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working papers

# DIFFERENTIABILITY OF THE VALUE FUNCTION WITHOUT INTERIORITY ASSUMPTIONS[*]

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**Abstract**

This paper studies first-order differentiability properties of the value function in concave dynamic programs. Motivated by economic considerations, we dispense with commonly imposed interiority assumptions. We suppose that the correspondence of feasible choices varies with the vector of state variables, and we allow the optimal solution to belong to the boundary of this correspondence. Under minimal assumptions we show that the value function is continuously differentiable. We then discuss this result in the context of several economic models.

*Keywords:* Constrained optimization, value and policy functions, differentiability, envelope theorem, shadow price.

# 1  Introduction

Dynamic optimization problems are often analyzed by the methods of dynamic programming – which build on properties of the value and policy functions. Although these methods have been extensively studied, we should note the following important gap that places this methodology really behind the static theory of constrained optimization: General results on the differentiability of the value function have only been established under suitable interiority conditions. Therefore, differentiability properties of this function – often needed for the characterization and computation of optimal solutions – cannot be invoked in full–fledged constrained optimization problems.

Constrained optimization is pervasive in economics. Constraints may come in the form of feasibility and technological conditions, individual rationality and incentive compatibility, transaction costs, borrowing limits and solvency restrictions, liquidity and collateral requirements, as well as many other financial frictions. Standard assumptions on utility and production functions do not prevent these constraints from being binding. Indeed, a vast body of research in economic dynamics has focussed on effects of these constraints on quantitative properties of equilibrium solutions.

In this paper we consider a general class of concave, discrete–time optimization problems in which the correspondence of feasible choices varies with the vector of state variables and the optimal solution may belong to the boundary of this correspondence. Under minimal assumptions we show that the value function is continuously differentiable. We then discuss this result in the context of some economic applications. We focuss on models of economic growth [Sargent (1980)], finance [Lucas (1978)], and dynamic contracts [Kehoe and Levine (1993), Kocherlakota (1996) and Thomas and Worrall (1988)]. In all these models the differentiability of the value function is key for the characterization, analysis, and computation of optimal solutions. For the models of Kocherlakota (1996) and Thomas and Worrall (1988), from differentiability properties of the Pareto frontier we establish some deep results on the structure of constrained efficient allocations.

There is an alternative characterization of optimal solutions in terms of the Euler equations and the transversality condition but this latter system of equations usually becomes rather awkward [e.g., Stokey, Lucas and Prescott (1989)]. Dynamic programming presents an attractive methodology for the analysis of optimization problems. The differentiability of the value function is essential to extend the powerful tools of differ-

ential calculus to the methodology of dynamic programming. In a recent paper, Araujo, Pascoa and Torres–Martínez (2006) study a family of non-smooth constrained optimization problems, and nicely merge dynamic programming methods with duality theory. We impose additional differentiability assumptions to get a sharper characterization of an optimal solution that is more suitable for computation, and we obtain an explicit expression for the derivative of the value function that may shed further insights into the determinants of equilibrium prices and quantities. Moreover, by the welfare theorems this derivative can be linked to equilibrium prices. Using the differentiability of the value function, we show *uniqueness* of the system of Kuhn–Tucker multipliers for a concave infinite-horizon optimization problem. This system of multipliers varies continuously with primitive parameters.

Most available results on the differentiability of the value function rest upon a weak interiority assumption which is generally subsumed under the following two conditions: (*i*) The optimal solution lies in the interior of the choice set [e.g., see Benveniste and Scheinkman (1979), and earlier, more restrictive results by Lucas (1978) and Mirman and Zilcha (1975)], and (*ii*) the choice set does not vary with the vector of states [e.g., see the seminal work of Danskin (1967), and Milgrom and Segal (2002) for further results and economic applications]. Both (*i*) and (*ii*) turn out to be mathematically equivalent if the policy function is continuous. Under these weak interiority conditions the differentiability of the value function can then be established by a well–known static argument in which this function is defined as the envelope of differentiable short–run return functions. For non–interior solutions, however, the static envelope construction breaks down. Thus, in the absence of (*i*) and (*ii*) the derivative of the value function may involve an infinite sum of discounted marginal utilities and returns, and to bound this derivative the asymptotic behavior of these discounted marginal quantities must be "well behaved".

In our analysis of the differentiability of the value function we introduce two further fundamental assumptions: A rank condition on the matrix of partial derivatives of the saturated constraints and an asymptotic condition on the behavior of discounted marginal utilities and returns. Both assumptions are indispensable and are only needed for non–interior solutions. The rank condition is familiar from the static theory [Gauvin and Dubeau (1982)]. The asymptotic condition is vacuously satisfied in static optimization problems.

The paper is structured as follows. In Section 2 we set out an abstract (reduced–

form) optimization problem, and recall some basic results from dynamic programming. In Section 3 we present our main results on the differentiability of the value function. In Section 4 we consider several economic applications to illustrate the applicability of our results and the role of our main assumptions. We conclude in Section 5 with some final comments. The Appendix contains all the proofs.

# 2  The Model and Preliminary Considerations

As in many other related papers, we lay out an abstract optimization framework that encompasses various economic models. For the sake of simplicity, our analysis will focuss on a deterministic discrete–time problem but our results can be extended to stochastic and continuous–time models.

The primitive elements of our optimization problem are given by a *correspondence of feasible choices* $\Gamma : X \longrightarrow 2^X$ on a *state space* $X \subset \mathbb{R}^n$, with $\Omega = \mathrm{Graph}(\Gamma)$ as the *choice set*, a *one–period return function* $U : \Omega \longrightarrow \mathbb{R}$, and a *discount factor* $0 < \beta < 1$. Then, for every initial condition $x_0$ the problem is to find a sequence $\{x_t\}_{t \geq 1}$ that solves the following discounted infinite–horizon program:

$$
\begin{aligned}
v(x_0) \;\; = \;\; & \sup_{\{x_t\}_{t \geq 1}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \\
& \text{s.t.} \quad x_{t+1} \in \Gamma(x_t), \qquad t = 0, 1, 2, \ldots, \\
& \qquad x_0 \text{ fixed.}
\end{aligned}
\tag{1}
$$

We shall often identify optimization problem (1) with the collection of its primitive elements $(X, \Gamma, \Omega, U, \beta)$.

## 2.1  Basic assumptions

The following basic assumptions will always be in force. Further specific assumptions will be introduced as we proceed with the analysis.

  **B1:** *The state space $X$ is a closed and convex set with nonempty interior.*

  **B2:** *The correspondence $\Gamma$ is continuous and compact–valued. Its graph $\Omega$ is a convex set.*

4

**B3:** *The utility function $U$ is concave and continuous. Moreover, for every fixed $x$ function $U(x, y)$ is strictly concave in $y$.*

## 2.2 Value and policy functions

These basic assumptions do not insure that the *value function* $v(x_0)$ in (1) takes on finite values. Hence, in the sequel we simply suppose that function $v : X \longrightarrow \mathbb{R}$ is well defined and continuous. Then, for every initial condition $x_0$ there exists a unique sequence $\{x_t\}_{t \geq 1}$ that solves the above discounted infinite–horizon program. As is well known, function $v$ is concave, and the unique fixed point of the *Bellman equation*

$$v(x) = \max_{y \in \Gamma(x)} \{U(x, y) + \beta v(y)\} \tag{2}$$

for all $x \in X$. The maximum value is attained at a unique point $y$ given by the *policy function*, $y = h(x)$. Function $h : X \longrightarrow X$ is continuous. A sequence $\{x_t\}_{t \geq 0}$ solves optimization problem (1) if and only if $x_{t+1} = h(x_t)$ for all $t \geq 0$. Hence, Bellman's equation characterizes completely the set of optimal solutions to the infinite–horizon program (1).

In applications it becomes of considerable interest to explore analytical properties of the value and policy functions. These analytical properties are often established by the *method of successive approximations*: For an initial guess $v_0$ let each function $v_T$ be defined recursively as

$$v_T(x) = \max_{y \in \Gamma(x)} \{U(x, y) + \beta v_{T-1}(y)\} \tag{3}$$

for all $T \geq 1$. Under mild regularity conditions [Le Van and Morhaim (2002) and Rincón–Zapatero and Rodríguez-Palmero (2003)], the sequence $\{v_T\}_{T \geq 0}$ converges uniformly to function $v$ on every compact set $K \subset X$. In the proofs of our main results we operate directly with the Bellman equation (2), but these results may also be established by approximating the value function $v$ by infinite sequences $\{v_T\}_{T \geq 0}$ generated recursively by (3).

# 3   Differentiability of the Value Function

Barring a few isolated studies [e.g., Sargent (1980), Huggett and Ospina (2001)], available results on the differentiability of class $C^1$ of the value function $v$ rely on the following

interiority assumption: If $y_0 = h(x_0)$ then there exists an open neighborhood $N(x_0)$ of point $x_0$ such that $y_0 \in \Gamma(x)$ for every $x \in N(x_0)$. In other words, this interiority assumption asserts that every optimal solution $y_0 = h(x_0)$ must be feasible for every point $x$ in an open neighborhood of $x_0$. The assumption is trivially satisfied if the correspondence $\Gamma(x)$ does not vary with $x$ [Danskin (1967)] or if the policy function is continuous and $y_0 \in \text{int}(\Gamma(x_0))$ [Benveniste and Scheinkman (1979)]. Under this interiority assumption, function $v$ can be defined as the envelope of short–run, $C^1$ functions and by concavity this function is of class $C^1$.

There are, however, many economic models displaying non–interior solutions and in which the feasible correspondence $\Gamma(x)$ depends on the initial state $x$. In growth theory, technological constraints are usually binding in models with adjustment costs, irreversible investment, or when the optimal consumption choice equals zero. There is also a growing, vast literature in macroeconomics and finance that considers models of heterogeneous agents with market frictions and borrowing and liquidity constraints. A main goal of this literature is to understand the effects of these restrictions on the evolution of equilibrium aggregates. Steady–state dynamics will only be affected if these restrictions bind infinitely often. As a matter of fact, the existence of a stochastic steady state may entail that borrowing constraints must always be binding in some states of nature [Huggett and Ospina (2001)].

As shown below, for non–interior solutions the derivative of the value function $v$ may be determined by an infinite sum of discounted marginal utilities and returns. Hence, in contrast to the interior case the derivative of $v$ cannot be calculated from a simple static problem, and to bound this derivative it becomes necessary to restrict the asymptotic behavior of the discounted marginal utilities and returns. A simple example below illustrates that the derivative of $v$ may be unbounded even if the derivatives of the utility and production functions are bounded.

