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SEQUENTIAL DECISIONS IN THE COLLEGE ADMISSIONS PROBLEM

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Abstract

This paper studies sequential mechanisms which mimic a matching procedures for many-to-one real life matching markets. We provide a family of mechanisms implementing the students´ optimal allocation in Subgame Perfect Equilibrium.

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1 Introduction

The college admissions problem models a contractual process in bilateral markets where monetary transfers are not relevant or can be embodied in agents' preferences. In this paper we analyze a mechanism whose rules are inspired by real-life allocation process. In particular, the one used by Spanish Universities in their admissions and the allocation of civil servants by Spanish Public Administration. We will show how the only matching supported by any Subgame Perfect Nash Equilibrium is the stable matching that is Pareto optimal from the students' point of view.

Early work, in particular Alĉalde and Barberà [2], have shown the generic non-existence of strategy-proof mechanisms selecting allocations to satisfy Pareto efficiency and individual rationality. The design of strategy-proof mechanisms selecting Pareto efficient allocations can be solved in a trivial way using "hierarchic dictatorships." For instance, consider that each college has a fixed number of vacant positions, and the students are ranked in some (exogenous) specific order. The first student is admitted to the best college (according to her preferences) that has a vacant position. The second student is admitted to the best college (according to her preferences) that has a vacant position, provided that the first student has already been allocated, and so on. We can consider the students' ranking as a dictatorship hierarchy.

This procedure is clearly strategy-proof. This is because (i) colleges have no possibility of rejecting students and (ii) each student obtains a position in her best college among the ones having a vacancy when she has to decide. This rule also selects allocations that are individually rational from the point of view of students, i.e., no student is allocated to a college she considers to be worse than the possibility of remaining unmatched. Nevertheless, since colleges, when faced with this mechanism have no possibility of rejecting students, it fails to satisfy individual rationality from the colleges' point of view. A way to recover this property is by allowing colleges to reject undesired students. These are the types of mechanisms this paper deals with.

Following Alcalde and Barberà [2], this mechanism has to fail in selecting Pareto efficient allocations and/or it has to be manipulable by some agent. Unfortunately, both negative properties are satisfied by the mechanism. Nevertheless, the matching rule we have described above satisfies two interesting properties, which leads us to recommend its application in some matching problems. The first property is that its description is very simple and mimics some real-live matching procedures. In fact, the admissions system for Spanish Universities is quite similar to the rule we described (see Romero-Medina [8]), and the allocation of civil servants in the Spanish Public Administration is done following a rule similar to the one described above. The second one is that the mechanism implements a subselection of the core correspondence. In fact, as we will see (Theorem 4) the only matching supported by any Subgame Perfect Nash equilibrium is the (only) stable matching that is Pareto optimal from the students' point of view. Moreover, our result is independent from the order in which students are ranked. This property highlights the robustness of our mechanism.

Given the symmetry that holds on both sides of the market, we also investigate the possibility of having a result similar to that of our Theorem 4, when the roles played by students and colleges are exchanged. This similarity holds in some matching mechanisms. For instance, Alcalde and Romero-Medina [4] show that, in two-stage game forms, in which all the agents on a given side of the market play simultaneously, the core correspondence is implemented in Subgame Perfect Nash Equilibrium, regardless of whether the first agents to play are the colleges or the students. Similar results have been provided by Alcalde, Pérez-Castrillo and Romero-Medina [3] for job markets. In this paper, we show that this symmetry does not hold if the first agents to play do so sequentially. In fact, the properties satisfied by the mechanism are not yet satisfied if the proposals are made by the colleges. In fact, (i) the matching supported by Subgame Perfect Nash Equilibria depends on the order in which colleges select their strategies and, consequently (ii) the equilibrium matching is not necessarily Pareto efficient from the colleges' point of view.

The rest of the paper is organized as follows. Section 2 introduces the basic model. Section 3 presents and analyzes the mechanism, called here the "students-sequentially-propose-and-colleges-choose" mechanism. Conclusions are presented in Section 4.

2 The model

We consider a college admissions problem with n colleges and m students. Let $C = \{c_1, ..., c_n\}$ and $S = \{s_1, ..., s_m\}$ be the set of colleges and students, respectively. Each college has preferences $P(c)$ over the set of groups of students. $P(c)$ is assumed to be a linear order on 2^S. Each student's preferences $P(s)$ is described by a linear order on $C \cup \{s\}$. A college admissions problem is fully described by a triplet $\left\{C, S; \underset{\sim}{P}\right\}$, where $\underset{\sim}{P}$

$= \{P(c_1), ..., P(c_n), P(s_1), ..., P(s_m)\}$ is a list containing a full description of the agents' preferences and is called a profile.

