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Mixing properties of crystallization processes

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Abstract

We are interested here in a birth-and-growth process where germs are born according to a Poisson point process with intensity measure invariant under space translations. The germs can be born in free space and then start growing until occupying the available space. In order to consider various ways of growing, we describe the crystals at each time through their geometrical properties. In this general framework, the crystallization process can be characterized by the random field giving for a point in the state space the first time this point is reached by a crystal. We prove under general conditions that this random field is mixing in the sense of ergodic theory and obtain estimates for the coefficient of absolute regularity.

Keywords: crystallization process, Poisson point process, ergodicity, alpha-mixing coefficient, absolute regularity.

MSC: 60, 60D, 60G.

1 Introduction

The crystallization process we consider here deals with germs $g = (x_g, t_g)$ that appear at random times t_g on random location x_g. The birth process $\mathcal{N}$ is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity measure denoted by Λ. Once a germ or crystallization center is born, the crystal is allowed to grow if its location is not yet occupied by another crystal and when two crystals meet the growth stops at meeting points but the growth continues at other points of the crystal. There are then many ways to describe crystal expansion. The first approach is to consider the random set (called crystallization state) that corresponds to the fraction of space occupied by crystals at a given time. In this case, crystallization is studied through the theory of set-valued processes.

Another way to describe crystal growth is the following. One can consider for a germ $g \in \mathbb{R}^d \times \mathbb{R}^+$ and a point $x \in \mathbb{R}^d$ the time $A_g(x)$ at which x is reached

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by the crystal associated to the germ g would it allowed to growth freely. The crystallization process is then characterized by the random field ξ giving for a location $x \in \mathbb{R}^d$ its crystallization time :

$$\xi(x) = \inf_{g \in \mathcal{N}} A_g(x). \tag{1}$$

We adopt in this paper the last definition and study the crystallization process through the random field ξ.

This model was introduced by Kolmogorov[3] and independently by Johnson and Mehl[4], and intensively studied by many authors. We mention here only a few papers which represent the main approaches and where one can find an exhaustive list of references: Møller[5] and also Micheletti and Capasso[6]. A very large part of these investigations deals with the study of probability distributions of typical cells of the mosaic once all the germs have finished their growth. Here, we are rather interested in estimation problems (such as the estimation of the parameters of the intensity measure Λ or other functionals like the number of crystals in the limit mosaic) in the case when only one realization can be observed on a sufficiently large domain compared to the mean size of crystals. Naturally, we suppose that the crystallization process is space homogeneous. More precisely, we assume that the intensity measure is defined as follows,

$$\Lambda = \lambda^d \times \mu, \tag{2}$$

where λ^d is the Lebesgue measure on $\mathbb{R}^d$ and μ is a measure on $\mathbb{R}^+$ finite on bounded Borel sets.

This article is mainly devoted to ergodic properties of the random field $\{\xi(x)_{x \in \mathbb{R}^d}\}$ defined by equation (1) which deliver a solid base for efficient estimation of parameters of the model and subsequent application to the study of its asymptotical normality.

Under the above hypothesis and rather general conditions on growth speed and geometrical shape of crystals, we demonstrate that the random field ξ is mixing in the sens of the ergodic theory. Moreover, under some additional assumptions, we obtain estimates of the absolute regularity coefficient.

Some application of theorem 1 on page 175 to the problem of parameter estimation can be found in Davydov and Illig (2009).

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[3] Kolmogorov, 1937, "Statistical theory of crystallization of metals".

[4] Johnson and Mehl, 1939, "Reaction Kinetics in Processes of Nucleation and Growth".

[5] Møller, 1986, "Random tessellations in $\mathbb{R}^d$";
Møller, 1989, "Random tessellations in $\mathbb{R}^d$";
Møller, 1992, "Random Johnson-Mehl tessellations";
Møller, 1995, "Generation of Johnson-Mehl crystals and comparative analysis of models for random nucleation".

[6] Micheletti and Capasso, 1997, "The stochastic geometry of polymer crystallization processes";
Capasso and Micheletti, 2003, "Stochastic geometry of spatially structured birth and growth processes".

2 Assumptions on the birth and growth processes

2.1 The birth process

Germs are born according to a Poisson point process on $E = \mathbb{R}^d \times \mathbb{R}^+$ denoted by $\mathcal{N}$. That is, germs are random points $g = (x_g, t_g)$ in E, where x_g is the location in the growth space $\mathbb{R}^d$ and t_g is the time of birth on the time axis $\mathbb{R}^+$. We suppose that the intensity measure Λ of $\mathcal{N}$ is the product equation (2) on page 170 of the Lebesgue measure λ^d on $\mathbb{R}^d$ and a measure μ on $\mathbb{R}^+$ such that $\mu([0,a]) < \infty$ for all $a > 0$. The most interesting cases to be considered[7] are those with a discrete measure μ and those with a density measure $\mu(dt) = \alpha t^{\beta-1}\lambda(dt)$ where α, $\beta > 0$ are parameters. Since the Lebesgue measure is translation-invariant, the Poisson point process $\mathcal{N}$ is space homogeneous and it is sufficient to consider sets around the origin. Thus, for any time t, we introduce the so-called causal cone:

$$K_t = \Big\{ g \in E \Big/ A_g(0) \leq t \Big\}$$

which consists of all the possible germs that can capture the origin before the time t.

The measure $\Lambda(K_t)$ of the causal cone K_t is denoted by $F(t)$. These set and function play important roles in the sequel.

2.2 Expansion of crystals

We say that a crystal is a free crystal if it originates from a germ born in a location not yet occupied by other crystals at the time of its birth. We associate with each germ g in E a function A_g:

$$\begin{aligned} A_g : \mathbb{R}^d &\to \mathbb{R}^+ \\ x &\mapsto A_g(x) \end{aligned}$$

where $A_g(x)$ is the time when x is reached by the crystal related to germ g and assumed to be free. As a consequence, a free crystal at time t is defined by the set

$$C_g(t) = \Big\{ x \Big/ A_g(x) \leq t \Big\}.$$

In the following we make several assumptions on the free crystals family $\{C_g / g \in \mathcal{N}\}$ and the functions family $\{A_g / g \in \mathcal{N}\}$. We also specify, when necessary, the link between assumptions and crystal growth.

We suppose that for any germ $g = (x_g, t_g)$ the associated free crystal at time $t \geq t_g$ equals to:

$$C_g(t) = x_g \oplus \Big[V(t) - V(t_g) \Big] K, \tag{3}$$

[7] See Møller, 1986, "Random tessellations in $\mathbb{R}^d$".

where K is a compact convex body containing 0 in its interior, $V(t)$ is an absolutely continuous function of t whose value is the distance achieved with function speed $v(t)$, and $\oplus$ represents the Minkowski summation of two sets A and B:

$$A \oplus B = \{x + y \,/\, x \in A,\ y \in B\}.$$

It is supposed (except for theorem 1 on page 175) that v is bounded and separated from zero:

$$0 < l \leq v(s) \leq L < \infty \tag{4}$$

almost everywhere.