## 3.1 Main results

To establish that function $v$ is of class $C^1$, we require differentiability of the return function $U$, and some regularity conditions for boundary solutions. These conditions include a qualification constraint transported from the static theory [Gauvin and Dubeau (1982)], and an asymptotic restriction on the discounted utility of a marginal unit invested

today. This latter condition is specific to the infinite–horizon model and it is only needed for those optimal paths $\{x_t\}_{t\geq 0}$ such that $(x_t, x_{t+1})$ belongs to the boundary of $\Omega$ for all $t \geq 0$.

**D1:** *For every point $x \in \text{int}(X)$ function $U$ is of class $C^1$ on some open neighborhood $N(x, h(x))$ of $(x, h(x))$.*

This assumption implies that at every point $(x, h(x))$ with $x \in \text{int}(X)$ the gradient vector $DU(x, h(x))$ does exist. The assumption is necessary for the derivative $Dv(x)$ to be well defined. Note that if $h(x)$ belongs to the boundary of $\Gamma(x)$ then $D1$ requires that $U$ admits a differentiable extension over some open neighborhood $N(x, h(x))$.

**D2:** $h(X) \subset \text{int}(X)$.

This simple assumption may be weakened in applications. The assumption may be innocuous if the set $X$ can be appropriately redefined, i.e., the domain could be restricted or expanded so that it is not optimal to reach its boundary. Furthermore, as shown below (see Figure 1) if $D2$ is violated, then the derivative of the value function may get unbounded at boundary points of $X$.

**D3:** *There is a finite collection of functions $g = (\ldots, g^i, \ldots)$ such that $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : g(x, y) \geq 0\}$. Each function $g^i : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is concave and of class $C^1$. Let $I(x) = \{i : g^i(x, h(x)) = 0\}$ denote the set of saturated constraints, and let $s$ denote the cardinality of $I(x)$. Then for all $x$ the rank of the matrix of partial derivatives $\{D_2 g^i(x, h(x)) : i \in I(x)\}$ must be equal to $s$.*

Hence, by $D3$ the graph of the technological correspondence $\Omega$ is a polyhedral set as it is defined by a finite number of concave functions. As is well understood from the static theory, the full rank condition entails that the number of choice variables cannot be less than that of saturated constraints. This constraint qualification is more restrictive than the usual Slater condition, which simply requires that the feasible set $\Gamma(x)$ has non–empty interior. The condition will imply that the Kuhn–Tucker multipliers are unique since the above matrix of partial derivatives can be inverted in a certain generalized sense. More specifically, let $D_2 g_s(x, h(x))$ refer to the matrix of partial derivatives of the saturated constraints $\{D_2 g^i(x, h(x)) : i \in I(x)\}$. Then, $D_2 g_s(x, h(x))$

has a *Moore–Penrose* inverse $D_2 g_s^+ = D_2 g_s^\top (D_2 g_s D_2 g_s^\top)^{-1}$ at every $(x, h(x))$, where $D_2 g_s^\top(x, h(x))$ denotes the transpose matrix. In all our computations below, we let $G(x_t, x_{t+1}) = [-D_2 g_s^+(x_t, x_{t+1})|\mathbf{0}_{n \times (m-s)}] D_1 g(x_t, x_{t+1})$, where for convenience the saturated constraints are listed first. It should be understood that if $x_{t+1} \in \text{int}(\Gamma(x_t))$ then $G(x_t, x_{t+1}) = \mathbf{0}$.

Let $\partial v(x)$ be the superdifferential of concave function $v$ at $x$. Recall that the superdifferential is well defined at every interior point $x$ [Rockafellar (1970)]. Moreover, function $v$ is differentiable at $x$ if and only if the superdifferential at $x$ is a singleton. Also, if $v$ is differentiable on a neighborhood $N(x)$ then it is continuously differentiable on $N(x)$. Let $\text{bd}(\Omega)$ denote the boundary of $\Omega$, and $\| \cdot \|$ the Euclidean norm.

**D4:** *Let $\{x_t\}_{t \geq 0}$ be an optimal solution. Assume that $(x_t, x_{t+1}) \in \text{bd}(\Omega)$ for all $t \geq 0$. Then*

*(i) If $X$ is a bounded set,*

$$\limsup_{t \to \infty} \beta \| G(x_t, x_{t+1}) \| < 1. \tag{4}$$

*(ii) If $X$ is an unbounded set,*

$$\liminf_{t \to \infty} \left( \beta^t \max_{q_t \in \partial v(x_t)} \left\| q_t \cdot \prod_{s=0}^{t-1} G(x_{t-s-1}, x_{t-s}) \right\| \right) = 0. \tag{5}$$

Assumption $D4$ will be discussed below in the context of some applications. $D4(i)$ can actually be restated in the weaker form: $\liminf_{t \to \infty} \beta^t \| \prod_{s=0}^{t-1} G(x_{t-s-1}, x_{t-s}) \| = 0$. For some simple models this limiting condition often reduces to $\beta(1 + r) < 1$, where $r$ is the constant interest rate. For an unbounded domain $X$ condition (4) may preclude growth, and hence it is replaced by the corresponding condition (5), which is a joint assumption on preferences and technology. Joint assumptions on preferences and technology are postulated in the endogenous growth literature to ensure the existence of an optimal path.

If there is a time $T$ such that $x_{T+1} \in \text{int}(\Gamma(x_T))$, then $D4$ is not needed since differentiability can be established from the static theory.

**Proposition 1** *Let a feasible optimization problem $(X, \Gamma, \Omega, U, \beta)$ satisfy B1–B4 and D1–D3. Let $\{x_t\}_{t \geq 0}$ be an optimal path with $x_0 \in \text{int}(X)$. Assume that there exists a first*

*time $T \geq 0$ such that $x_{T+1} \in \text{int}(\Gamma(x_T))$. Then, the value function $v$ is differentiable of class $C^1$ on a neighborhood $N(x_0)$ of $x_0$. The derivative of $v$ at $x_0$ is defined as*

$$Dv(x_0) = D_1 U(x_0, x_1)$$
$$+ \sum_{t=0}^{T-1} \beta^t \Big( \beta D_1 U(x_{t+1}, x_{t+2}) + D_2 U(x_t, x_{t+1}) \Big) \prod_{s=0}^{t-1} G(x_{t-s-1}, x_{t-s}). \tag{6}$$

Of course, for $T = 0$ we get the standard envelope theorem $Dv(x_0) = D_1 U(x_0, x_1)$. But it should be noted that for $T > 0$ this result requires the constraint qualification $D3$ – which applies to those optimal pairs $(x_t, x_{t+1})$ that belong to the boundary of $\Omega$.

**Theorem 1** *Let a feasible optimization problem $(X, \Gamma, \Omega, U, \beta)$ satisfy B1–B4 and D1– D4. Then, function $v : \text{int}(X) \longrightarrow \mathbb{R}$ is differentiable of class $C^1$. The derivative of $v$ at an interior point $x_0$ is defined as*

$$Dv(x_0) = \sum_{t=0}^{\infty} \beta^t \left( \Big( D_1 U(x_t, x_{t+1}) + D_2 U(x_t, x_{t+1}) G(x_t, x_{t+1}) \Big) \prod_{s=0}^{t-1} G(x_{t-s-1}, x_{t-s}) \right) \tag{7}$$

*for every optimal path $\{x_t\}_{t \geq 0}$ with $x_0 \in \text{int}(X)$.*

Our strategy of proof for Theorem 1 is to show that at every interior point $x_0$ the superdifferential $\partial v(x_0)$ is a singleton, and hence by concavity function $v$ is differentiable of class $C^1$ on $\text{int}(X)$. To this end, we prove the following technical result which can be viewed as a generalized version of the envelope theorem for concave functions (Theorem 7 and Proposition 3 in the Appendix): $q_0 \in \partial v(x_0)$ if and only if there exists some $q_1 \in \partial v(h(x_0))$ such that

$$q_0 = D_1 U(x_0, h(x_0)) + (D_2 U(x_0, h(x_0)) + \beta q_1) G(x_0, h(x_0)). \tag{8}$$

Then, using an iterative argument we substantiate uniqueness of $q_0$ as given by expression (7).

As already pointed out, for the differentiability of $v$ assumptions $D1$ and $D3$ are indispensable. Both assumptions are taken from the static optimization theory. Assumption $D2$ may be weakened in specific economic applications. $D4$ is also a sufficient condition to ensure that the infinite sum in (7) is well defined and equal to the derivative $Dv(x_0)$. In Section 4 below we will check $D3$–$D4$ in some economic applications.

We next show that a slightly weaker form of $D4$ is actually necessary for $v(x)$ to be well defined. For this simple result we assume that $G(x, h(x))$ and $D_1 U(x, h(x)) + D_2 U(x, h(x)) G(x, h(x))$ are non–negative numbers. As discussed below, these non–negativity conditions are satisfied in standard models of economic growth.

**Proposition 2** *Let $X \subset \mathbb{R}$. Let $\{x_t\}_{t \geq 0}$ be the optimal solution starting at $x_0 \in \text{int}(X)$. Assume that $(x_t, x_{t+1}) \in \text{bd}(\Omega)$ for every $t \geq 0$. Let $G(x_t, x_{t+1}) \geq 0$ and $D_1 U(x_t, x_{t+1}) + D_2 U(x_t, x_{t+1}) G(x_t, x_{t+1}) \geq 0$ for all $(x_t, x_{t+1})$. Then, under B1–B4 and D1–D3 we have:*
   *(i) If $X$ is a bounded set,*

$$\liminf_{t \to \infty} \beta G(x_t, x_{t+1}) \leq 1. \tag{9}$$

*(ii) If $X$ is an unbounded set, there exists a constant $\eta \geq 0$ such that*

$$\limsup_{t \to \infty} \left( \beta^t \min_{q_t \in \partial v(x_t)} q_t \cdot \prod_{s=0}^{t-1} G(x_{t-s-1}, x_{t-s}) \right) \leq \eta. \tag{10}$$

The necessity of these conditions stems from the concavity of the value function, since at an interior point $x_0$ every concave function must have well–defined directional derivatives [Rockafellar (1970)]. For some simple models condition (9) often reduces to $\beta(1+r) \leq 1$, where $r$ is the constant interest rate. Hence, our sufficient conditions (4)–(5) are slightly stronger than (9)–(10). These sufficient conditions ensure that the derivative of the value function $Dv(x_0)$ is given by expression (7).

## 3.2   Duality theory

As a simple application of Theorem 1 we now show that for every optimal path there exists a *unique* set of Kuhn-Tucker multipliers satisfying the Euler equations and the transversality condition. The existence of these multipliers can be established in a simple way by an induction argument on Bellman's equation [Weitzman (1973)], but uniqueness has remained an open issue because of the complexity involved in these equations. By the welfare theorems, the uniqueness of the multipliers entails that an optimal allocation is just supported by a unique price system.

