Career Concerns and Market Structure*

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Abstract

This paper analyzes the impact of market structure on career concerns. Effort increases the probability that a skilled agent achieves a one-time breakthrough. Wages are based on assessed ability and on expected output. For any wage, the agent works too little, too late. Under short-term contracts, effort and wages are single-peaked with seniority, due to the strategic substitutability of effort levels at different times. Both delay and underprovision of effort worsen if effort is observable. Commitment to wages by competing firms mitigates these inefficiencies. In that case, the optimal contract features piecewise constant wages and severance pay.

Keywords: career concerns, experimentation, career paths, up-or-out, reputation.

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1 Introduction

Career concerns are an important driver of incentives. This is particularly so in professional-service firms, such as law and consulting, but applies more broadly to environments in which creativity and originality are essential for success: pharmaceutical companies, biotechnology research labs, academia, etc. By the nature of research, learning and success display common patterns across these sectors: output measures are based on infrequent events, and promotion is related to peak performance. Consequently, the prevailing labor-market arrangements in these sectors display common institutional patterns: long-term contracts are in force; career paths often start with a probationary period that leads to an up-or-out decision; and wages are markedly lower during this period than after, with significant rigidity before the tenure decision. There are, however, other sectors (*e.g.*, creative arts and professional sports) in which agents are motivated by career concerns, their reputations are mostly tied to breakthrough performances, but short-term contracts are the norm (*i.e.*, future wages are not guaranteed).

Nearly all the existing literature on career concerns, starting with Holmström (1982/99), relies on two assumptions: rich measures of output are available throughout, and wages are set competitively at all times (spot contracts). The first assumption is at odds with our examples. The second precludes examining the interplay between career concerns and institutions. This makes it hard to interpret some of their findings. For instance, equilibrium effort decreases over time. Does this depend on the wage-setting rule, or does this follow from learning, independently of labor-market arrangements? Do probationary periods amplify reputation incentives?

In this paper, we investigate the interplay between career concerns and market structure by developing a framework that accommodates several alternative labor-market arrangements. Under spot contracts, ours is the first multiperiod career concerns model where output is not additively separable in the agent's talent and effort. The spot-contracts model uncovers a new dynamic link between effort levels at different times that explains rich equilibrium effort and wage patterns. We contrast the spot-contracts model with a model where firms compete by offering long-term contracts to which they commit. Thus, we disentangle the role of reputation from that of competition. Our main contribution is to show that regularities observed in practice (tenure, severance pay, signing bonuses) arise as optimal arrangements when firms compete with long-term contracts. To capture the features of research-intensive and creative industries, we make the following assumptions. The first two distinguish our model from Holmström.

(i) Success is rare. Across fields of research, the distribution of USPTO patents granted per inventor is heavily skewed to the left.[1] Other instances include working a breakthrough case in law or consulting, signing a record deal, or acting in a blockbuster movie.

(ii) Success is informative. Breakthroughs are defining moments in a professional's career. In other words, information is coarse: either an agent is successful, or he is not. Indeed, in many industries, there is growing evidence that the market rewards "star" professionals.[2]

(iii) Explicit output-contingent contracts are not used. While theoretically attractive, innovation bonuses in R&D firms are hard to implement due to complex attribution and timing problems. Junior associates in law and consulting firms receive fixed stipends. In the motion pictures industry, most contracts involve fixed payments rather than profit-sharing.[3]

The biotechnology & pharmaceutical industry provides a good example of a market that displays all three features. (i) Less than 2% of developed molecules eventually lead to a drug approved by the FDA, and the discovery and development process takes 10 to 15 years. Due to such high attrition rates and lengthy processes, researchers and clinical scientists working in drug discovery or development may not encounter success throughout their entire career.[4] (ii) In this context, taking a breakthrough drug to market sets apart the scientists working on the project from their peers, and the market recognizes such success quickly. Similarly, receiving a Research Project Grant by the NIH is a career-changing event for postdocs in the life sciences.[5] (iii) In biotechnology, both FDA drug approval and molecule patenting are delayed and noisy metrics of a drug's profitability, and so are rarely used (see Cockburn *et al.*, 1999). Few biotech companies offer scientists variable compensation in the form of stock options (see Stern, 2004).

[1]The NBER patent data analyzed by Trajtenberg et al. (2006) shows that over 60% of inventors who were awarded a patent by the USPTO over the period 1963-1999 were awarded only one, and 20% were awarded two.

[2]See Gittelman and Kogut (2003) and Zucker and Darby (1996) for evidence on the impact of "star scientists," and Caves (2003) for a discussion of A-list vs. B-list actors and writers.

[3]See Chisholm (1997).

[4]Ethiraj and Zhao (2012) find a success rate of 1.7% for molecules developed in 1990. The annual report by PhRMA (2012) shows even higher attrition rates in recent years.

[5]Anecdotal evidence abounds. For instance, consider the career progression of the leading researchers and clinical scientists working on famous drugs such as Novartis' Gleevec or Pfizer's Xeljanz. See Pray (2008) and LaMattina (2012) for detailed accounts.

And almost no large pharmaceutical company offers bonuses for drug approval to its scientists.[6]

More formally, information about ability is symmetric at the start.[7] Skill and output are binary and complements: only a skilled agent can achieve a high output, or *breakthrough*. The breakthrough time follows an exponential distribution, whose intensity increases with the worker's unobserved effort. Hence, effort increases not only expected output, but also the rate of learning, unlike in the Gaussian set-up. When a breakthrough obtains, the market recognizes the agent's talent and that is reflected in future earnings: the agent receives a constant exogenously specified compensation thereafter. The model allows for, but does not require, a penalty (representing diminished future earnings) for reaching an exogenous deadline without a breakthrough. The focus is on the relationship until a breakthrough occurs, or the deadline is reached.[8]

We contrast spot and long-term wage contracts. In either case, there is ongoing competition among firms who observe output and all offers. Thus, the agent reaps his entire marginal product. With spot contracts, the flow wage equals this marginal product at all times. Under long-term contracts, firms commit to a wage path, but the agent can leave at any time (the same horizon length applies to all firms, *i.e.* the "clock is not reset"). For this not to happen, the contract must perpetually deliver a continuation payoff above what the agent can get on the market. This "no-poaching" constraint implies that one must solve for the optimal contract in all possible continuation games, as competing firms' offers must themselves be immune to further poaching.[9]

Under any market structure, a tension emerges between competition and reputation-based incentives. Because of competition, the agent must be paid his full expected marginal product. As output-contingent payments are impossible, this typically involves positive wages even after prolonged failure. This implies very generally *underprovision and delay:* career concerns provide

[6] The one recent exception is Glaxo Smith Kline, whose proposal has been received with skepticism on several accounts. See Goodman (2013) and Shaywitz (2013) for more details.

[7] We shall also briefly discuss the case of asymmetric information. Alternatively, one could examine the consequences of overoptimism by the agent. In many applications, however, symmetrical ignorance appears like the more plausible assumption. See Caves (2003).

[8] In Section 4.1, we turn to the optimal design of an up-our-out arrangement. A probationary period is a hallmark of many occupations (law, accounting and consulting firms, etc.). Though alternative theories have been put forth (*e.g.*, tournament models), agency theory provides an appealing framework to analyze such systems (see Fama, 1980, or Fama and Jensen, 1983). Gilson and Mnookin (1989) offer a vivid account of associate career patterns in law firms, and the relevance of the career concerns model as a possible explanation.

[9] This "infinite regress" (in continuous time) raises technical challenges, restricting us to linear cost for formal results. Our main findings are confirmed numerically for more general cost functions, see below for an example.

insufficient incentives for effort independently from any particular equilibrium notion. *For any* wage path, the total amount of effort exerted is inefficiently low. In addition, whatever effort is provided, it is exerted too late: a social planner constrained to the same total amount of effort exerts it sooner. This backloading contrasts with the frontloading arising in Holmström. While both effects are due to positive wages, increasing wages throughout does not lead to lower effort at all times. Rather, it leads to lower aggregate effort, and effort being exerted later.

The dynamics of effort and wages under spot contracts are driven by the *strategic substitutability* between current and future effort. Substitutability is not a property of the technology (the arrival process of breakthroughs is memoryless) but of the market structure: if career concerns provide incentives for effort at some point in the worker's career, competitive wages at that time must reflect this increased productivity; in turn, this depresses incentives at all earlier times, as future wages make staying on the current job relatively more attractive. Thus, substitutability and competition introduce a tension between incentives at different stages in a worker's career.[10]

Just because future wages must be paid under competition does not imply that the timing of these payments is irrelevant. This is how long-term contracts can mitigate their adverse consequences. Because future wages paid in the event of persistent failure depress current incentives, it would be best to pay the worker his full marginal product *ex ante*. This payment being sunk, it would be equivalent, in terms of incentives, to no future payments for failure at all. Therefore, if the worker can commit to a *no-compete* clause, a simple signing bonus is optimal.

However, in most labor markets workers cannot commit to such clauses. It then follows that the firm will not offer such a bonus, anticipating the worker's incentive to quit right after cashing it in. Lack of commitment on the worker's side prevents the payment to precede the corresponding marginal product. Surprisingly, we show that, as far as current incentives are concerned, it is then best to pay him *as late as possible*. This follows from the value of learning: much later payments discriminate better than imminent ones between skilled and unskilled workers. Because effort and skill are complements, a skilled worker is likely to succeed by the end, and so less likely to be concerned by a payment that would only be made in the case of persistent failure. Higher effort being more valuable when the worker is skilled, this mitigates the pernicious effect of positive

[10]Substitutability does not arise in Holmström's additively separable model. It does not arise in Dewatripont, Jewitt and Tirole (1999a,b) either, because theirs is a two-period model (career concerns do not arise in the last period). Their analysis focuses on the strategic complementarity between expected and realized effort which generates, among others, equilibrium multiplicity. Here instead, the equilibrium is unique under mild conditions.

future wages precisely in the right circumstance. It also softens the no-poaching constraint, as the worker has fewer reasons to leave the firm when his payment is backloaded.

The following example illustrates our main results for the case of quadratic cost. It involves no termination penalty. Figure 1 shows wages w and effort u for two different horizon lengths, conditional on the worker not having had a breakthrough by a given time, holding all other parameters fixed.[11] These wages are not output-contingent, hence they are forfeited in case of success. Note the lump-sum payment M at the deadline, which is specified by the long-term contract. When the horizon is long enough, this is the only payment that a long-term contract specifies: the wage is zero until then (right panel). With a shorter deadline (left panel), the wage is also zero as we approach the deadline, but not necessarily at the beginning (when it actually is equal to the marginal product). Effort is also eventually nil, but it is positive in an initial phase, even if flow wages are zero. With spot contracts, there are no lump-sum payments (the marginal product being a flow), and flow wages decrease with time.[12] Effort is positive and single-peaked.

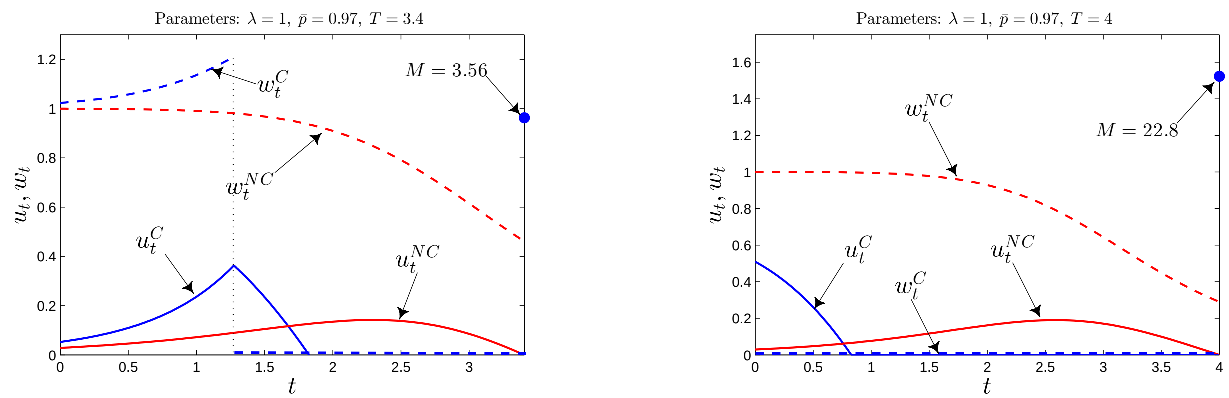

Figure 1: Effort and wages under spot and long-term contracts for two horizon lengths.

As shown in Figure 1, backloading wages is particularly valuable with a long deadline, in which case the only payment occurs as the horizon expires. Indeed, a "severance package" achieves the first-best asymptotically. With a short deadline, the wage is not *entirely* backloaded. Much later payments are preferable for current incentives than imminent ones, but much later payments not only depress current incentives, but also later incentives. Imminent payments, on the other

[11]In terms of the notation in Section 2, the parameters are $v = \lambda = 1, k = 0, c(u) = u^2/2$. Dashed lines are wages, solid lines are efforts. In the right panel, the wage under commitment w_t^C is identically zero.

[12]The pattern of spot wages might be more complicated in general, depending on cost, see Section 4.1.

hand, are no longer relevant for incentives to work after these are made. When the horizon is short, final payments are no longer as potent an instrument, and the trade-off sways towards earlier payments as well (the severance pay persists except for very short horizons).

The same forces explain why, with spot contracts, effort and wages are single-peaked, in contrast to several earlier models, in which they stochastically decrease over time. As is well-known, reputation provides incentives to exert effort. But suppose that these incentives are effective at some point during one's career. With spot contracts, the worker must be compensated for those. In turn, this depresses his incentives and his compensation at earlier stages.

To summarize somewhat loosely, long-term contracts backload payments, relative to spot contracts; hence, they frontload effort (as discussed, backloaded payments are great for early incentives, not so much for later ones). Several of these features are robust: effort is single-peaked under spot contracts, and long-term contracts frontload incentives relative to spot contracts. This is achieved via a "severance package" at the end of the probationary period, with zero wages in a phase that immediately precedes termination.[13]

Beyond these main results, we use our model to explore or revisit the role of other factors that shape career concerns. In particular, another feature that sets apart our model from Holmström's is that it allows for endogenous deadlines. In equilibrium, not only is effort single-peaked, it is decreasing at the deadline, and so is the wage. The worker quits too late relative to what would be optimal, but if he could commit to a deadline, he might choose a longer or a shorter one than without commitment, depending on the circumstances.

Our model also supports the notion that better monitoring need not help. Monitoring effort leads to weaker career concerns, as it disconnects the worker's effort from the firm's perception of ability. Aggregate effort and welfare are lower in all Markov equilibria of the model with observable effort. In particular, if there is no penalty for failure, effort is nil throughout. When a penalty exists, monitoring shifts the equilibrium effort pattern. Effort is now increasing over time, as it is delayed as much as possible given the deadline. Hence, wages are also more backloaded than without monitoring. Thus, better monitoring seems more in line with empirical patterns.

Finally, we explore the robustness of the findings to the stylized modeling assumptions. In turn, we consider the possibility of learning-by-doing; of more gradual learning about ability;

[13]Precisely, compensation involves a phase with wage equal to marginal product, followed by a phase with zero wages. The agent's product is backloaded into a final lump-sum, which might disappear for very short deadlines. See the end of Section 4.2 for a discussion.

and of an ability level that evolves over time.

The most closely related papers are Holmström, as mentioned, as well as Dewatripont, Jewitt and Tirole. In Holmström, skill and effort enter linearly and additively into the mean of the output that is drawn in every period according to a normal distribution. Wages are as in our baseline model: the worker is paid upfront the expected value of the output. Our model shares with the two-period model of Dewatripont, Jewitt and Tirole some features that are absent from Holmström's. In particular, effort and talent are complements. We shall discuss the relationship between the three models at length in the paper.

As Gibbons and Murphy (1992), our paper examines the interplay of implicit incentives (career concerns) and explicit incentives (termination penalty). It shares with Prendergast and Stole (1996) the existence of a finite horizon, and thus, of complex dynamics related to seniority. See also Bar-Isaac (2003) for reputational incentives in a model in which survival depends on reputation. The continuous-time model of Cisternas (2012a) extends the Gaussian framework to nonlinear environments, but maintains the additive separability of talent and action. Jovanovic (1979) and Murphy (1986) provide models of career concerns that are less closely related: the former abstracts from moral hazard and focuses on turnover when agents' types are match-specific; the latter studies executives' experience-earnings profiles in a model in which firms control the level of capital assigned to them over time.

The binary set-up is reminiscent of Mailath and Samuelson (2005), Bergemann and Hege (2005), and Board and Meyer-ter-Vehn (2013). The latter two papers use an exponential technology for output. However, in Bergemann and Hege (2005), the effort choice is binary and wages are not based on the agent's reputation, while Board and Meyer-ter-Vehn (2013) let the agent, who is privately informed, control the evolution of his type through his effort. Here instead, information is symmetric and types are fixed (Subsection 5.4 relaxes this assumption).[14] A theory of up-or-out contracts, based on asymmetric learning and promotion incentives, is investigated in Ghosh and Waldman (2010), while Ferrer (2011) studies how lawyers' career concerns impact litigation. Finally, the empirical work of Chevalier and Ellison (1999) provides evidence of the sensitivity of termination to performance, while Johnson (2011) and Kolstad (2012) quantify the effect of individual and market learning on physicians' incentives and career paths.

[14] Board and Meyer-ter-Vehn (2010) study the Markov-perfect equilibria of a game in which effort affects the evolution of the player's type both under symmetric and asymmetric information.