In this case

$$\int_0^\infty v(s)\,\mathrm{d}s = \infty$$

and it guarantees that each bounded volume will be completely crystallized within a finite time.

We denote by $p_{x,K}$ the unique positive number such that $\frac{x}{p_{x,K}} \in \partial K$. Then, a point x is reached at time t by the free crystal born in x_g at time t_g if

$$\Big(V(t) - V(t_g)\Big) p_{x-x_g,K} = \Big|x - x_g\Big|.$$

As $V(t)$ is invertible,

$$t = A_g(x) = V^{-1}\left(\frac{\left|x - x_g\right|}{p_{x-x_g,K}} + V(t_g)\right).$$

Let us mention several useful properties of the families $\{C_g\}$ and $\{A_g\}$.

1. For each $t \geq t_g$ and $h \leq 0$

$$C_g(t+h) = C_g(t) \oplus [V(t+h) - V(t)]K.$$

2. If $K = B(0,1)$ and $v(s) = c > 0$, we get the classical model of linear and isotropic expansion of crystals.

3. Crystal growth is space homogeneous: for all germ $g = (x_g, t_g)$,

$$C_g(t) = C_{(0,t_g)}(t) + x_g,\ \forall\, t \in \mathbb{R}^+.$$

4. The functions $x \mapsto A_g(x)$ are continuous and for $t \geq t_g$

$$\partial C_g(t) = \Big\{x \Big/ A_g(x) = t\Big\}.$$

5. Let $m = \inf\{\|x\| \,/\, x \in \partial K\}$, $M = \sup\{\|x\| \,/\, x \in \partial K\}$. Then $\forall\, x \in \mathbb{R}^d$

$$t_g + \frac{1}{ML}|x| \le A_{(0,t_g)}(x) \le t_g + \frac{1}{ml}|x|. \tag{5}$$

6. It is easy to see that under our hypothesis the causal cone K_t has the following structure: its horizontal section $K_t(s)$ at the level s, $0 \le s \le t$, is the set $-C_g$ symmetric to the set C_g with $g = (0, t-s)$. Hence

$$F(t) = \Lambda(K_t) = \lambda^d(K) \int_0^t [V(t-s)]^d \, m(\mathrm{d}s).$$

2.3 Crystallization process

As it was already said, the main object of our studying is the process

$$\xi(x) = \inf_{g \in \mathcal{N}} A_g(x)$$

It contains in itself all information on development of crystallization and many important characteristics can be expressed directly in terms of ξ. So, for example, the crystallized part of a window $W \subset \mathbb{R}^d$ by the time t is $Z_W = \{x \in W \,/\, \xi(x) \le t\}$, and its volume is equal to $\int_W \mathbf{1}_{[0,t]}(\xi(x))\lambda^d(\mathrm{d}x)$.

It is easy to understand that transition to a time scale $s = V(t)$ turns our process ξ into a process ζ with linear growth associated with K. Indeed, under such a transformation the germ $g = (x_g, t_g)$ pass to $g' = (x_g, V^{-1}s_g)$ and the crystal $C_g(t)$ will be transformed to

$$C'_{g'}(s) = x_g \oplus \Big[V(t) - V(t_g)\Big]K = x_g \oplus (s - s_g)K.$$

In some cases it is convenient to use this circumstance. For example, as the set $\{x \,/\, \xi(x) \le t\}$ representing crystallized part of space at the time t by process ξ coincides with a set $\{x \,/\, \zeta(x) \le s\}$, final mosaics (Jonson-Mehl tessellations) for ξ and ζ coincide. However, for our purposes it is more preferable to work directly with process ξ.

It should be noted also an interesting relation with the Boolean model. Let B be the epigraph of the function $A_{(0,0)}(x)$, $B = \{(x,t) \subset \mathbb{R}^d \times \mathbb{R}_+ \,/\, A_{(0,0)}(x) \le t\}$. We will suppose that the growth rate of crystals is constant, for simplicity we take $v(s) = 1$. Then the set B is the cone $\{cK \,/\, c \ge 0\}$. The corresponding Boolean model is defined by the random set

$$\Psi = \cup_{x \in \mathcal{N}} (B \oplus x),$$

and it is clear that the boundary of Ψ is exactly the graph of ξ:

$$\xi(x) = \inf\{t \,/\, (x,t) \in \Psi\}.$$

We would like to stress that noted compliance will take place only in a case when growth rate doesn't depend on locations of a crystal. Figure 1 below illustrates distinction which arises in opposite case: part a) represents the boundary of the union Ψ of two zones crystallized by the crystals growing with different speeds from the germs g and h; part b) shows the crystallization process corresponding to the same two crystals.

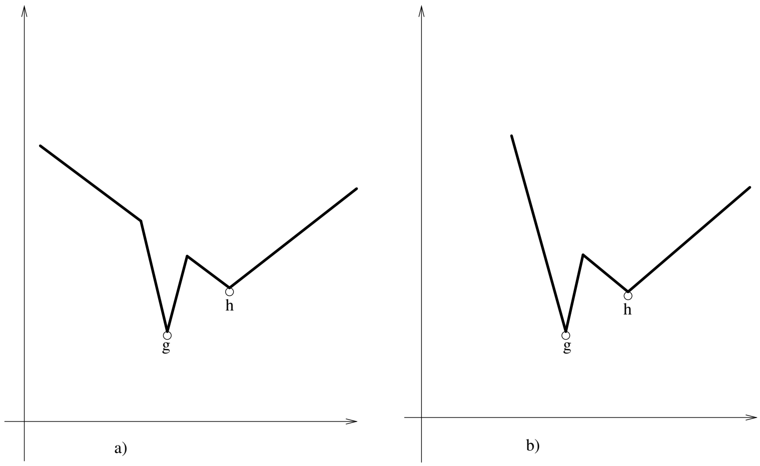

Figure 1: $d = 1$, dynamic of two crystals with different rate of growth
a) Boolean model: union of 2 crystals
b) Crystallization process

3 Results

We assume, without loss of generality, that the random field $\xi = (\xi(x))_{x\in\mathbb{R}^d}$ defined by equation (1) on page 170 is a canonical random field on $(\Omega, \mathcal{F}, \mathbb{P})$. Namely, we suppose that $\Omega = \mathbb{R}^T$ with $T = \mathbb{R}^d$, $\mathcal{F}$ is the σ-algebra generated by the cylinders and $\mathbb{P}$ is the distribution of ξ so that for all $\omega \in \Omega$, $\xi(x,\omega) = \omega(x)$. As Lebesgue measure λ^d on $\mathbb{R}^d$ is translation-invariant, we deduce that ξ is homogeneous. This means that $\mathbb{P}$ is invariant under the translations

$$S_h(\omega)(x) = \omega(x-h),\ h \in \mathbb{R}^d.$$

3.1 Mixing

We precise here what we call a mixing random field.