An allocation for such a problem, or *matching*, is a mapping μ from $C \cup S$ into $2^S \cup C$ satisfying

(i) for all $c \in C$, $\mu(c) \in 2^S$,

(ii) for all $s \in S$, $\mu(s) \in C \cup \{s\}$, and

(iii) for each pair $(c, s) \in C \times S$, $[\mu(s) = c \Longleftrightarrow s \in \mu(c)]$.

>From here on we will consider C and S to be fixed sets, thus we can identify a college admissions problem $\left\{C, S; \underset{\backsim}{P}\right\}$ with the preference profile $\underset{\backsim}{P}$.[1] Let $\mathcal{M}$ be the set of all possible matchings μ. Finally, $\underset{\backsim}{\mathbb{P}}$ denotes the set of (potential) matching markets.

Let $\underset{\backsim}{P}$ be a matching market. Given a set of students $A \subseteq S$, we denote by $Ch_c(A)$ the maximal element on 2^A under the linear order $P(c)$.

Definition 1 *A matching μ is said to be individually rational for $\underset{\backsim}{P}$ iff*

(i) $Ch_c(\mu(c)) = \mu(c)$ for all $c \in C$, and

(ii) for all $s \in S$, $c \in C$ $\quad [sP(s)c \Longrightarrow s \notin \mu(c)]$.

Definition 2 *Let μ be a matching for $\underset{\backsim}{P}$. We say that μ is blocked by a pair $(c, s) \in C \times S$ iff*

(i) $c\ P(s)\ \mu(s)$, and

[1] For the sake of simplicity, we will employ the same notation for a preference profile and the related college admissions problem. The context will be made precise if $\underset{\backsim}{P}$ denotes a matching problem or simply a preference profile.

(ii) $s \in Ch_c(\mu(c) \cup \{s\})$.

A pair (c, s) which satisfies the above two conditions is called a blocking pair for μ.

Definition 3 *Let μ be a matching for $\underset{\sim}{P}$. We say that μ is (pair-wise) stable if it is individually rational and there is no pair blocking it.*

Finally, we assume that colleges' preferences, with regard to groups of students, are substitutive. That is, for any two students $s \neq s'$ if s belongs to $Ch_c(A)$, then she will also belong to $Ch_c(A \setminus \{s'\})$. This assumption is usual in the literature and guarantees non-emptiness of the set of stable allocations. (See Theorem 6.5 in [10]). Note that when colleges' preferences are substitutive, the set of (pair-wise) stable allocations coincides with the core of the related college admissions problem.[2] That is, given a stable allocation, no group of agents can find a matching to improve the utility of all its members without being matched with agents outside this group. Furthermore, if colleges' preferences satisfy substitutability, the set of stable allocations has a latticial structure. This property guarantees (i) the existence of a single stable allocation which is Pareto optimal from the point of view of students and, (ii) the existence of a single allocation which is Pareto optimal from the point of view of colleges (when restricted to the set of stable matchings).

The concept of implementation that we are going to use throughout the paper is well-known in the literature. We next formalize this for the Subgame Perfect Nash Equilibrium (SPE). Let $\mathcal{E}_k$ be the set of strategies for agent k and let $\mathcal{E} = \underset{x \in C \cup S}{\times} \mathcal{E}_x$ be the set of strategy profiles. Associated with each strategy profile $\tilde{e} \in \mathcal{E}$ we can define a message profile $m(\tilde{e})$,

[2] Proposition 6.4 in Roth and Sotomayor [10] establishes that stability and pair-wise stability are equivalent concepts in college admissions problems with substitutive preferences.

or simply $\tilde{m}$, which describes the action taken by each individual when the agents choose such strategies. A matching mechanism is described by the set of strategies allowed to each agent, and an outcome function γ that assigns a matching to each profile of messages. We say that a matching mechanism *implements* a solution concept, say χ, in Subgame Perfect Nash Equilibria if (i) for any $\tilde{e}$, Subgame Perfect Equilibrium of the game $\Gamma := \left\{C, S; \underset{\backsim}{P}; \gamma\right\}$, $\gamma(m(\tilde{e}))$ belongs to $\chi\left(\underset{\backsim}{P}\right)$ and (ii) for each μ in $\chi\left(\underset{\backsim}{P}\right)$ there exists a SPE for Γ, say $\tilde{e}'$, such that $\gamma(m(\tilde{e}')) = \mu$.