   Let us rewrite Bellman's equation as

$$v(x) = \max_y \{U(x, y) + \beta v(y) + \lambda(x) g(x, y)\}$$

where $\lambda(x)$ is a non-negative vector of Kuhn-Tucker multipliers. By our generalized version of the envelope theorem (see Theorem 7 below) the derivative of the value function $Dv(x)$ is given by

$$Dv(x) = D_1 U(x, h(x)) + \lambda(x) D_1 g(x, h(x)) \tag{11}$$

for every $\lambda(x)$ such that

$$D_2 U(x, y) + \beta Dv(y) + \lambda(x) D_2 g(x, y) = 0. \tag{12}$$

Observe that the above expression (8) readily follows from (11) after substituting out $\lambda(x)$ from (12). Moreover, from these equations we can see informally the role of assumption $D3$: If the matrix of derivatives of the saturated constraints $D_2 g_s(x, y)$ has full rank then (12) implies that the vector of multipliers $\lambda(x)$ is unique. Consequently, if $v$ is differentiable at $h(x)$ then there is a unique multiplier $\lambda(x)$ and so the superdifferential $\partial v(x)$ must contain a unique vector.

Let $\{x_t\}_{t \geq 0}$ be an optimal path. If the value function $v$ is differentiable, then by conditions (11) and (12) evaluated over these optimal values we can derive the following system of Euler equations

$$D_2 U(x_{t-1}, x_t) + \lambda_{t-1} D_2 g(x_{t-1}, x_t) + \beta[D_1 U(x_t, x_{t+1}) + \lambda_t D_1 g(x_t, x_{t+1})] = 0, \tag{13}$$

where $\lambda_t \geq 0$ and $g(x_t, x_{t+1}) \geq 0$ with $\lambda_t g(x_t, x_{t+1}) = 0$, for all $t = 1, 2, \cdots$. To this system of equations we also need to append a transversality condition. For simplicity, let us assume that $X$ is a compact set. Then, let

$$\lim_{T \to \infty} \beta^T[D_1 U(x_T, x_{T+1}) + \lambda_T D_1 g(x_T, x_{T+1})] = 0. \tag{14}$$

As is well known [cf. Benveniste and Scheinkman (1982)], both (13)–(14) are sufficient conditions for the characterization of an optimal path $\{x_t\}_{t \geq 0}$. [1]

**Theorem 2** *Assume that $X$ is a compact set. Under the conditions of Theorem 1, for every optimal path $\{x_t\}_{t \geq 0}$ there exists a unique system of Kuhn-Tucker multipliers $\{\lambda_t\}_{t \geq 0}$ satisfying (13)–(14).*

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[1]The extension of our uniqueness result below to an unbounded domain $X$ requires some further mild regularity conditions. The non–negativity conditions of Proposition 2 allow for a simple extension of the transversality condition to unbounded domains, e.g., see Benveniste and Scheinkman (1982).

This result can be viewed as an envelope theorem for concave infinite-horizon optimization. Indeed, by (11)–(12) we can construct a system of Kuhn-Tucker multipliers $\{\lambda_t\}_{t\geq 0}$ that satisfies the Euler equations (13) and the transversality condition (14). Then, Theorem 2 completes the other direction: The system of multipliers $\{\lambda_t\}_{t\geq 0}$ satisfying the Euler equations (13) and the transversality condition (14) is unique and can be generated by the derivative of the value function $Dv$ as given by (11)–(12).

## 3.3  Sensitivity

In many economic applications it is of interest to establish that the derivative of the value function varies continuously with perturbations of the model. By (11)–(12) the derivative of the value function determines the dynamics of the state variables and the corresponding system of shadow values. We now prove a simple result on the continuity of the derivative of the value function with respect to approximations of the return function. (Similar results on the continuity of the derivative of the value function will also hold for other primitive components.) By Theorem 2 this continuity result applies to the unique system of Kuhn-Tucker multipliers $\{\lambda_t\}_{t\geq 0}$ satisfying the Euler equations (13) and the transversality condition (14).

For simplicity, we focuss on perturbations on the return function $U$ under the sup norm. For given two functions $U$ and $U_n$ let $\|U - U_n\| = \sup_{(x,y)\in\Omega} \|U(x,y) - U_n(x,y)\|$. Convergence in the sup norm is known as uniform convergence in the space of functions. Let $v_n$ refer to the value function of an optimization problem $(X, \Gamma, \Omega, U_n, \beta)$

**Theorem 3** *Let all feasible optimization problems $(X, \Gamma, \Omega, U, \beta)$ and $\{(X, \Gamma, \Omega, U_n, \beta)\}_{n\geq 0}$ satisfy assumptions B1–B4 and D1–D4. Assume that the sequence of functions $\{U_n\}_{n\geq 0}$ converges uniformly to function $U$. Then, the sequence of value functions $\{v_n\}_{n\geq 0}$ converges uniformly to the original value function $v$, and the sequence of derivative functions $\{Dv_n\}_{n\geq 0}$ converges uniformly to the derivative $Dv$ on every compact set $K \subset X$.*

This theorem is a strengthening of Theorem 1 as it also ensures the continuity of the derivative with respect to primitive parameters. As in our preceding analysis, concavity is essential for this result. Observe that if a sequence of $C^1$ concave functions $\{U_n\}_{n\geq 0}$ converges uniformly to a $C^1$ function $U$, the sequence of derivative functions $\{DU_n\}_{n\geq 0}$ converges uniformly to the derivative function $DU$ on every compact set $K \subset X$ [cf.

Rockafellar (1970, Theorem 25.7)].[2] Then, we show that the corresponding sequence of value functions $\{v_n\}_{n \geq 0}$ converges uniformly to the value function $v$, and the sequence of derivative functions $\{Dv_n\}_{n \geq 0}$ converges uniformly to the derivative function $Dv$ on every compact set $K \subset X$.

# 4  Applications

Envelope theorems are often encountered in the analysis of dynamic economic models. We study the differentiability of the value function in some simple models of economic growth, finance, and dynamic contracts. Other areas of economics using recursive optimization include monetary theory, taxation, labor, and industrial organization. It should also be clear that these results readily extend to stochastic models with a finite number of states of uncertainty at each date.

## 4.1  A simple growth model

We first consider the differentiability of the value function in the standard one-sector model with irreversible investment. This problem was originally studied by Sargent (1980). We show that the derivative of the value function is well defined at every positive capital stock, but it may get unbounded as this stock approaches zero. Let

$$
\begin{aligned}
v(x_0) \quad &= \quad \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad &c_t + x_{t+1} = f(x_t), \\
&x_{t+1} \geq (1-\delta)x_t, \quad t = 0, 1, 2, \ldots, \\
&x_0 \text{ fixed}, \quad 0 < \beta < 1.
\end{aligned}
$$

The utility function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and the production function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are both increasing, strictly concave and of class $C^1$. Moreover, $f(0) = 0$ and $\lim_{x \to \infty} \beta f'(x) < 1$. From these primitive functions, the return function $U : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is defined as $U(x, y) = u(f(x) - y)$ and the feasible correspondence $\Gamma : \mathbb{R} \longrightarrow 2^{\mathbb{R}_+}$ as $\Gamma(x) = [(1-\delta)x, f(x)]$. The graph of correspondence $\Gamma(x)$ is bordered by two concave

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[2]Actually, the uniform convergence of the sequence of derivatives $\{DU_n\}_{n \geq 0}$ on a compact set $K$ holds under *pointwise* convergence of the sequence $\{U_n\}_{n \geq 0}$; see Rockafellar (1970, Theorem 25.7).

functions $g^1(x, y) = f(x) - y \geq 0$ and $g^2(x, y) = -(1 - \delta)x + y \geq 0$. Both constraints depend on state variable $x$. Hence, the standard envelope theorem cannot be applied if any of these constraints is saturated.

Let us first consider boundary solutions of the form $y = h(x) = f(x)$, where $c = 0$ and $G(x, y) = f'(x)$. Then, as the optimal policy is monotone, condition (4) in $D4$ reads simply as

$$\lim_{t \to \infty} \beta f'(x_t) < 1.$$

But $D4$ is trivially satisfied since an optimal solution $\{x_t\}_{t \geq 0}$ cannot stay at the upper boundary at all times: There must be a time $T$ such that $c_T > 0$.[3] Also, for $h(x) = f(x)$ we have $D_1 U(x, h(x)) + D_2 U(x, h(x))G(x, h(x)) = u'(0)f'(x) - u'(0)f'(x) = 0$. Hence, the non-negativity restriction of Proposition 2 is always satisfied. As this value is equal to zero it follows from (8) that

$$v'(x) = \beta v'(h(x))f'(x). \tag{15}$$

Moreover, by (12) we must have $u'(0) \leq \beta v'(f(x))$. Then, iterating on (15) we get

$$v'(x_0) = \beta^T v'(x_T)(f^T)'(x_0), \tag{16}$$

where $T$ is the first time that $c_T > 0$. By the envelope theorem $v'(x_T) = u'(c_T)f'(x_T)$ for $x_{T+1} > (1 - \delta)x_T$,.

So far, our discussion for boundary solutions of the form $h(x) = f(x)$ has centered on positive initial conditions $x > 0$. We now illustrate that in the absence of $D2$ for $x = 0$ the derivative $v'(0)$ may be unbounded even if $u'(0)$ and $f'(0)$ are finite. Suppose that $\beta f'(0) > 1$ and $\lim_{x \to \infty} \beta f'(x) < 1$. Then, there is a unique, stable steady-state solution $(x^*, c^*)$. Let us furthermore assume that $u'(0) < \beta u'(c^*)f'(0)$. (This last condition is satisfied if the curvature of $u$ is sufficiently small.) Under the postulated conditions $\beta f'(0) > 1$ and $u'(0) < \beta u'(c^*)f'(0)$ it is readily seen (Figure 1) that for every $x$ in a small neighborhood of 0 we have $h(x) = f(x)$, and so (15) holds true. Let $T(x_0)$ be the first time that optimal consumption $c_T > 0$ for initial condition $x_0$. Then $T(x_0)$ converges to $\infty$ as $x_0$ goes to 0. As $\beta f'(0) > 1$ and $c_T \leq c^*$, we get from (16) that $v'(x_0)$ gets unbounded as $x_0$ converges to zero.

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[3]Note that the related condition $\lim_{x \to \infty} \beta f'(x) < 1$ ensures existence of a steady-state solution $(x^*, c^*)$.