Definition 1 – A random field $\xi = (\xi(x))_{x\in\mathbb{R}^d}$ is **mixing** if for all $A, B \in \mathcal{F}$,

$$\mathbb{P}\Big(A \cap S_h^{-1}(B)\Big) \xrightarrow[|h|\to\infty]{} \mathbb{P}(A)\mathbb{P}(B).$$

Remark 1 – Note that every mixing random field in the sense of definition 1 is ergodic.

Theorem 1 – *Under the only assumption equation* (3) *the random field* $\xi = (\xi(x))_{x\in\mathbb{R}^d}$ *defined by equation* (1) *on page 170 is mixing.*

Proof. Let us consider the dynamical system $\{\mathbb{K}, \mathcal{K}, Q, (T_h)\}$, where

- $\mathbb{K}$ is the family of locally finite configurations of $\mathbb{R}^d \times \mathbb{R}_+$;
- $\mathcal{K}$ is the σ-algebra generated by the applications $\pi_K : \mathbb{K} \to \mathbb{R}_+$,
- $\pi_K(\varkappa) = \mathrm{card}(\varkappa \cup K)$, K being a compact subset of $\mathbb{R}^d \times \mathbb{R}_+$;
- Q is the law of p.p.p. $\mathcal{N}$;
- $(T_h)_{h\in\mathbb{R}^d}$ is the group of translations, $T_h(\varkappa) = \varkappa - h$.

Let $\varphi : \mathbb{K} \to \Omega$ be the application defined by

$$\varphi(\varkappa)(x) = \inf_{g\in\varkappa} A_g(x).$$

It is easy to see that for all $\varkappa$ and all h

$$S_h(\varphi(\varkappa)) = \varphi(T_h(\varkappa)),$$

where $S_h : \Omega \to \Omega$, $S_h(f)(x) = f(x-h)$.

It means the system $\{\Omega, \mathcal{F}, \mathbb{P}, (S_h)\}$ is a factor-system with respect to $\{\mathbb{K}, \mathcal{K}, Q, (T_h)\}$. As the last one is evidently mixing, by well known fact from the ergodic theory[8] the system $\{\Omega, \mathcal{F}, \mathbb{P}, (S_h)\}$ is also mixing. □

3.2 Absolute regularity

For a subset T of $\mathbb{R}^d$, we denote by $\mathcal{F}_T$ the σ-field generated by the random variables $\xi(x)$ for all x in T. For two disjoint sets T_1 and T_2 in $\mathbb{R}^d$ and the two σ-fields $\mathcal{F}_{T_1}$ and $\mathcal{F}_{T_2}$, the absolute regularity coefficient is

$$\beta(T_1, T_2) = \left\|\mathcal{P}_{T_1\cup T_2} - \mathcal{P}_{T_1} \times \mathcal{P}_{T_2}\right\|_{\mathrm{var}},$$

[8] See e.g. theorem 3 from chapter 10, §1, of Kornfeld, Fomin, and Sinai, 1982, *Ergodic Theory*.

where $\|\nu\|_{\text{var}}$ is the total variation norm of a signed measure ν and $\mathcal{P}_T$ is the distribution of the restriction $\xi_{|T}$ in the set $\mathcal{C}(T)$ of continuous real-valued functions defined on T. Note that $\mathcal{C}(T_1 \cup T_2)$ is canonically identified to $\mathcal{C}(T_1) \times \mathcal{C}(T_2)$ when $T_1 \cap T_2 = \varnothing$. The strong mixing coefficient is defined as follows,

$$\alpha(T_1, T_2) = \sup_{\substack{A \in \mathcal{F}_{T_1} \\ B \in \mathcal{F}_{T_2}}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

The process ξ is said to be absolutely regular (respectively α-mixing) if the absolute regularity coefficient (respectively strong mixing coefficient) converges to zero when the distance between T_1 and T_2 tends to infinity with T_1 and T_2 belonging to a certain class of sets.

Remark 2 – It is well known that

$$\alpha(T_1, T_2) \leq \frac{1}{2}\beta(T_1, T_2)$$

so that absolute regularity of the process ξ implies α-mixing.

Keeping in mind that in our context the process ξ is (strictly) stationary, when $d = 1$, one usually chooses $T_1 = (-\infty, 0]$ and $T_2 = [r, +\infty)$ with $r > 0$ whereas in the case $d \geq 2$, there are several sorts of sets to be considered. The results we obtain in this paper when $d \geq 2$ deal with quadrant domains as represented on figure 2 on the facing page and enclosed cube domains as represented on figure 3 on page 179.

3.3 Dimension $d = 1$

In this case $T_1 = (-\infty, 0]$ and $T_2 = [r, +\infty)$. We denote by $\beta(r)$ the coefficient $\beta(T_1, T_2)$.

Theorem 2 – *Suppose that the dimension d is equal to* 1. *If equations* (3) *and* (4) *on page 171 and on page 172 are satisfied, the process ξ has the absolute regularity property and for all $r > 0$,*

$$\beta(r) \leq 8\mathrm{e}^{-F(C_1 r)}.$$

Here $F(t) = \Lambda(K_t)$, $C_1 = \frac{1}{2ML}$ with L, M the constants from equations (4) *and* (5) *on page 172 and on page 173.*

3.4 Dimension $d \geq 2$

We obtain first an upper bound for the absolute regularity coefficient when the two quadrants T_1 and T_2 are separated by a $2r$-width band. As the random field ξ is homogeneous, we can choose $T_1 = \prod_{i=1}^d (-\infty, 0]$ and $T_2 = \prod_{i=1}^d [a_i, +\infty)$. We denote by L_1, (respectively L_2) the hyper-plane orthogonal to $e = \frac{1}{\sqrt{d}}(1, \ldots, 1)$ and containing

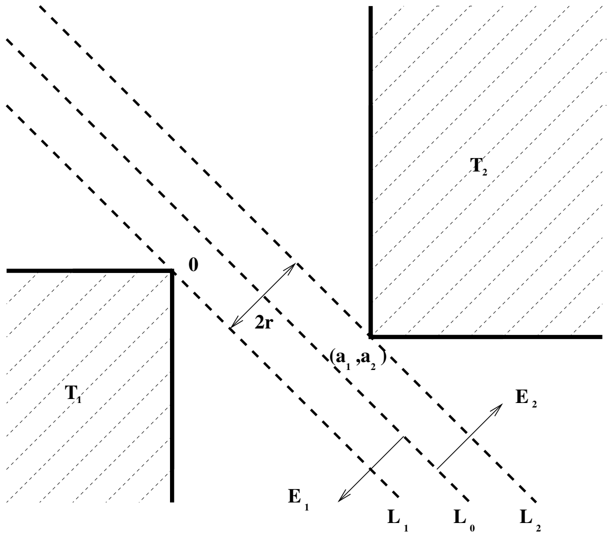

Figure 2: Quadrant domains for $d = 2$

the point $(0,\ldots,0)$ (respectively $(a_1,\ldots,a_d)$) as represented on figure 1 on page 174 when $d = 2$. The distance between the hyper-planes L_1 and L_2 equals $2r =< e,a >$. Since $< e,a >$ is positive, we can introduce the hyper-plane L_0 situated at equal distance between L_1 and L_2. Finally, we denote by E_1 (respectively E_2) the open half-space delimited by L_0 and containing L_1 (respectively L_2).

