly from our definition of a voting mechanism by permitting the outcome to be stochastic. The resulting mechanism is symmetric and neutral. The same conclusions would be obtained in the case of the deterministic mechanism: D is chosen whenever the two representatives disagree. This mechanism is not neutral ($\mathcal{W}$ is not proper).

[79]We remind that $\lambda_{IC}(k, l) = \frac{1}{2^{2n}}\binom{n}{k}\binom{n}{l}$ and $\lambda_{IAC^*}(k, l) = \frac{1}{(n+1)^2}$.

$$\lambda_{IAC}(x,y) = \frac{\Gamma(12)}{\Gamma(x+1)\Gamma(12-x)} \times \frac{\Gamma(12)}{\Gamma(y+1)\Gamma(12-y)} \times \frac{\Gamma(x+y+1)\Gamma(23-x-y)}{\Gamma(24)}$$

which is depicted on figure 23 below.

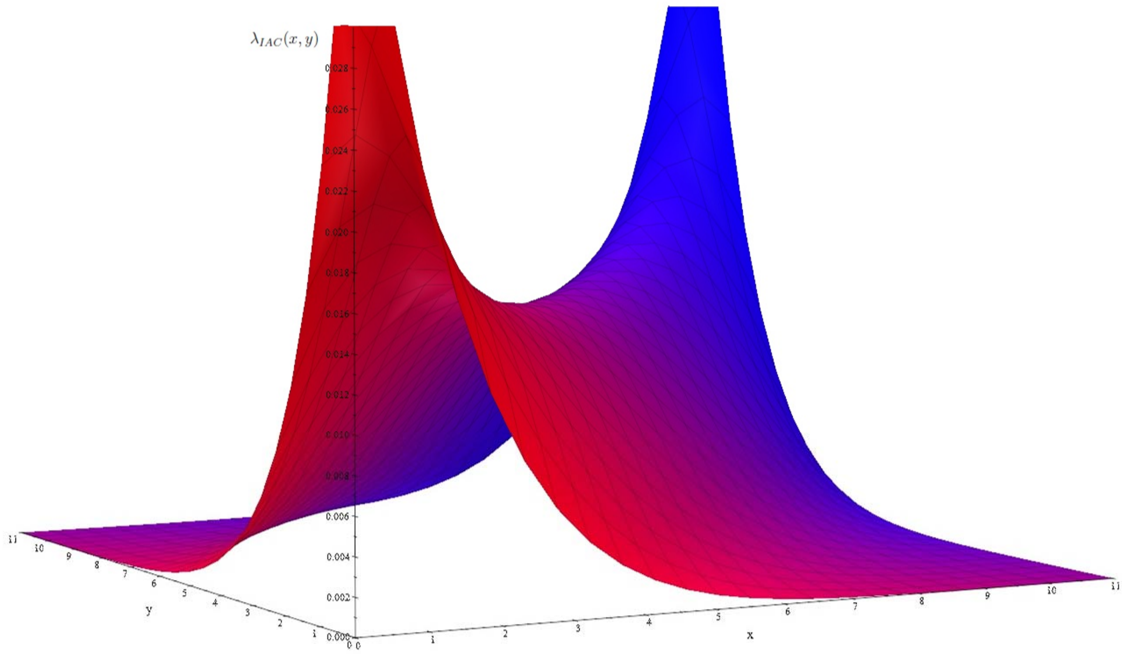

Figure 23: Graph of $\lambda_{IAC}(x,y)$.

Here is a sample of numerical values for $\lambda(k,r)$:

$$\begin{aligned}
\lambda(5,5) &= \frac{11!}{5!(6)!} \times \frac{11!}{5!(6)!} \times \frac{(10)!(12)!}{(23)!} = 1.4351 \times 10^{-2} \\
\lambda(3,3) &= \frac{11!}{3!(8)!} \times \frac{11!}{3!(8)!} \times \frac{(6)!(16)!}{(23)!} = 1.5864 \times 10^{-2} \\
\lambda(1,1) &= \frac{11!}{1!(10)!} \times \frac{11!}{1!(10)!} \times \frac{(2)!(20)!}{(23)!} = 2.2774 \times 10^{-2} \\
\lambda(2,8) &= \frac{11!}{2!(9)!} \times \frac{11!}{8!(3)!} \times \frac{(10)!(12)!}{(23)!} = 6.1017 \times 10^{-4}
\end{aligned}$$

These computations suggest the following conjectures. Among the $(m+1)^2$ possible realizations, the bivariate random variable $(\widetilde{k},\widetilde{r})$ seems to be mostly concentrated on the main diagonal i.e. basic events of the type (k,k). Further, more mass is placed on the extremes of the diagonal as compared to the center. Let us have first a look at diagonal events. To ease the algebra, let us depart temporarily (for the sake of these computations) from our assumption that m is odd and assume instead that m is even. The probability of the event $\left(\frac{m}{2},\frac{m}{2}\right)$ is equal to:

$$\frac{m!}{(\frac{m}{2})!(\frac{m}{2})!}\times\frac{m!}{(\frac{m}{2})!(\frac{m}{2})!}\times\frac{(m)!(m)!}{(2m)!}\times\frac{1}{2m+1}$$

We will use repeatedly the first order Stirling's formula:

$$m!\simeq\sqrt{2\pi m}\left(\frac{m}{e}\right)^m$$

Then:

$$\left(\frac{m}{2}\right)!\simeq\sqrt{\pi m}\left(\frac{m}{2e}\right)^{\frac{m}{2}}$$

Therefore:

$$\frac{m!}{(\frac{m}{2})!(\frac{m}{2})!}\simeq\sqrt{\frac{2}{\pi m}}\,(2)^m$$

and

$$\left(\frac{m!}{(\frac{m}{2})!(\frac{m}{2})!}\right)^2\simeq\frac{2}{\pi m}\,(2)^{2m}$$

On the other hand:

$$(2m)!\simeq\sqrt{4\pi m}\left(\frac{2m}{e}\right)^{2m}$$

Therefore:

$$\frac{(m)!(m)!}{(2m)!}\simeq\frac{2\pi m}{\sqrt{4\pi m}}\left(\frac{1}{2}\right)^{2m}=\sqrt{\pi m}\left(\frac{1}{2}\right)^{2m}$$

Collecting all the terms, we obtain that the probability of the event $\left(\frac{m}{2},\frac{m}{2}\right)$ is approximately (i.e. when m is large) equal to:

$$\sqrt{\pi m}\times\frac{2}{\pi m}\times\frac{1}{2m+1}\simeq\frac{1}{\sqrt{\pi}}\times\frac{1}{m^{\frac{3}{2}}}$$

In contrast, note that under IAC^*, this probability is equal to:

$$\left(\frac{1}{m+1}\right)^2 \simeq \frac{1}{m^2}$$

At the other extreme, what is the probability of the event $(1,1)$? We get:

$$\frac{m!}{(1)!(m-1)!} \times \frac{m!}{(1)!(m-1)!} \times \frac{(2)!(2m-2)!}{(2m)!} \times \frac{1}{2m+1}$$

$$= \frac{2m^2}{2m(2m-1)(2m+1)} \simeq \frac{1}{4m}$$

And for $(0,0)$ we get:

$$\frac{1}{2m+1} \simeq \frac{1}{2m}$$

Finally for $(2,2)$, we get:

$$\frac{m^2(m-1)^2}{4} \times \frac{24}{(2m+1)(2m)(2m-2)(2m-2)(2m-3)} \simeq \frac{3}{16m}$$