2 The model

2.1 Set-up

We consider the incentives of an agent (or *worker*) to exert effort (or *work*). Time is continuous, and the horizon finite: $t \in [0, T]$, $T > 0$. Most results carry over to the case $T = \infty$, as shall be discussed, and the case of endogenous deadlines T will be studied in detail in Section 4.1.2.

The game (or *project*) can end before $t = T$, if the agent's effort is successful. Specifically, there is a binary state of the world. If the state is $\omega = 0$, the agent is bound to fail, no matter how much effort he exerts. If the state is $\omega = 1$, a success (or *breakthrough*) arrives at a time that is exponentially distributed, with an intensity that increases in the instantaneous level of effort exerted by the agent. The state can be interpreted as the agent's ability, or skill. We will refer to the agent as a high- (resp., low-) ability agent if the state is 1 (resp. 0). The prior probability of state 1 is $p^0 \in (0, 1)$, which measures occupational harshness.

Effort is a (measurable) function from time to the interval $[0, \bar{u}]$, where $\bar{u} \in \bar{\mathbb{R}}$ represents an upper bound (possibly infinite). If a high-ability agent exerts effort u_t over the time interval $[t, t + \mathrm{d}t)$, the probability of a success over that time interval is $(\lambda + u_t)\mathrm{d}t$. The parameter $\lambda \geq 0$ can be interpreted as the luck of a talented agent. Alternatively, it measures the minimum effort level that the principal can force upon the agent by direct oversight, *i.e.*, the degree of contractibility of the worker's effort. Formally, the instantaneous arrival rate of a breakthrough at time t is given by $\omega \cdot (\lambda + u_t)$. Note that, unlike in Holmström's model, but as in the model of Dewatripont, Jewitt and Tirole, work and talent are complements.

As long as the game has not ended the agent receives a flow wage w_t. For now, let us think of this wage as an exogenous (integrable, non-negative) function of time only that accrues to the agent as long as the game has not ended. Eventually, equilibrium constraints will be imposed on this function, and this wage will reflect the market's expectations of the agent's effort and ability, given that the market values a success. This value is normalized to one.

In addition to receiving this wage, the agent incurs a cost of effort: exerting effort level u_t over the time interval $[t, t + \mathrm{d}t)$ entails a flow cost $c(u_t)\mathrm{d}t$. We shall consider two cases: in the convex case, we assume that $\bar{u} = \infty$, c is increasing, thrice differentiable and convex, with $c(0) = 0$, $\lim_{u \to 0} c'(u) = 0$, $\lim_{u \to \infty} c'(u) = \infty$, $c'' > 0$ and $c''' \geq 0$.[15] In the linear case, $\bar{u} < \infty$

[15] The assumption that c' is convex is only required for three results: Lemma 2.1, equilibrium uniqueness and

and $c(u) = \alpha \cdot u$, where $\alpha > 0$. The linear case is not a special case of what is called the convex one, but it yields similar results, while allowing for illustrations and sharper characterizations.

Achieving a success is desirable on two accounts: first, a known high-ability agent can expect a flow outside wage of $v \geq 0$, so that this outside option v is a (flow) opportunity cost for him that is incurred as long as no success has been achieved.[16] The outside option of the low-ability agent is normalized to 0. Second, we allow for a fixed penalty of $k \geq 0$ for reaching the deadline (*i.e.*, for not achieving a success by time T). This might represent diminished career opportunities to workers with such poor records. Alternatively, this penalty might be an adjustment cost, or the difference between the wage he could have hoped for had he succeeded, and the wage he will receive until retirement. Note that the special case with no penalty ($k = 0$) is allowed. There is no discounting.[17]

The worker's problem can then be stated as follows: to choose $u : [0, T] \to [0, \bar{u}]$, measurable, to maximize his expected sum of rewards, net of the outside wage v:

$$\mathbb{E}_u \left[\int_0^{T \wedge \tau} \left[w_t - v \chi_{\omega=1} - c(u_t) \right] \mathrm{d}t - \chi_{\tau \geq T} k \right],^{18}$$

where $\mathbb{E}_u$ is the expectation conditional on the worker's strategy u and τ is the time at which a success occurs (a random time that is exponentially distributed, with instantaneous intensity at time t equal to 0 if the state is 0, and to $\lambda + u_t$ otherwise) and χ_A is the indicator of event A.

Of course, at time t effort is only exerted, and the wage collected, conditional on the event that no success has been achieved. We shall omit to say so explicitly, as those histories are the only nontrivial ones. Given his past effort choices, the agent can compute his belief p_t that he is of high ability by using Bayes' rule. It is standard to show that, in this continuous-time environment, Bayes' rule reduces to the ordinary differential equation

$$\dot{p}_t = -p_t (1 - p_t)(\lambda + u_t), \ p_0 = p^0. \tag{1}$$

By the law of iterated expectations, we can then rewrite our objective as

$$\int_0^T e^{-\int_0^t p_s(\lambda + u_s) \mathrm{d}s} \left[w_t - p_t v - c(u_t) \right] \mathrm{d}t - k e^{-\int_0^T p_t(\lambda + u_t) \mathrm{d}t}.$$

single-peakedness of equilibrium wage in Section 4 (Theorem 4.2).

[16]A natural case is the one in which v equals the flow value of success given that the agent has established that $\omega = 1$. Because successes arrive at rate λ and are worth 1, $v = \lambda$ in that case.

[17]At the beginning of the appendix, we explain how to derive the objective function from its discounted version as discounting vanishes. Values and optimal policies converge pointwise.

[18]Stating the objective as a net payoff ensures that the program is well-defined even when $T = \infty$.

The exponential term captures the probability of reaching time t without a breakthrough. Using eqn. (1), or equivalently, observing that

$$\mathbb{P}[\tau \geq t] = \frac{\mathbb{P}[\omega = 0 \cap \tau \geq t]}{\mathbb{P}[\omega = 0|\tau \geq t]} = \frac{\mathbb{P}[\omega = 0]}{\mathbb{P}[\omega = 0|\tau \geq t]} = \frac{1-p_0}{1-p_t},$$

the problem simplifies to the maximization of

$$\int_0^T \frac{1-p_0}{1-p_t}\left[w_t - c(u_t) - v\right] \mathrm{d}t - \frac{1-p_0}{1-p_T}k, \text{[19]} \tag{2}$$

given w, over all measurable $u : [0, T] \to [0, \bar{u}]$, subject to (1).

Considering this last maximization, there appears to be two drivers to the worker's effort. First, if the wage falls short of the outside option (*i.e.*, if $w_t - v$ is negative), he has an incentive to exert high effort to stop incurring this flow deficit. Achieving this is more realistic when the belief p is high, so that this incentive should be strongest early on, when he is still sanguine about the project. Second, if the penalty is strictly positive, there is an incentive to succeed so as to avoid paying it. This incentive should be most acute when the deadline looms close, as success becomes unlikely to arrive without effort. Taken together, this suggests an effort pattern that is a convex function of time. However, this ignores that, in equilibrium, the wage reflects the agent's expected effort. As a result, we shall show that the worker's effort pattern is the exact opposite of what this first intuition suggests.

2.2 The social planner

Before solving the agent's problem, we start by analyzing the simpler problem faced by a social planner. What is the expected value of a breakthrough? Recall that the value of a realized breakthrough is normalized to one. But a breakthrough only arrives with instantaneous probability $p_t(\lambda + u_t)$, as it occurs at rate $\lambda + u_t$ only if $\omega = 1$. Therefore, the planner maximizes

$$\int_0^T \frac{1-p_0}{1-p_t}\left[p_t(\lambda + u_t) - v - c(u_t)\right] \mathrm{d}t - k\frac{1-p_0}{1-p_T}, \tag{3}$$

[19]Note that we have replaced $p_t v$ by the simpler v in the bracketed term inside the integrand. This is because

$$\int_0^T \frac{p_t}{1-p_t} v\mathrm{d}t = \int_0^T \frac{v}{1-p_t}\mathrm{d}t - vT,$$

and we can ignore the constant vT, at least until Section 4.1.2, where the deadline is endogenized.

over all measurable $u : [0, T] \to [0, \bar{u}]$, given (1). As for most of the optimization programs considered in this paper, we apply Pontryagin's maximum principle to get a characterization. The proof of the next lemma and of all formal results can be found in the appendix. For $\bar{u} < \infty$, a strategy u is *extremal* if it only takes extreme values: $u_t \in \{0, \bar{u}\}$, for all t.

Lemma 2.1 *The optimum exists. At any optimum:*

1. *effort u is monotone (in t); it is non-increasing if and only if the deadline exceeds some finite length;*

2. *in addition, in the case of linear cost, the optimal strategy is extremal and maximum effort precedes zero effort if and only if $v > \alpha\lambda$;*

3. *if effort is non-increasing, so is the marginal product $p(\lambda + u)$; if it is non-decreasing, then the marginal product is single-peaked in the convex cost case, and piecewise decreasing with at most one upward jump in the linear cost case.*

Monotonicity of effort can be roughly understood as follows, in the linear cost case. There are two reasons why effort can be valuable: because it helps reduce the time over which the waiting cost v is incurred, and because it helps avoid paying the penalty k. The latter encourages late effort, the former early effort, provided the belief is high. But, in the absence of discounting, it makes little sense to work early if one plans on stopping before working eventually again: it is then better to postpone exerting this effort to this later stage where no effort is planned. Hence, if effort is exerted eventually, it is exerted only at the end. Conversely, if the penalty does not motivate late effort, effort is only exerted at the beginning.

Because the belief p is decreasing over time, note that the marginal product is decreasing whenever effort is decreasing, but the converse need not hold (as the product $p(\lambda + u)$ might vary in either direction). The interval over which the marginal product is non-decreasing can be empty, or the entire horizon. Conversely, it is straightforward to construct examples in which effort is increasing, and the marginal product is first increasing, then decreasing. Note that, for the critical deadline mentioned in the first part of the lemma, effort is constant.

With linear cost, whether effort is non-increasing or non-decreasing depends on the sign of $v - \alpha\lambda$ only. This does not contradict the first part of the lemma: for long enough deadlines, effort is constant (and 0) if $v \leq \alpha\lambda$, and first maximal then zero if $v > \alpha\lambda$. Note that neither the initial belief (p^0), nor the terminal cost (k) affect whether maximum effort is exerted first or last. Of course, they affect the total amount of effort, but given this amount, they do not affect its timing. The role of the sign $v - \alpha\lambda$ in the ordering of these intervals can be seen as follows: consider exerting some bit of effort now or at the next instant (thus, keeping the total amount of planned effort fixed); by waiting, a loss $v\mathrm{d}t$ is incurred; on the other hand, with probability $\lambda\mathrm{d}t$, the marginal cost of this effort, α, will be saved. Therefore, if $v > \alpha\lambda$, it is socially desirable to work early than late, if at all. We shall maintain this assumption from now on.

Assumption 2.2 *In the linear cost case, the parameters α, v and λ are such that*

$$v > \alpha\lambda.$$

Under this assumption, effort can be efficient even far from the deadline. An example of such a path is given by the left panel in Figure 2. The right panel gives the corresponding path for the value of output ($i.e.$, $p_t(\lambda + u_t)$).

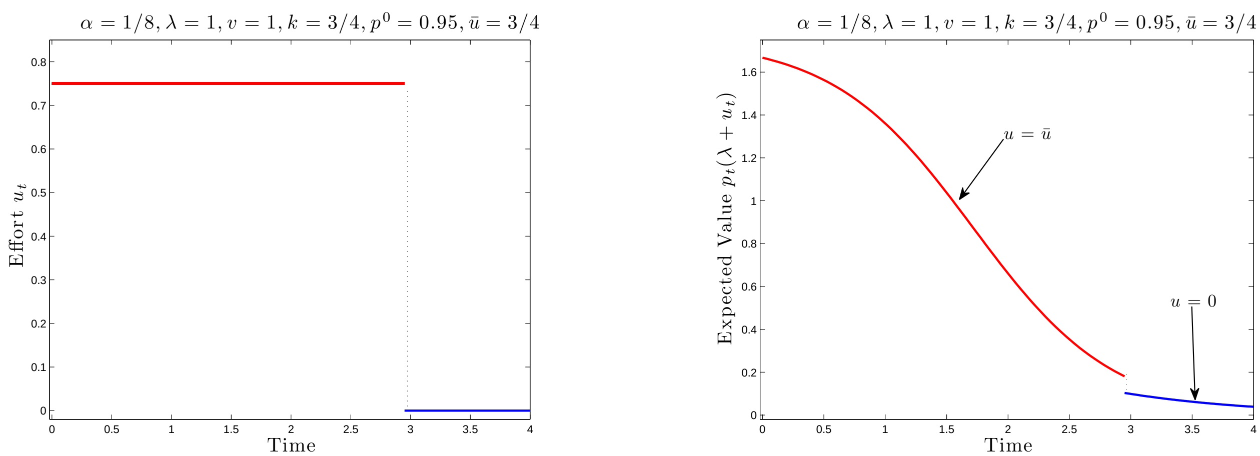

Figure 2: Effort and expected value at the social optimum

Whether effort is exerted at the deadline depends on how pessimistic the planner is at that point. By standard arguments (see Appendix A), full effort is exerted then if and only if

$$p_T\,(1 + k) \geq \alpha.^{20} \tag{4}$$

[20] In the convex case, the social planner exerts an effort level that solves $p_T\,(1 + k) = c'\,(u_T)$.

This states that the expected marginal social gains from effort (success and penalty avoidance) should exceed the marginal cost. If the social planner becomes too pessimistic, he "gives up" before the end. Note that the flow loss v no longer plays a role at that time, as the terminal (lump-sum) penalty overshadows any such flow cost.

It is straightforward to solve for the switching belief in the linear case. This belief decreases in α and increases in v and k: the higher the cost of failing, or the lower the cost of effort, the longer effort is exerted. More generally, we have the following result.

Lemma 2.3

1. *Both in the convex and linear cost case, the final belief decreases with the deadline.*
2. *Total effort exerted increases with the deadline*
   (a) *in the linear case, if and only if $\lambda(1+k) < v$;*
   (b) *in the convex case, if*
$$\max_u [(\lambda + u)(1 + k) - c(u)] < v.$$

Hence, total effort need not increase with the deadline; the sufficient condition given in the convex case (implying $\lambda(1+k) < v$) is not necessary; weaker, but less concise conditions can be given for the convex case, as well as examples in which total effort decreases with the deadline.

3 The role of wages

In this section, we take the wage path as entirely exogenous. This allows us to provide an analysis of reputational incentives that is not tied to any particular equilibrium notion.

Consider an arbitrary *exogenous* (integrable) wage path $w : [0, T] \to \mathbb{R}_+$. The agent's problem given by (2) differs from the social planner's in two respects: the agent disregards the expected value of a success (in particular, at the deadline), which increases with effort; and he takes into account future wages, which are less likely to be pocketed if more effort is exerted. We start with a technical result.

Lemma 3.1 *A solution to the maximization problem* (2) *exists. With convex cost, the optimal trajectory p is unique; with linear cost, if p_1 and p_2 are optimal trajectories, and $p_{1,t} \neq p_{2,t}$ over some interval $[a, b] \in [0, T]$, then $w_t = v - \alpha\lambda$ (a.e.) on $[a, b]$. Furthermore, $p_{1,T} = p_{2,T}$.*

That is, there is a unique solution (in terms of trajectories and hence control) in the convex-cost case, and multiplicity (both in terms of instantaneous effort and cumulative effort) in the linear-cost case is confined to time intervals over which the wage is equal to a particular constant. While this last case might appear non-generic, it plays an important role in the equilibrium analysis nonetheless.

3.1 Level of effort

What determines the instantaneous *level* of effort? Transversality implies that, at the deadline, the agent exerts an effort level that solves

$$p_T k = c'(u_T).^{21}$$

This is similar to the social planner's trade-off, except that the worker does not take into account the lump-sum value of success (compare with eqn. (4)). Hence, given p_T, his effort level is smaller.

It follows from Pontryagin's theorem that the amount of effort put in at time t solves

$$c'(u_t) = -\int_t^T (1-p_t)\frac{p_s}{1-p_s}\left[w_s - c(u_s) - v\right]\mathrm{d}s + (1-p_t)\frac{p_T}{1-p_T}k. \tag{5}$$

The left-hand side is the instantaneous marginal cost of effort. The marginal benefit (right-hand side) can be understood as follows. Conditioning throughout on reaching time t, the expected flow utility over some interval ds at time $s \in (t,T)$ is

$$\mathbb{P}\left[\tau \geq s\right](w_s - c(u_s) - v)\,\mathrm{d}s.$$

From (**??**), recall that

$$\mathbb{P}\left[\tau \geq s\right] = \frac{1-p_t}{1-p_s} = (1-p_t)\left(1+\frac{p_s}{1-p_s}\right);$$

that is, effort at time t affects the probability that time s is reached only through the likelihood ratio $p_s/(1-p_s)$. From (1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{p_t}{1-p_t} = -\frac{p_t}{1-p_t}(\lambda + u_t),$$

[21] In the linear case, this must be understood as: the agent chooses $u = \bar{u}$ if and only if $p_T k \geq \alpha$, and chooses $u = 0$ otherwise.

and so a slight increase in u_t decreases the likelihood ratio at time s precisely by $-p_s/(1-p_s)$. Combining, such an increase changes expected revenue from time s by an amount

$$-(1-p_t)\frac{p_s}{1-p_s}[w_s - c(u_s) - v]\,\mathrm{d}s,$$

and integrating over s (including $s = T$) yields the result.

The trade-off captured by eqn. (5) illustrates a key feature of career concerns in this model. Information is coarse: either a success is observed or not. This structure only allows the agent to affect the probability that the relationship terminates.[22] As is intuitive, increasing the wedge between the future rewards from success and failure $(v - w_s)$ encourages high effort, *ceteris paribus*. Higher wages in the future depress incentives to exert effort today, as they reduce this wedge. Given eqn. (5), it is straightforward to prove the following lemma, whose proof is omitted.