Theorem 3 – *Suppose that $d \geq 2$. If equations* (3) *and* (4) *on page 171 and on page 172 are satisfied and T_1 and T_2 are the quadrant domains previously described, then*

$$\beta(T_1,T_2) \leq 8\sum_{k=1}^{\infty} k^{d-1}\exp\left\{-F\left(\frac{2r}{dH^2}k\right)\right\}, \tag{6}$$

where $F(t)$ is the measure of K_t and $H = ML(1+\frac{1}{ml})+1$ with M, m and L, l the constants from equations (4) *and* (5) *on page 172 and on page 173.*

Before proving the theorem, we give an estimate of the majorant series in equation (6) for two typical cases.

Example 1 – Suppose that

$$F(t) \geq (d+\delta)\ln t - \ln\gamma$$

for some δ, $\gamma > 0$.

This inequality is fulfilled in particular if we take $K = B(0,1)$, $v(t) = 1$, $t \geq 0$, and for $t > 0$

$$\mu([0,t]) \geq \frac{d+\delta}{2^d c_d} t^{-d} \ln t,$$

where $c_d = \lambda^d(B(0,1))$. Then $\mathrm{e}^{-F(t)} \leq \gamma\, t^{-(d+\delta)}$ and we obtain a polynomial estimation of the sum:

$$\sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(Ck)} \leq \gamma'\, C^{-(d+\delta)}$$

with

$$\gamma' = \gamma \sum_{k=1}^{\infty} k^{-(1+\delta)}.$$

Example 2 – If we rather suppose that $F(t) \geq \gamma\, t^{\delta} - c$ with δ, γ, $c > 0$, then $\mathrm{e}^{-F(t)} \leq C_1\, \mathrm{e}^{-\gamma\, t^{\delta}}$ with $C_1 = \mathrm{e}^c$. It is true if as before $K = B(0,1)$, $v(t) = 1$, $t \geq 0$, and if for $t > 0$

$$\mu([0,t]) \geq \gamma c_d^{-1} t^{\delta - d}.$$

In this case we get a super-exponential estimation of the sum:

$$\sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(Ck)} \leq C_2\, \mathrm{e}^{-\gamma\, C^{\delta}},$$

with

$$C_2 = C_1 \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-\gamma\, C^{\delta}(k^{\delta}-1)}.$$

Evidently, C_2 still bounded when $C \to +\infty$.

We give now an upper bound for the absolute regularity coefficient $\beta(T_1, T_2)$ in the case of enclosed cube domains separated by a $2r$-width polygonal band. As the random field ξ is homogeneous, we consider centered domains $T_1 = [-a,a]^d$ and $T_2 = ([-b,b]^d)^c$ as represented on figure 3 on the facing page for $d = 2$.

Theorem 4 – *Suppose that $d \geq 2$. If equations* (3) *and* (4) *on page 171 and on page 172 are satisfied and T_1, T_2 are the enclosed domains previously described with $b \geq 2(2H-1)a$, then*

$$\beta(T_1, T_2) \leq 4(1 + d\, 2^d) \sum_{k=1}^{\infty} k^{d-1} \exp\Big\{-F\Big(\frac{2r}{dH^2} k\Big)\Big\},$$

where F and H are the same as in theorem 3 on page 177.

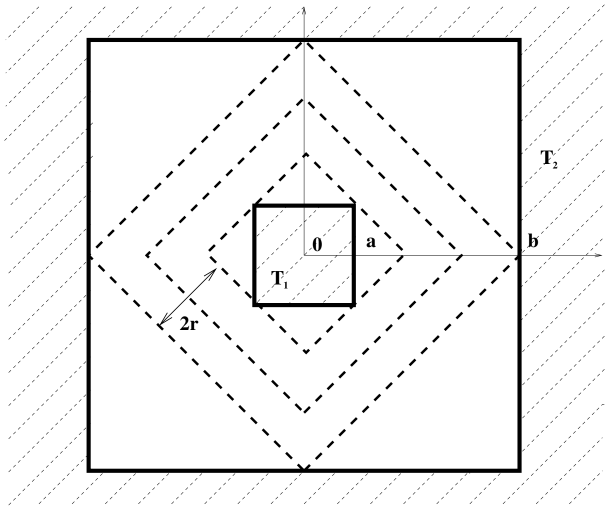

Figure 3: Sketch for $d = 2$

3.5 Lower bounds

As a conclusion, we give a lower bound for the β-coefficient in the context of examples 1 and 2 on page 177 and on page 178. These lower bounds are similar to the upper bounds in theorems 2 to 4 on pages 176–178. Thus, the upper bounds obtained in this article appear to be sufficiently precise.

For simplicity we will consider only a one-dimensional case. Fix positive numbers r, ρ and set $a = \frac{r}{\rho}$, $A = \{\xi(0) > a\}$ and $B = \{\xi(x) > a\}$ with $|x| = r$. It is clear that

$$\beta(r) \geq 2|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

Since ξ is space homogeneous, we obtain that

$$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}\{\mathcal{N} \cap K_a = \emptyset\} = \mathrm{e}^{-F(a)}.$$

To compute $\mathbb{P}(A \cap B)$, we remark that by definition $K_t \subset K_s$ if $t \leq s$. Then for all $h \in \mathbb{R}^d$, $|h| \leq \rho t$, we have

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{P}\{\mathcal{N} \cap K_a = \emptyset,\ \mathcal{N} \cap (K_a + x) = \emptyset\}\\ &\geq \mathbb{P}\Big\{\mathcal{N} \cap K_{(1+\rho)a} = \emptyset\Big\}\\ &= \mathrm{e}^{-F((1+\rho)a)}.\end{aligned}$$

As $r = \rho a$,

$$\beta(r) \geq \left| e^{-2F\left(\frac{r}{\rho}\right)} - e^{-F\left(\frac{(1+\rho)}{\rho} r\right)} \right|. \tag{7}$$

We compute the minoration term in inequality equation (7) for examples 1 and 2 on page 177 and on page 178. In the case of example 1, where $F(t) = (d+\delta)\ln(t) - \ln(\gamma)$ with δ, $\gamma > 0$, we obtain that

$$e^{-2F\left(\frac{r}{\rho}\right)} = \gamma^2 \rho^{2(d+\delta)} r^{-2(d+\delta)}$$

and

$$e^{-F\left(\frac{(1+\rho)}{\rho} r\right)} = \gamma \left(\frac{\rho}{\rho+1}\right)^{d+\delta} r^{-(d+\delta)}.$$

Thus, for r sufficiently large,

$$\beta(r) \geq \kappa_1 r^{-(d+\delta)}$$

with $\kappa_1 > 0$.