Note finally that if we take an entry very distant from the diagonal like $(0,m)$, we get a very low probability:

$$\frac{(m)!(m)!}{(2m+1)!} \simeq \frac{1}{2m+1} \frac{2\pi m}{\sqrt{4\pi m}} \frac{1}{2^m}$$

The total mass on the diagonal is given by:

$$\sum_{k=0}^{m} \binom{m}{k}\binom{m}{k} \times \frac{(2k)!(2m-2k)!}{(2m+1)!} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)}$$

When $m = 11$, we obtain approximately 0.25851[80]:

Let us now evaluate the probability that the outcome of the Electoral College differs from the popular vote. Since $2m$ is even, we need to specify how ties are broken in majority voting in the popular vote and in the second tier. To proceed to that evaluation, we first compute the probability that the two districts vote the same. The probability that the two districts vote D is given by:

$$\sum_{k=\frac{m+1}{2}}^{m} \sum_{r=\frac{m+1}{2}}^{m} \lambda_{IAC}(k,r) = \sum_{k=\frac{m+1}{2}}^{m} \sum_{r=\frac{m+1}{2}}^{m} \frac{m!}{k!(m-k)!} \times \frac{m!}{r!(m-r)!} \times \frac{(k+r)!(2m-k-r)!}{(2m+1)!}$$

[80]In contrast, in the IAC^* case, it is equal to $\frac{1}{12} = 8.3333 \times 10^{-2}$

The Electoral College outcome differs from the popular vote if and only if the two districts disagree. The probability of election inversion is then:

$$\frac{1}{2}-\sum_{k=\frac{m+1}{2}}^{m}\sum_{r=\frac{m+1}{2}}^{m}\frac{m!}{k!(m-k)!}\times\frac{m!}{r!(m-r)!}\times\frac{(k+r)!(2m-k-r)!}{(2m+1)!}$$

For both IC and IAC^{*} this probability is equal to $\frac{1}{4}=25\%$ which is quite large. In contrast, the number above is much smaller for IAC. For instance when $m=11$, we obtain:

$$\sum_{k=6}^{11}\sum_{r=6}^{11}\frac{11!}{k!(11-k)!}\times\frac{11!}{r!(11-r)!}\times\frac{(k+r)!(22-k-r)!}{(23)!}=0.421\,07$$

and the probability of election inversion equals to $0.5-0.42107=7.893\%$

From above, the IAC probability that the Electoral College delivers a clear majority outcome is here 84.214% to be compared to the probability obtained in the popular vote case which is equal to 95.652%[81]. In contrast, under IC and IAC^{*} this probability is simply equal to 50%.

So far we can compute unconditional probabilities of electoral outcomes like (D,D). We could also be interested in knowing how this probability changes given that the total of D votes is x. For all $x\geq m+1$, this probability is equal to:

$$(2m+1)\int_{0}^{1}\sum_{k=\frac{m+1}{2}}^{x-\frac{m+1}{2}}\frac{m!}{k!(m-k)!}\times\frac{m!}{(x-k)!(k+m-x)!}p^{x}(1-p)^{2m-x}dp=$$

$$\frac{x!(2m-x)!}{(2m)!}\sum_{k=\frac{m+1}{2}}^{x-\frac{m+1}{2}}\frac{m!}{k!(m-k)!}\times\frac{m!}{(x-k)!(k+m-x)!}$$

Let $\Psi(x)$ be the conditional (with respect to x) expectation of the number of representatives (i.e. seats in the upper tier): this corresponds to what political scientist call a seats/votes curve (Tufte (1973)). For all $x\geq m$, the probability that the electoral outcome is (D,R) or (R,D) given x is[82]:

$$1-\frac{x!(2m-x)!}{(2m)!}\sum_{k=\frac{m+1}{2}}^{x-\frac{m+1}{2}}\frac{m!}{k!(m-k)!}\times\frac{m!}{(x-k)!(k+m-x)!}$$

Therefore for all $x\geq m$:

[81] Precisely $1-\frac{1}{23}$.

[82] The probability of the outcome (R,R) is equal to 0.

$$\Psi(x) = 1 + \frac{x!(2m-x)!}{(2m)!} \sum_{k=\frac{m+1}{2}}^{x-\frac{m+1}{2}} \frac{m!}{k!(m-k)!} \times \frac{m!}{(x-k)!(k+m-x)!}$$

Consider now the case where $x < m$. In such case, there is at most one D representative. The probability that the electoral outcome is (D, R) or (R, D) given x is:

$$2\frac{x!(2m-x)!}{(2m)!} \sum_{k=\frac{m+1}{2}}^{x} \frac{m!}{k!(m-k)!} \times \frac{m!}{(x-k)!(k+m-x)!}$$

and therefore[83]:

$$\Psi(x) = 2\frac{x!(2m-x)!}{(2m)!} \sum_{k=\frac{m+1}{2}}^{x} \frac{m!}{k!(m-k)!} \times \frac{m!}{(x-k)!(k+m-x)!}$$

For the sake of illustration, consider the case where $m = 11$. We obtain for instance

$$\Psi(15) = (0 \times 0) + (1 \times (1 - 0.93653)) + (2 \times 0.93653) \simeq 1.93653$$

$$\Psi(13) = (0 \times 0) + (1 \times (1 - 0.613)) + (2 \times 0.613) \simeq 1.613$$

For the sake of visualization, we may normalize the two quantities in $[0, 1]$. For the sample of two values that we have considered, we obtain[84]:

$$\Psi(59.1\%) = 80.60\% \text{ and } \Psi(68.2\%) = 96.83\%$$

Proceeding similarly on the opposite side and a sample of three values[85], we obtain:

$$\Psi(9) = 0.38700, \Psi(8) = 0.18266 \text{ and } \Psi(7) = 6.3467 \times 10^{-2}$$

i.e. after normalization:

$$\Psi(31.82\%) = 3.17\%, \Psi(36.36\%) = 9.2\% \text{ and } \Psi(41\%) = 19.35\%$$

It would be interesting to depict the entire votes/seats curve over the finite grid:

$$\left\{\frac{0}{22}, \frac{1}{22}, \ldots, \frac{21}{22}, \frac{22}{22}\right\}$$

[83] The event (D, D) has probability 0.

[84] On the horizontal axis, we have $\frac{13}{22} = 0.59091$, $\frac{15}{22} = 0.68182$ while on the vertical axis we get: $\frac{1.613}{2} = 0.8065$ and $\frac{1.93653}{2} = 0.96827$.

[85] Namely: $x = 9$, $x = 8$ and $x = 7$.

and to extend by linear interpolation.