Lemma 3.2 *Consider the case of convex cost, and fix $T > 0$. Let w, w' be two wage paths, and denote by p, p' the corresponding beliefs. Then $w_t < w'_t$ for all $t \in [0, T]$ implies that $p_T < p'_T$.*

That is, if wages are higher throughout, total effort, as measured by terminal belief, is lower. However, because of the transversality condition, the effort paths must cross at some point. That is, increasing wages throughout depresses total effort, but it does not depress instantaneous effort at all times. In fact, instantaneous effort is eventually higher under the higher wage path.

Higher wages far in the future have a smaller effect on current-period incentives for two reasons, as is clear from eqn. (5). The relationship is less likely to last until then, and conditional on reaching these times, the agent's effort is less likely to be productive (as the probability of a high type then is very low).[23] Hence, as we shall see in Section 4.2, it is not true that pushing wages back, holding the total wage bill constant, necessarily depresses total effort.

Similarly, a higher penalty for termination or a lower cost of effort provide stronger incentives.

[22]This is a key difference with Holmström's model in which signals and posterior beliefs are one-to-one. Although the log-likelihood ratio is linear in effort, as is the principal's posterior belief in Holmström's model, here there is no scope for future wage to adjust linearly in output, so as to provide incentives that would be independent of the wage level itself. As we will see in Section 4, the level of future compensation does affect incentives to exert effort in equilibrium.

[23]Note also that, although learning is valuable, the value of information cannot be read off this first-order condition directly: the maximum principle is an "envelope theorem," and as such does not explicitly reflect how future behavior adjusts to current information.

3.2 Timing of effort

To understand how effort is allocated over time, let us differentiate eqn. (5). (See the proof of Proposition 3.3 for the formal argument.) We obtain:

$$p_t \cdot \underbrace{c\left(u_{t+\mathrm{d}t}\right)}_{\text{cost saved}} + \underbrace{p_t\left(v-w_t\right)}_{\text{wage premium}} + \underbrace{c''\left(u_t\right)\dot{u}_t}_{\text{cost smoothing}} = \underbrace{p_t\left(\lambda+u_t\right)}_{\text{Pr. of success at } t} \cdot c'\left(u_t\right). \tag{6}$$

The right-hand side captures the gains from shifting an effort increment $\mathrm{d}u$ from the time interval $[t, t+\mathrm{d}t)$ to $[t+\mathrm{d}t, t+2\mathrm{d}t)$ (*backloading*): the agent saves the marginal cost of this increment $c'(u_t)\mathrm{d}u$ with instantaneous probability $p_t(\lambda+u_t)\mathrm{d}t$ –the probability with which this additional effort will not have to be carried out. The left-hand side measures the gains from exerting this increment early instead (*frontloading*): the agent increases by $p_t\mathrm{d}u$ the probability that the cost of tomorrow's effort $c(u_{t+\mathrm{d}t})\mathrm{d}t$ is saved. He also increases at that rate the probability of getting the "premium" $(v-w_t)\mathrm{d}t$ an instant earlier. Last, if effort increases at time t, frontloading improves the workload balance, which is worth $c''(u)\mathrm{d}u\mathrm{d}t$. This yields the arbitrage eqn. (6).[24]

With linear cost, cost-smoothing is irrelevant, and because this is the only term that is not proportional to the belief p_t, the condition simplifies: frontloading effort is preferred if the wage premium exceeds the value of "luck" in cost units:

$$v - w_t \geq \alpha\lambda. \tag{7}$$

That the belief is irrelevant to the timing of effort (absent the cost-smoothing motive) is intuitive: if the state is 0, the cost of the effort increment is incurred either way, so that the comparison can be conditioned on the event that the state is 1.

Eqn. (7) is instructive about effort dynamics. First, note that, unless $w = v - \alpha\lambda$ holds identically over some interval, effort is extremal. Second, suppose that w is increasing. Then the left-hand side decreases over time, and the agent prefers frontloading up to some critical time, after which backloading becomes optimal (the critical time might be 0 or T). This does not imply that his effort is non-increasing; rather, if he puts in low effort, he must do so over some intermediate time interval. Similarly, if wages decrease over time, the agent first backloads, then frontloads effort. That is, if he ever puts in high effort, he will do so in some intermediate phase.

[24]Note that all these terms are "second order" terms. Indeed, to the first order, it does not matter whether effort is slightly higher over $[t, t+\mathrm{d}t)$ or $[t+\mathrm{d}t, t+2\mathrm{d}t)$. Similarly, while doing such a comparison, we can ignore the impact of the change on later revenues, as it is the same under both scenarios.

The same observations can be made by considering eqn. (6) for the convex case, though effort will not be extremal. The next proposition formalizes this discussion.

Proposition 3.3

1. *If w is decreasing, u is a quasi-concave function of time; if w is increasing, u is quasi-convex; if w is constant, u is monotone.*

2. *With linear cost and strictly monotone wages, the optimal strategy is extremal.*

To conclude, even when wages are monotone, the worker's incentives need not be so. Not surprisingly then, equilibrium wages, as determined in Section 4, will not be either.

3.3 Comparison with the social planner

Note that eqn. (7) reduces to the social planner's trade-off when $w_t = 0$ (see Lemma 2.1.2). Hence, the social planner's arbitrage condition coincides with the agent's if there were no wages. The same holds with convex cost, although the social planner internalizes the value of possible success at future times. This is because the corresponding term in eqn. (3) can be "integrated out,"

$$\int_0^T \frac{1-p_0}{1-p_t} p_t \left(\lambda + u_t\right) \mathrm{d}t = -(1-p_0) \int_0^T \frac{\dot{p}_t}{(1-p_t)^2} \mathrm{d}t = (1-p_0) \ln \frac{1-p_T}{1-p_0},$$

so that it only affects the final belief, and hence the transversality condition. But the agent's and the social planner's transversality conditions do not coincide, even when $w_s = 0$. As mentioned, the agent fails to take into account the value of a success at the last instant. Hence, his incentives at T, and hence his strategy for the entire horizon, differ from the social planner's. The agent works too little, too late.

The next proposition formalizes this discussion. Given w, denote by p^* the belief trajectory solving the agent's problem, and p^{FB} the corresponding trajectory for the social planner.

Proposition 3.4 *Consider the convex cost case, and fix $T > 0$ and $w > 0$.*

1. *The agent's aggregate effort is lower than the planner's, i.e., $p_T^* > p_T^{FB}$. Furthermore, instantaneous effort at any t is lower than the planner's, given the current belief p_t^*.*

2. *Suppose the planner's aggregate effort is constrained so that* $p_T = p_T^*$. *Then the planner's optimal trajectory* p *lies below the agent's trajectory, i.e., for all* $t \in (0, T)$, $p_t^* > p_t$.

The first part states that aggregate effort is too low, but also instantaneous effort, *given* the agent's belief. Nevertheless, as a function of calendar time, effort might be higher for the agent at some dates, because the agent is more optimistic than the social planner at that point. The next example (Figure 3) illustrates this phenomenon in the case of equilibrium wages.

The second part of the proposition states that, even fixing the aggregate effort, this effort is allocated too late relative to the first-best: the prospect of collecting future wages encourages "procrastination." The same is true in the linear case (although the inequality can be weak: if the agent's effort is maximum throughout, he is working just as much as the social planner).

4 Equilibrium

This section "closes" the model by considering alternative labor market arrangements. First, the case of short-term contracts; second, the case of long-term contracts. Finally, we examine short-term contracts when effort is observed.

4.1 Short-term contracts

Suppose now that the wage is set by a principal (or *market*) without commitment power. This is the type of contracts considered in the literature on career concerns. The principal does not observe the agent's past effort, only the lack of success. Non-commitment motivates the assumption that wage equals expected marginal product, *i.e.*,

$$w_t = \mathbb{E}_t[p_t(\lambda + u_t)],$$

where p_t and u_t are the agent's belief and effort, respectively, at time t, given his private history of past effort (as long as he has had no successes so far), and the expectation reflects the principal's beliefs regarding the agent's history (in case the agent mixes).[25] Given Lemma 3.1, the agent will not use a chattering control (*i.e.*, a distribution over measurable functions u), but rather a

[25] A lot is buried in this assumption. In discrete time, if $T < \infty$, and under assumptions that guarantee uniqueness of the equilibrium (see below), non-commitment *implies* that wage is equal to marginal product in equilibrium, by a backward induction argument, assuming that the agent and the principal share the same prior.

single function (unless the cost is linear and $w = v - \alpha\lambda$ over some interval, but even then the multiplicity is limited to the distribution of effort over this interval).[26] Therefore, we may write

$$w_t = \hat{p}_t(\lambda_t + \hat{u}_t), \tag{8}$$

where $\hat{p}_t$ and $\hat{u}_t$ denote the belief and anticipated effort at time t, as viewed from the principal.

In equilibrium, expected effort must coincide with actual effort.

Definition 4.1 *An equilibrium is a measurable function u and a wage path w such that:*

1. *u is a best-reply to w given the agent's private belief p, which he updates according to (1);*
2. *the wage equals the marginal product, i.e. (8) holds for all t;*
3. *beliefs are correct on the equilibrium path, that, is, for every t,*

$$\hat{u}_t = u_t,$$

and therefore, also, $\hat{p}_t = p_t$ at all $t \in [0, T]$.

Note that, if the agent deviates, the market will typically hold incorrect beliefs.

To understand the structure of equilibria, consider the following example, illustrated in Figure 3. Suppose that the principal expects the agent to put in the efficient amount of effort, which decreases over time in this example. Accordingly, the wage paid by the firm decreases as well. The agent's best-reply, then, is quasi-concave: effort first increases, and then decreases (see left panel). The agent puts in little effort at the start, as he has no incentive "to kill the golden goose." Once wages come down, effort becomes more attractive, so that the agent increases his effort, before fading out as pessimism sets in. The market's expectation does not bear out: marginal product is single-peaked. In fact, it would decrease at the beginning if effort was sufficiently flat.

Alternatively, this is the outcome if a sequence of short-run principals (at least two at every instant), whose information is symmetric and no worse than the agent's, compete through prices for the agent's services. We shall follow the literature by directly *assuming* that wage is equal to marginal product.

[26] If there are such time intervals (as equilibrium existence requires for many parameter values), the multiplicity of best-replies over this interval is of no importance: the expected effort at any time during this interval, as well as the aggregate effort over this interval will be uniquely determined, and the agent is indifferent over all effort levels over this time interval; the multiplicity does not affect wages, effort or beliefs before or after such an interval.

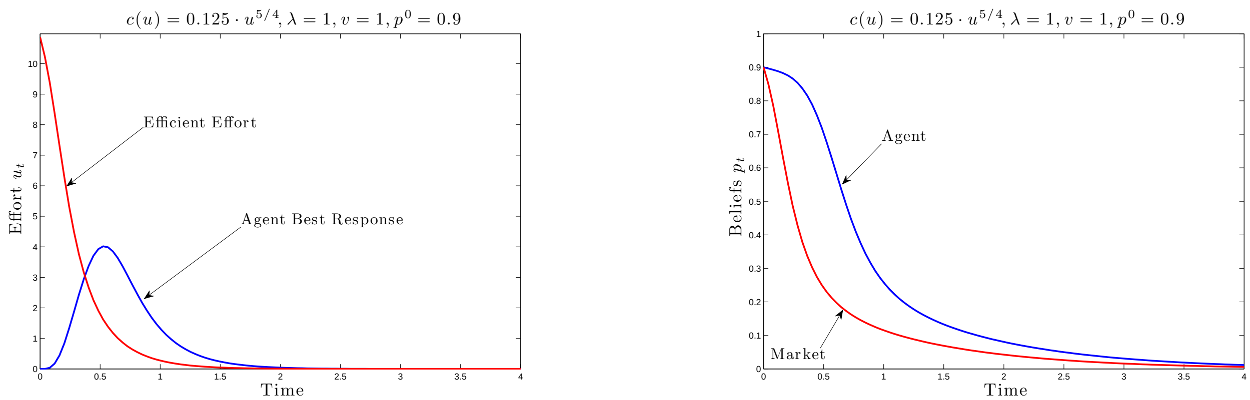

Figure 3: Agent's best-reply and beliefs to the efficient wage scheme

Eventually the agent exerts more effort than the social planner would. This is because the agent is more optimistic at those times, having worked less in the past (see right panel). Effort is always too low given the actual belief of the agent, but not necessarily given calendar time.

As this example makes clear, effort, let alone wage, is not monotone in general. However, it turns out that the equilibrium structure remains simple enough.

Theorem 4.2

1. *An equilibrium exists. It is unique in the linear case if $\alpha < k$, and in the convex case if*

$$c''(0) \geq \frac{1}{\lambda}\left(\frac{v}{\lambda} - k\right)\frac{p^0}{1-p^0}. \tag{9}$$

2. *In every equilibrium, (on path) effort is single-peaked, and the wage is non-decreasing in at most one interval. In the convex case, the wage is single-peaked.*

The proof is in Appendix C. A sketch for uniqueness is as follows: from Lemma 3.1, the agent's best-reply to any wage yields a unique path p; *given* the value of the belief p_T, we argue there is a unique path of effort and beliefs consistent with the equilibrium restriction on wages; we then look at the time it takes, along the equilibrium path, to drive beliefs from p^0 to p_T and show that it is strictly increasing. Thus, given T, there exists a unique value of p_T that can be reached in equilibrium. Condition (9) is sufficient to establish the last step in the convex case.[27]

[27] The uniqueness result contrasts with the multiplicity found in Dewatripont, Jewitt and Tirole. Although Holmström does not discuss uniqueness in his model, his model admits multiple equilibria.

Equilibrium wages are not single-peaked in general for the linear case, and single-peakedness in the convex case relies on our assumption that the marginal cost is convex (as does the uniqueness proof). Figure 4 illustrates that this is not true otherwise (note that the cost is convex, but not the marginal cost). The mode of the wage lies to the left of the mode of effort: if the wage is increasing over time, it must be that effort is increasing, but not conversely.

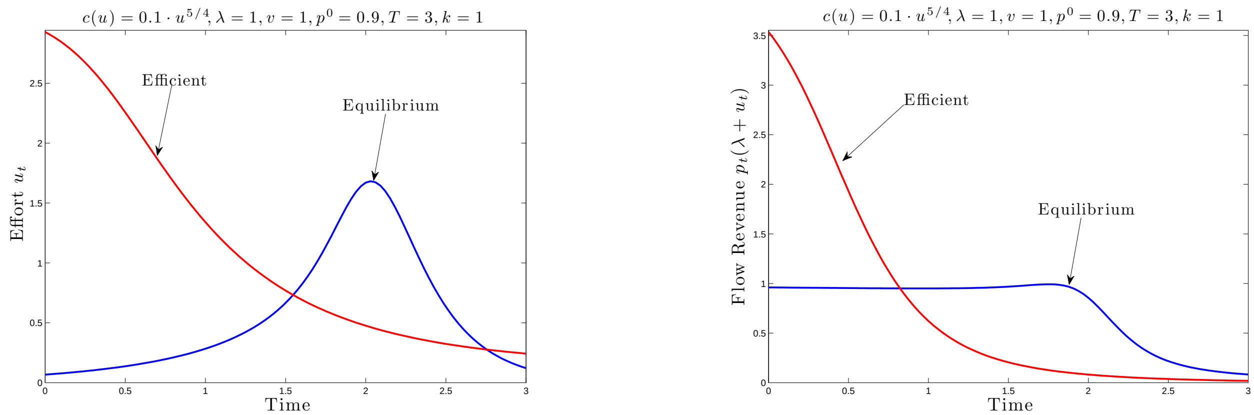

Figure 4: Effort and wages with convex costs

As stated, simple conditions guarantee uniqueness. For example, it obtains whenever the penalty k is large enough. We have been unable to construct any example of multiple equilibria.

How does the structure depend on parameters? Fixing all other parameters, if initial effort is increasing for some prior p^0, then it is increasing for higher priors –in fact, effort is increasing throughout if the prior were 1. Although effort can be decreasing for other motives (a long enough deadline, say), growing pessimism plays a role in turning increasing into decreasing effort.

Numerical simulations suggest that the payoff is single-peaked (with possibly interior mode) in p^0. This is a recurrent theme in the literature on reputation: uncertainty is the lever for reputational incentives. (Recall however that the payoff is net of the outside option, which is not independent of p^0; otherwise, it is increasing in p^0.)

A more precise description of the overall structure of the equilibrium can be given in the case of linear cost. Effort is first zero, then interior, then maximum, and finally 0. Therefore, in line with the results on convex cost, effort is single peaked, but wage is first decreasing before being single-peaked as well. Depending on parameters, any of these time intervals might be empty. The reader is referred to the appendix to a formal description (see Section 4, Proposition C.1).

Note that we have not specified the worker's equilibrium strategy entirely, as we have not described his behavior following his own (unobservable) deviations. Yet it is not difficult to describe the worker's optimal behavior off-path, as it is the solution of the optimization problem studied before, for the belief that results from the agent's history, given the wage path.

One might wonder whether the penalty k really hurts the worker. After all, it endows him with some commitment. In the linear cost case, simple algebra shows a higher k leads to higher amount of total effort; furthermore, if parameters are such that working at some point is optimal, then the optimal (*i.e.*, payoff-maximizing) termination penalty is strictly positive.