3 The "students-sequentially-propose-and-colleges-choose mechanism

We shall now introduce the mechanism. But first, we shall fix the order in which the students will play. Without loss of generality, let us assume that s_1 is the first to play, s_2 is the second and so on. Let us define the following $m+1$ stage game form. At stage i, $i = 1, \ldots, m$, student s_i selects a college. Thus, each student's message space coincides with the set of colleges (and her option of being unmatched). At stage $m+1$, the last stage, colleges simultaneously select the set of students they want to admit, one set of students for each college. Thus, each college message space coincides with 2^S. Finally, the outcome function, denoted by Φ^{SsC}, selects the matching defined as follows:

$\Phi^{SsC}(\tilde{m}) = \mu_{\tilde{m}}$, where for any s in S,

$$\mu_{\tilde{m}}(s) = \begin{cases} m(s) & \text{if } s \in m(m(s)) \\ s & \text{otherwise} \end{cases}$$

and, for each c in C,

$$\mu_{\tilde{m}}(c) = \{s \in m(c) \mid c = m(s)\}$$

where $\tilde{m}$ is a list containing a full description of agents' messages.

Theorem 4 shows that our mechanism yields to stable outcomes. But only one stable outcome can be reached by a Subgame Perfect Equilibrium of this mechanism. This is the optimal stable matching from the point of view of students.

Next, we will note that the students-sequentially-propose-and-colleges-choose mechanism implements, in SPE, the stable solution which is optimal from the students' point of view.

Theorem 4 *Let $\tilde{e}$ be a SPE for $\Gamma^{SsC} := \left\{C, S; \underset{\backsim}{P}; \Phi^{SsC}\right\}$, and $\tilde{m}$ be the vector of messages that agents state in $\tilde{e}$. Then $\Phi^{SsC}(\tilde{m}) = \mu^S\left(\underset{\backsim}{P}\right)$, the optimal stable allocation from the students' point of view.*

Proof. We shall now show this result in a constructive way. First, we present some properties that any SPE has to satisfy. Then we will argue that agents' messages will lead to the *students' optimal stable* matching.

In order to characterize the set of SPE, we will apply backward induction. At stage $m+1$, given the students' messages, each college c has a best response, namely, $m^*(c) = \arg\max P(c)$ on $\{s \mid m(s) = c\}$.[3] At stage m, given messages for students other than s_m,

[3] Notice that such a strategy is not the unique best response. In fact, any best response for college c can be expressed by the union of such a set with any set of students S' such that $c \neq m(s')$ for all s' in S'. Nevertheless, all these messages are strategically equivalent. Since we are interested in equilibrium payoffs rather than equilibria strategies, we do not pay attention to these strategies.

$(\bar{m}(s_1), \ldots, \bar{m}(s_{m-1}))$, and knowing the colleges' behavior, this agent's best reply is[4]

$$m^*(s_m) = \arg\max P(s_m) \text{ on}$$
$$\{c \mid s_m \in Ch_c(\{s \in S\backslash \{s_m\} \mid \bar{m}(s) = c\} \cup \{s_m\})\}.$$

Notice that such a message coincides with $\mu^S\left(s_m;\ \underset{\backsim}{P}^m\right)$, the s_m mate at the students' optimal stable matching for market $\underset{\backsim}{P}^m$, where $P^m(c) = P(c)$, for any college c; $P^m(s_m) = P(s_m)$; and $P^m(s_i) = m(s_i)$ for any student s_i in $S\backslash\ \{s_m\}$[5] .

In order to apply an inductive argument, let's assume that s_k message, $m^*(s_k)$ coincides with s_k's mate at the students' optimal stable matching for market $\underset{\backsim}{P}^{k-1}$, where $P^{k-1}(c) = P(c)$, for any college c; $P'(s_h) = P(s_h)$ for any $h \geq k$; and $P'(s_i) = m(s_i)$ for any for any $h < k$. Note that $\mu^S\left(s_k;\ \underset{\backsim}{P}^{k-1}\right)$ coincides with $\mu^S\left(s_k;\ \underset{\backsim}{P}^{k-1'}\right)$ for any profile $\underset{\backsim}{P}^{k-1}$where $P^{k-1'}(c) = P^{k-1}(c)$ for any college, and $P^{k-1'}(s) \in \{P^{k-1}(s), P(s)\}$ for each student s. Therefore, the message of student s_{k-1} should coincide with $\mu^S\left(s_{k-1};\ \underset{\backsim}{P}^{k-2}\right)$, where $P^{k-2}(c) = P(c)$ for any college c; $P^{k-2}(s_h) = m(s_h)$ for all $h < k-1$; and $P^{k-2}(s_h) = P(s_h)$ for each $h \geq k-1$.