In contrast, let us see how the votes/seats curve looks like in the IAC^* case. In such case, the probability of having two seats given $x \geq m$ is equal to :

$$\frac{1}{(m+1)^2} \sum_{k=\frac{m+1}{2}}^{x-\frac{m+1}{2}} 1 = \frac{x-m+1}{(m+1)^2}$$

Since the probability that the total of D votes is equal to x is:

$$\frac{2m-x+1}{(m+1)^2}$$

we deduce that the conditional probability of the (D, D) outcome is:

$$\frac{x-m+1}{2m-x+1}$$

Therefore the IAC^* seats/votes curve Φ looks like:

$$\Phi(x) = (0 \times 0) + (1 \times \frac{3m-2x}{2m-x+1}) + (2 \times \frac{x-m+1}{2m-x+1})$$

When $m = 11$ and $x \geq 11$, we obtain after normalization[86]:

$$\Phi\left(59.1\%\right) = 65\% \text{ and } \Phi\left(68.2\%\right) = 81.75\%$$

The IAC^* curve seems closer to the diagonal!

9.2.2 The IAC Model with Three Equipopulated States

We now assume that the country is divided into three equipopulated states and we denote again by m the number of voters in each state and assume (unless otherwise specified) that m is an odd integer. Moving from two to three increases drastically the complexity of the analysis. We are not going to report results on the geography of votes as described by the trivariate distribution $\lambda_{IAC}\left(k, r, l\right)$. The probability of an election inversion is studied in de Mouzon et al. (2017) where it is shown that this probability tends to 0 as the inverse of the square root of the population. In this subsection, we first illustrate, through elementary computations, the fact that the popular vote and the Electoral College are indistinguishable from the perspective of influence.

We denote respectively by $Piv_{ind}(m)$ and $Piv_d(m)$ the probabilities of being pivotal for the Electoral College and for the popular vote.

[86]Before normalization: $\Phi(13) = 1.3$ and $\Phi(15) = 1.635$.

It is well known that:

$$Piv_d(m) = \frac{1}{3m}$$

In contrast:

$$Piv_{ind}(m) =$$

$$2\sum_{k=\frac{m+1}{2}}^{m}\sum_{r=0}^{\frac{m-1}{2}}\frac{m!}{k!(m-k)!}\times\frac{m!}{r!(m-r)!}\frac{(m-1)!}{\frac{m-1}{2}!\frac{m-1}{2}!}\frac{(k+r+\frac{m-1}{2})!(3m-1-k-r-\frac{m-1}{2})!}{(3m)!}$$

When $m = 11$, we obtain:

$$Piv_{ind}(11) =$$

$$2\sum_{k=6}^{11}\sum_{r=0}^{5}\frac{11!}{k!(11-k)!}\times\frac{11!}{r!(11-r)!}\frac{(10)!}{5!5!}\frac{(k+r+5)!(32-k-r-5)!}{(33)!}$$

$$= 3.030\,3\times 10^{-2} = \frac{1}{33} = Piv_d(11)$$

An equivalent way to present the computation of pivotality for a voter in state 1 is to use the Bayes formula. The probability for a voter to be pivotal in state 1 is $\frac{1}{m}$. For the representative of state 1 to be pivotal in the Electoral College, it must be the case that the two other representatives have different opinions. The difficulty is that this second event is not independent from the first one. The posterior $\widetilde{p}$ on the Bernoulli parameter p is not anymore uniform. When we learn that there is an equal split in state 1, we tend to revise the prior in direction of $\frac{1}{2}$. What is this revised prior exactly? From Bayes's formula we obtain the density:

$$g(p) \equiv \Pr(\widetilde{p} = p \mid S = \frac{m-1}{2}) = m\binom{m-1}{\frac{m-1}{2}}p^{\frac{m-1}{2}}(1-p)^{\frac{m-1}{2}}$$

When $m = 11$, we obtain:

$$g(p) = 11\binom{10}{5}p^5(1-p)^5$$

The graph of the density is shown of figure 24.

The probability mass is more concentrated around $\frac{1}{2}$ with $g(0.5) = 11\binom{10}{5}(0.5)^5(0.5)^5 = 2.\ 707$. Note also that now the probability that any two voters vote D is not equal to $\frac{1}{3}$ anymore but to:

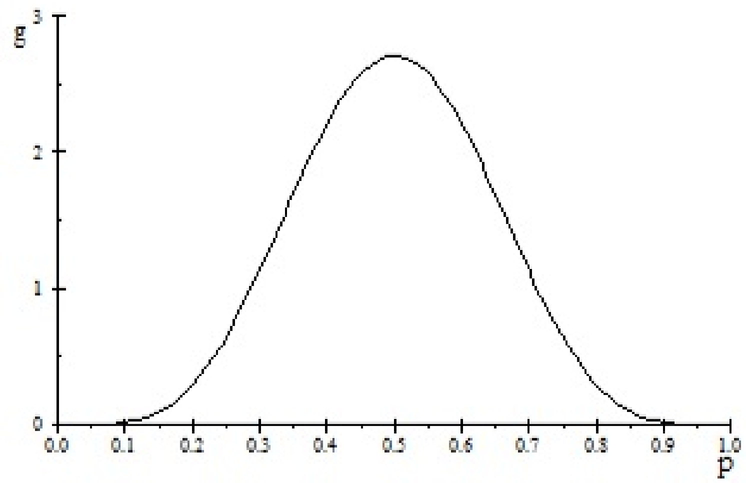

Figure 24: Graph of the density of g.

$$\int_0^1 11\binom{10}{5}p^7(1-p)^5dp = 0.26923$$

much closer to the independent case $\frac{1}{4}$!

We also note that the conditional probability that states 2 and 3 have different majority votes is equal to:

$$22\sum_{k=6}^{11}\sum_{r=0}^{5}(\int_0^1 \frac{11!}{k!(11-k)!}\times\frac{11!}{r!(11-r)!}\binom{10}{5}p^{5+k+r}(1-p)^{27-k-r}dp) = 0.333\,33$$

a number quite different from $1-0.84214 = 0.15786$, the value of the unconditional probability which was determined in the subsection dedicated to the two states setting.

The IAC equivalence between the popular vote and the Electoral College in terms of influence does not hold for utilities. For the sake of illustration, we are going to show this in the case where $m = 11$.