Finally, holding $\lambda + \bar{u}$ constant, we can interpret λ as a degree of contractibility of effort. Appending a linear cost to this effort, we can ask whether aggregate effort increases with the level of contractible effort. Numerically, it appears that the optimal choice is always extremal, but choosing a high λ can be counter-productive: forcing the worker to maintain a high effort level prevents him from scaling it back as it should be when the project appears to be unlikely to succeed. This lack of flexibility can be more costly than the benefits from direct oversight.

4.1.1 Discussion

The key driver behind the equilibrium structure, as described in Theorem 4.2, is the strategic substitutability between effort at different dates. If more effort is expected "tomorrow," wages tomorrow will be higher in equilibrium, which depresses incentives, and hence effort "today." There is substitutability between effort at different dates for the social planner as well, because higher effort tomorrow makes effort today less useful, but wages create an additional channel.

This substitutability appears to be new to the literature on career concerns. As we have mentioned, in the model of Holmström, the optimal choices of effort today and tomorrow are entirely independent, and because the variance of posterior beliefs is deterministic with Gaussian signals, the optimal choice of effort is deterministic as well. Dewatripont, Jewitt and Tirole emphasize the complementarity between expected effort and incentives for effort (at the same date): if the agent is expected to work hard, failure to achieve a high signal will be particularly detrimental to tomorrow's reputation, which provides a boost to incentives today. Substitutability between effort today and tomorrow does not appear in their model, because it is primarily focused on two periods, and at least three are required for this effect to appear. With two periods only, there are no reputation-based incentives to exert effort in the second (and final) period anyhow.[28]

Conversely, complementarity between expected and actual effort at a given time is not discernible in our model, because time is continuous. But this complementarity appears in discrete time versions of it, and three-period examples can be constructed that illustrate this point.

As a result of this novel effect, effort and wage dynamics display original features. Both in Holmström's and in Dewatripont, Jewitt and Tirole's models, the wage is a supermartingale. Here instead, effort can be first increasing, then decreasing, and wages can be decreasing first, increasing then, and decreasing again. These dynamics are not driven by the deadline.[29] They are not driven either by the fact that, with two types, the variance of the public belief need not be monotone.[30] The same pattern emerges in examples with an infinite horizon, and a prior $p^0 < 1/2$ that guarantees that this variance only decreases over time, see Figure 5. As equation (5) makes clear, the provision of effort is tied to the capital gain that the agent obtains if he breaks through. Viewed as an integral, this capital gain is too low early on, it increases over time, and then declines again, for a completely different reason. Indeed, this wedge depends on two components: the wage gap, and the impact of effort on the (expected) arrival rate of a success. Therefore, high initial wages would depress the first component, and hence kill incentives to exert effort early on. The latter component declines over time, so that eventually effort fades out.

Similarly, one might wonder whether the possibility of non-increasing wages in this model is driven by the fact that effort and wage paths are truly conditional paths, inasmuch as they assume that the agent has not succeeded. Yet it is not hard to provide numerical examples which illustrate that the same phenomenon arises for the unconditional flow payoff (v in case of a past success), though the increasing cumulative probability that a success has occurred by a given time, leading to higher payoffs (at least if $w_t < v$) dampens the downward tendency.

We have assumed –as is usually done in the literature– that the agent does not know his own skill. The analysis of the game in which the agent is informed is simple, as there is no scope for signaling. An agent who knows that his ability is low has no reason to exert any effort, so

[28]It is worth noting that this substitutability does not require the multiplicative structure that we have assumed. If instead, we had posited that instantaneous success probability is given by $\lambda \chi_{\omega=1} + u_t$, effort would be similarly single-peaked, as is readily verified.

[29]This is unlike for the social planner, for which we have seen that effort is non-increasing with an infinite horizon, while it is monotone (and possibly increasing) with a finite horizon.

[30]Recall that, in Holmström's model, this variance decreases (deterministically) over time, which plays an important role in his results.

we focus on the high-skilled agent. Because of the market's declining belief, the same dynamics arise, and this agent's effort is single-peaked (in general, it is not monotone). The high-skilled agent's effort need not vanish, though expected effort from the market's point of view does so.

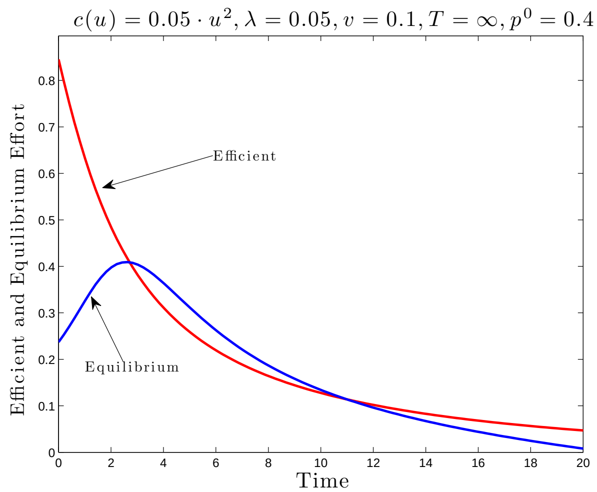

Figure 5: The same pattern in the case of $T = \infty$, $p^0 < 1/2$

Dynamics of the complexity just described are not often observed in practice: while it is difficult to ascertain effort patterns, wages do typically go up over time. See Abowd, Kramarz and Margolis (1999), Murphy (1986) and Topel (1991) among others, and Hart and Holmström (1987) and Lazear and Gibbs (2007) for surveys. Lazear (1981) obtains a positive impact of wages on seniority by (among others) assuming that the worker's outside option is increasing over time, and also derives the optimal deadline, or retirement age (Lazear, 1979). Nevertheless, it is not hard to find examples in which compensation goes down with repeated failure. For instance, the income of junior analysts is proportional to the number of job assignments they are being given, so that struggling analysts end up earning lower wages. Similar patterns arise in the art and entertainment industries.

4.1.2 Endogenous deadlines

As a minimal departure from the competitive benchmark that has been considered, we endogenize the deadline: the worker decides when to quit the profession. We assume (for now) that he has no commitment power. The principal anticipates the quitting decision, and takes this into account while determining the agent's equilibrium effort, and therefore, the wage he should be paid.

More specifically, in each interval $[t, t+\mathrm{d}t)$ such that the agent has not quit yet, the principal pays a wage $w_t\mathrm{d}t$, then the agent decides how much effort to exert over this time interval, and at the end of it, whether to stay or leave, an observable choice. This raises the issue of the principal's beliefs if the agent were to deviate and stay when expected to leave. For simplicity, we adopt passive beliefs. That is, if the agent is supposed to drop out at some time but fails to, the principal does not revise his belief regarding past effort choices, ascribing the failure to quit to a mistake (this implies that he expects the agent to quit at the next opportunity).[31]

Endogenous deadlines do not affect the pattern of effort and wage. With convex cost, effort is *always* decreasing at the deadline (*i.e.*, at the agent's optimal quitting time). This implies that the wage is decreasing at the end (but not necessarily at the beginning). Hence, effort is single-peaked and wages are first decreasing and then single-peaked. The belief at the deadline is too high relative to the social planner's at the first-best deadline. Furthermore, both effort and the worker's marginal product are decreasing throughout in the first-best solution.

How about if the worker could commit to the deadline (but still not to effort levels)? The optimal deadline with commitment can be either shorter or longer than without. In either case, however, the deadline is set so as to increase aggregate effort, and so increase wages. This may require increasing the deadline –so as to increase the duration over which higher effort levels are sustained, even if that means quitting at a point where staying is unprofitable– or decreasing the deadline –so as to make high effort levels credible. Figure 6 illustrates the two possibilities.

We summarize our results in Proposition 4.3, the proof of which can be found in the working paper.

Proposition 4.3 *With convex cost,*

1. *effort is always decreasing at the optimal deadline without commitment;*
2. *the belief of the planner at the deadline is lower than the agent's at the optimal deadline without commitment;*
3. *the deadline with commitment can be shorter or longer than without.*

[31]In the linear cost case, this means that we fix the off-equilibrium beliefs to specify $\hat{u}_t = \bar{u}$ if $p_t > p^*$, where p^* is the lowest belief at which it would be optimal for the agent to exert maximum effort if he anticipated quitting at the end of the interval $[t, t+\mathrm{d}t)$ (see Appendix B for p^* in closed-form), and $\hat{u}_t = 0$ otherwise. In other words, the market does not react to a failure to quit, anticipates the agent quitting immediately afterwards and expects instantaneous effort to be determined as if $p = p_T$ were the terminal belief.

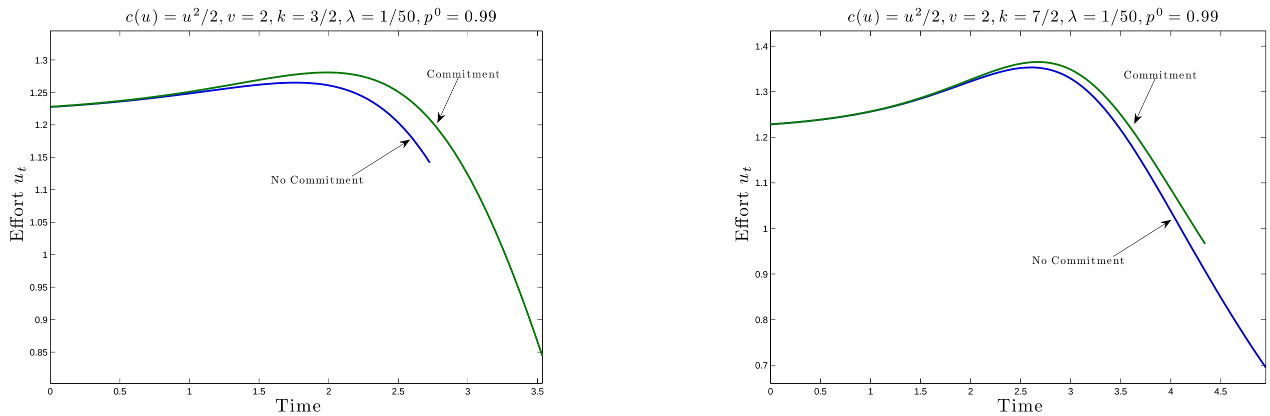

Figure 6: Setting the deadline with commitment can push it higher or lower than without (the curves stop at the respective deadlines).

4.2 Long-term contracts

Under spot contracts, the wage is equal to the worker's marginal product. This is a reasonable premise in a number of industries, in which lack of transparency or volatility in the firm's revenue stream might inhibit commitment by the firm to a particular wage scheme. Alternatively, it is sometimes argued that competition for the agent's services leads to a similar outcome. Our model does not substantiate such a claim: if the principal can commit to a wage path, matters change drastically, even under competition.

In particular, if the principal could commit to a breakthrough-contingent wage scheme, the moral hazard problem would be solved entirely: under competition, the principal would offer the agent the value of a breakthrough, 1, whenever a success occurs, and nothing otherwise.

If at least the principal could commit to a time-contingent wage scheme that involved payments after a breakthrough (with payments possibly depending on the agent staying with the firm, but not on the realization of output), the moral hazard would also be mitigated. If promised payments at time t in the case of no breakthrough are also made if a breakthrough has occurred, all disincentives due to wages are eliminated.

Here, we examine a weaker form of commitment. The agent cannot be forced to stay with a principal (he can leave at any time). Once a breakthrough occurs, the agent moves on (*e.g.*, to a different industry or position), and the firm is unable to retain him in this event. The principal can commit to a wage path that is conditional on the agent working for her firm. Thus, wages can only be paid in the continued absence of a breakthrough. Until a breakthrough occurs, other firms, who are symmetrically informed (they observe the wages paid by all past employers), compete by offering wage paths. The same deadline applies to all wage paths, *i.e.* the tenure clock is not reset. For instance, the deadline could represent the agent's retirement age, so that switching firms does not affect the horizon.

For the remainder of this subsection, we restrict attention to the linear cost case. We write the principal's problem as of maximizing the agent's welfare subject to constraints. Formally, we solve the following optimization problem $\mathcal{P}$.[32] The principal chooses $u : [0, T] \to [0, \bar{u}]$ and $w : [0, T] \to \mathbb{R}_+$, integrable, to maximize $W(0, p^0)$, where, for any $t \in [0, T]$,

$$W(t, p_t) := \max_{w,u} \int_t^T \frac{1 - p_t}{1 - p_s} (w_s - v - \alpha u_s) \, \mathrm{d}s - k \frac{1 - p_t}{1 - p_T},$$

such that, given w, the agent's effort is optimal,

$$u = \arg\max_u \int_t^T \frac{1 - p_t}{1 - p_s} (w_s - v - \alpha u_s) \, \mathrm{d}s - k \frac{1 - p_t}{1 - p_T},$$

and the principal offers as much to the agent at later times than the competition could offer at best, given the equilibrium belief,

$$\forall \tau \geq t : \int_\tau^T \frac{1 - p_\tau}{1 - p_s} (w_s - v - \alpha u_s) \, \mathrm{d}s - k \frac{1 - p_\tau}{1 - p_T} \geq W(\tau, p_\tau) ; \tag{10}$$

finally, the firm's profit must be non-negative,

$$0 \leq \int_t^T \frac{1 - p_t}{1 - p_s} (p_s(\lambda + u_s) - w_s) \mathrm{d}s.$$

Note that competing principals are subject to the same constraints as the principal under consideration: because the agent might ultimately leave them as well, they can offer no better than $W(\tau, p_\tau)$ at time τ, given belief p_τ. This leads to an "infinite regress" of constraints, with the value function appearing in the constraints themselves. To be clear, $W(\tau, p_\tau)$ is not the continuation payoff that results from the optimization problem, but the value of the optimization

[32] We are not claiming that this optimization problem yields the equilibrium of a formal game, in which the agent could deviate in his effort scheme, leave the firm, and competing firms would have to form beliefs about the agent's past effort choices, etc. Given the well-known modeling difficulties that continuous time raises, we view this merely as a convenient shortcut. Among the assumptions that it encapsulates, note that there is no updating based on an off-path action (*e.g.*, switching principals) by the agent.

problem if it started at time τ.[33] Because of the constraints, the solution is not time-consistent, and dynamic programming is of little help. Fortunately, this problem can be solved, as shown in Appendix C.2–at least as long as $\bar{u}$ and v are large enough. Formally, we assume that

$$\bar{u} \geq \left(\frac{v}{\alpha\lambda} - 1\right) v - \lambda, \text{ and } v \geq \lambda(1+k).^{34} \tag{11}$$

Before describing its solution, let us provide some intuition. Recall the first-order condition (5) that determines the agent's effort. Clearly, the lower the future total wage bill, the stronger the agent's incentives to exert effort, which is inefficiently low in general. Now consider two times $t < t'$: to provide strong incentives at time t', it is best to frontload any promised payment to times before t', as such payments will no longer matter at that time. Ideally, the principal would pay what he owes upfront, as a "signing bonus." However, this violates the constraint (10), as an agent left with no future payments would leave right after cashing in the signing bonus.

But from the perspective of incentives at time t, backloading promised payments is better. To see this formally, note that the coefficient of the wage w_s, $s > t$, in eqn. (5) is (up to the factor $(1-p_t)$) the likelihood ratio $p_s/(1-p_s)$, as explained before eqn. (5). Alternatively,

$$(1-p_t)\frac{p_s}{1-p_s} = \mathbb{P}\left[\omega = 1 | \tau \geq s\right] \mathbb{P}\left[\omega = 1\right] = \mathbb{P}\left[\omega = 1 \cap \tau \geq s\right];$$

that is, effort at time t is affected by wage at time $s > t$ inasmuch as time s is reached and the state is 1: otherwise effort plays no role anyhow.

In terms of the firm's profit (or the agent's payoff), the coefficient placed on the wage at time s (see (2)) is

$$\mathbb{P}\left[\tau \geq s\right],$$

i.e., the probability that this wage is paid (or collected). Because players grow more pessimistic over time, the former coefficient decreases faster than the latter: backloading payments is good for incentives at time t. Of course, to provide incentives with later payments, those must be increased, as a breakthrough might occur until then, which would void them; but it also decreases the probability that these payments must be made in the same proportion. Thus, what matters is not the probability that time s is reached, but the fact that reaching those later times is

[33] Harris and Holmström (1982) impose a similar condition in a model of wage dynamics under incomplete information. However, because their model abstracts from moral hazard, constraint (10) reduces to a non-positive continuation profit condition.

[34] We do not know whether these assumptions are necessary for the result.

indicative of state 0, which is less relevant for incentives. Hence, later payments depress current incentives less than earlier payments.

To sum up: from the perspective of time t, backloading payments is useful; from the point of view of $t' > t$, it is detrimental, but frontloading is constrained by (10). Note that, as $T \to \infty$, the planner's solution tends to the agent's best response to a wage of $w = 0$. Hence, the firm can approach first best by promising a one-time payment arbitrarily far in the future (and wages equal to marginal product thereafter). This would be almost as if $w = 0$ for the agent's incentives, and induce efficient effort. The lump sum payment would then be essentially equal to $p^0/(1-p^0)$.

Note finally that, given the focus on linear cost, there is no benefit in giving the agent any "slack" in his incentive constraint at time t; otherwise, by frontloading slightly future payments, incentives at time t would not be affected, while incentives at later times would be enhanced. Hence, the following result should come as no surprise.

Theorem 4.4 *The following is a solution to the optimization problem $\mathcal{P}$, for some $t \in [0, T]$. Maximum effort is exerted up to time t, and zero effort is exerted afterwards. The wage is equal to $v - \alpha\lambda$ up to time t, so that the agent is indifferent between all levels of effort up to then, and it is 0 for all times $s \in (t, T)$; a lump-sum is paid at time T.*[35]

The proof is in Appendix C.2, and it involves several steps: we first conjecture a solution in which effort is first full (and the agent is indifferent), then nil; we relax the objective in program $\mathcal{P}$ to maximization of aggregate effort, and constraint (10) to a non-positive continuation profit constraint; we verify that our conjecture solves the relaxed program, and finally that it also solves the original program. In the last step we show that (a) given the shape of our solution, maximizing total effort implies maximizing the agent's payoff, and (b) the competition constraint (10) is slack at all times $t > 0$.