If constraint $g^2$ is saturated, then $G(x, y) = 1 - \delta$. Hence, condition (4) in $D4$ is always satisfied for this kind of boundary solutions as $\beta(1 - \delta) < 1$. Differentiating Bellman's equation (2) for $h(x) = (1 - \delta)x$ we get $v'(x) = u'(f(x) - (1 - \delta)x)[f'(x) - (1 - \delta)] + \beta(1 - \delta)v'((1 - \delta)x)$. Consequently, if over an optimal path $\{x_t\}_{t \geq 0}$ constraint $g^2(x_t, x_{t+1})$ is saturated at all times, then the derivative of the value function is determined by an infinite sum of marginal values. More precisely, $v'(x_0) = \sum_{t=0}^{\infty} \beta^t (1 - \delta)^t u'(c_t)(f'(x_t) - (1 - \delta))$. For deterministic models only very restrictive conditions ensure that an optimal solution $\{x_t\}_{t \geq 0}$ will always be at the lower boundary $g^2(x, y) \geq 0$. But for some stochastic models [Sargent (1980) and Christiano and Fisher (2000)] after a large negative shock the irreversibility constraint may be binding. Our results can readily be extended to cover these stochastic models.

To complete the argument that $D4$ holds true in this model, we now observe that there are no other boundary solutions: There is no optimal path $\{x_t\}_{t \geq 0}$ in which constraint $g^1(x, y) \geq 0$ is saturated at some time periods and constraint $g^2(x, y) \geq 0$ is saturated at some other periods. Indeed, if $\beta f'(0) > 1$ then there exists a globally stable interior steady-state solution $(x^*, c^*)$. And if $\beta f'(0) \leq 1$ then all optimal paths $\{x_t\}_{t \geq 0}$ converge to $x = 0$, and at most constraint $g^2(x, y) \geq 0$ is saturated.

It is now readily seen that the above assumptions $D1$ and $D3$–$D4$ are all satisfied in the present model. $D2$ is not satisfied in this model as $f(0) = 0$. However, as suggested in our commentary after $D2$ if the domain can be changed slightly to $X = [\underline{x}, \infty)$ where $\underline{x} > 0$ is an arbitrarily small number, then $D2$ simply requires $\beta f'(0) > 1$. But even if $\beta f'(0) \leq 1$ assumption $D2$ is not really needed since for this model our method of proof directly yields the differentiability of $v$ at every interior point $x_0$. Therefore, by a straightforward extension of Theorem 1 we have that the value function $v$ is differentiable at every interior point $x_0$.

## 4.2 Growing economies

Suppose now that $f(x) = Ax + (1 - \delta)x$. If $A + (1 - \delta) > 1$, then every optimal path $\{x_t\}_{t \geq 0}$ with $x_0 > 0$ must be unbounded. For simplicity, let $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\infty > \sigma > 0$. It is easy to check that the objective in the above one-sector growth model will be unbounded if $\beta(A + (1 - \delta))^{1-\sigma} > 1$. In short, in growing economies the existence of an optimal path is usually achieved under joint restrictions on preferences and productivity.

For these economies $D4(i)$ is not suitable, and $D4(ii)$ is the corresponding condition that allows for growth over time. If $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and the production function is linear,[4] then the value function $v$ is homogeneous of degree $1 - \sigma$ and the superdifferential of $v$ is homogeneous of degree $-\sigma$. Under this homogeneity property of the superdifferential, condition $D4(ii)$ can be easily checked. Indeed, for $A+(1-\delta) > 1$ with $\beta(A+(1-\delta))^{1-\sigma} < 1$, condition $D4(ii)$ will be satisfied for every optimal solution $\{x_t\}_{t \geq 0}$ that grows at a rate $g$ with $1 + g \leq Ax + (1 - \delta)$ since $\beta(1 + g)^{1-\sigma} < 1$. Therefore, $D4(ii)$ allows for positive growth at the cost of imposing joint restrictions on preferences and technologies. These joint restrictions are usually postulated for the optimization problem to be well defined.

## 4.3 Finance

Since the influential work of Lucas (1978), the derivative of the value function has been widely used for the pricing of assets and the study of their dynamic properties. Our results are useful to integrate various types of financial constraints and market frictions. Most of our remarks on the differentiability of the value function for models of economic growth can be applied directly to asset pricing models. Hence, our discussion will be brief.

As already pointed out, the standard envelope theorem can be applied for interior solutions or for fixed choice sets such as for predetermined borrowing limits, short-sale constraints, and further exogenous bounds on asset trading. Our work is meant to deal with financial constraints which vary endogenously with the vector of state variables. For instance, think of models with cash-in-advance constraints, reserve and collateral requirements, and various other liquidity restrictions. Since most of these models allow for active asset trading and portfolio reallocation, the non-interiority restrictions imposed by these constraints may only last for one period until the portfolio can be reallocated. Hence, there could be no boundary solutions in the sense of $D4$, and the differentiability of the value function follows from Proposition 1. There are, however, endogenous financial restrictions in which $D4$ may become relevant. These restrictions may arise from solvency and participation constraints, reputation and learning. Kiyotaki and Moore (1997) are concerned with propagation effects of collateral requirements arising from a

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[4]Linearity of the production function is needed for the existence of a balanced growth path.

participation constraint. In this model the optimal strategy for the investor is always to reinvest all marketable dividends and to consume the non-marketable part. Thus, even though consumption is positive the optimal solution is always at the boundary of feasible investment. Alvarez and Jermann (2000) present an asset pricing model with a non-default participation constraint as set forth by Kehoe and Levine (1993). This participation constraint is analyzed in the following application for a consumption allocation problem.

## 4.4 Constrained efficient allocations

Constrained efficient allocations are of interest for the characterization and computation of equilibrium solutions in non-optimal economies [Cass (1984) and Kehoe and Levine (1993)]. We study the differentiability of the Pareto frontier for the models of Kocherlakota (1996) and Thomas and Worrall (1988). The proof of Thomas and Worrall (1988) replicates the envelope argument of Benveniste and Scheinkman (1979). This method of proof, however, is of limited application for this family of models since the choice set may vary with the state variables. Koeppl (2003) considers a simple version of Kocherlakota's model in which the value function is not always differentiable.

As an extension of the above results, here we offer a complete analysis of differentiability. The differentiability of the value function rests on a certain number of constraints not being saturated/binding.[5] Roughly speaking, for Kocherlakota's model the value function fails to be differentiable at those utility levels in which the maximum possible number of constraints are binding. By the concavity of the value function we then show that for every point in a subset of full measure of the Pareto frontier there is some state $s$ such that none of the participation constraints are binding and the value function is continuously differentiable. Hence, non-differentiability of the value function can only occur at "switching" points where a maximum possible combination of constraints are saturated/binding. This information is valuable to understand the dynamics of efficient allocations in constrained optimization problems. We would like to remark that these results can be extended to some other models with incentive and participation constraints

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[5]As before, we say that a constraint $g^i(x, h(x)) \geq 0$ is saturated if $g^i(x, h(x)) = 0$. A constraint $g^i(x, h(x)) \geq 0$ is binding if there is an associated Kuhn-Tucker multiplier $\lambda^i(x) > 0$, for $\lambda^i(x)$ defined as in (12). Note that by the Kuhn-Tucker theorem if a constraint is binding then it is saturated, but a saturated constraint may not be binding.

[e.g., the unemployment insurance model of Hopenhayn and Nicolini (1996) and several other models discussed in Ljungqvist and Sargent (2004)].

The special structure of the optimization allows for a direct proof without resorting to assumption $D4$. In pure exchange economies this assumption usually holds trivially. We identify the lack of differentiability of the value function with a failure of assumption $D3$. All the constraints are analytically quite similar as they entail certain utility requirements. Hence, the matrix of partial derivatives as defined in $D3$ becomes singular at points in which a maximum number of constraints become saturated. We should note that these singularities on the matrix of derivatives may be circumvented by seemingly marginal changes in the formulation of the model. Thus, two nearly equal formulations may yield quite different outcomes regarding the differentiability of the value function and the dynamics of optimal allocations. The models of Kocherlakota (1996) and Thomas and Worrall (1988) differ in the timing for the participation constraints and the degree of risk aversion. What turns out to be crucial for the differentiability of the value function in this setting is the timing of events rather than the presence of a risk neutral agent.

We start with a simple version of Kocherlakota's model. Consider an exchange economy with two agents 1 and 2. There are $s = 1, \ldots, S$ states at each date $t = 1, 2, \ldots$. For simplicity, assume that the aggregate quantity of output $\overline{\omega}$ is constant, but individual endowments $\omega^i(s) > 0$, for $i = 1, 2$, are subject to idiosyncratic shocks. For present purposes there is no restriction of generality to suppose that the endowment streams $\{\{\omega_t^i(s)\}_{t \geq 0}\}_{i=1,2}$ are symmetrically distributed and follow an *iid* process. Let $\pi(s) > 0$ denote the time-invariant probability for each state $s$.

Both agents have identical preferences represented by the discounted expected objective $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t(s))$, where $E_0$ denotes conditional expectation at time $t = 0$. The one-period utility $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is increasing, strictly concave, and continuously differentiable.

Let $U_{\text{aut}} = E_0 \sum_{t=0}^{\infty} \beta^t u(\omega_t^i(s))$ be the reservation utility generated by the consumption of the endowment stream $\{\omega_t^i(s)\}_{t \geq 0}$ for every agent $i$. We then require that at each date and state an individual consumption allocation must at least secure the agent's reservation utility. This individual rationality constraint seems a minimal requirement for loan repayments and other contractual obligations so that no agent will have incentives to renege the contract at some future date [Kehoe and Levine (1993)]. As most of the literature, we trace down the set of efficient consumption allocations by a social planning

program in which the value function $V(U_0)$ assigns the maximum utility to an agent (say agent 2) over all possible utility levels $U_0$ of the other agent. More precisely, let

$$V(U_0) = \max_{\{(c_t^1(s), c_t^2(s))\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^2(s))$$

$$\text{s.t.} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^1(s)) \geq U_0,$$

$$u(c_t^1(s)) + E_t \sum_{\tau=1}^{\infty} \beta^\tau u(c_{t+\tau}^1(s)) \geq u(\omega^1(s)) + \beta U_{\text{aut}} \text{ for all } t \text{ and } s,$$

$$u(c_t^2(s)) + E_t \sum_{\tau=1}^{\infty} \beta^\tau u(c_{t+\tau}^2(s)) \geq u(\omega^2(s)) + \beta U_{\text{aut}} \text{ for all } t \text{ and } s,$$

$$c_t^1(s) + c_t^2(s) = \overline{\omega} \text{ for all } t \text{ and } s,$$

where $E_t$ is the expectations operator at time $t$. The graph of function $V(U_0)$ defines the Pareto frontier, that is, all feasible non-dominated utility pairs $(U_0, V(U_0))$. Let $U_{\max} = V(U_{\text{aut}})$. In the sequel we assume that the Pareto frontier is non-degenerate, that is, $U_{\max} > U_{\text{aut}}$. Then, one readily checks that function $V : [U_{\text{aut}}, U_{\max}] \longrightarrow \mathbb{R}$ is well defined, decreasing, concave and continuous.