The expected utility of a voter from state 1 is twice the sum of three terms:

The expected utility that he votes democrat, that the two other states vote democrat and state 1 votes republican

$$2\sum_{k=6}^{11}\sum_{r=6}^{11}\sum_{i=0}^{4}\frac{11!}{k!(11-k)!}\times\frac{11!}{r!(11-r)!}\times\frac{10!}{i!(10-i)!}\times\frac{(k+r+i+1)!(32-k-r-i)!}{(34)!}$$

$$= 1.540\,0\times 10^{-2}$$

The expected utility that he votes democrat, state 1 votes democrat and either state 2 or state 3 but not both votes democrat:

$$2\sum_{k=6}^{11}\sum_{r=0}^{5}\sum_{i=5}^{10}\frac{11!}{k!(11-k)!}\times\frac{11!}{r!(11-r)!}\times\frac{10!}{i!(10-i)!}\times\frac{(k+r+i+1)!(32-k-r-i)!}{(34)!}$$

$$=5.1778\times 10^{-2}$$

and the expected utility that he votes democrat and all states vote democrat:

$$\sum_{k=6}^{11}\sum_{r=6}^{11}\sum_{i=5}^{10}\frac{11!}{k!(11-k)!}\times\frac{11!}{r!(11-r)!}\times\frac{10!}{i!(10-i)!}\times\frac{(k+r+i+1)!(32-k-r-i)!}{(34)!}$$

$$=0.31052$$

Therefore, the expected utility of a voter from state 1 is

$$2(0.310520+5.1778\times 10^{-2}+1.5400\times 10^{-2})=0.7554$$

In contrast, the expected utility for the popular vote mechanism is:

$$2\sum_{k=16}^{32}\frac{32!}{k!(32-k)!}\frac{(k+1)!(32-k)!}{(34)!}=0.75758$$

The Weber ratio is :

$$\frac{0.7554-0.5}{0.75758-0.5}\simeq 99.154\%$$

This is an interesting case where the pivotality and utilitarian criteria conflict[87] in the comparison of two mechanisms : one criterion declares them equivalent while the other one declares one mechanism superior to the other one.

9.3 Appendix 3: Election Inversions in MRPV under IC: An Alternative Estimation through Orthant Probabilities

We derive an analytic form of the probability inversions when r is large and take advantage of it to suggest a program delivering numerical approximations. We will denote by Z_{ik} the democrat vote of voter i in state k where $i=1,...,2r+1$ and $k=1,...,K$.

[87] On this, the reader could consult Le Breton and Van Der Straeten (2015).

$$Z_{ik} = \begin{cases} 1 \text{ if } i \text{ votes } D \\ 0 \text{ if } i \text{ votes } R \end{cases}$$

To evaluate $\Xi(K, L)$ we examine separately the following two cases:

Case 1: Consider first the case where $L \geq Q$. An election inversion occurs when the coalition votes D and the country votes R or when the coalition votes R and the country votes D. Consider the first event and assume without loss of generality that the interstate compact consists of the first L states.

The event is described by the following two inequalities:

$$X = \sum_{k=1}^{L} \sum_{i=1}^{2r+1} Z_{ik} \geq \frac{L(2r+1)}{2}$$

$$Y = \sum_{k=1}^{K} \sum_{i=1}^{2r+1} Z_{ik} \leq \frac{K(2r+1)}{2}$$

or equivalently:

$$\widehat{X} = \frac{\sum_{k=1}^{L} \sum_{i=1}^{2r+1} \left(Z_{ik} - \frac{1}{2}\right)}{\frac{1}{2}\sqrt{L(2r+1)}} \geq 0$$

$$\widehat{Y} = \frac{\sum_{k=1}^{K} \sum_{i=1}^{2r+1} \left(Z_{ik} - \frac{1}{2}\right)}{\frac{1}{2}\sqrt{K(2r+1)}} \leq 0.$$

From the multivariate central limit theorem, we deduce that when r tends to ∞, the bivariate vector $\widehat{Z} = \left(\widehat{X}, \widehat{Y}\right)$ tends to a Gaussian vector $N(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$ where:

$$\rho = \frac{\frac{L}{4}(2r+1)}{\frac{1}{4}\sqrt{LK}(2r+1)} = \sqrt{\frac{L}{K}}.$$

From Stieltjes's formula (Gupta (1963)), we deduce that:

$$\text{Prob } (\widehat{X} \geq 0, \widehat{Y} \geq 0) = \frac{1}{4} + \frac{\arcsin \sqrt{\frac{L}{K}}}{2\pi}.$$

Exploiting the symmetry of $\widehat{Z}$, we deduce that the the probability Prob $(\widehat{X} \geq 0, \widehat{Y} \leq 0)+$ Prob $(\widehat{X} \leq 0, \widehat{Y} \geq 0)$ of an inverted election is equal to $1 - 2$Prob $(\widehat{X} \geq 0, \widehat{Y} \geq 0)$ i.e.

$$\Xi_{IC}(K, L) = \frac{1}{2} - \frac{\arcsin \sqrt{\frac{L}{K}}}{\pi}.$$

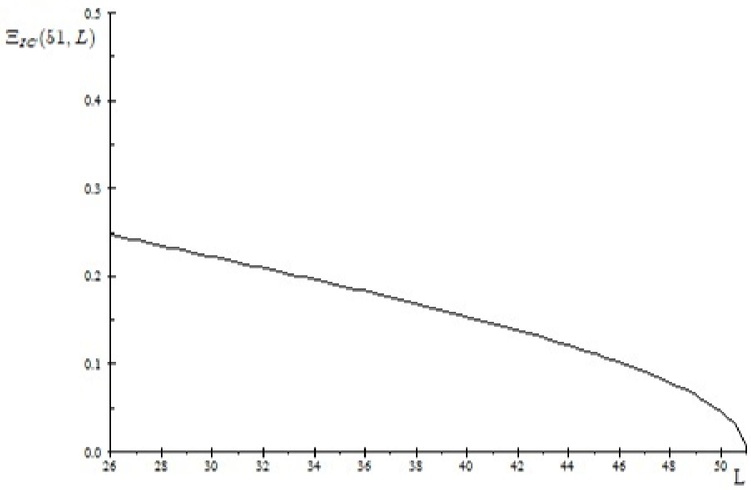

Figure 25: Graph of $\Xi_{IC}(51, L)$.

The graph of the function $\Xi_{IC}(51, L)$ over the range $[26, 51]$ is depicted on figure 25 below[88]:

Case 2: Consider now the case where $L < Q$. An election inversion occurs when either (i) the coalition votes D and k states outside the coalition where $k \geq Q - L$ vote D while the country votes R or (ii) when the coalition votes R and k states outside the coalition where $k \geq Q - L$ vote R while the country votes D, or (iii) when the coalition votes D and k states outside the coalition where $k \geq Q$ vote R while the country votes D or (iv) when the coalition votes R and k states outside the coalition where $k \geq Q$ vote D while the country votes R. We consider now the $(K - L + 2)-$dimensional random vector $\widehat{Z} = \left(\widehat{X}, \widehat{X}_{D+1}, ..., \widehat{X}_K, \widehat{Y}\right)$ where:

$$\widehat{X}_k = \frac{\sum_{i=1}^{2r+1}\left(Z_{ik} - \frac{1}{2}\right)}{\frac{1}{2}\sqrt{(2r+1)}} \text{ for all } k = L+1, ..., K.$$

From the multivariate central limit theorem, we deduce that when r tends to ∞, the vector $\widehat{Z}$ tends to a Gaussian vector $N(0, \Omega)$ where:

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & . & 0 & \sqrt{\frac{L}{K}} \\ 0 & 1 & 0 & . & 0 & \sqrt{\frac{1}{K}} \\ 0 & 0 & 1 & . & . & . \\ . & . & . & . & 0 & . \\ 0 & 0 & . & 0 & 1 & \sqrt{\frac{1}{K}} \\ \sqrt{\frac{L}{K}} & \sqrt{\frac{1}{K}} & . & . & \sqrt{\frac{1}{K}} & 1 \end{pmatrix}$$

[88] For the sake of illustration: $\Xi\,(101, 51) = \frac{1}{2} - \frac{\arcsin\sqrt{\frac{51}{101}}}{\pi} = 0.248\,42$ and $\Xi(101, 75)\frac{1}{2} - \frac{\arcsin\sqrt{\frac{75}{101}}}{\pi} = 0.169\,38$.