Hence, under one-sided commitment, high effort might be exerted throughout. This happens if T is short and $k > 0$. When $\bar{u}$ is high enough (precisely, when (11) holds), the agent produces revenue that exceeds the flow wage collected as time proceeds: the liability recorded by the principal grows over time, shielding it from the threat of competition. This liability will eventually be settled via a lump-sum payment at time T that can be interpreted as severance pay. If the horizon is longer or $k = 0$, the lump-sum wipes out incentives close to the deadline, as in our

[35] The wage path that solves the problem is not unique in general.

introductory example, and effort is zero in a terminal phase. Thus, a phase with no effort exists if and only if the deadline is long enough. The two cases are illustrated in Figure 7 below.

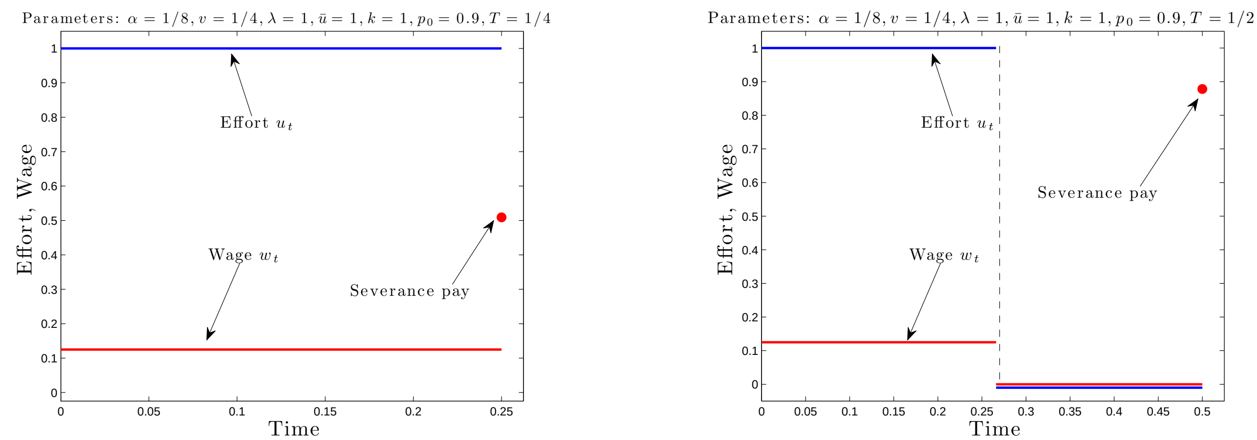

Figure 7: Wages and effort under commitment, for two horizon lengths.

As mentioned, all these results are proved for the case of linear cost. It is all the more striking that the result obtains in this "linear" environment. Indeed, rigidity and severance pay are usually attributed to risk aversion by the agent (see Azariadis, 1975, among many others). Commitment provides an alternative rationale.

Nonetheless, it is desirable to know to which extent the findings are valid for general cost specification. Our earlier numerical example with quadratic costs (Figure 1 in the introduction) provides some important hints. In general, the equilibrium involves two phases: in a first phase, the agent is paid his marginal product; in a second phase, he is not paid at all, and the excess product of his effort is backloaded into a final lump payment. If the horizon is long enough, the phase in which wages are equal to marginal product disappears and all payments are backloaded. If the horizon is very short and k is low enough, the backloading phase disappears, and the structure is then identical to spot contracts: the agent is paid his marginal product throughout.

The principal faces a simple choice: paying as soon as possible, which requires setting the wage equal to the marginal product to satisfy the no-poaching constraint; or backloading entirely. Early payments are bad for current incentives, but best for later incentives because these payments will be sunk by then; backloading is best for current incentives, but depresses later incentives. Depending on the horizon length, one or the other solution is chosen. If the horizon is very long, backloading takes over: by the time the deadline is near, the principal already owes the worker a large enough amount that the competition constraint is ineffective even though the large payment promised at the deadline eliminates the incentives to work close to T.

However, full backloading is not the solution if the relationship begins close to the deadline: with a very short horizon, the imminent lump-sum payment reduces the agent's incentives to work, and hence the marginal product. Then there is not enough time for the principal to accumulate a liability sufficient to make his contract immune to poaching. The principal can then do no better than offer a contract replicating spot wages, which is immune to itself.[36]

To sum up, with a long horizon, commitment alleviates the unavoidable delay in effort provision that comes with compensating a worker for persistent failure by backloading compensation.

5 Robustness

Undoubtedly, our model has stylized features. Output is not observed at all; all uncertainty is resolved after one breakthrough, there is no learning-by-doing, and the quality of the project cannot change over time. We start discussing the case in which effort is observable. Predictably, this changes conclusions significantly. We then briefly consider three variations (multiple breakthroughs, learning-by-doing, changing types), each of which embeds our baseline model as a special case. We argue that none of these features is critical to our main findings.

5.1 Observable effort

The inability to contract on effort might be attributable to the subjectivity in its measurement rather than to the impossibility of monitoring it. Co-workers usually have a fairly good idea of their colleagues' work ethic. The evaluation of a worker's immediate superior affects his future compensation. To understand incentives in such an environment, we assume here that effort is observed. We revert to spot contracts: the firm pays upfront the value of output. Because effort is monitored, the firm and agent beliefs coincide on and off path. The flow wage is given by

$$w_t = p_t(\lambda + \hat{u}_t),$$

[36]The only difference with the linear case is that, with linear cost, a constant flow wage $w = v - \alpha\lambda$ induces the agent to exert maximal effort and generates positive flow profits for the firm. Thus, there is always some excess marginal product that gets backloaded. This allows the principal to accumulate sufficient liabilities even with short horizons. If, in addition, k is sufficiently large, the agent can be induced to exert maximal effort throughout.

where p_t is the belief and $\hat{u}_t$ is expected effort. We assume linear cost. The agent maximizes

$$V(p_0, 0) := \int_0^T \frac{1-p_0}{1-p_t} \left[p_t(\lambda + \hat{u}_t) - \alpha u_t - v\right] \mathrm{d}t - k\frac{1-p_0}{1-p_T}.$$

In contrast to (2), the revenue is no longer a function of time only, as effort affects future beliefs, thus wages. Hence, effort is a function of t and p. We focus on equilibria in Markov strategies

$$u : [0,1] \times [0,T] \to [0,\bar{u}],$$

such that $u(p,t)$ is upper semi-continuous and the value function $V(p,t)$ piecewise differentiable.[37]

Lemma 5.1 *Fix a Markov equilibrium. If $u = 0$ on some open set $\Omega \subset [0,1] \times [0,T]$, then also $u(p',t') = 0$ if the equilibrium trajectory that starts at (p',t') intersects Ω.*

Hence, if the agent ever exerts effort, he is willing to do so at any later time (on path). Thus, in any equilibrium involving extremal effort levels only, there are at most two phases: the worker exerts no effort, and then full effort. This is the opposite of the socially optimal policy, which frontloads effort (see Lemma 2.1). The agent can only be trusted to put in effort if he is "back to the wall," so that effort remains optimal at any later time, no matter what he does; if the market paid for effort, yet the agent was expected to let up later on, then he would gain by deviating to no effort, pocketing the high wage in the process; because such a deviation makes everyone more optimistic, it would only increase his incentives to exert effort (and so his wage) at later times.

This does not imply that the equilibrium is unique, as the next theorem establishes.

Theorem 5.2 *Given $T > 0$, there exists continuous, distinct, non-increasing $\underline{p}, \bar{p} : [0,T] \to [0,1]$, with $\underline{p}_t \le \bar{p}_t$ and $\underline{p}_T = \bar{p}_T$, such that:*

1. *all Markov equilibria involve maximum effort above $\bar{p}$:*

$$p_t > \bar{p}_t \Rightarrow u(p,t) = \bar{u};$$

2. *all Markov equilibria involve no effort below $\underline{p}$:*

$$p_t \le \underline{p}_t \Rightarrow u(p,t) = 0;$$

[37]That is, there exists a partition of $[0,1] \times [0,T]$ into closed S_i with non-empty interior, such that V is differentiable on the interior of S_i, and the intersection of any S_i, S_j is empty or a smooth 1-dimensional manifold.

3. *these bounds are tight: there exists a Markov equilibrium $\underline{\sigma}$ (resp. $\bar{\sigma}$) in which effort is either* 0 *or* $\bar{u}$ *if and only if* p *is below or above* $\underline{p}$ *(resp.* $\bar{p}$*).*

The proof of Theorem 5.2 provides a description of these belief boundaries. These boundaries might be as high as one, in which case effort is never exerted at that time: indeed, there is t^* (independent of T) such that effort is zero at all times $t < T - t^*$ (if $T > t^*$). The threshold $\underline{p}$ is decreasing in the cost α, and increasing in v and k. Considering the equilibrium with maximum effort, the agent works more, the more desirable success is.[38] Figure 8 illustrates these dynamics.

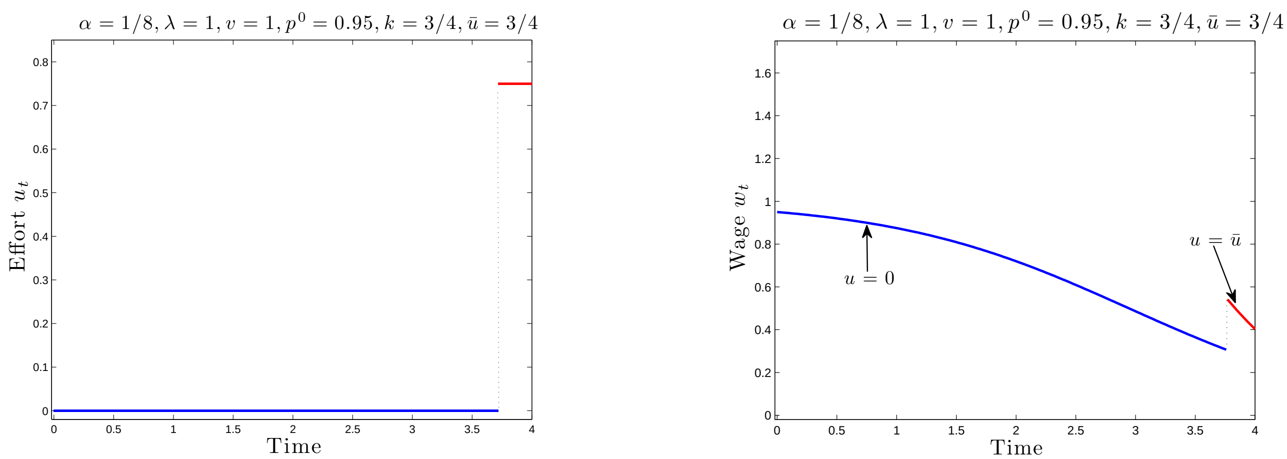

Figure 8: Effort and wages in the observable case

In any extremal equilibrium, wages decrease over time, except for an upward jump when effort jumps up to $\bar{u}$. In the interior-effort equilibrium described in the proof (in which effort is continuous throughout), wages decrease throughout. Further comparative statics are in appendix.

Equilibrium multiplicity has a simple explanation. Because the firm expects effort only if the belief is high and the deadline is close, such states (belief and times) are desirable for the agent, as the higher wage more than outweighs the effort cost. Yet low effort is the best way to reach those states, as effort depresses beliefs: hence, if the firm expects the agent to shirk until a high boundary is reached (in (p, t)-space), the agent has strong incentives to shirk to reach it; if the firm expects shirking until an even higher boundary, this would only reinforce this incentive.

[38] While $\underline{\sigma}$ and $\bar{\sigma}$ provide upper and lower bounds on equilibrium effort (in the sense of (1)–(2)), these equilibria are not the only ones. Other equilibria exist that involve only extremal effort, with switching boundary in between $\underline{p}$ and $\bar{p}$; there are also equilibria in which interior effort levels are exerted at some states.

Let us turn to a comparison with the case of non-observable effort, as described in subsection 4.1. Along the equilibrium path, the dynamics of effort look very different when one compares the social planner's solution to the agent's optimum under either observability assumption. Yet effort can be ranked across those cases. To do so, the key is to describe effort in terms of the state (p, t), *i.e.*, the public belief and calendar time. In all equilibria, effort is lower under observability.

Proposition 5.3 *The maximum effort region for the observable case is contained in the full effort region(s) for the non-observable case.*

This confirms that observability depresses incentives: the highest effort equilibrium with observability entails less effort than without. Recall from Proposition 3.4 that the (interior or full) effort region in the non-observable case is itself contained in the effort region for the social planner.

Having the worker quit when it is best for him (without commitment to the deadline) reinforces our comparison between observable and non-observable effort.[39]

5.2 Multiple breakthroughs

Suppose that one success does not resolve all uncertainty. Specifically, there are three states, $\omega = 0, 1, 2$, two consecutive projects, and an infinite horizon. The first one can be completed if and only if the state is not 0; assume arrival rates of $\lambda_1 + u_t$ and $\lambda_2 + u_t$, respectively, conditional on $\omega = 1, 2$; if the first project is completed, an observable event, the agent tackles the second, which can be completed only if $\omega = 2$; assume again an arrival rate of $\lambda_2 + u_t$ if $\omega = 2$.

Such an extension can be solved by "backward induction." Once the first project is completed, the continuation game reduces to the game of Section 4. The value function of this problem then serves as continuation payoff to the first stage. While this value function cannot be solved in closed-form, it is easy to derive the solution numerically. The following example illustrates the structure of the solution. The parameters are $v = 1, \alpha = 1/2, \mathbb{P}[\omega > 0] = 0.85, \mathbb{P}[\omega = 2 \mid \omega > 0] = 0.75, \lambda_2 = 1, \lambda_1 = 0.6, c(u) = u^2/8$. See Figure 9. The left panel shows effort and wages during the first stage. As is clear, the same pattern as in our model emerges: effort is single-peaked, and as a result, wages can be first decreasing, then single-peaked.

[39] See the working paper for details. Non-Markov equilibria exist. Defining them in our environment is problematic, but it is clear that threatening the agent with reversion to the Markov equilibrium $\bar{\sigma}$ provides incentives for effort extending beyond the high-effort region defined by $\underline{\sigma}$ –in fact, beyond the high-effort region in the unobservable case. The planner's solution remains out of reach, as punishments are restricted to beliefs below $\underline{p}$.

The right panel shows how efforts and beliefs evolve before and after the first success. The green curves represent the equilibrium belief that $\omega = 2$, before and after the success (the light green curve is the belief as long as no success has occurred, and the dark green one the belief right after a success has occurred); the blue curves are equilibrium effort (the light blue curve is effort as long as no success has occurred, the dark blue one is the effort right after a success). Note that effort at the start of the second project is also single-peaked as a function of the time at which this project is started (the later it is started, the more pessimistic the agent at that stage, though his belief has obviously jumped up given the success).

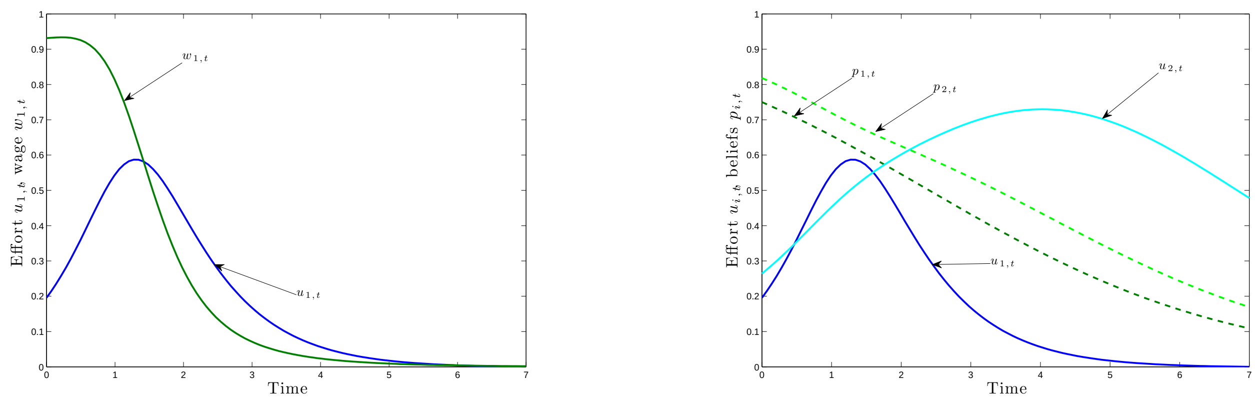

Figure 9: Efforts and beliefs with two breakthroughs

5.3 Learning-by-doing

Memorylessness is a convenient but stark property of the exponential distribution. It implies that past effort plays no role in the probability of instantaneous breakthrough. In many applications, agents learn from the past not only about their skill levels, but about the best way to achieve a breakthrough. While a systematic analysis of learning-by-doing is beyond the scope of this paper, we can gain some intuition from numerical simulations. We model human capital accumulation as in Doraszelski (2003). The evolution of human capital is given by

$$\dot{z}_t = u_t - \delta z_t, \quad z_0 = 0,$$

while its productivity is

$$h_t = u_t + \rho z_t^{\phi}.$$

That is, the probability of success over the interval $[t, t + \mathrm{d}t)$ is $(\lambda + h_t)\mathrm{d}t$, given human capital h_t and effort u_t. Here δ, ρ and ϕ are positive constants that measure how fast human capital depreciates, its importance relative to instantaneous effort, and the returns to scale from human capital.[40]

Figure 10 illustrates the results. Not surprisingly, the main new feature is a spike of effort at the beginning, whose purpose is to build human capital. This spike might lead to decreasing initial effort, before it becomes single-peaked, though this need not be the case. Beyond this new twist, features from the baseline model appear quite robust.