By the optimality principle applied to this optimization problem, we get the following version of Bellman's equation:

$$V(U_0) = \max_{\{c_s, U_s\}} \quad \sum_{s=1}^{S} \pi_s [u(\overline{\omega} - c_s) + \beta V(U_s)]$$

$$\text{s.t.} \quad \sum_{s=1}^{S} \pi_s [u(c_s) + \beta U_s] \geq U_0, \qquad (P_1)$$

$$u(c_s) + \beta U_s \geq u(\omega_s^1) + \beta U_{\text{aut}} \text{ for all } s, \qquad (P_2)$$

$$u(\overline{\omega} - c_s) + \beta V(U_s) \geq u(\overline{\omega} - \omega_s^1) + \beta U_{\text{aut}} \text{ for all } s, \qquad (P_3)$$

$$U_s \in [U_{\text{aut}}, U_{\max}] \text{ for all } s.$$

Note that the value function $V$ also appears in constraint $(P_3)$, but this does not complicate substantially the application of our methods. Assuming that $V$ is differentiable and that all optimal solutions $\{c_s, U_s\}$ lie in the interior, we get the following system of first-order conditions

$$0 = -\pi_s u'(\overline{\omega} - c_s) + \lambda \pi_s u'(c_s) + \mu_s u'(c_s) - \nu_s u'(\overline{\omega} - c_s), \qquad (17)$$

$$0 = \beta \pi_s V'(U_s) + \lambda \beta \pi_s + \mu_s \beta + \nu_s \beta V'(U_s), \qquad (18)$$

where $\lambda$, $\{\mu_s\}_s$, and $\{\nu_s\}_s$ are Kuhn-Tucker multipliers corresponding to the constraints $(P_1)$, $(P_2)$, and $(P_3)$, respectively, for all $s$.

For given $U_0$, let $S_2(U_0)$ be the subset of states $s$ where constraint $(P_2)$ is saturated, and $S_2^b(U_0)$ the subset of states $s$ where constraint $(P_2)$ is binding. That is, $S_2(U_0)$ is the subset of states $s$ where constraint $(P_2)$ holds with equality at the optimal solution

$\{c_s, U_s\}$, and $S_2^b(U_0)$ is the subset of states $s$ with some $\mu_s > 0$. Analogously, let $S_3(U_0)$ be the subset of states $s$ where constraint $(P_3)$ is saturated and $S_3^b(U_0)$ be the subset of states $s$ where constraint $(P_3)$ is binding. Under the present assumptions, $(P_1)$ will always be binding. Also, it is easy to show that the intersection $S_2(U_0) \cap S_3(U_0)$ is empty [cf., Kocherlakota (1996)]. Hence, for each $s$ there is at most one constraint $(P_2)$ or $(P_3)$ that is saturated.

**D3':** $S_2(U_0) \cup S_3^b(U_0) \neq S$ and $S_2^b(U_0) \cup S_3(U_0) \neq S$.

We first show that both $S_2(U_0)$ and $S_3(U_0)$ are always proper subsets of $S$. Moreover, the value function $V$ is differentiable at an interior point $U_0$ if and only if $D3'$ is satisfied.

**Theorem 4** *Let $U_0 \in \text{int}([U_{\text{aut}}, U_{\text{max}}])$. Then, $S_2(U_0) \neq S$ and $S_3(U_0) \neq S$. The value function $V$ is differentiable of class $C^1$ at $U_0$ if and only if $D3'$ is satisfied. At points of differentiability, the derivative*

$$V'(U_0) = -\frac{u'(\overline{\omega} - c_s)}{u'(c_s)},$$

*for every state $s$ where none of the constraints $(P_2)$–$(P_3)$ are binding.*

**Remark 1** *As one can see from the method of proof of Theorem 4, the full rank condition in D3 is equivalent to the condition: $S_2(U_0) \cup S_3(U_0) \neq S$. But one should bear in mind that this qualification constraint is a sufficient condition. Given the special structure of our optimization problem, D3 can be sharpened. The necessary and sufficient condition for Kocherlakota's model is $D3'$. Under this latter condition there is a unique $\lambda$ as determined by the equations system (17)–(18) and so the derivative $V'(U_0) = -\lambda$. Note that $D3'$ can be defined as: (i) $S_2(U_0) \cup S_3(U_0) \neq S$ or (ii) for $S_2(U_0) \cup S_3(U_0) = S$ there must be some non-binding constraint in $S_2(U_0)$ and some other non-binding constraint in $S_3(U_0)$. Hence, $D3'$ is slightly weaker than $S_2(U_0) \cup S_3(U_0) \neq S$. We would like to emphasize that the condition $S_2^b(U_0) \cup S_3^b(U_0) \neq S$ leaves out some cases in which $V$ is not differentiable, and hence this condition is not adequate for our purposes. A main advantage of working with a necessary and sufficient condition for the differentiability of $V$ is that the condition must hold generically. That is, by concavity the value function $V$ is differentiable at almost all $U_0$. Hence, $D3'$ must be satisfied at almost all $U_0$. As illustrated presently, this result allows us to prove that the value function $V$ is of class $C^1$*

*over certain regions of the domain and to study further dynamic properties of constrained efficient allocations.*

Following Koeppl (2003), let us discuss these results for the model with two states, $s = a, b$. Suppose that $\omega^1(a) > \omega^2(b)$, so that $a$ is the good state for agent 1 and $b$ is the good state for agent 2. We first observe that $S_2(U_0) \neq S$ and $S_3(U_0) \neq S$ for every interior point $U_0$. Suppose not. Thus, let $S_2(U_0) = S$. Then, $\pi_a[u(c_a) + \beta U_a] + \pi_b[u(c_b) + \beta U_b] = U_{\text{aut}}$, which violates $(P_1)$, since we have assumed that $U_0 > U_{\text{aut}}$. Therefore, $S_2(U_0) \neq S$ and $S_3(U_0) \neq S$. Also, observe that if no constraint is binding over the states $s = a, b$, then the multipliers $\{\mu_s\}_s$, and $\{\nu_s\}_s$ are always equal to zero. By (18), we get that $-\lambda = V'(U_0) = V'(U_a) = V'(U_b)$. Hence, the consumption allocation is always constant $\{(c, \overline{\omega} - c)\}$ and so it must be first-best efficient.[6]

Suppose now that no first-best consumption allocation is attainable. Then, for every interior utility level $U_0$ at least one constraint $(P_2)$ or $(P_3)$ must be binding at some state $s$. Let $\widehat{U}$ be the fixed point $\widehat{U} = V(\widehat{U})$ where both agents are assigned the same utility level. Consider a utility level $U_{\text{aut}} < U_0 < \widehat{U}$. If the value function $V$ is differentiable, then we claim that $S_2^b(U_0) = \{a\}$ and $S_3(U_0)$ is empty. Note that if function $V$ is differentiable, by Theorem 4 we must have $S_2^b(U_0) \cup S_3(U_0) \neq S$; moreover, since $U_0 < \widehat{U}$, constraint $(P_2)$ must be binding at state $a$ so that agent 1 can make consumption transfers to agent 2. Hence, by Theorem 4 the set $S_3(U_0)$ is empty and the derivative $V'(U_0) = -\frac{u'(\overline{\omega} - c_b)}{u'(c_b)}$. This function varies continuously with $U_0$ as optimal consumption $c_b$ is a continuous function of $U_0$. Therefore, by a standard result [cf. Rockafellar (1970, Theorem 25.1)] we get that the value function $V$ is continuously differentiable at every point $U_0$ such that $U_{\text{aut}} < U_0 < \widehat{U}$. In an analogous manner we can establish the differentiability of $V$ at every point $U_0$ such that $\widehat{U} < U_0 < U_{\text{max}}$. In this latter case, $S_2(U_0)$ is empty and $S_3^b(U_0) = \{b\}$, and the derivative $V'(U_0) = -\frac{u'(\overline{\omega} - c_a)}{u'(c_a)}$. Since no first-best allocation is attainable, it follows that at point $\widehat{U}$ we must have $S_2^b(\widehat{U}) \cup S_3^b(\widehat{U}) = S$, and by Theorem 4 the value function $V$ fails to be differentiable.

Let us now examine the dynamics of consumption allocations. Let $U_{\text{aut}} < U_0 < \widehat{U}$. Then, as already discussed, $S_2^b(U_0) = \{a\}$ and $S_3(U_0)$ is empty. By (18), we have $-\lambda = V'(U_0) = V'(U_b) > V'(U_a)$. Consequently, at state $a$ agent 1 transfers consumption to

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[6]We stress that a first-best, Pareto-efficient allocation may be compatible with $S_2(U_0) \cup S_3(U_0) = S$, but it is not compatible with some constraint being binding. Therefore, by Theorem 4 the value function $V$ is differentiable at every $U_0$ achieving a first-best allocation.

agent 2 with the promise of a higher utility $U_a > U_0$, whereas at state $b$ the same utility level is preserved, $U_b = U_0$. Therefore, the sequence of promised utilities converges monotonically to $\widehat{U}$ with probability one. As a matter of fact, convergence may be achieved in finite time since the value function is not differentiable at $\widehat{U}$. Let us summarize these results:

**Theorem 5** *Assume that there are two states $s = a, b$. Suppose that no first-best consumption allocation is attainable. Let $\widehat{U} = V(\widehat{U})$. For every point $U_0 \in \mathrm{int}([U_{\mathrm{aut}}, \widehat{U}])$ the set $S_2^b(U_0) = \{a\}$ and $S_3(U_0)$ is empty. The value function $V$ is differentiable of class $C^1$ at $U_0$ and the derivative $V'(U_0) = -\frac{u'(\overline{\omega} - c_b)}{u'(c_b)}$. Analogously, for every point $U_0 \in \mathrm{int}([\widehat{U}, U_{\max}])$ the set $S_2(U_0)$ is empty and $S_3^b(U_0) = \{b\}$. The value function $V$ is differentiable of class $C^1$ at $U_0$ and the derivative $V'(U_0) = -\frac{u'(\overline{\omega} - c_a)}{u'(c_a)}$. Finally, at point $\widehat{U}$ the set $S_2^b(\widehat{U}) = \{a\}$ and $S_3^b(\widehat{U}) = \{b\}$ and the value function $V$ is not differentiable.*

Finally, let us turn our attention to the model of Thomas and Worrall (1988). We consider a version of this model in which both agents are risk averse. Here, the value function is differentiable. The technical reason is that assumption $D3$ is always satisfied because of a slightly different timing for the participation constraints.