The probability of an election inversion $\Xi_{IC}(K,L)$ is then expressed as follows[89]:

$$\Xi_{IC}(K,L) = 2\sum_{k=Q-L}^{K-L-1}\binom{K-L}{k}P_{I_kJ_k}(\Omega) + 2\sum_{k=Q}^{K-L}\binom{K-L}{k}P_{\widetilde{I}_k\widetilde{J}_k}(\Omega)$$

where:

$$I_k = \{1, L+1, ..., L+k\}, J_k = \{L+k+1, ..., K-L+1, K-L+2\}$$

$$\widetilde{I}_k = \{1, L+k+1, ..., K-L+1, K-L+2\}$$

$$\text{and } \widetilde{J}_k = \{L+1, ..., L+k\}.$$

In contrast to case 1, we do not have a closed form. This is due to the fact that in general (Gupta (1963)) no closed form expressions of the orthant probabilities of a Gaussian vector are known.

Note however that the above analytical expressions suggest a Monte Carlo random approximation of these numbers based on the following idea. Draw a random sample of T observations of the vector $\widehat{Z} : \widehat{Z^1}, \widehat{Z}^2, ..., \widehat{Z}^T$ and define, for all $t = 1, ..., T$, the corresponding sequence of Bernoulli random variables $\widehat{W_k^1}, \widehat{W_k}^2, ..., \widehat{W_k}^T$ and $\widehat{V_k^1}, \widehat{V_k}^2, ..., \widehat{V_k}^T$ as follows:

$$\widehat{W_k^t} = \begin{cases} 1 \text{ if } \widehat{Z}_j \geq 0 \text{ for all } j \in I_k \text{ and } \widehat{Z}_j \leq 0 \text{ for all } j \in J_k \\ 0 \text{ otherwise} \end{cases}$$

$$\widehat{V_k^t} = \begin{cases} 1 \text{ if } \widehat{Z}_j \geq 0 \text{ for all } j \in \widetilde{I}_k \text{ and } \widehat{Z}_j \leq 0 \text{ for all } j \in \widetilde{J}_k \\ 0 \text{ otherwise} \end{cases}$$

Then, from the Glivenko-Cantelli 's theorem, we deduce that $\frac{\sum_{t=1}^{T}\widehat{W_k^t}}{T}$ and $\frac{\sum_{t=1}^{T}\widehat{V_k^t}}{T}$ converge respectively almost surely to $P_{I_kJ_k}(\Omega)$ and $P_{\widetilde{I}_k\widetilde{J}_k}(\Omega)$. The computations then can be time consuming. For instance when $K = 51$ and $L = 10$, we need to compute 25 probabilities of the first type and 16 probabilities of the second type.

9.4 Appendix 4: Election Inversions in MRVP under IAC*: An Alternative Estimation through Uniform Orthant Probabilities

Like in the IC case, we derive an analytic form of the probability inversions if r is large and suggest another program to obtain numerical approximations. A notable difference with the IC

[89] Let z to be a $m-$dimensional Gaussian vector with mean 0 and variance-covariance matrix Ω. Let $M = \{1, ..., m\}$ and consider $I, J \subseteq M$ such that $I \cap J = \emptyset$. $P_{IJ}(\Omega)$ denotes the probability of the event $\{z_j \geq 0 \text{ for all } j \in I \text{ and } z_j \leq 0 \text{ for all } j \in J\}$.

case is that the asymptotic behavior of $\frac{\sum_{i=1}^{2r+1} Z_{ik}}{2r+1}$ describing the frequency of the democrat vote differs completely from its behavior under IC. Under IC, this ratio converges almost surely (and thus in law) to the degenerate distribution $\delta_{\frac{1}{2}}$[90]and, in section 3, we have exploited the fact that the difference between $\frac{\sum_{i=1}^{2r+1} Z_{ik}}{2r+1}$ and $\frac{1}{2}$ multiplied by $2\sqrt{(2r+1)}$ converges in law to a unit Gaussian. Under IAC^*, using Chamberlain and Rothschild (1981) we deduce instead that for all $k = 1, ..., K$:

$$\frac{\sum_{i=1}^{2r+1} Z_{ik}}{2r+1} \text{ converges in law when } r \to \infty \text{ towards the uniform distribution } \mathcal{U}_k \text{ on } [0,1]$$

where the random variables $(\mathcal{U}_k)_{1 \leq k \leq K}$ are independent. Of course the random variable $\frac{\sum_{k=1}^{K} U_k}{K}$ describing the democrat vote in the all country is not independent from them. Once adjusted to take into account these technical differences, the reasoning is similar to the one developed in the IC setting.

Case 1: Consider first the case where $L \geq Q$. An election inversion occurs when the coalition votes D and the country votes R or when the coalition votes R and the country votes D. Consider the first event and assume without loss of generality that the coalition consists of the first L states.

The event is described by the following two inequalities:

$$\widehat{X} = \sum_{k=1}^{L} \mathcal{U}_k \geq \frac{L}{2}$$

$$\widehat{Y} = \sum_{k=1}^{K} \mathcal{U}_k \leq \frac{K}{2}$$

If K (and therefore L) is large, we deduce from the multivariate central limit theorem, that the bivariate vector $\widehat{Z} = \left(\widehat{X}, \widehat{Y}\right)$ behaves like a Gaussian vector:

$$N(\begin{pmatrix} \frac{L}{2} \\ \frac{K}{2} \end{pmatrix}, \begin{pmatrix} \frac{L}{12} & \frac{L}{12} \\ \frac{L}{12} & \frac{K}{12} \end{pmatrix})$$

i.e.

$$\rho = \sqrt{\frac{L}{K}}$$

like in the IC case. When K is not large, we cannot use the Gaussian approximation. We could develop instead a Monte Carlo method to compute this probability. To do so, we could

[90] δ_x denotes the Dirac mass in x.

emulate T random draws of a $K-$dimensional vector of independent uniform random variables. Then, we calculate $\widehat{X}^t$ and $\widehat{Y}^t$ for all $t = 1, ..., T$ and we define the Bernoulli random variable $\widehat{W^t}$

$$\widehat{W^t} = \begin{cases} 1 \text{ if } \widehat{X}^t \geq \frac{L}{2} \text{ and } \widehat{Y}^t \leq \frac{K}{2} \\ 0 \text{ otherwise} \end{cases}$$

Then, from the Glivenko-Cantelli's theorem, we deduce that $\frac{\sum_{t=1}^{T} \widehat{W_k^t}}{T}$ converges almost surely to the probability of the event $\left\{ \widehat{X} \geq \frac{L}{2} \text{ and } \widehat{Y} \leq \frac{K}{2} \right\}$.