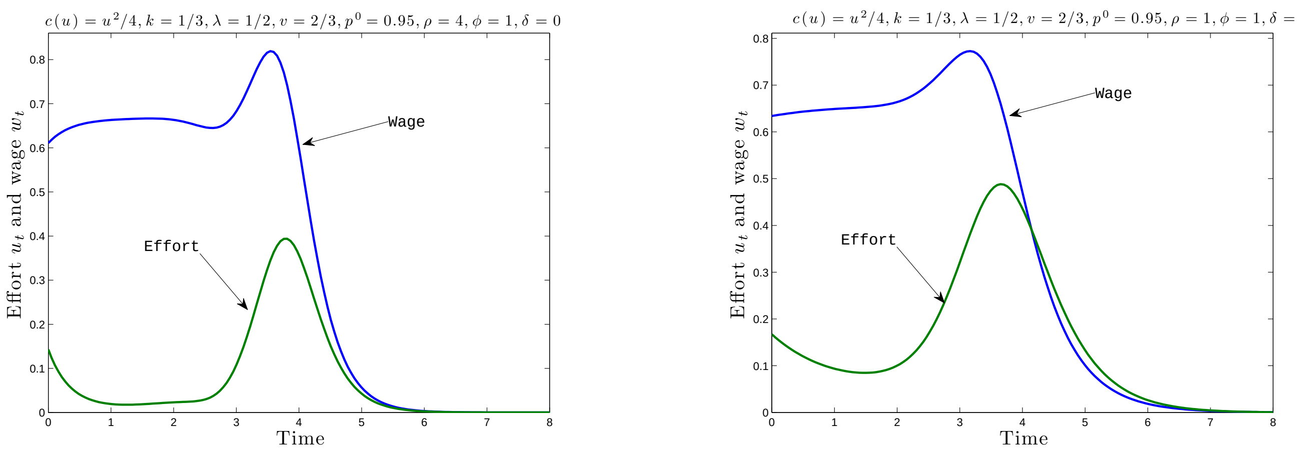

Figure 10: Two possible configurations with learning-by-doing

5.4 Changing state

Suppose finally that, unbeknownst to the agent and the principal, the state of the world is reset at random times, exponentially distributed at rate $\rho > 0$; whenever it is reset, the state is reset to 1 with probability $p^* \in (0, 1)$.[41] This is the counterpart of the stationary version of Holmström for Gaussian noise. Specifically, suppose that with instantaneous probability $\rho > 0$ the ability is

[40] Cisternas (2012b) introduces human-capital accumulation in the Holmström framework by letting the agent's skill follow a diffusion with an endogenous drift component.

[41] This specification bears a close similarity to Board and Meyer-ter-Vehn (2010, 2013), though it also differs in some key respects; among others, the state-resetting process is exogenous and information is symmetric (on path).

reset, in which case it is high with probability p^*. Such an event is unobserved by all parties. As before, a breakthrough ends the game, and the environment remains the same as before, with linear cost (and $v > \alpha\lambda$, as before) and an infinite horizon. (Thus, the baseline model with linear cost and $T = \infty$ is a special case in which $\rho = 0$.) By Bayes' rule, the (agent's) belief p obeys

$$\dot{p}_t = \rho\,(p^* - p_t) - p_t\,(1 - p_t)\,(\lambda + u_t)\,, \quad p_0 = p^0,$$

and this is the same as the principal's belief in equilibrium. The equilibrium is unique, and effort and belief converge to some limiting value, which is independent of the prior, and decreasing in the eventual effort level u, as follows. (The proof of the following is available from the authors.)

Proposition 5.4 *There exists $\alpha\lambda < \underline{v} < \bar{v}$ and $0 < \underline{p} < \bar{p} < p^*$ such that, if:*

1. *$v > \bar{v}$, effort is eventually maximum, and p tends to a limit below $\underline{p}$;*
2. *$v \in (\underline{v}, \bar{v})$, effort is eventually interior, with p tending to a limit in $(\underline{p}, \bar{p})$;*
3. *$v < \underline{v}$, effort is eventually zero, and p tends to a limit above $\bar{p}$.*

The higher the value, the more effort is exerted, the lower is the asymptotic belief. This eventual belief is non-decreasing in p^* and ρ and non-increasing in $\bar{u}$. It is decreasing in λ when effort is extremal, but increasing otherwise. Asymptotic effort (or stationary effort if $p^0 = p^*$) is decreasing in α, the marginal cost of effort, and in λ, the "luck" component of the arrival of breakthroughs. Comparative statics of effort with respect to p^* and ρ are ambiguous.

Note that, if the prior belief is below the limiting value, effort and hence wage can be increasing over time. (It is easy to construct examples in which wage increases throughout, see right panel of Figure 11, but it need not be so, see left panel.) It would be interesting to consider the game that does not end with a success, but rather continues with a value reset at the prior of 1 (which immediately starts declining towards p^*), but we have not pursued this here.

6 Concluding remarks

There are several extensions that we think could potentially yield interesting insights.

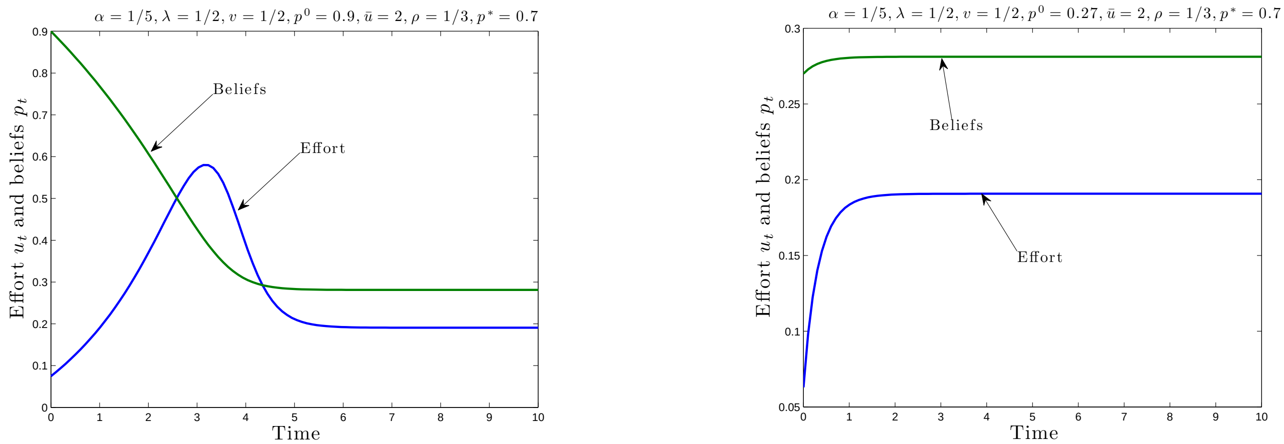

Figure 11: Changing states: two possible configurations

Policy Design We have not touched upon the issue of corrective measures that can be taken to remedy the inefficiency of the different market structures that have been considered. Various arrangements could be studied, that differ in terms of flexibility and coerciveness: taxes, pensions, or even, in the extreme case, slavery, etc. For instance, an obvious institution to examine within our framework is apprenticeship length (see Malcomson, Maw and McCormick, 2003).

Search equilibrium We have not addressed either the optimal timing for a firm to lay off the worker. To examine this issue, it is necessary to introduce some frictions: with short-run contracts, the firm breaks even at all times, so that it has nothing to lose or gain by firing the worker. Yet this is an important question, in light of the rigid tenure policies adopted by many professional service firms. Why not keep the employee past the probationary period, adjusting the wage for the diminished incentives and lower assessed ability?[42] Firms have a cost of hiring (or firing) workers –possibly due to the delay in filling a vacancy– but derive a surplus in excess of the competitive wage. Studying the efficiency properties and the characteristics of the labor market (composition of the working force, duration of unemployment) would be an interesting undertaking. A natural starting point would be a market in which marginal productivities are heterogenous, and there are matching frictions, as in Postel-Vinay and Robin (2002).

[42]See Gilson and Mnookin (1989) for a discussion of this puzzle for the case of law firms.

Partnerships Our framework is suitable for an analysis of reputation in partnerships, which is relevant for professional service firms. In law or consulting, projects are assigned to teams, combining partners with junior associates. The team pursues conflicting objectives: incentivizing both partner and associate, and eliciting information about the associate's ability. How should profits be shared? Is it optimal to combine workers whose assessed abilities differ?

Another issue is yardstick competition: here, there is no distinction between the agent's skill and the project's feasibility. In practice, the market learns about both. This also occurs through the progress of other agents' work on related tasks. Yardstick competition affects incentives. Relatedly, workers of different perceived skills might choose different types of projects; more challenging projects, or tougher environments, might foster learning of very high skilled workers, but be redhibitory for workers with lower perceived skills. Examining how the market allocates employees and firms, and how this differs from the efficient match is an interesting problem.

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Online Appendix

Throughout this appendix, we shall use the log-likelihood ratio

$$x_t := \ln \frac{1-p_t}{p_t}$$

of state $\omega = 0$ vs. $\omega = 1$. We set $x^0 := \ln(1-p^0)/p^0$. Note that x increases over time and, given u, follows the O.D.E.

$$\dot{x}_t = \lambda + u_t,$$

with $x_0 = x^0$. We shall also refer to x_t as the "belief," hoping that this will create no confusion.

We start by explaining how the objective function can be derived as the limit of a discounted version of our problem. Suppose that V is the value of a success and $V_L = V - k$ is the value of failure. Given the discount rate r, the agent's payoff is given by

$$\left(1+e^{-x_0}\right) V_0 = \int_0^T e^{-rt}\left(1+e^{-x_t}\right)\left(\frac{\dot{x}_t}{1+e^{x_t}} V + w_t - c\left(u_t\right)\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right) V_L,$$

where V_0 is his *ex ante* payoff.
Integrating by parts we obtain

$$\begin{aligned}
\left(1+e^{-x_0}\right) V_0 &= \int_0^T e^{-rt} e^{-x_t} \dot{x}_t V \mathrm{d}t + \int_0^T e^{-rt}\left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right)\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right)(V-k) \\
&= -e^{-rt} e^{-x_t} V\Big|_0^T + \int_0^T e^{-rt}\left(\left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right)\right) - e^{-x_t} rV\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right)(V-k) \\
&= e^{-x_0} V - e^{-rT} e^{-x_T} V + \int_0^T e^{-rt}\left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right) - \frac{rV}{1+e^{x_t}}\right) \mathrm{d}t + e^{-rT}\left(1+e^{-x_T}\right)(V-k),
\end{aligned}$$

so that as $r \to 0$ (and defining v as $rV \to v$) we obtain

$$\left(1+e^{-x_0}\right)\left(V_0 - V\right) = \int_0^T \left(1+e^{-x_t}\right)\left(w_t - c\left(u_t\right) - \frac{v}{1+e^{x_t}}\right) \mathrm{d}t - k\left(1+e^{-x_T}\right).$$

Similarly, one can show the social planner's payoff is given by

$$\left(1+e^{-x_0}\right)\left(V_0 - V\right) - e^{-x_0} + k = -\int_0^T \left(1+e^{-x_t}\right)\left(c\left(u_t\right) + \frac{v}{1+e^{x_t}}\right) \mathrm{d}t - (1+k) e^{-x_T}.$$

A Proofs for Section 2

Proof of Lemma 2.1. In both the linear and convex cases, existence and uniqueness of a solution follow as special case of Lemma 3.1, when $w = 0$ identically (the transversality condition must be adjusted). To see that the social planner's problem is equivalent to this, note that (whether the cost is convex or linear), the "revenue" term of the social planner's objective satisfies

$$\int_0^T (1+e^{-x_t})\frac{\lambda+u_t}{1+e^{x_t}}\mathrm{d}t = \int_0^T \dot{x}_t e^{-x_t}\mathrm{d}t = e^{-x^0} - e^{-x_T},$$

and so this revenue only affects the necessary conditions through the transversality condition at T.

Let us start with the linear case. The social planner maximizes

$$\int_0^T (1+e^{-x_t})\left(\frac{\lambda+u_t}{1+e^{x_t}} - \alpha u_t - v\right)\mathrm{d}t - ke^{-x_T}, \text{ s.t. } \dot{x}_t = \lambda + u_t.$$

We note that the maximization problem cannot be abnormal, since there is no restriction on the terminal value of the state variable. See Note 5, Ch. 2, Seierstad and Sydsæter (1987). The same holds for all later optimization problems.

It will be understood from now on that statements about derivatives only hold almost everywhere.

Let γ_t be the costate variable. The Hamiltonian for this problem is

$$H(x,u,\gamma,t) = e^{-x_t}(\lambda+u_t) - (1+e^{-x_t})(v+\alpha u_t) + \gamma_t(\lambda+u_t).$$

Define $\phi_t := \partial H/\partial u_t = (1-\alpha)e^{-x_t} - \alpha + \gamma_t$. Note that given x_t and γ_t, the value of ϕ_t does not depend on u_t. Pontryagin's principle applies, and yields

$$u_t = \bar{u} \ \ (u_t = 0) \Leftrightarrow \phi_t := \frac{\partial H}{\partial u_t} = (1-\alpha)e^{-x_t} - \alpha + \gamma_t > (<)0,$$

as well as

$$\dot{\gamma}_t = e^{-x_t}(\lambda - v + (1-\alpha)u_t), \gamma_T = ke^{-x_T}.$$

Differentiating ϕ_t with respect to time, and using the last equation gives

$$\dot{\phi}_t = e^{-x_t}(\alpha\lambda - v), \phi_T = (1+k-\alpha)e^{-x_T} - \alpha.$$

Note that ϕ is either increasing or decreasing depending on the sign of $\alpha\lambda - v$. Therefore, the planner's solution is either maximum effort–no effort, or no effort–maximum effort, depending on the sign of this expression. Finally, the marginal product $p(\lambda+u)$ is decreasing if effort maximum–zero. If effort is zero–maximum, the marginal product is decreasing, jumps up, and then decreases again.

Consider now the convex case. Applying Pontryagin's theorem (and replacing the revenue term by its expression in terms of x_t and x^0, as explained above) yields as necessary conditions

$$\dot{\gamma}_t = -e^{-x}(c(u)+v), \gamma_t = (1+e^{-x_t})c'(u_t),$$

where γ_t is the co-state variable, as before. Differentiate the second expression with respect to time, and use the first one to obtain

$$\dot{u} \quad = \quad \frac{(\lambda+u)\,c'(u) - c(u) - v}{c''(u)\,(1+e^x)}, \tag{12}$$

in addition to $\dot{x} = \lambda + u$ (time subscripts will often be dropped for brevity). Let

$$\phi(u) := (\lambda + u) c'(u) - c(u) - v.$$

Note that $\phi(0) = -v < 0$, and $\phi'(u) = (\lambda + u) c''(u) > 0$, and so ϕ is strictly increasing and convex. Let $u^* \geq 0$ be the unique solution to

$$\phi(u^*) = 0,$$

and so ϕ is negative on $[0, u^*]$ and positive on $[u^*, \infty)$. Accordingly, $u < u^* \implies \dot{u} < 0$, $u = u^* \implies \dot{u} = 0$ and $u > u^* \implies \dot{u} > 0$. Given the transversality condition

$$(1 + e^{x_T}) c'(u_T) = 1 + k,$$

we can then define $x_T(x^0)$ by

$$x_T(x^0) = \frac{1}{\lambda + u^*} \left[\ln \left(\frac{1+k}{c'(u^*)} - 1 \right) - x^0 \right],$$

and so effort is decreasing throughout if $x_T > x_T(x^0)$, increasing throughout if $x_T < x_T(x^0)$, and equal to u^* throughout otherwise. The conclusion then follows from the proof of Lemma 2.3, which establishes that the belief x_T at the deadline is increasing in T.