For example 2 where $F(t) = \gamma t^\delta - c$ with γ, δ,, $c > 0$, we find that

$$e^{-2F\left(\frac{r}{\rho}\right)} = e^{2c} e^{-\frac{2\gamma}{\rho^\delta} r^\delta}$$

and

$$e^{-F\left(\frac{(1+\rho)}{\rho} r\right)} = e^{c} e^{-\frac{\gamma(1+\rho)^\delta}{\rho^\delta} r^\delta}.$$

Finally, if $\rho < 2^{\frac{1}{\delta}} - 1$, then for r sufficiently large,

$$\beta(r) \geq \kappa_2 e^{-\gamma\left(\frac{1+\rho}{\rho}\right)^\delta r^\delta}$$

with $\kappa_2 > 0$.

4 Proofs

4.1 Approach

In order to obtain upper bounds for the absolute regularity coefficient $\beta(T_1, T_2)$, we approximate the restrictions of ξ on T_1 and T_2 by two independent random fields and apply the following lemma.

Lemma 1 – *Let us consider a random field* $(\xi(x))_{x\in\mathbb{R}^d}$ *and two disjoint subsets* T_1 *and* T_2 *of* $\mathbb{R}^d$*. If there exists two random fields* $(\eta_1(x))_{x\in\mathbb{R}^d}$ *and* $(\eta_2(x))_{x\in\mathbb{R}^d}$ *and two positive constants* δ_1 *and* δ_2 *such that:*

1. η_1 *and* η_2 *are independent;*
2. $\mathbb{P}\{\xi(x) = \eta_i(x),\ \forall\, x \in T_i\} \geq 1 - \delta_i$ *for* $i = 1, 2$*,*

then

$$\beta(T_1, T_2) \leq 4\,(\delta_1 + \delta_2).$$

Proof. Let us denote by $\mathcal{P}_1$ the distribution of the restriction $\xi_{|T_1}$ of ξ to T_1, by $\mathcal{P}_2$ the distribution of the restriction $\xi_{|T_2}$ of ξ to T_2, by $\mathcal{Q}_1$ the distribution the restriction $\eta_{1|T_1}$ of η_1 to T_1, and by $\mathcal{Q}_2$ the distribution of the restriction $\eta_{2|T_2}$ of η_2 to T_2. From item 2 we have for $i = 1, 2$, that

$$\|\mathcal{P}_i - \mathcal{Q}_i\|_{\text{var}} \leq 2\delta_i.$$

Now, we denote by $\mathcal{P}$ the distribution of ξ on $T_1 \cup T_2$ and $\mathcal{Q}$ the distribution of η on $T_1 \cup T_2$ with η defined as follows:

$$\eta(x) = \begin{cases} \eta_1(x) & x \in T_1, \\ \eta_2(x) & x \in T_2. \end{cases}$$

We have

$$\mathbb{P}\{\xi(x) = \eta(x),\ \forall\, x \in T_1 \cup T_2\} = \mathbb{P}(D_1 \cap D_2),$$

where $D_i = \{\xi(x) = \eta_i(x),\ \forall\, x \in T_i\}$, $i = 1, 2$.

But,

$$\mathbb{P}(D_1 \cap D_2) = 1 - \mathcal{P}(D_1^c \cup D_2^c) \geq 1 - \mathbb{P}(D_1^c) - \mathbb{P}(D_2^c)$$

and since $\mathbb{P}(D_i) \geq 1 - \delta_i$ for $i = 1, 2$,

$$\mathbb{P}\{\xi(x) = \eta(x),\ \forall\, x \in T_1 \cup T_2\} \geq 1 - (\delta_1 + \delta_2).$$

By previous arguments

$$\|\mathcal{P} - \mathcal{Q}\|_{\text{var}} \leq 2\,(\delta_1 + \delta_2).$$

Finally, we have that

$$\|\mathcal{P} - \mathcal{P}_1 \times \mathcal{P}_2\|_{\text{var}} \leq \|\mathcal{P} - \mathcal{Q}\|_{\text{var}} + \|\mathcal{Q} - \mathcal{Q}_1 \times \mathcal{Q}_2\|_{\text{var}} + \|\mathcal{Q}_1 \times \mathcal{Q}_2 - \mathcal{P}_1 \times \mathcal{P}_2\|_{\text{var}}.$$

As η_1 and η_2 are independent,

$$\|\mathcal{Q} - \mathcal{Q}_1 \times \mathcal{Q}_2\|_{\text{var}} = 0.$$

Moreover,

$$\|\mathcal{P}_1 \times \mathcal{P}_2 - \mathcal{Q}_1 \times \mathcal{Q}_2\|_{\text{var}} \le \|\mathcal{P}_1 - \mathcal{Q}_1\|_{\text{var}} + \|\mathcal{P}_2 - \mathcal{Q}_2\|_{\text{var}} \le 2\,(\delta_1 + \delta_2)$$

and

$$\|\mathcal{P} - \mathcal{Q}\|_{\text{var}} \le 2\,(\delta_1 + \delta_2).$$

Thus, we derive that

$$\|\mathcal{P} - \mathcal{P}_1 \times \mathcal{P}_2\|_{\text{var}} \le 4\,(\delta_1 + \delta_2).$$ □

4.2 Proof of theorem 2

We introduce for any subset T of $\mathbb{R}$, the process ξ_T defined as follows

$$\xi_T(x) = \inf_{\substack{g \in \mathcal{N} \\ x_g \in T}} A_g(x), \; x \in \mathbb{R}^d. \tag{8}$$

The proof of theorem 2 on page 176 is based on the two lemmas.