Case 2: Consider now the case where $L < Q$. An election inversion occurs when either (i) the coalition votes L and k states outside the coalition where $k \geq Q - L$ vote D while the country votes R or (ii) when the coalition votes R and k states outside the coalition where $k \geq Q - L$ vote R while the country votes D, or (iii) when the coalition votes D and k states outside the coalition where $k \geq Q$ vote R while the country votes D or (iv) when the coalition votes R and k states outside the coalition where $k \geq Q$ vote D while the country votes R. We consider now the $(K - L + 2)-$dimensional random vector $\widehat{Z} = \left(\widehat{X}, \widehat{X}_{L+1}, ..., \widehat{X}_K, \widehat{Y} \right)$ where:

$$\widehat{X} = \sum_{k=1}^{L} \mathcal{U}_k$$

$$\widehat{X}_k = \mathcal{U}_k \text{ for all } k = L + 1, ..., K$$

$$\widehat{Y} = \sum_{k=1}^{K} \mathcal{U}_k$$

For all $Q - L \leq k \leq K - L - 1$, the first event denoted E_k that we are considering is described by the inequalities:

$$\widehat{X} \geq \frac{L}{2}$$

$$\widehat{X}_j \geq \frac{1}{2} \text{ for all } j = L + 1, ..., L + k$$

$$\widehat{X}_j \leq \frac{1}{2} \text{ for all } j = L + 1 + k, ..., K - L + 1$$

$$\widehat{Y} \leq \frac{K}{2}$$

For all $Q \leq k \leq K - L$, the second event denoted F_k that we are considering is described by the inequalities:

$$\widehat{X} \geq \frac{L}{2}$$

$$\widehat{X}_j \leq \frac{1}{2} \text{ for all } j = L+1, ..., L+k$$

$$\widehat{X}_j \geq \frac{1}{2} \text{ for all } j = L+1+k, ..., K-L+1$$

$$\widehat{Y} \leq \frac{K}{2}$$

The IAC^* probability of an election inversion $\Xi_{IAC^*}(K, L)$ is expressed as follows:

$$\Xi_{IAC^*}(K, L) = 2 \sum_{k=Q-L}^{K-L-1} \binom{K}{k} Q\left(E_k\right) + 2 \sum_{k=Q-L}^{K-L} \binom{K}{k} Q\left(F_k\right)$$

where:

$$Q\left(E_k\right) = Prob\left(\begin{array}{c} \widehat{X} \geq \frac{L}{2},\ \widehat{X}_j \geq \frac{1}{2} \text{ for all } j = L+1, ..., L+k \text{ and } \widehat{X}_j \leq \frac{1}{2} \\ \text{for all } j = L+1+k, ..., K-L+1 \end{array}\right) \text{ and}$$

$$Q\left(F_k\right) = Prob\left(\begin{array}{c} \widehat{X} \geq \frac{L}{2},\ \widehat{X}_j \leq \frac{1}{2} \text{ for all } j = L+1, ..., L+k,\ \widehat{X}_j \geq \frac{1}{2} \\ \text{for all } j = L+1+k, ..., K-L+1 \text{ and } \widehat{Y} \leq \frac{K}{2} \end{array}\right)$$

The computation of the probabilities $Q\left(E_k\right)$ and $Q\left(F_k\right)$ which appear in the above summation can be done through a Monte Carlo technique quite similar to the one described in the IC case. The unique real change is that the Gaussian distributions are replaced by uniform distributions. Precisely, we draw a random sample of T observations of a $K-$dimensional vector $\widehat{X^t}$ whose coordinates $\widehat{X_k^t}$ are independent and identically distributed as uniform on $[0, 1]$ and compute the corresponding $\frac{\widehat{X^t}}{L}$ and $\frac{\widehat{Y^t}}{K}$. Let $\widehat{Z^t} = \left(\frac{\widehat{X^t}}{L}, \widehat{X^t}, \widehat{Y^t}\right)$ and define, for all $t = 1, ..., T$, exactly as in the IC case, two sequences of Bernoulli random variables $\widehat{W_k^1}, \widehat{W_k^2}, ..., \widehat{W_k^T}$ and $\widehat{V_k^1}, \widehat{V_k^2}, ..., \widehat{V_k^T}$as follows:

$$\widehat{W_k^t} = \begin{cases} 1 \text{ if } \widehat{Z}_j \geq \frac{1}{2} \text{ for all } j \in I_k \text{ and } \widehat{Z}_j \leq \frac{1}{2} \text{ for all } j \in J_k \\ 0 \text{ otherwise} \end{cases}$$

$$\widehat{V_k^t} = \begin{cases} 1 \text{ if } \widehat{Z}_j \geq \frac{1}{2} \text{ for all } j \in \widetilde{I}_k \text{ and } \widehat{Z}_j \leq \frac{1}{2} \text{ for all } j \in \widetilde{J}_k \\ 0 \text{ otherwise} \end{cases}$$

Then, from the Glivenko-Cantelli's theorem, we deduce that $\frac{\sum_{t=1}^{T} \widehat{W_k^t}}{T}$ and $\frac{\sum_{t=1}^{T} \widehat{V_k^t}}{T}$converge respectively almost surely to $Q\left(E_k\right)$ and $Q\left(F_k\right)$. The computations then can be time consuming like in the IC case.

9.5 Appendix 5: The Analytics of Total Utility in MRPV under IAC*

Since the Penrose formula does not hold in the IAC^* case, we cannot derive information on the utilities of the voters from what we know about their powers. The purpose of this section is to revisit some of the analysis undertaken in the previous subsections from the perspective of individual and total utilities rather than individual and total powers. Along the same lines as before, we will first derive analytical formulas that suggest a way to program the numerical computation of utilities in the asymptotic setting i.e. when r tends to ∞. Then we present the simulations derived from the program described in Appendix 1.

The theoretical formula distinguishes two cases.

Case 1: Consider first the case where $L \geq Q$ and a voter in the first state (say the first) of the interstate compact and assume that this voter has voted D. Using the notations introduced earlier, the conditional probability that the interstate compact votes D is the probability of the event:

$$\sum_{k=1}^{L} \sum_{i=1}^{2r+1} (Z_{ik} \mid Z_{11} = 1) > \frac{L(2r+1)}{2}$$

When r is large, this event writes:

$$\sum_{k=2}^{L} \mathcal{U}_k + \frac{\sum_{i=1}^{2r+1} (Z_{i1} \mid Z_{11} = 1)}{2r+1} > \frac{L}{2}$$

Consider the last term. From Bayes's formula, the conditional density of p over $[0, 1]$ is equal to $2p$. Therefore using an argument similar to Chamberlain and Rothschild (1981), we deduce that:

$$\frac{1 + \sum_{i=2}^{2r+1} (Z_{i1} \mid Z_{11} = 1)}{2r+1}$$

converges in law to V_1 the probability with density $2p$ on $[0, 1]$. This implies that $\frac{\sum_{k=1}^{L} \sum_{i=1}^{2r+1} (Z_{ik} | Z_{11}=1)}{2r+1}$ converges in law towards $V_1 + \sum_{k=2}^{L} U_k$. This is the convolution of L random variables : the last $L - 1$ are uniform while the first one is slightly biased towards D in the sense that Prob $(V_1 \geq \frac{1}{2}) > \frac{1}{2}$. Precisely:

$$Prob(V_1 \geq \frac{1}{2}) = \int_{\frac{1}{2}}^{1} 2pdp = \left[p^2\right]_{\frac{1}{2}}^{1} = \frac{3}{4}$$

The computation amounts to compute the probability of the event $\left\{V_1 + \sum_{k=2}^{L} \mathcal{U}_k \geq \frac{L}{2}\right\}$.