We now turn to the marginal product $p(\lambda + u)$. In terms of x, the marginal product is given by

$$\begin{aligned} w(x) &:= \frac{\lambda + u(x)}{1 + e^x}, \text{ and so} \\ w'(x) &= \frac{u'(x)}{1 + e^x} - w(x) \frac{e^x}{(1 + e^x)}, \end{aligned}$$

so that $w'(x) = 0$ is equivalent to

$$u'(x) = w(x) e^x.$$

Notice that $u'(x) \leq 0$ implies $w'(x) < 0$. Conversely, if $u'(x) > 0$, consider the second derivative of $w(x)$. We have

$$w''(x) = -\frac{e^x}{1 + e^x} w'(x) + \frac{1}{1 + e^x} \left(u''(x) - w(x) e^x - u'(x) e^x \right),$$

so that when $w'(x) = 0$ we have

$$w''(x) = \frac{u''(x) - u'(x)}{1 + e^x}.$$

From equation (12) we obtain an expression for the derivative of u with respect to x:

$$u'(x) = \frac{(\lambda + u) c'(u) - c(u) - v}{c''(u)(1 + e^x)(\lambda + u)}.$$

Let $g(u) = v + c(u) - (\lambda + u) c'(u)$ and study $u''(x)$ when $w'(x) = 0$. We have

$$\begin{aligned}
u''(x) &= \frac{u'}{1+e^x} + \frac{g\left((1+e^x)\left(c'' + (\lambda+u) c'''\right) u'(x) + e^x c''(\lambda+u)\right)}{\left(c''(u)(1+e^x)(\lambda+u)\right)^2} \\
&= \frac{c''(\lambda+u) u'(x)}{c''(1+e^x)(\lambda+u)} - \frac{u'(x)\left((1+e^x)\left(c''+(\lambda+u)c'''\right)u'(x) + e^x c''(\lambda+u)\right)}{c''(1+e^x)(\lambda+u)} \\
&= -\frac{u'(x)\left((1+e^x)\left(c''+(\lambda+u)c'''\right)u'(x) + e^x c''(\lambda+u) - c''(\lambda+u)\right)}{c''(1+e^x)(\lambda+u)} \\
&= -\frac{u'(x)\left(\left(2c''+(\lambda+u)c'''\right)e^x(\lambda+u) - c''(\lambda+u)\right)}{c''(1+e^x)(\lambda+u)}.
\end{aligned}$$

We therefore consider the quantity

$$\begin{aligned}
u''(x) - u'(x) &= -\frac{u'(x)\left(\left(2c''+(\lambda+u)c'''\right)e^x - c'' + c''(1+e^x)\right)}{c''(1+e^x)} \\
&= -\frac{u'(x)\left(3c''+(\lambda+u)c'''\right)e^x}{c''(1+e^x)} < 0,
\end{aligned}$$

if as we have assumed, $c''+(\lambda+u)c''' > 0$. Therefore, we have a single-peaked (at most increasing then decreasing) marginal product. □

Proof of Lemma 2.3. We shall use the necessary conditions obtained in the previous proof. Part (1) is almost immediate. Note that in both the linear and convex case, the necessary conditions define a vector field $(\dot{u}, \dot{x})$, with trajectories that only define on the time left before the deadline and the current belief. Because trajectories do not cross (in the plane $(-\tau, x)$, where τ is time-to-go and x is the belief), and belief x can only increase with time, if we compare two trajectories starting at the same level x^0, the one that involves a longer deadline must necessarily involve as high a terminal belief x as the other (as the deadline expires).

(2) In the linear case, it is straightforward to solve for the switching time (or switching belief) under Assumption 2.2. For all terminal beliefs $x_T > x^*$, for which no effort is exerted at the deadline, the switching belief between equilibrium phases is determined by

$$(1+k-\alpha)e^{-x_T} - \alpha = \int_x^{x_T} e^{-s}\frac{\alpha\lambda - v}{\lambda}\mathrm{d}s,$$

which gives as value of x (as a function of t)

$$x(t) = \ln\Big((1+k-v/\lambda)e^{-\lambda(T-t)} - (\alpha - v/\lambda)\Big) - \ln\alpha.$$

This represents a frontier in (t, x) space that the equilibrium path will cross from below for sufficiently long deadlines. Consistent with the fact that, in the optimum, a switch to zero effort is irreversible, when $u_t = 0$ and $\dot{x}_t = \lambda$, the path leaves this locus (*i.e.*, it holds that $x'(t) < \lambda$).

The switching belief $x(t)$ decreases in T: the longer the deadline, the longer maximum effort will be exerted (recall that x measures pessimism). This belief decreases in α and increases in v and k: the higher the cost of failing, or the lower the cost of effort, the longer effort is exerted. These are the comparative statics mentioned in the text before Lemma 2.3.

Furthermore, by differentiating, the boundary $x(\cdot)$ satisfies $x'(t) < 0$ (resp. > 0) if and only if $1 + k < v/\lambda$. In that case, total effort increases with T: considering the plane $(-\tau, x)$, where τ is time-to-go and x is the belief, increasing the deadline is equivalent to increasing τ, *i.e.* shifting the initial point to the left; if $x' < 0$, it means that the range of beliefs over which high effort is exerted (which is 1-to-1 with time spent exerting maximum effort, given that $\dot{x} = \lambda + \bar{u}$) increases. If instead x' is positive, total effort decreases with T, by the same argument.

Consider now the convex case. Note that

$$\frac{1}{x'(u)} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\dot{u}}{\dot{x}} = \frac{(\lambda + u)\, c'(u) - c(u) - v}{c''(u)\,(1 + e^{x})\,(\lambda + u)},$$

along with

$$(1 + e^{x_T})\, c'(u_T) = 1 + k,$$

which is the transversality condition, can be integrated to

$$\phi(u) = \frac{(k+1)\phi(u_T)}{1 + k - c'(u_T)} \frac{1}{1 + e^{-x}}.$$

Note also that, defining $g(u) := \frac{\phi(u)}{1+k-c'(u)}$,

$$g'(u) = \frac{(\lambda + u)\, c''(u)}{1 + k - c'(u)} + \frac{\phi(u)}{(1 + k - c'(u))^2} c''(u),$$

which is of the sign of

$$\psi(u) := (\lambda + u)(1 + k) - c(u) - v,$$

which is strictly concave, negative at ∞, and positive for u small enough if and only if $1 + k > v/\lambda$.

Note that increasing T is equivalent to increasing x_T, which in turn is equivalent to decreasing u_T, because the transversality condition yields

$$\frac{\mathrm{d}u_T}{\mathrm{d}x_T} = -\frac{c'(u)\, e^{x}}{(1 + e^{x})\, c''(u)} < 0.$$

Because ϕ is increasing, u increases when u_T decreases if and only if ψ is decreasing at u.

So if $\max_u[(\lambda + u)(1 + k) - c(u)] < v$, ψ is negative for all u, and it follows that u increases for fixed x; in addition to all values of x that are visited in the interval $[x^0, x_T]$, as T increases, additional effort accrues at time T; overall, it is then unambiguous: total effort increases.

On the other hand, if $1 + k > v/\lambda$, then if the deadline is long enough for effort to be small throughout, effort at $x < x_T$ decreases as T increases, but since an additional increment of effort is produced at time T, it is unclear. A simple numerical example shows that total effort can then decrease. □

B Proofs for Section 3

Proof of Lemma 3.1. We address the two claims in turn.

Existence: Note that the state equation is linear in the control u, while the objective's integrand is concave in u. Hence the set $N(x,U,t)$ is convex (see Thm. 8, Ch. 2 of Seierstad and Sydsæter, 1987). Therefore, the Filippov-Cesari existence theorem applies.

Uniqueness: We can write the objective as, up to constant terms,

$$\int_0^T (1+e^{-x_t})(w_t - v - c(u_t))\mathrm{d}t - ke^{-x_T},$$

or, using the likelihood ratio $l_t := p_t / (1-p_t) > 0$,

$$J(l) := \int_0^T (1+l_t)(w_t - v - c(u_t))\ \mathrm{d}t - kl_T.$$

Consider the linear case. Letting $g_t := w_t - v + \alpha\lambda$, we rewrite the objective in terms of the likelihood ratio as

$$\int_0^T l_t g_t \mathrm{d}t - (k-\alpha) l_T + \alpha \ln l_T + \text{ Constant.}$$

Because the first two terms are linear in l while the last is strictly concave, it follows that there exists a unique optimal terminal odds ratio $l_T^* := l_T$. Suppose that there exists two optimal trajectories l_1, l_2 that differ. Because $l_{1,0} = l_{2,0} = p^0 / \left(1-p^0\right)$ and $l_{1,T} = l_{2,T} = l_T^*$, yet the objective is linear in l_t, it follows that every feasible trajectory l with $l_t \in [\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\}]$ is optimal as well.[43] Consider any interval $[a,b] \subset [0,T]$ for which $t \in [a,b] \implies \min\{l_{1,t}, l_{2,t}\} < \max\{l_{1,t}, l_{2,t}\}$. Consider any feasible trajectory l with $l_t \in [\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\}]$ for all t, $l_t \in (\min\{l_{1,t}, l_{2,t}\}, \max\{l_{1,t}, l_{2,t}\})$ for $t \in [a,b]$ and associated control such that $u_t \in (0, \bar{u})$ for $t \in [a,b]$. Because there is an open set of variations of u that must be optimal in $[a,b]$, it follows from Lemma 2.4.ii of Cesari (1983) that $g_t = 0$ (a.e.) on $[a,b]$.

Consider now the convex case. Suppose that there are two distinct optimal trajectories l_1 and l_2, with associated controls u_1 and u_2. Assume without loss of generality that

$$l_{1,t} < l_{2,t} \text{ for all } t \in (0,T].$$

We analyze the modified objective function

$$\tilde{J}(l) := \int_0^T (1+l_t)(w_t - v - \tilde{c}_t(u_t))\mathrm{d}t - kl_T,$$

in which we replace the cost function $c(u_t)$ with

$$\tilde{c}_t(u) := \begin{cases} \alpha_t u & \text{if} \quad u \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}] \\ c(u) & \text{if} \quad u \notin [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}], \end{cases}$$

where

$$\alpha_t := \frac{\max\{c(u_{1,t}), c(u_{2,t})\} - \min\{c(u_{1,t}), c(u_{2,t})\}}{\max\{u_{1,t}, u_{2,t}\} - \min\{u_{1,t}, u_{2,t}\}}.$$

[43]Feasibility means that $\dot{l}_t \in [l_t\lambda, l_t(\lambda + \bar{u})]$ for all t.

(If $u_{1,t} = u_{2,t} =: u_t$ for some t, set α_t equal to $c'(u_t)$). Because $\tilde{c}_t(u) \geq c(u)$ for all t, u, the two optimal trajectories l_1 and l_2, with associated controls u_1 and u_2, are optimal for the modified objective $\tilde{J}(l)$ as well. Furthermore, $\tilde{J}(l_1) = J(l_1)$ and $\tilde{J}(l_2) = J(l_2)$.

We will construct a feasible path l_t and its associated control $u_t \in [\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ which attains a higher payoff $\tilde{J}(l)$ and therefore a strictly higher payoff $J(l)$. Suppose $u_t \in [u_{1,t}, u_{2,t}]$ for all t. Letting $g_t := w_t - v + \alpha\lambda - \dot{\alpha}_t$, we rewrite the modified objective as

$$\int_0^T l_t g_t \mathrm{d}t - \int_0^T \dot{\alpha}_t \ln l_t \mathrm{d}t - (k - \alpha_T) l_T + \alpha_T \ln l_T + \text{ Constant.}$$

We now consider a continuous function $\varepsilon_t \geq 0$ and two associated variations on the paths l_1 and l_2,

$$\begin{aligned} l'_{1,t} &:= (1 - \varepsilon_t) l_{1,t} + \varepsilon_t l_{2,t} \\ l'_{2,t} &:= (1 - \varepsilon_t) l_{2,t} + \varepsilon_t l_{1,t}. \end{aligned}$$

Because l_1 and l_2 are optimal, for any ε_t it must be the case that

$$\begin{aligned} \tilde{J}(l_1) - \tilde{J}(l'_1) &\geq 0 \\ \tilde{J}(l_2) - \tilde{J}(l'_2) &\geq 0. \end{aligned}$$

We can write these payoff differences as

$$\begin{aligned} \int_0^T \varepsilon_t (l_{1,t} - l_{2,t}) g_t \mathrm{d}t + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{2,t} - l_{1,t}}{l_{1,t}} \mathrm{d}t - (k - \alpha_T)\varepsilon_T (l_{1,T} - l_{2,T}) - \alpha_T \varepsilon_T \frac{l_{2,T} - l_{1,T}}{l_{1,T}} + o(\|\varepsilon\|) &\geq 0 \\ \int_0^T \varepsilon_t (l_{2,t} - l_{1,t}) g_t \mathrm{d}t + \int_0^T \dot{\alpha}_t \varepsilon_t \frac{l_{1,t} - l_{2,t}}{l_{2,t}} \mathrm{d}t - (k - \alpha_T)\varepsilon_T (l_{2,T} - l_{1,T}) - \alpha_T \varepsilon_T \frac{l_{1,T} - l_{2,T}}{l_{2,T}} + o(\|\varepsilon\|) &\geq 0. \end{aligned}$$

Letting

$$\rho_t : l_{1,t}/l_{2,t} < 1 \text{ for all } t > 0,$$

we can sum the previous two conditions (up to the second order term). Finally, integrating by parts, we obtain the following condition,

$$\int_0^T \left[\frac{\dot{\varepsilon}_t}{\varepsilon_t}\left(2 - \rho_t - \frac{1}{\rho_t}\right) + \dot{\rho}_t \frac{1 - \rho_t^2}{\rho_t^2}\right] \alpha_t \varepsilon_t \mathrm{d}t \geq 0,$$

which must hold for all ε_t. Using the fact that $\dot{\rho} = \rho(u_2 - u_1)$ we have

$$\int_0^T \left[-\frac{\dot{\varepsilon}_t}{\varepsilon_t}(1 - \rho_t) + (u_{2,t} - u_{1,t})(1 + \rho_t)\right] \alpha_t \varepsilon_t \frac{1 - \rho_t}{\rho_t} \mathrm{d}t \geq 0. \tag{13}$$

We now identify bounds on the function ε_t so that both variations l'_1 and l'_2 are feasible and their associated controls lie in $[\min\{u_{1,t}, u_{2,t}\}, \max\{u_{1,t}, u_{2,t}\}]$ for all t. Consider the following identities

$$\begin{aligned} \dot{l}'_1 &= -l'_{1,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t (l_{2,t} - l_{1,t}) - \lambda l'_{1,t} - (1 - \varepsilon_t) u_{1,t} l_{1,t} - \varepsilon_t u_{2,t} l_{2,t} \\ \dot{l}'_2 &= -l'_{2,t}(\lambda + u_t) \equiv \dot{\varepsilon}_t (l_{1,t} - l_{2,t}) - \lambda l'_{2,t} - \varepsilon_t u_{1,t} l_{1,t} - (1 - \varepsilon_t) u_{2,t} l_{2,t}. \end{aligned}$$

We therefore have the following expressions for the function $\dot{\varepsilon}/\varepsilon$ associated with each variation

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t}-u_t)\,\frac{1-\varepsilon_t}{\varepsilon_t} l_{1,t} + l_{2,t}\,(u_{2,t}-u_t)}{l_{2,t}-l_{1,t}}, \tag{14}$$

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{(u_{1,t}-u_t)\,l_{1,t} + \frac{1-\varepsilon_t}{\varepsilon_t} l_{2,t}\,(u_{2,t}-u_t)}{l_{1,t}-l_{2,t}}. \tag{15}$$

In particular, whenever $u_{2,t} > u_{1,t}$ the condition

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{1-\varepsilon_t}{\varepsilon_t}\frac{l_{2,t}\,(u_{2,t}-u_{1,t})}{l_{2,t}-l_{1,t}}, \frac{l_{1,t}\,(u_{2,t}-u_{1,t})}{l_{2,t}-l_{1,t}}\right]$$

ensures the existence of two effort levels $u_t \in [u_{1,t}, u_{2,t}]$ that satisfy conditions (14) and (15) above. Similarly, whenever $u_{1,t} > u_{2,t}$ we have the bound

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} \in \left[-\frac{l_{1,t}\,(u_{1,t}-u_{2,t})}{l_{2,t}-l_{1,t}}, \frac{1-\varepsilon_t}{\varepsilon_t}\frac{l_{2,t}\,(u_{2,t}-u_{1,t})}{l_{2,t}-l_{1,t}}\right].$$

Note that $\dot{\varepsilon}_t/\varepsilon_t = 0$ is always contained in both intervals.

Finally, because $\rho_0 = 1$ and $\rho_t < 1$ for all $t > 0$, we must have $u_{1,t} > u_{2,t}$ for $t \in [0, t^*)$ with $t^* > 0$. Therefore, we can construct a path ε_t that satisfies

$$(u_{2,t}-u_{1,t})\,\frac{1+\rho_t}{1-\rho_t} < \frac{\dot{\varepsilon}_t}{\varepsilon_t} < 0 \quad \forall t \in [0, t^*),$$

with $\varepsilon_0 > 0$, and $\varepsilon_t \equiv 0$ for all $t \geq t^*$. Substituting into condition (13) immediately yields a contradiction. □

Proof of Proposition 3.3. Consider the convex case. Applying Pontryagin's theorem yields eqn. (6). It also follows that the effort and belief (x, u) trajectories satisfy

$$c''(u)\,(1+e^x)\,\dot{u} = (\lambda+u)\,c'(u) - c(u) + w_t - v \tag{16}$$

$$\dot{x} = \lambda + u \tag{17}$$

with boundary conditions

$$x_0 = x^0 \tag{18}$$

$$ke^{-x_T} = \left(1+e^{-x_T}\right)c'(u_T). \tag{19}$$

Differentiating (16) further, we obtain

$$\begin{aligned}(c''(u)\,(1+e^x))^2\,u''_t &= ((\lambda+u)\,c''(u)\,u'_t + w'_t)\,c''(u)\,(1+e^x) \\ &\quad -((\lambda+u)\,c'(u) + w_t - c(u) - v)\,(c'''(u)\,u'_t\,(1+e^x) + e^x\,(\lambda+u)\,c''(u)).\end{aligned}$$

So that when $u'_t = 0$ we obtain

$$c''(u)\,(1+e^x)\,u''_t = w'_t.$$

This immediately implies the first conclusion.