Let $U_{\mathrm{aut}}^i(s) = u(\omega^i(s)) + E_1 \sum_{t=1}^{\infty} \beta^t u(\omega_t^i(s))$ where $E_1$ denotes the expectations operator at time 1. Hence, $U_{\mathrm{aut}}^i(s)$ is the reservation utility for agent $i = 1, 2$ once state $s$ is known. Then, consider the following recursive program

$$V(U_s) = \max_{\{c_s, U_{s'}\}} \quad u(\overline{\omega} - c_s) + \beta \sum_{s'=1}^{S} \pi_{s'}[V(U_{s'})]$$
$$\text{s.t.} \quad u(c_s) + \beta \sum_{s'=1}^{S} \pi_{s'} U_{s'} \geq U_s, \qquad (P_1')$$
$$U_{s'} \geq U_{\mathrm{aut}}^1(s') \text{ for all } s', \qquad (P_2')$$
$$V(U_{s'}) \geq U_{\mathrm{aut}}^2(s') \text{ for all } s'. \qquad (P_3')$$

In this latter version of the model the suggested interpretation is that contracting takes place at the beginning of the period but after the state $s$ is known, whereas in Kocherlakota's model contracting takes place before the state $s$ is known. Under this seemingly minor change in the timing of contracting, the value function $V(U_s)$ is always continuously differentiable.

**Theorem 6** *For every $s$ the value function $V$ is differentiable of class $C^1$ at every interior point $U_s$ of the feasible domain $[U_{\mathrm{aut}}^1(s)), V(U_{\mathrm{aut}}^1(s))]$. The derivative*

$$V'(U_s) = -\frac{u'(\overline{\omega} - c_s)}{u'(c_s)}.$$

Hence, a variation in the timing of events changes the structure of the constraints so that $D3$ is always satisfied and the value function is differentiable. Of course, these changes will influence the set of efficient allocations and the dynamics of the system. From the perspective of economic theory none of these models can be dismissed. Both models seem plausible and logically consistent. Indeed, the choice of timing for the participation constraints has to be dictated by the economic application.

# 5 Concluding Remarks

In this paper we consider a general class of concave, infinite-horizon constrained optimization problems in which the choice set may depend on the vector of state variables. Under minimal assumptions we show that the value function is continuously differentiable.

We dispense with commonly imposed interiority assumptions. These interiority assumptions are quite restrictive in economic applications and are not needed in the static optimization theory. For our optimization problem the derivative of the value function may be defined by an infinite sum of future discounted marginal utilities and returns, and so the usual static envelope argument breaks down. Furthermore, an example above (see Figure 1) illustrates that the derivative of the value function may be unbounded even if the utility and production functions have bounded derivatives.

To circumvent these interiority restrictions we postulate two further fundamental assumptions: A qualification constraint on the saturated constraints and an asymptotic condition on the behavior of marginal utilities and returns. As is well known from the static theory, the qualification constraint guarantees the existence of a unique set of Kuhn-Tucker multipliers. The uniqueness of these multipliers is necessary for the existence of the derivative of the value function (see our technical Theorem 7 in the Appendix). An application on constrained efficient allocations in Section 4 illustrates that the constraint qualification may not hold in some economic models and the value function may not be differentiable. As shown in Proposition 2 the asymptotic condition on the behavior of marginal utilities and returns is also unavoidable. Under this asymptotic condition the derivative of the value function is bounded and can be expressed as an infinite sum of discounted marginal utilities and returns.

We may then conclude from the present work that in dynamic optimization problems under mild regularity assumptions the value function is differentiable of class $C^1$.

Moreover, this function will not generally be differentiable of class $C^2$ since the optimal decision rule may display kinks at points in which additional constraints get saturated [cf., Santos (1991)].

The differentiability of the value function is often invoked in the analysis of optimal solutions. Moreover, as shown above this differentiability property ensures uniqueness of the system of Kuhn-Tucker multipliers for infinite–horizon optimization, and the continuity of these multipliers with respect to initial conditions and primitive parameters. These shadow values appear as additional state variables in proofs of existence of Markov equilibria for dynamic games of monetary and fiscal policy [Kydland and Prescott (1980) and Phelan and Stacchetti (2001)] and for competitive economies with heterogeneous agents and market frictions [Miao and Santos (2005)]. For the computation of these equilibria for every initial condition it is useful to have uniqueness and continuity of the system of multipliers.

There are several directions in which this research can be extended. First, in Section 4 our results were directly applied to a stochastic model with a finite number of states of uncertainty. It should be possible to consider more general stochastic frameworks. Note that in stochastic models the interiority assumption is usually very restrictive, since after a bad shock the optimal solution may reach the boundary [Christiano and Fisher (2000) and Kehoe, Levine and Woodford (1992)]. Second, concavity plays a crucial role in our results. For non-concave problems the optimal solution may not be unique, and the differentiability of the value function is a very demanding property. A considerable body of theoretical research has focussed on some other regularity properties of the value function such as Lipschitz continuity [cf. Askri and Le Van (1998) and Clarke (1990)] and supermodularity [cf. Amir, Mirman, and Perkins (1991)]. Finally, this research was motivated by various economic applications. As discussed in Section 4 the differentiability of the value function is key for the study of equilibrium solutions in models of economic growth, finance, and dynamic contracts. But these exercises do not exhaust the range of possible economic applications: There are many other related dynamic models for which the differentiability of the value function should be of interest.

# 6  Appendix

All results should be understood to hold under the assumptions in the text.

## 6.1  Proofs for Section 3

In this part we apply basic results from convex analysis to the Bellman equation

$$v(x) = \max_{y \in \Gamma(x)} \{U(x,y) + \beta v(y)\}$$

for all $x \in X$. For convenience, we write $\varphi(x,y) = U(x,y) + \beta v(y)$.

Note that function $v$ has a well defined superdifferential $\partial v(x_0)$ at every interior point $x_0$ in $X$ [Rockafellar (1970)]. Hence, in what follows $x_0$ refers to an interior point. For our first preliminary results we will work with the normal cone $N_\Omega(x_0, y_0)$ of the convex set $\Omega$ at point $(x_0, y_0)$ :

$$N_\Omega(x_0, y_0) = \{\xi \in \mathbb{R}^{2n} : \xi \cdot (x - x_0, y - y_0) \leq 0, \forall (x,y) \in \Omega\}.$$

**Lemma 1** $q_0 \in \partial v(x_0)$ if and only if there exists $\xi \in \partial \varphi(x_0, h(x_0))$ such that $((q_0, 0) - \xi) \in -N_\Omega(x_0, h(x_0))$.

**Proof.** We follow Aubin (1993, Problem 35). Define the indicator function of set $\Omega$ as

$$I_\Omega(x,y) = \begin{cases} 0, & (x,y) \in \Omega \\ -\infty, & (x,y) \notin \Omega \end{cases}$$

Note that function $I_\Omega$ is concave and upper semicontinuous. Now, rewrite Bellman's equation as

$$v(x) = \max_{y \in \mathbb{R}^n} (\varphi + I_\Omega)(x,y)$$

for all $x \in X$. This is an unconstrained optimization problem. By Aubin (1993, Prop. 4.3), $q_0 \in \partial v(x_0)$ if and only if $(q_0, 0) \in \partial(\varphi + I_\Omega)(x_0, y_0)$. Moreover,

$$\partial(\varphi + I_\Omega)(x_0, h(x_0)) = \partial\varphi(x_0, h(x_0)) + \partial I_\Omega(x_0, h(x_0)) = \partial\varphi(x_0, h(x_0)) - N_\Omega(x_0, h(x_0)).$$

Therefore, $q_0 \in \partial v(x_0)$ if and only if there exists $\xi \in \partial\varphi(x_0, h(x_0))$ such that $((q_0, 0) - \xi) \in -N_\Omega(x_0, h(x_0))$.  *Q.E.D.*

**Corollary 1** $q_0 \in \partial v(x_0)$ *if and only if there exists $p_0 \in \partial_2 \varphi(x_0, h(x_0))$ such that $(q_0 - D_1\varphi(x_0, h(x_0)), -p_0) \in -N_\Omega(x_0, h(x_0))$.*

**Proof.** By $D1$, function $\varphi(x, y)$ is differentiable with respect to $x$. Hence, $\partial_1\varphi(x, y) = D_1\varphi(x, y)$, and so $\partial\varphi(x, y) = \{\partial_1\varphi(x, y)\} \times \partial_2\varphi(x, y)$ for all $(x, y)$. Let $\xi = (\xi_1, \xi_2)$, $\xi_i \in \partial_i\varphi(x_0, h(x_0))$, $i = 1, 2$. Then, the corollary is a straightforward consequence of our previous lemma for $p_0 = \xi_2$. Q.E.D.

From these preliminary results we now derive a generalized envelope theorem for constrained, non-smooth optimization.

**Theorem 7** $q_0 \in \partial v(x_0)$ *if and only if there exists $p_0 \in \partial_2\varphi(x_0, h(x_0))$ such that $q_0 = D_1\varphi(x_0, h(x_0)) + p_0 G(x_0, h(x_0))$.*

**Proof.** As is well known [e.g., Clarke (1990, Corollary 2, p. 56)] by $D3$ we must have

$$-N_\Omega(x_0, y_0) = \left\{ (q, p) \in \mathbb{R}^n \times \mathbb{R}^n : (q, p) = \sum_{i \in I(x)} \lambda^i (D_1 g^i(x_0, y_0), D_2 g^i(x_0, y_0)), \lambda^i \geq 0 \right\}.$$

Then, by our previous lemma

$$q_0 - D_1\varphi(x_0, y_0) = \sum_{i \in I(x)} \lambda^i D_1 g^i(x_0, h(x_0,)) \tag{19}$$

$$-p_0 = \sum_{i \in I(x)} \lambda^i D_2 g^i(x_0, h(x_0)) \tag{20}$$

for some $\lambda^i \geq 0$, for all $i \in I(x)$. Let $\lambda = (\ldots, \lambda^i, \ldots)$. Hence, from (20) we get that $\lambda = -p_0 D_2 g_s^+(x_0, h(x_0))$. To complete the proof we substitute this expression for $\lambda$ into (19) and let $G(x_0, h(x_0)) = [-(D_2 g_s^+(x_0, h(x_0))|\mathbf{0}] D_1 g(x_0, h(x_0))$. Q.E.D.

This static result is now extended to a finite number of periods.