Let us denote $\xi(L)$ this probability. Certainly, $\xi(L)$ is larger than $\frac{1}{2}$ and decreasing with L. For instance for $L = 2$, we obtain:

$$\xi(2) = \int_0^1 \int_{1-x}^1 2y dy dx = \frac{2}{3}$$

The expected utility of any voter in the interstate compact is equal to:

$$2 \times \frac{1}{2} \times \xi(L) = \xi(L)$$

while the utility of any voter outside the interstate compact is equal to $\frac{1}{2}$. Up to the multiplicative constant $2r + 1$, the total expected utility is therefore:

$$U(L) = L(2r + 1)\xi(L) + \frac{(K - L)(2r + 1)}{2}$$

Case 2: Consider now the case where $L < Q$. The analysis is as above. Consider first, without loss of generality, a voter in the first state of the interstate compact who votes D. The conditional probability that the interstate compact votes D is equal to $\xi(L)$. Therefore the probability that the country selects D is equal to:

$$\psi(L) = \xi(L) \times \sum_{k=Q-L}^{K-L} \binom{K-L}{k} \frac{1}{2^{K-L}} + (1 - \xi(L)) \sum_{k=Q}^{K-L} \binom{K-L}{k} \frac{1}{2^{K-L}}$$

The utility of a voter outside the interstate compact is computed as follows. Consider a voter in state $L + 1$ voting D. The conditional probability that state $L + 1$ vote D is equal to $\frac{3}{4}$. The probability that the country selects D is equal to:

$$\varphi(L) = \frac{3}{4} \times \frac{1}{2} \times \sum_{k=Q-L-1}^{K-L-1} \binom{K-L-1}{k} \frac{1}{2^{K-L-1}} + \frac{3}{4} \times \frac{1}{2} \times \sum_{k=Q-1}^{K-L-1} \binom{K-L-1}{k} \frac{1}{2^{K-L-1}}$$

$$+\frac{1}{4} \times \frac{1}{2} \times \sum_{k=Q-L}^{K-L-1} \binom{K-L-1}{k} \frac{1}{2^{K-L-1}} + \frac{1}{4} \times \frac{1}{2} \times \sum_{k=Q}^{K-L-1} \binom{K-L-1}{k} \frac{1}{2^{K-L-1}}$$

Up to the multiplicative constant $2r + 1$, the total expected utility is now:

$$U(L) = L\psi(L) + (K - L)\varphi(L)$$

This exact formula paves the road for a numerical computation. To compute $\psi(L)$ and $\varphi(L)$, it is enough to compute $\xi(L)$. To obtain a numerical approximation of $\xi(L)$, one way to proceed is to emulate T random draws of a $L-$dimensional random vector $(V_1, U_2, ..., U_L)$ with independent marginals distributed as follows: the first is distributed according to the continuous density $2p$ while the $L-1$ remaining ones are distributed uniformly. For each draw t and each value of L we define $\xi(t,L)$ to be equal to 1 if $V_1 + \sum_{k=2}^{L} U_k > \frac{L}{2}$ and 0 otherwise. Since the empirical average $\frac{\sum_{t=1}^{T}\xi(t,L)}{T}$ converges almost surely to $\xi(L)$, we obtain an approximation of $\xi(L)$ if T is large enough.

9.6 Appendix 6: The Necessity of Conditioning in MNPV

In this appendix, we illustrate the necessity of conditioning in the the simplest case: $K = 3$ and $L = 1$. In such case the representative (here a unique state is part of the coalition) of state 1 is decisive if the votes in states 2 and 3 are opposed. Let us compute the conditional probability of the complementary event. The number of preference profiles for which the votes of the $3m-1$ voters are divided equally among D and R and the vector of votes of the representatives from states 2 and 3 is (D, D) is equal to:

$$\sum_{k=0}^{\frac{m-3}{2}} \binom{m-1}{k} \left(\sum_{l=\frac{m+1}{2}}^{m-1-k} \binom{m}{l} \binom{m}{\frac{3m-1}{2}-k-l} \right)$$

while the total number of preference profiles for which we have a tie among the $3m-1$ voters is equal to:

$$\binom{3m-1}{\frac{3m-1}{2}}$$

The IC conditional probability of the (D, D) outcome for states 2 and 3 is therefore:

$$\psi(m) \equiv \frac{\sum_{k=0}^{\frac{m-3}{2}} \binom{m-1}{k} \left(\sum_{l=\frac{m+1}{2}}^{m-1-k} \binom{m}{l} \binom{m}{\frac{3m-1}{2}-k-l} \right)}{\binom{3m-1}{\frac{3m-1}{2}}}$$

We have computed $\psi(m)$ for a sample of values of m to obtain:

m	3	5	7	9	15	31	41	51
$\psi(m)$	12.857%	14.569%	15.223%	15.567%	16.26%	16.364%	16.439%	16.484%

Table 2 Values of ψ

These values suggest that, when m tends to ∞, the function tends to a limit which is distant from $\frac{1}{4}$. When IC is replaced by IAC^* a similar difference shows up. The total number of anonymous configurations attached to the event that we consider is equal to:

$$\sum_{k=0}^{\frac{m-3}{2}}\left(\sum_{l=\frac{m+1}{2}}^{m-1-k}1\right)=\sum_{k=0}^{\frac{m-3}{2}}\left(\frac{m-3}{2}-k\right)=\frac{(m-1)(m-3)}{8}$$

Since the total number of anonymous configurations is equal to $m(m+1)^2$, we obtain that the IAC^* probability that the votes of the $3m-1$ voters are divided equally among D and R and the vector of votes of the representatives from states 2 and 3 is (D,D) is equal to:

$$\frac{(m-1)(m-3)}{8m(m+1)^2}$$

On the other hand, we know from Le Breton et al. (2016) that, when m is large, the IAC^* probability that the national electorate is divided equally between D and R is approximately equal to:

$$\frac{9}{12m}$$

The IAC^* conditional probability of the (D,D) outcome for states 2 and 3 is therefore:

$$\frac{(m-1)(m-3)}{6(m+1)^2}\simeq\frac{1}{6}=0.16667$$

which is again quite distant from 0.25.

The influence of a voter resident in a state outside the coalition is instead the sum of two terms[91]. Like any voter from a state inside the interstate, he is decisive in situations where the national vote is tied and the block representative from the interstate is decisive in the upper tier. But in addition to that, he is also decisive when his vote is decisive within his state and his state representative is decisive in the upper tier. *This implies that the influence of a voter outside the interstate is larger than the influence of a voter inside the interstate*[92]. We are now in a situation quite different from the situation prevailing when the interstate was allocating its votes to the popular regional winner. We have moved from a situation where the externality of the insiders on the outsiders was negative to a situation where it is positive. It is always better to be outside the interstate than inside.