Finally, given that the messages that students other than s_1 have to be $m^*(s_i) = \mu^S\left(s_i; \underset{\backsim}{P}\right)$, she will select the best college she can, provided the actions she expected from others students. Since $\mu^S\left(\underset{\backsim}{P}\right)$ is the only stable matching which is Pareto optimal from the students' point of view, and taking into account her colleagues' best responses, her decision

[4] Note that, when $m^*(s_m) = s_m$, such a strategy is equivalent to declaring any college c such that $cP(s_m)s_m$. Nevertheless, since this college will not select that student, both messages are strategically equivalent. In order to provide a simple proof of our result, we assume that, whenever $m^*(s_m) = s_m$, this student's message will be her "stay unmatched" option.

[5] For simplicity, we identify student s_i's preference with college c_j, i.e. $P(s_i) = c_j$, whenever this college is the only one for which $c_j P(s_i) s_i$ holds.

should be to select her mate in the students' optimal stable matching, $\mu^S\left(s_1; \underset{\backsim}{P}\right)$.

Consequently, in equilibrium, each student's message will coincide with her mate in the students' optimal stable matching. ∎

4 Final Remarks

This paper introduces a mechanism that implements a particular selection from the core, namely the students' optimal stable matching. Thus, we provide a positive answer to the implementability of a selection from the core in matching markets. Notice that Kara and Sönmez [7] show that no selection of the core can be implemented in Nash Equilibrium.

In the particular case of marriage markets (colleges have only one position each), a symmetrical result for Theorem 4 can be stated by exchanging the role of students and colleges. In the many-to-one matching problem, the framework this paper deals with, Alcalde and Romero-Medina [4] show that when students play simultaneously, the role played by colleges and students can be exchanged without affecting the results. They implement the core correspondence, no matter whether it is the colleges or the students who made the offers (i.e. play at the first stage). Nevertheless, a similar result can not be established if offers are made sequentially. We shall now conclude this paper by providing an example that shows the above-mentioned asymmetry.

Example 5 *Let* $\left\{C, S; \underset{\backsim}{P}\right\}$ *be a three colleges-four students market. The following table* *summarizes agents preferences.*

$$
\begin{aligned}
&P(s_1) = c_3c_2c_1\\
&P(s_2) = c_2c_1c_3\\
&P(s_3) = c_2c_1c_3\\
&P(s_4) = c_3c_1c_2\\
&P(c_1) = (s_3s_4)(s_2s_4)(s_2s_3)(s_1s_4)(s_1s_3)(s_1s_2)\,s_4s_3s_2s_1\\
&P(c_2) = s_4s_3s_2s_1\\
&P(c_3) = s_3s_4s_1s_2
\end{aligned}
$$

Let us consider the "colleges-sequentially-propose-and-students-choose" mechanism, Γ^{CsS}. This is a symmetrical version of Γ^{SsC} in which it is the colleges who are to make proposals in a sequential way. We will see that two interesting features which are satisfied by the family of mechanisms Γ^{SsC} are not satisfied by a mechanism in Γ^{CsS}. First, some SPE outcomes can be unstable, relative to agents' preferences. In order to show this, let us suppose that the order in which colleges sequentially decide is c_1 , c_2 and c_3. There is a SPE with messages $m(c_1) = (s_1s_2)$, $m(c_2) = (s_4)$, $m(c_3) = (s_3)$, $m(s_1) = (c_1)$, $m(s_2) = (c_1)$, $m(s_3) = (c_3)$ and $m(s_4) = (c_2)$. Notice that $\Gamma^{CsS}(\tilde{m})$ is unstable because the pair $\{c_1s_3\}$ blocks it. Secondly, the SPE outcomes set depends on the order in which colleges make their decisions. Indeed, let us consider the order for colleges in which c_2 proposes first, then c_3, and finally c_1 is the last to make a proposal. In such a case the only SPE outcome is $\mu(c_1) = (s_1s_2)$, $\mu(c_2) = (s_3)$ and $\mu(c_3) = (s_4)$, which is different from the SPE outcome when college c_1 is the first to play.

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