**Proposition 3** $q_0 \in \partial v(x_0)$ *if and only if for every $T \geq 1$ there exists some $q_T \in \partial v(x_T)$ such that*

$$q_0 = \sum_{t=0}^{T-1} \beta^t \big( D_1 U(x_t, x_{t+1}) + D_2 U(x_t, x_{t+1}) G(x_t, x_{t+1}) \big) \prod_{s=0}^{t-1} G(x_{t-1-s}, x_{t-s})$$

$$+ \beta^T q_T \prod_{t=0}^{T-1} G(x_{T-t}, x_{T+1-t}).$$

**Proof.** Fix an arbitrary $q_0 \in \partial v(x_0)$. By the previous envelope theorem, $q_0 \in \partial v(x_0)$ if and only if there exists $p_0 \in D_2 U(x_0, x_1) + \beta \partial v(x_1)$ such that $q_0 = D_1 U(x_0, x_1) + p_0 G(x_0, x_1)$. Then, there is some $q_1 \in \partial v(x_1)$ such that

$$q_0 = D_1 U(x_0, x_1) + (D_2 U(x_0, x_1) + \beta q_1) G(x_0, x_1). \tag{21}$$

Now, the same argument can be applied to $q_1$. Hence, there exists $p_1 \in D_2 U(x_1, x_2) + \beta \partial v(x_2)$ such that $q_1 = D_1 U(x_1, x_2) + p_1 G(x_1, x_2)$. Therefore, $q_1 = D_1 U(x_1, x_2) + (D_2 U(x_1, x_2) + \beta q_2) G(x_1, x_2)$ for some $q_2 \in \partial v(x_2)$. Plugging in this last value for $q_1$ into (21), we have

$$\begin{aligned}
q_0 &= D_1 U(x_0, x_1) + D_2 U(x_0, x_1) G(x_0, x_1) \\
&\quad + \beta D_1 U(x_1, x_2) G(x_0, x_1) + \beta D_2 U(x_1, x_2) G(x_1, x_2) G(x_0, x_1) \\
&\quad + \beta^2 q_2 G(x_1, x_2) G(x_0, x_1).
\end{aligned}$$

Proceeding inductively, the result holds for every $T \geq 1$.   *Q.E.D.*

**Proposition 4** *Let $\{x_t\}_{t \geq 0}$ be an optimal sequence. If $v$ is differentiable at $x_T$ for some $T \geq 0$, then $v$ is differentiable at $x_0$. The derivative of $v$ at point $x_0$ is defined as*

$$\begin{aligned}
Dv(x_0) &= \sum_{t=0}^{T} \beta^t \big( D_1 U(x_t, x_{t+1}) + D_2 U(x_t, x_{t+1}) G(x_t, x_{t+1}) \big) \prod_{s=0}^{t-1} G(x_{t-s-1}, x_{t-s}) \\
&\quad + \beta^T Dv(x_T) \prod_{t=0}^{T-1} G(x_{T-t}, x_{T-t+1}).
\end{aligned}$$

**Proof.** If $v$ is differentiable at $x_T$ then $\partial v(x_T) = \{q_T\}$ is a singleton. Hence, by the previous proposition there exists a unique vector $q_0 \in \partial v(x_0)$. Therefore, $v$ differentiable at $x_0$ and the derivative $Dv(x_0)$ is given by the above expression.   *Q.E.D.*

PROOF OF PROPOSITION 1. By the envelope theorem [Benveniste and Scheinkman (1979)] the value function is differentiable of class $C^1$ on a neighborhood $N(x_T)$ of $x_T$, and the derivative $Dv(x_T) = D_1 U(x_T, x_{T+1})$. Hence, by Proposition 4 this function is differentiable of class $C^1$ on some neighborhood $N(x_0)$ of $x_0$. Moreover, this proposition also gives the value of the derivative in (6) for $Dv(x_T) = D_1 U(x_T, x_{T+1})$.   *Q.E.D*

PROOF OF THEOREM 1. Assume first that $X$ is bounded, and hence that it is a compact set. As already pointed out, function $v$ is differentiable at $x_0$ if $\partial v(x_0)$ is a singleton. By way of contradiction, let $q_0, \widetilde{q}_0 \in \partial v(x_0)$. Then, by Proposition 3, for every $T \geq 0$ there exist $q_{T+1}, \widetilde{q}_{T+1} \in \partial v(x_{T+1})$ such that

$$\|q_0 - \widetilde{q}_0\| \leq \beta^T \left\| \prod_{t=0}^{T-1} G(x_t, x_{t+1}) \right\| \|q_T - \widetilde{q}_T\|. \tag{22}$$

We next show that this expression converges to zero as $T$ goes to $\infty$. Note that $h(X)$ is a compact set, and by $D2$ we have that $h(X) \subset int(X)$. Hence, $\partial v(h(X))$ is a compact set [Rockafellar (1970)] and all vectors $q_T, \widetilde{q}_T$ must belong to $\partial v(h(X))$. Then, by condition $D4(i)$ the above expression (22) converges to zero and so $q_0 = \widetilde{q}_0$. This proves that $\partial v(x_0)$ is singleton at every interior point $x_0$. Hence, $v$ is differentiable of class $C^1$ on $int(X)$. The value of the derivative $Dv(x_0)$ is obtained by letting $T$ go to $\infty$ in Proposition 4.

If $X$ is an unbounded set, then $D4(ii)$ applies, and the differentiability of the value function is obtained in a similar way. $Q.E.D$

PROOF OF PROPOSITION 2. Suppose that $X$ is a compact set. Under the non-negativity conditions, it follows from Proposition 3 that for $q_0 = \max \partial v(x_0)$ we must have

$$q_0 \geq \beta^T q_T \prod_{t=0}^{T-1} G(x_{T-t}, x_{T-t+1}) \geq 0 \tag{23}$$

for some $q_T \in \partial v(x_T)$, for all $T \geq 1$. Since $v$ is strictly concave and $h(X) \subset int(X)$ every sequence of vectors $\{q_T\}_{T \geq 0} \subset \partial v(h(X))$ must be bounded below by a positive constant. Hence, the sequence $\{\beta^{T+1} \prod_{t=0}^{T} G(x_{T-t}, x_{T-t+1})\}_{T \geq 0}$ must be bounded. As $X \subset \mathbb{R}$, this last condition implies that $\liminf_{t \to \infty} \beta G(x_t, x_{t+1}) \leq 1$.

If $X$ is unbounded the proof proceeds in a similar way. For this purpose we should note that by Proposition 3 inequality (23) must be satisfied for some $q_T \in \partial v(x_T)$, and so it must always hold for the minimum value $q_T \in \partial v(x_T)$. $Q.E.D$

PROOF OF THEOREM 2. Suppose that $\{\overline{x}_t\}_{t \geq 0}$ is the optimal path starting at $\overline{x}_0$. For this optimal path assume that there are two sequences of Kuhn-Tucker multipliers $\{\lambda_t\}_{t \geq 0}$ and $\{\lambda'_t\}_{t \geq 0}$ that satisfy equations (13) and (14). For all initial conditions $x_0$ and

all feasible sequences $x_{t+1} \in \Gamma(x_t)$ for $t = 1, 2, \cdots, n$, let

$$v_{\lambda,n}(x_0) = \max_{\{x_t\}_{t=1}^{n+1}} \sum_{t=0}^{n} \beta^t U(x_t, x_{t+1}) + \beta^{n+1} \alpha_{n+1} x_{n+1}, \qquad (24)$$

where $\alpha_{n+1} = D_1 U(\overline{x}_{n+1}, \overline{x}_{n+2}) + \lambda_{n+1} D_1 g(\overline{x}_{n+1}, \overline{x}_{n+2})$, and

$$v_{\lambda',n}(x_0) = \max_{\{x_t\}_{t=1}^{n+1}} \sum_{t=0}^{n} \beta^t U(x_t, x_{t+1}) + \beta^{n+1} \alpha'_{n+1} x_{n+1}, \qquad (25)$$

where $\alpha'_{n+1} = D_1 U(\overline{x}_{n+1}, \overline{x}_{n+2}) + \lambda'_{n+1} D_1 g(\overline{x}_{n+1}, \overline{x}_{n+2})$. Note that the added linear parts $\alpha_{n+1}$ and $\alpha'_{n+1}$ are chosen so that at point $\overline{x}_0$ the optimal solution is $\{\overline{x}_t\}_{t=0}^{n+1}$ for both optimization problems, and for this optimal solution $\{\lambda_t\}_{t=0}^{n}$ is the sequence of associated Kuhn-Tucker multipliers under (24), and $\{\lambda'_t\}_{t=0}^{n}$ is the sequence of associated Kuhn-Tucker multipliers under (25). By $D3$, each sequence of multipliers is unique.

By the same methods as the proof of Theorem 1, we can readily see that functions $v_{\lambda,n}$ and $v_{\lambda',n}$ are concave and of class $C^1$. Moreover, by (14) and the definitions of $\alpha_{n+1}$, $\alpha'_{n+1}$, the sequences of functions $\{v_{\lambda,n}\}_{n\geq 1}$ $\{v_{\lambda',n}\}_{n\geq 1}$ converge uniformly to function $v$. Hence, the sequences of derivative functions $\{Dv_{\lambda,n}\}_{n\geq 1}$ and $\{Dv_{\lambda',n}\}_{n\geq 1}$ converge uniformly to function $Dv$ on every compact set $K \subset \text{int}(X)$ [Rockafellar (1970, Theorem 25.7)]. Observe that $Dv_{\lambda,n}(\overline{x}_0) = D_1 U(\overline{x}_0, \overline{x}_1) + \lambda_0 D_1 g(\overline{x}_0, \overline{x}_1)$, and $Dv_{\lambda',n}(\overline{x}_0) = D_1 U(\overline{x}_0, \overline{x}_1) + \lambda'_0 D_1 g(\overline{x}_0, \overline{x}_1)$, for all $n$. The convergence of these derivatives to a unique common value implies that $\lambda_0 D_1 g(\overline{x}_0, \overline{x}_1) = \lambda'_0 D_1 g(\overline{x}_0, \overline{x}_1)$. Moreover, by the same argument it follows that $\lambda_1 D_1 g(\overline{x}_1, \overline{x}_2) = \lambda'_1 D_1 g(\overline{x}_1, \overline{x}_2)$. Then, by condition $D3$ applied to (13) we get uniqueness of the multiplier, $\lambda_0 = \lambda'_0$. $Q.E.D$