[91]Under the presumption that the probability of the joint occurrence of the two events is of second order magnitude as compared to the the probability of each of them separately.

[92]Quite interestingly, the asymmetry between the two types of citizens is identical to the asymmetry explored by Le Breton and Lepelley (2014) in their analysis of the so-called French law of "double vote" which has prevailed for some time during the French period of Restauration.

9.7 Appendix 7: Decomposition of the Probability of being Pivotal when $L \geq \frac{K+1}{2}$ and m is Large

Note that if $L \geq \frac{K+1}{2}$, then only BI and IO do not vanish in the decomposition. Take an arbitrary elector i in state $L+1$.We are in the BI case iff i is pivotal in state $L+1$ and pivotal in the popular vote. This event can be equivalently formulated as i is pivot in state $L+1$ and the population of $(K-1)m$ voters living outside state $L+1$ is divided equally between the two camps. Let us evaluate sequentially the probability of that event for the three models.

IC. In such case, the probability of that event is equal to:

$$\frac{\binom{m-1}{\frac{m-1}{2}}}{2^{m-1}} \times \frac{\binom{(K-1)m}{\frac{(K-1)m}{2}}}{2^{(K-1)m}}$$

If m is large[93] we obtain from Stirling:

$$\sqrt{\frac{2}{\pi m}} \times \sqrt{\frac{2}{\pi(K-1)m}} = \frac{2}{\pi m} \times \frac{1}{\sqrt{K-1}} \simeq \frac{2}{\pi m} \times \frac{1}{\sqrt{K}}$$

On the other hand, the probability of being pivotal in the popular vote is equal to:

$$\frac{\binom{Km-1}{\frac{Km-1}{2}}}{2^{Km-1}} \simeq \sqrt{\frac{2}{\pi m}} \times \frac{1}{\sqrt{K}}$$

Therefore, the BI share in the decomposition is approximatively equal to:

$$\sqrt{\frac{2}{\pi m}}$$

IAC. In such case, using the classical Beta formula, we obtain that the probability of the event of double pivotality is equal to:

$$m \int_0^1 \binom{(K-1)m}{\frac{(K-1)m}{2}} \binom{m-1}{\frac{m-1}{2}} p^{\frac{Km-m}{2}} (1-p)^{\frac{Km-m}{2}} p^{\frac{m-1}{2}} (1-p)^{\frac{m-1}{2}} dp$$

$$= m \binom{(K-1)m}{\frac{(K-1)m}{2}} \binom{m-1}{\frac{m-1}{2}} \times \frac{(\frac{Km-1}{2})!(\frac{Km-1}{2})!}{Km!}$$

If m is large[94] we deduce from Stirling that is is approximatively equal to:

[93] When $m = 3$, the Stirling approximation leads to 0.46066 while the true percentage is 0.5033. This is the number that comes out of *SimuElect* on figure 20.

[94] When $m = 3$, the Stirling approximation leads to 0.46066 while the true percentage is 0.5033. This is the number that comes out of *SimuElect* on figure 20.

$$\frac{1}{K} \times \sqrt{\frac{2}{\pi(K-1)m}} 2^{(K-1)m} \times \sqrt{\frac{2}{\pi m}} 2^{m-1} \times \frac{1}{\sqrt{\frac{2}{\pi(Km-1)}} 2^{Km-1}}$$

$$\simeq \frac{1}{K} \times \sqrt{\frac{2K}{m\pi(K-1)}} \simeq \sqrt{\frac{2}{m\pi K(K-1)}}$$

On the other hand, the probability of being pivotal in the popular vote is equal to:

$$\frac{1}{Km}$$

Therefore, the BI share in the decomposition is approximatively equal to:

$$\sqrt{\frac{2K}{m\pi(K-1)}} \simeq \sqrt{\frac{2}{m\pi}}$$

We have derived the surprising conclusion that the BI share behaves similarly under IC and IAC when m gets large. This similarity is confirmed by the simulations derived from *SimuElect*.

IAC^*. In such case, the probability of the event of double pivotality is equal to:

$$\frac{1}{m} \times \phi(K-1,(K-1)m)$$

where $\phi(K-1,(K-1)m)$ denotes the probability that the population of $(K-1)m$ voters outside state $L+1$ is divided equally among the two camps. From Le Breton, Lepelley and Smaoui (2016), we deduce that:

$$\phi(K-1,(K-1)m) \simeq \frac{c_{K-1}}{(K-1)m}$$

On the other hand, the probability of being pivotal in the popular vote is equal to:

$$\frac{c_K}{Km-1}$$

Therefore, the BI share in the decomposition is approximatively equal to:

$$\frac{c_{K-1} \times K}{mc_K \times (K-1)} \simeq \frac{1}{m}$$

Now the BI share tends to 0 asymptotically more rapidly than under IC and IAC. This is confirmed by the simulations.

9.8 Appendix 8: No Unanimous Support for Reforming the Electoral College with a Simple Biased Probabilistic Model

The purpose of this additional appendix is to illustrate the importance of the unbiaisedness property which is common to the three probabilistic models which have been considered in that paper. We have argued that, from an a priori normative perspective, the use any such model was appropriate. In contrast, from a positive perspective, this assumption is hardly defendable as there are states where a large majority of citizens vote either democrat or republican over all elections. This raised the question is : when the statistical description of citizen preferences displays some biaises, are the implications of an interstate compact on election inversions and the welfare of citizens derived in this paper still valid ? We now show through a stylistic example that the answer to that question is negative.

To make our point in the simplest way, we consider three equipopulated states with 101 voters in each state and the statistical model described as follows:

$$\begin{cases} \text{All voters in state 1 vote democrat.} \\ \text{In states 2 and 3 (the swing/battleground states),} \\ \qquad \text{only partitions } (51,50) \text{ and } (50,51) \text{ have a positive probability} \\ \qquad \text{and they are all equiprobable.} \end{cases}$$

Let us examine in turn the expected payoffs of the voters under the popular vote and the Electoral College.

Under the popular vote, the democrat candidate wins with probability 1. The expected utility of a voter in state 1 is then equal to 1. The expected utility of a voter in states 2 and 3 is instead $\frac{1}{2} \times \frac{51}{101} + \frac{1}{2} \times \frac{50}{101} = \frac{1}{2}$.

Under the Electoral College, there are four possible state electoral outcomes: (D, D, D), (D, D, R), (D, R, D) and (D, R, R). They all occur with probability $\frac{1}{4}$. Therefore, the expected utility of a voter in state 1 is $\frac{3}{4}$. Instead, the expected utility of a voter in states 2 and 3 is equal to:

$$\frac{1}{4} \times \frac{51}{101} + \frac{1}{4} \times \frac{51}{101} + \frac{1}{4} \times \frac{50}{101} + \frac{1}{4} \times \frac{51}{101} = \frac{203}{404} > \frac{202}{404} = \frac{1}{2}$$

The swing states do not benefit from reforming the electoral college. To conclude, note that the probability of an election inversion is equal to 25%.