Lemma 2 – *Under the assumptions of theorem 2 on page 176, for all $r > 0$, we have that*

$$\mathbb{P}\Big\{\xi(x) = \xi_{(-\infty,MLr]}(x), \; \forall\, x \le 0\Big\} \ge 1 - \mathrm{e}^{-F(r)}$$

with $\xi_{(-\infty,MLr]}$ defined by relation equation (8) *with $T = (-\infty, MLr]$.*

Proof. Let us show first that

$$\{\xi(0) \le r\} \subset \Big\{\xi(x) = \xi_{(-\infty,MLr]}(x), \; \forall x \le 0\Big\}. \tag{9}$$

Suppose that $\xi(0) \le r$ and prove that for all $x \le 0$

$$\inf_{\substack{g \in \mathcal{N} \\ x_g \le MLr}} A_g(x) \le \inf_{\substack{g \in \mathcal{N} \\ x_g > MLr}} A_g(x). \tag{10}$$

For all $g = (x_g, t_g) \in E$ such that $x_g > MLr$ we have due to equation (5) on page 173

$$A_g(0) \ge t_g + \frac{|x_g|}{ML} > r.$$

Since $\xi(0) \le r$, we then deduce that

$$\xi(0) = \inf_{\substack{g\in\mathcal{N}\\ x_g\le MLr}} A_g(0).$$

Therefore, there exists $g_0 \in \mathcal{N}$ such that $x_{g_0} \le MLr$ and $A_{g_0}(0) = \xi(0)$. But then for all $g = (x_g, t_g) \in E$ such that $x_g > MLr$ we have $A_g(0) \ge A_{g_0}(0)$ which gives

$$A_g(x) \ge A_{g_0}(x),\ \forall\, x \le 0,$$

and equation (10) on page 182 follows. It means that for all $x \le 0$

$$\xi(x) = \xi_{(-\infty,MLr]}(x)$$

and equation (9) on page 182 is then proved. Finally,

$$\mathbb{P}\Big\{\xi(x) = \xi_{(-\infty,MLr]}(x),\ \forall x \le 0\Big\} \ge \mathbb{P}\{\xi(0) \le r\}$$

and

$$\mathbb{P}\{\xi(0) \le r\} \ge 1 - \mathrm{e}^{-\Lambda(K_{0,r})}. \qquad \square$$

Thanks to symmetry arguments, we have also the following lemma.

Lemma 3 – *Under the same assumptions as in theorem 2 on page 176, for all $r > 0$, we have that*

$$\mathbb{P}\Big\{\xi(x) = \xi_{[MLr,+\infty)}(x),\ \forall\, x \ge 2MLr\Big\} \ge 1 - \mathrm{e}^{-F(r)}$$

where $\xi_{[MLr,+\infty)}$ is defined by relation equation (8) *with $T = [MLr, +\infty)$.*

We turn back to the demonstration of theorem 2 on page 176.

Proof (of theorem 2 on page 176). Let $t > 0$ and consider r such that $2MLr = t$. Lemmas 2 and 3 on page 182 and on this page allow us to apply lemma 1 on page 181 with $\eta_1 = \xi_{(-\infty,MLr]}$, $\eta_2 = \xi_{[MLr,+\infty)}$, $T_1 = (-\infty, 0]$, $T_2 = [2MLr, +\infty)$ and $\delta_1 = \delta_2 = \mathrm{e}^{-F(r)}$. We obtain then that

$$\beta(t) \le 4\,(\delta_1 + \delta_2) = 8\,\mathrm{e}^{-F(\frac{t}{2ML})}. \qquad \square$$

4.3 Proof of theorem 3

To prove theorem 3 on page 177, we approximate the process ξ on the sets T_1 and T_2. Thus, we introduce for all $r > 0$ the following random fields:

$$\eta_r^1(x) = \inf_{\substack{g\in\mathcal{N}\\ x_g\in E_1}} A_g(x),\ x \in \mathbb{R}^d, \tag{11}$$

$$\eta_r^2(x) = \inf_{\substack{g\in\mathcal{N} \\ x_g\in E_2}} A_g(x), \; x \in \mathbb{R}^d. \tag{12}$$

For $r > 0$ we denote by ξ_r the random field defined as follows:

$$\xi_r(x) = \inf_{\substack{g\in\mathcal{N} \\ |x_g|\le r}} A_g(x), \; x \in \mathbb{R}^d, \tag{13}$$

and we set $\xi_r^y(x) = \xi_r(x-y)$.

The proof of theorem 3 on page 177 is then based on three lemmas. The following lemma is in some sense an analogue of lemma 2 on page 182.

Lemma 4 – *Let* $H = ML(1+\frac{1}{ml})+1$. *Under the assumptions of theorem 3 on page 177, for all* $r > 0$,

$$\mathbb{P}\{\xi(x) = \xi_{Hr}(x) \,/\, |x| \le r\} \ge 1 - \mathrm{e}^{-F(r)}$$

with ξ_{Hr} *defined by equation equation* (13).

Proof. It is sufficient to show that

$$\{\xi(0) \le r\} \subset \{\xi(x) = \xi_{Hr}(x) \,/\, |x| \le r\}.$$

If $\xi(0) \le r$, then there exists g_0 such that $\xi(0) = A_{g_0}(0)$. Hence by equation (5) on page 173 for all x

$$\xi(x) \le A_{g_0}(x) \le A_{(0,r)}(x) \le r + \|x\|\frac{1}{ml}.$$

Therefore

$$\max_{\|x\|\le r} \xi(x) \le r(1+\frac{1}{ml}).$$

Now if germ g is such that $\left\|x_g\right\| > Hr$, then, again by equation (5) on page 173,

$$A_g(x) \ge A_{(x_g,0)}(x) \ge \left\|x_g - x\right\|\frac{1}{RL}.$$

For $\|x\| \le r$ the last inequality gives

$$A_g(x) > (H-1)\frac{r}{RL} = r + \|x\|\frac{1}{ml}.$$

It means that $\xi(x) = \xi_{Hr}(x)$ for all $x, \|x\| \le r$. □

Lemma 5 – *Under the assumptions of theorem 3 on page 177, for all $r > 0$,*

$$\mathbb{P}\Big\{\xi(x) = \eta_r^1(x) \Big/ x \in T_1\Big\} \geq 1 - \sum_{k=1}^{\infty} k^{d-1}\, \mathrm{e}^{-F(Ck)}$$

with η_r^1 defined by equation (11) *on page 183, $C = \frac{2r}{dH^2}$, $H = ML(1 + \frac{1}{ml}) + 1$.*

Proof. We use notation $B(x,r)$ for closed ball with center x and radius r. Let $G = \frac{r}{H}$. We split the set T_1 into d-dimensional cubes denoted by $A_{\overline{k}}$, where for all $\overline{k} = (k_1, \dots, k_d) \in (-\mathbb{N})^d$,

$$A_{\overline{k}} = \prod_{i=1}^{d} \left[\frac{2G}{\sqrt{d}}(k_i - 1), \frac{2G}{\sqrt{d}}(k_i) \right].$$

Each cube $A_{\overline{k}}$ is centered in $x_{\overline{k}} = (\frac{G}{\sqrt{d}}(2k_i - 1))_{i=1,\dots,d}$ and has diameter equal to $2G$. Remark also that the distance between $x_{\overline{k}}$ and L_1 equals $s_{\overline{k}}$ with

$$s_{\overline{k}} = G + \left| \left\langle \frac{2G}{\sqrt{d}} \overline{k}, e \right\rangle \right| = G \left(1 + \frac{2}{d} \left| \sum_{i=1}^{d} k_i \right| \right).$$