PROOF OF THEOREM 3. Let us first prove that $\{v_n\}_{n\geq 0}$ converges uniformly to $v$. The proof is standard. Pick an initial condition $x_0$. For this initial condition $x_0$, let $\{x_t\}_{t\geq 0}$ be the optimal solution for optimization problem $(X, \Gamma, \Omega, U, \beta)$, and let $\{x_{nt}\}_{t\geq 0}$ be the optimal solution for optimization problem $(X, \Gamma, \Omega, U_n, \beta)$. Without loss of generality, assume that $v(x_0) > v_n(x_0)$. Then,

$$v(x_0) - v_n(x_0) = \sum_{t=0}^{\infty} \beta^t (U(x_t, x_{t+1}) - U_n(x_{nt}, x_{nt+1})) \leq \sum_{t=0}^{\infty} \beta^t (U(x_t, x_{t+1}) - U_n(x_t, x_{t+1}))$$

$$\leq \frac{1}{1-\beta} \|U - U_n\|.$$

Hence, $\{v_n\}$ converges uniformly to $v$. By Theorem 1, all functions $v_n$ and $v$ are differentiable. As these functions are also concave, by Rockafellar (1970, Theorem 25.7) the

sequence of derivative functions $\{Dv_n\}_{n\geq 0}$ converges uniformly to $Dv$ on every compact set $K \subset \text{int}(X)$. *Q.E.D*

## 6.2 Proofs for Section 4

PROOF OF THEOREM 4. The proof for $S_2(U_0) \neq S$ and $S_3(U_0) \neq S$ has already been discussed in the text. Hence, let us first show that if $S_2(U_0) \cup S_3^b(U_0) = S$ then the value function $V$ is not differentiable at $U_0$. For concreteness, we suppose that $S_3^b(U_0) = \{1, \ldots, \overline{s}\}$ and $S_2(U_0) = \{\overline{s}+1, \ldots, S\}$. We argue by contradiction. If the value function $V$ is differentiable at $U_0$, then considering the non–smooth form of the first–order optimality conditions, these can be expressed as the linear system

$$
\begin{pmatrix}
\pi_1 u'(c_1) & -u'(\overline{\omega}-c_1) & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\pi_{\overline{s}} u'(c_{\overline{s}}) & 0 & \cdots & -u'(\overline{\omega}-c_{\overline{s}}) & 0 & \cdots & 0 \\
\pi_{\overline{s}+1} u'(c_{\overline{s}+1}) & 0 & \cdots & 0 & u'(c_{\overline{s}+1}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\pi_S u'(c_S) & 0 & \cdots & 0 & 0 & \cdots & u'(c_S) \\
\pi_1 & q_1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\pi_{\overline{s}} & 0 & \cdots & q_{\overline{s}} & 0 & \cdots & 0 \\
\pi_{\overline{s}+1} & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\pi_S & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\lambda \\ \nu_1 \\ \vdots \\ \nu_{\overline{s}} \\ \mu_{\overline{s}+1} \\ \vdots \\ \mu_S
\end{pmatrix}
=
\begin{pmatrix}
\pi_1 u'(\overline{\omega}-c_1) \\ \vdots \\ \pi_S u'(\overline{\omega}-c_S) \\ -\pi_1 q'_1 \\ \vdots \\ -\pi_S q'_S
\end{pmatrix}
$$

where $q_s \in \partial V(U_s)$ for $s \in S_3^b(U_0)$ and $q'_s \in \partial V(U_s)$ for $s \in S$. We will prove first that $q_s = -u'(\overline{\omega}-c_s)/u'(c_s)$ for every $s \in S_3^b(U_0)$. Suppose not, that is, $q_s u'(c_s)+u'(\overline{\omega}-c_s) \neq 0$ for some $s \in S_3^b(U_0)$. Then, a contradiction is attained as follows. Solving for $\nu_s$ from the system we obtain $\nu_s = -\pi_s \frac{q'_s u'(c_s)+u'(\overline{\omega}-c_s)}{q_s u'(c_s)+u'(\overline{\omega}-c_s)}$. The non–negativity of the multiplier $\nu_s$ impose $q_s \neq q'_s$. Solving now for $\lambda$, $\lambda = \frac{u'(\overline{\omega}-c_s)}{\pi_s u'(c_s)}(\nu_s + \pi_s)$, and substituting the value of $\nu_s$ we get $\lambda = \pi_s(q_s - q'_s)\frac{u'(\overline{\omega}-c_s)}{q_s u'(c_s)+u'(\overline{\omega}-c_s)}$. As we are supposing in our reasoning that $V$ is differentiable at $U_0$, $\lambda$ must be unique, thus $q_s = q'_s$, in contradiction with our previous supposition $q_s \neq q'_s$. In consequence, $q_s = -u'(\overline{\omega} - c_s)/u'(c_s)$ for every $s \in S_3^b(U_0)$, as claimed. Now, we will show that $q_s = q'_s$ for every $s \in S_3^b(U_0)$ and that also $q'_s = -u'(\overline{\omega} - c_s)/u'(c_s)$ for $s \in S_2$. The proof of these claims is based in the following observation. Since that the first order conditions hold at the solution, the

30

linear system defined above is solvable. Thus the Rouche–Frobenius Theorem implies that the independent term, $(\pi_1 u'(\overline{\omega} - c_1), \ldots, \pi_S u'(\overline{\omega} - c_S), -\pi_1 q_1', \ldots, -\pi_S q_S')^{\top}$, must be linearly dependent of the vector columns of the matrix of the system. Given the easy structure of the matrix, it is readily seen that the unique possibility is $q_s = q_s'$ for every $s \in S_3^b(U_0)$ and $q_s' = -u'(\overline{\omega} - c_s)/u'(c_s)$ for $s \in S_2$, as claimed.

Based in these observations, it is immediate to see that the rank of the matrix system is $S$, lesser that the number of unknowns, $S+1$, as the last $S$ rows are linearly dependent of the first $S$ rows. Hence, we must have infinitely many solutions, which are of the form $\lambda = -(1 + \frac{\nu_{s'}}{\pi_{s'}})q_{s'}$ for $\nu_{s'} \geq 0$ and all $s' \in S_3^b(U_0)$, and $\mu_s = -\pi_s(\lambda + q_s)$ for $\mu_s \geq 0$ and all $s \in S_2(U_0)$. More specifically, let $\lambda_0 = V'(U_0)$. Then, by (17)–(18) we have $\lambda_0 = -q_{s'} + \nu_{s'} \frac{u'(\overline{\omega} - c_{s'})}{\pi_{s'} u'(c_{s'})}$ for $s' \in S_3^b(U_0)$. Hence, the interval $[\lambda_0 - \nu_{s'} \frac{u'(\overline{\omega} - c_{s'})}{\pi_{s'} u'(c_{s'})}, \infty]$ determines the set of feasible $\lambda$ satisfying (17)–(18). Let $\Delta(s') = \min_{s'} \nu_{s'} \frac{u'(\overline{\omega} - c_{s'})}{\pi_{s'} u'(c_{s'})}$ over all $s' \in S_3^b(U_0)$. Note that $\Delta(s') > 0$. It follows that all $\lambda$ in $[\lambda_0 - \Delta(s'), \infty]$ are feasible solutions under (17)–(18) for *all* $s' \in S_3^b(U_0)$. In the same way, we may define $\Delta(s) = \min_s \frac{\mu_s}{\pi_s}$ over all $s \in S_2(U_0)$. Then, $\Delta(s) \geq 0$ and all $\lambda$ in $[0, \lambda_0 + \Delta(s)]$ are feasible solutions under (17)–(18) for *all* $s \in S_2(U_0)$. By Theorem 7, the superdifferential $\partial V(U_0) = \{-\lambda : -\lambda_0 - \Delta(s) \leq -\lambda \leq -\lambda_0 + \Delta(s')\}$. By assumption, this is a non-degenerate interval. Hence, we have reached a contradiction to the assumed differentiability of $V$ at $U_0$. Therefore, if $D3'$ is not satisfied then function $V$ is not differentiable at $U_0$.

Finally, If $D3'$ is satisfied, then $\Delta(s') = \Delta(s) = 0$, as either $S_2(U_0) \cup S_3(U_0) \neq S$ or for $S_2(U_0) \cup S_3(U_0) = S$ there must be one non-binding constraint in $S_2(U_0)$ and another non-binding constraint in $S_3(U_0)$. By Theorem 7, the superdifferential $\partial V(U_0)$ reduces to a unique vector $\lambda_0$, and so $V$ is differentiable at $U_0$. Moreover, the corresponding expression for the derivative follows from the first–order condition (17). *Q.E.D*

PROOF OF THEOREM 6. Although it is easy to check that the qualification constraint always holds true in this model, we shall offer a direct proof of this result. Note that the value function $V(U_0)$ is concave, and hence this function is differentiable almost everywhere. By similar arguments to Theorem 7, at a point of differentiability the derivative $V'(U_s) = -\frac{u(\overline{\omega} - c_s)}{u'(c_s)}$. Since $c_s$ is a continuous function of $U_s$, at points of existence the derivative $V'(U_s)$ varies continuously with $U_s$. Therefore, by a standard result [Rockafellar (1970, Theorem 25.1)] the value function is differentiable of class $C^1$ over the whole domain $\mathrm{int}([U_{\mathrm{aut}}^1(s)), V(U_{\mathrm{aut}}^1(s))])$ for each $s$. *Q.E.D*

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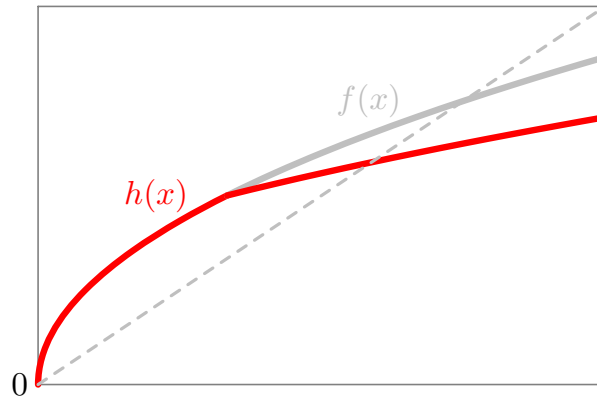
Figure 1: The derivative of the value function may be unbounded even if the utility and production functions have bounded derivatives. If for all $x$ in a small neighborhood of $x = 0$ the policy function $h(x) = f(x)$ and $\beta f'(0) > 1$ then $v'(x)$ gets unbounded as $x$ converges to 0. This example does not satisfy $D2$, and hence it illustrates that for points in the boundary of the domain $X$ some regularity condition is needed for the derivative of the value function to be well defined. As already remarked, in this example $D2$ would hold if the domain $X$ is restricted to $\{x : \underline{x} \leq x < \infty\}$ for every positive constant $\underline{x}$ sufficiently close to 0.