Denote by p the probability $\mathbb{P}\{\xi(x) = \eta_r^1(x) / x \in T_1\}$ and note that

$$p = \mathbb{P}\left\{ \bigcap_{\overline{k} \in (-\mathbb{N})^d} B_{\overline{k}} \right\}, \tag{14}$$

with

$$B_{\overline{k}} = \Big\{ \xi(x) = \eta_r^1(x) \Big/ x \in A_{\overline{k}} \Big\}.$$

From lemma 4 on page 184, we obtain for all $r > 0$ that

$$\mathbb{P}\Big\{ \xi(x) = \xi_{Hr}^{x_{\overline{k}}}(x) \Big/ \|x - x_{\overline{k}}\| \leq r \Big\} \geq 1 - \mathrm{e}^{-F(r)},$$

with $\xi_{Hr}^{x_{\overline{k}}}$ defined by relation equation (13). Take $r = G + \frac{s_{\overline{k}}}{H}$. Then, $A_{\overline{k}} \subset B(x_{\overline{k}}, r)$ and

$$\Big\{ \xi(x) = \xi_{Hr}^{x_{\overline{k}}} = (x) \Big/ \|x - x_{\overline{k}}\| \leq r \Big\} \subset \Big\{ \xi(x) = \xi_{Hr}^{x_{\overline{k}}}(x) \Big/ x \in A_{\overline{k}} \Big\}.$$

Moreover $B(x_{\overline{k}}, Hr)$ is included in the half-space E_1. Consequently,

$$\Big\{ \xi(x) = \xi_{Hr}^{x_{\overline{k}}}(x) \Big/ x \in A_{\overline{k}} \Big\} \subset \Big\{ \xi(x) = \eta_r^1(x) \Big/ x \in A_{\overline{k}} \Big\}.$$

Denoting by $p_{\overline{k}}$ the probability $\mathbb{P}(B_{\overline{k}})$, we finally obtain that

$$p_{\overline{k}} \geq 1 - \mathrm{e}^{-F\left(G+\frac{s_{\overline{k}}}{H}\right)}. \tag{15}$$

On the other hand, equation (14) on page 185 implies that

$$p = 1 - \mathbb{P}\Big(\bigcup_{\overline{k}\in(-\mathbb{N})^d} B^c_{\overline{k}}\Big).$$

From equation (15), we deduce that

$$p \geq 1 - \sum_{\overline{k}\in(-\mathbb{N})^d} \mathrm{e}^{-F\left(G+\frac{s_{\overline{k}}}{H}\right)}. \tag{16}$$

Now, we obtain an upper bound for the sum in equation (16) as follows:

$$\begin{aligned}\sum_{\overline{k}\in(-\mathbb{N})^d} \mathrm{e}^{-F\left(G+\frac{l_{\overline{k}}}{H}\right)} &= \sum_{n=0}^{\infty} \#\left\{\overline{k} \Big/ \left|\sum_{i=1}^{d} k_i\right| = n\right\} \mathrm{e}^{-F\left(G+\frac{G}{H}(1+\frac{2}{d}n)\right)} \\ &\leq \sum_{n=0}^{\infty} (n+1)^{d-1} \mathrm{e}^{-F\left(G\left(1+\frac{1}{H}(1+\frac{2}{d}n)\right)\right)}.\end{aligned}$$

Since $G\Big(1+\frac{1}{H}(1+\frac{2}{d}n)\Big) \geq C(m+1)$ with $C = \frac{2G}{dH}$ when $d \geq 2$, we finally derive that

$$p \geq 1 - \sum_{n=1}^{\infty} n^{d-1} \mathrm{e}^{-F(Cn)}. \qquad \square$$

Symmetry arguments lead to the following lemma.

Lemma 6 – *Under the assumptions of theorem 3 on page 177, for all $a > 0$,*

$$\mathbb{P}(\xi(x) = \eta_r^2(x),\ x \in T_2) \geq 1 - \sum_{m=1}^{\infty} m^{d-1} \mathrm{e}^{-F(Cm)}$$

with η_r^2 defined by equation (12) *on page 184, $C = \frac{2G}{dH}$, $G = \frac{r}{H}$, $H = ML(1+\frac{1}{ml})+1$.*

Proof (of theorem 3 on page 177). Lemmas 5 and 6 on page 185 and on the current page enable us to make use of lemma 1 on page 181 with $\eta_1 = \eta_1^r$, $\eta_2 = \eta_2^r$, $\delta_1 = \delta_2 = \sum_{k=1}^{\infty} k^{d-1}\mathrm{e}^{-F(Ck)}$ and T_1, T_2 the quadrant domains. We then have that

$$\beta(T_1, T_2) \leq 4(\delta_1+\delta_2) = 8\sum_{k=1}^{\infty} k^{d-1}\mathrm{e}^{-F(Ck)}. \qquad \square$$

4.4 Proof of theorem 4

The proof of theorem 4 on page 178 make use of the same kind of arguments as in the proof of theorem 3 on page 177. Therefore, we introduce some notations in order to define the random fields η_1^r and η_2^r approximating ξ respectively on T_1 and T_2. Thus, we denote by $e_1, \ldots, e_d$ the d vectors of the canonical base in $\mathbb{R}^d$ and consider the set $A = \{\alpha \,/\, (\alpha_1, \ldots, \alpha_d),\, \alpha_i = \pm 1\}$ which cardinal equals $\#A = 2^d$. For all i, the hyper-plane $e_i^\perp$ separates the set $\mathbb{R}^d$ into two open half-space E_i^ϵ with $\epsilon = \pm 1$ and ϵe_i contained in E_i^ϵ. For all $\alpha \in A$, we introduce the quadrant:

$$\mathcal{Z}_\alpha = \bigcap_{i=1}^{d} E_i^{\alpha_i}$$

and for all $i = 1, \ldots, d$ the translated quadrant:

$$\mathcal{Z}_{\alpha,i} = \mathcal{Z}_\alpha \oplus \alpha_i \, b \, e_i. \tag{17}$$

Observe that

$$T_2 = \bigcup_{\alpha \in A} \bigcup_{i=1}^{d} \mathcal{Z}_{\alpha,i}.$$

On the other hand, let us define for all $\alpha \in A$, the normed vector of $\mathcal{Z}_\alpha$:

$$d_\alpha = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} \alpha_i \, e_i.$$

To separate the sets T_1 and T_2 by a $2r$-width polygonal band, the quantity $r = \frac{(b-2a)\sqrt{d}}{4}$ must be positive. Thus, we assume that $b > 2a$. In this case, we consider the hyper-planes

$$\begin{aligned}
L_\alpha^0 &= d_\alpha^\perp + \frac{(b+2a)\sqrt{d}}{4}\, d_\alpha \\
L_\alpha^2 &= L_\alpha^0 + r\, d_\alpha = d_\alpha^\perp + \frac{b}{2}\sqrt{d}\, d_\alpha \\
L_\alpha^1 &= L_\alpha^0 - r\, d_\alpha = d_\alpha^\perp + a\sqrt{d}\, d_\alpha
\end{aligned}$$

as represented on figure 4 on the next page for $d = 2$ and $\alpha = (1,1)$.

We introduce now, for all α in A, the open half-space S_α^2 delimited by the hyper-plane L_α^0 and containing the quadrants $\mathcal{Z}_{\alpha,i}$ for $i = 1, \ldots, d$. At last, we consider the set S_2 containing T_2:

$$S_2 = \bigcup_{\alpha \in A} S_\alpha^2.$$

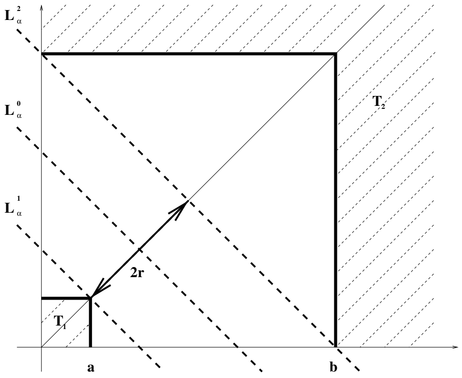

Figure 4: Sketch for $d = 2$

Then, we introduce for all $\alpha \in A$, the random field:

$$\eta_\alpha(x) = \inf_{g \in S^2_\alpha} A_g(x), \quad x \in \mathbb{R}^d$$

and approximate ξ on T_2 by the following random field:

$$\eta^r_2(x) = \inf_{g \in S_2} A_g(x), \quad x \in \mathbb{R}^d. \tag{18}$$

Lemma 7 – *Under assumptions of theorem 4 on page 178*

$$\mathbb{P}\{\xi(x) = \eta^r_2(x),\ x \in T_2\} \geq 1 - d\, 2^d \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(C\,k)}$$

where C is the constant of theorem 4 on page 178 and η^r_2 is defined by equation (18).

Proof. As for all $\alpha \in A$ and all $i = 1, \ldots, d$ the sets $\mathcal{Z}_{\alpha,i}$ defined by equation (17) on page 187 are quadrants included in S^2_α, ξ can be approximate by η_α on each $\mathcal{Z}_{\alpha,i}$ by lemma 5 on page 185 so that:

$$\mathbb{P}\{\xi(x) = \eta_\alpha(x),\ \forall\, x \in \mathcal{Z}_{\alpha,i}\} \geq 1 - \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(C\,k)}.$$

Since for all $x \in \mathbb{R}^d$

$$\xi(x) \leq \eta^r_2(x) \leq \eta_\alpha(x),$$

we deduce for all $\alpha \in A$ and all $i = 1, \ldots, d$ that

$$\mathbb{P}\{\xi(x) = \eta^r_2(x),\ \forall\, x \in \mathcal{Z}_{\alpha,i}\} \geq 1 - \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(C\,k)}.$$

Finally, we get

$$\mathbb{P}\{\xi(x) = \eta_2^r(x),\ x \in T_2\} \geq 1 - d\, 2^d \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(C\,k)}. \qquad \square$$

Consider now for all α in A the open half-space $S_\alpha^1 = (S_\alpha^2)^c \backslash L_\alpha^0$. We also introduce the intersection

$$S_1 = \bigcap_{\alpha \in A} S_\alpha^1$$

on which ξ can be approximated by the following random field:

$$\eta_1^r(x) = \inf_{g \in S_1} A_g(x), \quad x \in \mathbb{R}^d. \tag{19}$$

Lemma 8 – *Under assumptions of theorem 4 on page 178 and if* $b \geq 2(2H-1)a$, *then*

$$\mathbb{P}\{\xi(x) = \eta_1^r(x),\ x \in T_1\} \geq 1 - \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(C\,k)},$$

where C is the constant of theorem 4 on page 178 and η_1^r is defined by equation (19).

Proof. We consider the centered open ball $B_1 = B(0, a\sqrt{d})$ included in T_1 and the ball $B_2 = B(0, a\sqrt{d} + r')$ with $r' \leq r$ so that B_2 is contained in S_1. If we denote by R the radius of B_1 and assume that $RH = a\sqrt{d} + r'$ with H the constant of theorem 4 on page 178, we find that

$$r' = (H-1)\,a\sqrt{d} \leq r = \frac{(b-2a)\sqrt{d}}{4}$$

and finally that b must be such that $b \geq 2(2H-1)a$. Since $H \geq 1$, it follows that $b > 2a$. We introduce the random fields η_{B_2}:

$$\eta_{B_2}(x) = \inf_{g \in B_2} A_g(x), \quad x \in \mathbb{R}^d,$$

and remark that for all $x \in \mathbb{R}^d$

$$\xi(x) \leq \eta_1^r(x) \leq \eta_{B_2}(x). \tag{20}$$

As by lemma 4 on page 184,

$$\mathbb{P}\Big\{\xi(x) = \eta_{B_2}(x),\ \forall x \in B_1\Big\} \geq 1 - \mathrm{e}^{-F(R)},$$

it follows from equation (20) on page 189 that

$$\mathbb{P}\{\xi(x) = \eta_1^r(x),\ \forall x \in B_1\} \geq 1 - \mathrm{e}^{-F(R)}.$$

As $B_1 \subset T_1$, we also have that

$$\{\xi(x) = \eta_1^r(x),\ x \in B_1\} \subset \{\xi(x) = \eta_1^r(x),\ x \in T_1\}.$$

Therefore

$$\mathbb{P}\{\xi(x) = \eta_1^r(x),\ \forall x \in T_1\} \geq 1 - \mathrm{e}^{-F(R)}.$$

Finally, as $H \geq 1$, we have that $R \geq C$ with $C = \frac{2R}{dH}$ and $\mathrm{e}^{-F(R)} \leq \mathrm{e}^{-F(C)}$. Since $\mathrm{e}^{-F(C)} \leq \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(Ck)}$, we get

$$\mathbb{P}\{\xi(x) = \eta_1^r(x),\ \forall x \in T_1\} \geq 1 - \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(Ck)}. \qquad \square$$

Proof (of theorem 4 on page 178). We apply again lemmas 1, 7 and 8 on page 181, on page 188 and on page 189 with $\eta_1 = \eta_1^r$, $\eta_2 = \eta_2^r$, $\delta_1 = \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(Ck)}$, $\delta_2 = d\, 2^d\, \delta_1$ and T_1, T_2 the enclosed domains. We then have that

$$\beta(T_1, T_2) \leq 4(\delta_1 + \delta_2) = 4(1 + d\, 2^d) \sum_{k=1}^{\infty} k^{d-1} \mathrm{e}^{-F(Ck)}. \qquad \square$$

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