In the linear case, mimicking the proof of Lemma 2.1, Pontryagin's principle applies, and yields the existence of an absolutely continuous function $\gamma : [0, T] \to \mathbb{R}$ such that

$$\gamma_t - \alpha \left(1 + e^{-x_t}\right) > (<) 0 \Rightarrow u_t = \bar{u} \ \ (u_t = 0) .$$

as well as

$$\dot{\gamma}_t = e^{-x_t}(w_t - \alpha u_t - v), \gamma_T = k e^{-x_T}.$$

Define ϕ by $\phi_t := \gamma_t - \alpha(1 + e^{-x_t})$. Note that $\phi_t > 0$ (resp. < 0) $\Rightarrow u_t = \bar{u}$ (resp. $= 0$). Differentiating ϕ_t with respect to time, and using the last equation gives

$$\dot{\phi}_t = e^{-x_t}(\alpha\lambda + w_t - v), \phi_T = (k - \alpha)e^{-x_T} - \alpha.$$

(This is the formal derivation of eqn. (7).) Observe now that if w is monotone, so is $\alpha\lambda + w_t - v$, and hence $\dot{\phi}$ changes signs only once. Conclusion 1 follows for the linear case. If it is strictly monotone, ϕ is equal to zero at most at one date t, and so the optimal strategy is extremal, yielding the second conclusion of the lemma. □

Proof of Proposition 3.4. Recall from the proof of Lemma 2.3 that

$$\frac{1}{x'(u)} = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\dot{u}}{\dot{x}} = \frac{(\lambda + u) c'(u) - c(u) - v}{c''(u)(1 + e^x)(\lambda + u)}$$

must hold for the optimal trajectory (in the (x, u)-plane) for the social planner. Denote this trajectory x^{FB}. The corresponding law of motion for the agent's optimum trajectory x^* given w is

$$\frac{1}{x'(u)} = \frac{(\lambda + u) c'(u) - c(u) + w_t - v}{c''(u)(1 + e^x)(\lambda + u)}.$$

(Note that, not surprisingly, time matters). This implies that (in the (x, u)-plane) the trajectories x^{FB} and x^* can only cross one way, if at all, with x^* being the flatter one. Yet the (decreasing) transversality curve of the social planner, implicitly given by

$$(1 + e^{x_T}) c'(u_T) = 1 + k,$$

lies above the (decreasing) transversality curve of the agent, which is defined by

$$(1 + e^{x_T}) c'(u_T) = k.$$

Suppose now that the trajectory x^{FB} ends (on the transversality curve) at a lower belief x_T^{FB} than x^*: then it must be that effort u was higher throughout along that trajectory than along x^* (since the latter is flatter, x^{FB} must have remained above x^* throughout). But since the end value of the belief x is simply $x^0 + \int_0^T u_s \mathrm{d}s$, this contradicts $x_T^{FB} < x_T^*$.

It follows that for a given x, the effort level u is higher for the social planner.

The same reasoning implies the second conclusion: if $x_T^{FB} = x_T^*$, so that total effort is the same, yet the trajectories can only cross one way (with x^* being flatter), it follows that x^* involves lower effort first, and then larger effort, *i.e.* the agent backloads effort. □

C Section 4

First, we describe and prove the omitted characterization in the case of linear cost. Proposition C.1 describes the equilibrium in the linear case, before turning to Theorem 4.2.

Proposition C.1 *With linear cost, any equilibrium path consists of at most four phases, for some* $0 \leq t_1 \leq t_2 \leq t_3 \leq T$*:*

1. *during* $[0, t_1]$*, no effort is exerted;*
2. *during* $(t_1, t_2]$*, effort is interior, i.e.* $u_t \in (0, \bar{u})$*;*
3. *during* $(t_2, t_3]$*, effort is maximal;*
4. *during* $(t_3, T]$*, no effort is exerted.*

Any of these intervals might be empty.[44]

Proof of Proposition C.1. We prove the following:

1. If there exists $t \in (0, T)$ such that $\phi_t > 0$, then there exists $t' \in [t, T]$ such that $u_s = \bar{u}$ for $s \in [t, t']$, $u_s = 0$ for $s \in (t', T]$.
2. If there exists $t \in (0, T)$ such that $\phi_t < 0$, then either $u_s = 0$ for all $s \in [t, T]$ or $u_s = 0$ for all $s \in [0, t]$,

which implies the desired decomposition. For the first part, note that either $u_s = \bar{u}$ for all $s > t$, or there exists t'' such that both $\phi_{t''} > 0$ (so in particular $u_{t''} = \bar{u}$) and $\dot{\phi}_{t''} < 0$. Because p_t decreases over time, and $u_s \leq u_{t''}$ for all $s > t''$, it follows that $w_s < w_{t''}$, and so $\dot{\phi}_s < \dot{\phi}_{t''} < 0$. Hence ϕ can cross 0 only once for values above t, establishing the result. For the second part, note that either $u_s = 0$ for all $s \geq t$, or there exists $t'' \geq t$ such that $\phi_{t''} < 0$ (so in particular $u_{t''} = 0$) and $\dot{\phi}_{t''} > 0$. Because p_t decreases over time, and $u_s \geq u_{t''}$ for all $s < t''$, it follows that $w_s \geq w_{t''}$, and so $\dot{\phi}_s > \dot{\phi}_{t''} > 0$. For all $s < t''$, $\phi_s < 0$ and $\dot{\phi}_s > 0$. Hence, $u_s = 0$ for all $s \in [0, t]$.□

C.1 Proofs for Subsection 4.1

Proof of Theorem 4.2. We study the linear and convex cases in turn.

Proof of Theorem 4.2 (Linear case). We start by establishing uniqueness.
Uniqueness: Assume an equilibrium exists, and note that, given a final belief x_T, the pair of differential equations for ϕ and x (along with the transversality condition) admit a unique solution, pinning down, in particular, the effort exerted by, and the wage received by the agent. Therefore, if two (or more) equilibria existed for some values (x_0, T), it would have to be the case that each of them is associated with a different terminal belief x_T.

[44] Here and elsewhere, the choices at the extremities of the intervals are irrelevant, and our specification is arbitrary in this respect.

However, we shall show that, for any x_0, the time it takes to reach a terminal belief x_T is a continuous, strictly increasing function $T(x_T)$; therefore, no two different terminal beliefs can be reached in the same time T. The details of this step can be found in the working paper and are omitted.

Existence: We have established that the time necessary to reach the terminal belief is a continuous and strictly increasing function. Therefore, the terminal belief reached in equilibrium is itself given by a strictly increasing function

$$x_T(T) : \mathbb{R}_+ \to [x_0, \infty).$$

Since there exists a unique path consistent with optimality for each terminal belief, given a deadline T we can establish existence by constructing the associated equilibrium outcome, and in particular, the equilibrium wage path. Existence and uniqueness of an optimal strategy for the worker, after any (on or off-path) history, follow then from Lemma 3.1.

Proof of Theorem 4.2 (Convex case). We proceed as in the linear case.
Uniqueness: Fix T. The two differential equations obeyed by the (x, u)-trajectory are

$$\begin{aligned} \dot{x} &= \lambda + u \\ \dot{u} &= \frac{(\lambda+u)\, c'(u) - c(u) + \frac{\lambda+u}{1+e^x} - v}{c''(u)\,(1+e^x)}. \end{aligned}$$

We have, using that $\mathrm{d}x = (\lambda+u)\,\mathrm{d}t$,

$$\frac{\mathrm{d}u}{\mathrm{d}x} =: f(u,x) = \frac{(\lambda+u)\, c'(u) - c(u) + \frac{\lambda+u}{1+e^x} - v}{(\lambda+u)\, c''(u)\,(1+e^x)}. \tag{20}$$

Recall also that the transversality curve is given by

$$(1+e^x)\, c'(u) = k,$$

and so

$$\frac{\mathrm{d}u_T}{\mathrm{d}x_T} = -\frac{c'(u)\, e^x}{(1+e^x)\, c''(u)} < 0.$$

Note that the slope of $u(x)$ at the deadline T is at most first positive then negative. To see this, differentiate the numerator in (20) and impose (20) equal to zero. We obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x_T}\frac{\mathrm{d}u(x_T)}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x_T}\left[(\lambda+u(x_T))\, c'(u(x_T)) - c(u(x_T)) + \frac{\lambda+u(x_T)}{1+e^{x_T}}\right] \\ &= \left((\lambda+u(x_T))\, c''(u(x_T)) + \frac{1}{1+e^{x_T}}\right)\frac{\mathrm{d}u_T}{\mathrm{d}x_T} - \frac{(\lambda+u(x_T))\, e^{x_T}}{(1+e^{x_T})^2} < 0. \end{aligned}$$

Suppose now we had

$$u'(x) > \frac{\mathrm{d}u_T}{\mathrm{d}x_T}, \text{ at } x = x_T,$$

so that the trajectory does not cross the transversality line from above. Then we would be done. A path leading to a higher x_T lies below one leading to the lower x_T so later beliefs take longer to reach.

Denote the difference in the slopes of the effort and transversality lines by

$$\Delta(u,x) := \frac{(\lambda+u)\,c'(u) - c(u) + \frac{\lambda+u}{1+e^x} - v}{(\lambda+u)\,c''(u)\,(1+e^x)} + \frac{c'(u)\,e^x}{(1+e^x)\,c''(u)}.$$

Note that $\Delta = 0 \Rightarrow \Delta'(x_T) < 0$ so our trajectory crosses transversality (at most) first from below then from above.

More generally, the time required to reach terminal belief x_T is given by

$$T = \int_{x_0}^{x_T} \frac{1}{\lambda + u(x)} \mathrm{d}x = \int_{x_0}^{x_T} \left(\lambda + u(x_T) - \int_x^{x_T} u'(x)\,\mathrm{d}x\right)^{-1} \mathrm{d}x.$$

Differentiating with respect to x_T we obtain

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}x_T} &= \frac{1}{\lambda+u(x_T)} + \left(u'(x_T) - \frac{\mathrm{d}u_T}{\mathrm{d}x_T}\right)\int_{x_0}^{x_T} \frac{1}{(\lambda+u(x))^2}\mathrm{d}x \\ &= \frac{1}{\lambda+u(x_T)} + \Delta(u_T,x_T)\int_{x_0}^{x_T} \frac{1}{(\lambda+u(x))^2}\mathrm{d}x.\end{aligned}$$

Clearly, if the function $u(x)$ crosses the transversality line from below ($\Delta > 0$) then we are done: a path leading to a higher x_T lies below one leading to the lower x_T so later beliefs take longer to reach. Imposing transversality and simplifying we obtain that a necessary condition for $\Delta > 0$ for all x_T is

$$k \geq v/\lambda.$$

Because we do not wish to assume that, note that the function $f(u,x)$ in (20) has the following properties

$$\begin{aligned} f(u,x) &\leq 0 \Rightarrow f_u(u,x) > 0 \\ f(u,x) &\geq 0 \Rightarrow f_x(u,x) < 0. \end{aligned}$$

In words, a trajectory at $(x, u + \mathrm{d}u)$ comes down not as fast as a trajectory at (x,u) if (x,u) is such that $\dot{u} \leq 0$. Conversely, a trajectory at $(x - \mathrm{d}x, u)$ climbs faster than a trajectory at (x,u) if (x,u) is such that $\dot{u} \geq 0$.

Therefore, consider a trajectory $u(x)$ such that $\Delta(x_T) < 0$. As we increase x_T, the new trajectory lies everywhere above the original one. For a small increase in x_T, because the trajectory changes continuously, the two properties of $f(u,x)$ ensure that the vertical distance between the two trajectories is maximized at x_T.

We then have the condition

$$\frac{\mathrm{d}T}{\mathrm{d}x_T} > \frac{1}{\lambda+u(x_T)} + (x_T - x_0)\frac{\Delta(u_T,x_T)}{(\lambda+u(x_T))^2}.$$

Using transversality and rewriting $\Delta(u_T,x_T)$ we obtain the condition

$$c''(u(x_T))(\lambda+u(x_T)) \geq \frac{x_T - x_0}{1+e^{x_T}}\left(\frac{c(u(x_T))+v}{\lambda+u(x_T)} - k - \frac{1}{1+e^{x_T}}\right). \tag{21}$$

Note that (21) clearly holds at $x_T = x_0$. Furthermore, if $c''(0) > 0$, (21) also holds in the limit for $x_T \to \infty$. Finally, since $c''(u)(\lambda+u)$ was assumed increasing, a sufficient condition for (21) to be satisfied is given by

$$c''(0) > \frac{1}{\lambda}\left(\frac{v}{\lambda} - k\right)h(x_0) \geq \frac{1}{\lambda}\left(\frac{v}{\lambda} - k\right)e^{-x_0},$$

which is the condition for uniqueness. Existence is established as in the linear case.

Single-peakedness: Single-peakedness of effort is almost immediate. Substituting the equilibrium expression $w_t = (\lambda + u_t) / (1 + e^{x_t})$ in the boundary value problem (16). Differentiating u'_t further, we obtain

$$u'_t = 0 \Rightarrow c''(u)\left(1 + e^{-x}\right) u''_t = -(w_t)^2,$$

which implies that the function u is at most first increasing then decreasing.

We now argue that the wage is single-peaked. In terms of x, the wage is given by

$$\begin{aligned} w(x) &= \frac{\lambda + u(x)}{1 + e^x}, \text{ and so} \\ w'(x) &= \frac{u'(x)}{1 + e^x} - \frac{\lambda + u(x)}{(1 + e^x)^2} e^x, \end{aligned}$$

so that $w'(x) = 0$ is equivalent to

$$u'(x) = w(x) e^x.$$

As in the proof of Lemma 2.1, when $w'(x) = 0$ we have

$$w''(x) = \frac{u''(x) - u'(x)}{1 + e^x}.$$

Furthermore, we know that

$$u'(x) = \frac{(\lambda + u) c'(u) - c(u) + \frac{\lambda + u}{1 + e^x} - v}{c''(u)(1 + e^x)(\lambda + u)}.$$

Mimicking the proof of Lemma 2.1, we conclude that $w'(x) = 0$ implies

$$u''(x) - u'(x) = -\frac{u'(x)\left(3c'' + (\lambda + u) c'''\right) e^x}{c''(1 + e^x)} < 0,$$

if as we have assumed, $c'' + (\lambda + u) c''' > 0$. Therefore, we also have single-peaked (at most increasing then decreasing) wages. (More generally, if $c''' < 0$ but $3c'' + (\lambda + u) c'''$ is increasing in u then the wage can be increasing on at most one interval.) □

C.2 Proofs for Subsection 4.2

Proof of Theorem 4.4. The proof is divided in several steps. Consider the maximization program $\mathcal{P}$ in the text: we begin by conjecturing a full-zero (or "FO") solution, *i.e.* a solution in which the agent first exerts maximum effort, then no effort; we show this solution solves a relaxed program; and finally we verify that it also solves the original program.

C.2.1 Candidate solution

Consider the following compensation scheme: pay a wage $w_t = 0$ for $t \in [0, t_0] \cup [t_1, T]$, a constant wage $w_t = v - \alpha\lambda$ for $t \in [t_0, t_1]$, and a lump-sum L at $t = T$. The agent exerts maximal effort for $t \leq t_1$ and zero thereafter. Furthermore, the agent is indifferent among all effort levels for $t \in [t_0, t_1]$.

For short enough deadlines, there exists a payment scheme of this form that induces full effort throughout, *i.e.* $t_0 > 0$ and $t_1 = T$, and leaves the agent indifferent between effort levels at T. Whenever this is the case, we take this to be our candidate solution. The conditions that pin down this solution are given by indifference at T and by zero profits at $t = 0$. Recall the definition of ϕ_t from the proof of Proposition 3.3. The conditions are then given by

$$\phi_T = (k - \alpha - L)\, e^{-x_T} - \alpha = 0, \tag{22}$$

$$\int_0^{t_0} \left(1 + e^{-x_s}\right) \frac{\lambda + u}{1 + e^{x_s}} \mathrm{d}s + \int_{t_0}^{T} \left(1 + e^{-x_s}\right) \left(\frac{\lambda + \bar{u}}{1 + e^{x_s}} - v + \alpha\lambda\right) \mathrm{d}s - \left(1 + e^{-x_T}\right) L = 0. \tag{23}$$

As T increases, $t_0 \to 0$. Let T^* denote the longest deadline for which this solution induces full effort throughout. The threshold T^* is the unique solution to (22) and (23) with $x_T = x_0 + (\lambda + \bar{u})\, T$ and $t_0 = 0$.

Lemma C.2 *The candidate solution is the unique compensation scheme that induces full effort on* $[0, T^*]$.

Proof of Lemma C.2. Delaying any payment from t to t' would induce the agent to shirk at t' because he is now indifferent for $t \leq t_1$. Anticipating payments while preserving zero profits ex ante would lead the agent to shirk at t. To see this, notice that, if the firm wants to hold the ex-ante profit level constant and shift wages across time periods, it can do so by setting

$$\Delta w_1 = -\frac{1 + e^{-x_2}}{1 + e^{-x_1}} \Delta w_2.$$

Then by construction,

$$\Delta w_1 + \Delta w_2 = -\left(e^{-x_1} \Delta w_1 + e^{-x_2} \Delta w_2\right).$$

Therefore, by delaying payments (in a profit-neutral way, and without affecting effort), incentives at time t can be increased. Consider the function

$$\phi_t = \phi_T - \int_t^T e^{-x_s} \left(w_s - v + \alpha\lambda\right) \mathrm{d}s$$

and two times t_1 and t_2. Indeed, if $\Delta w_2 > 0$, then $\Delta w_1 < 0$ and $\Delta w_1 + \Delta w_2 > 0$, which increases ϕ_1. Conversely, anticipating payments reduces incentives ϕ_1. □

For $T > T^*$, we cannot obtain full effort throughout. Our candidate solution is then characterized by $t_0 = 0$, $t_1 < T$, indifference at $t = T$, and zero profits at $t = 0$. The final belief is given by $x_T = x_t + \lambda\,(T - t) + \bar{u}\,(t_1 - t)$. It is useful to rewrite our three conditions in beliefs space. We have

$$(k - \alpha - L)\, e^{-x_T} - \alpha + (v/\lambda - \alpha)\left(e^{-x_1} - e^{-x_T}\right) = 0, \tag{24}$$

$$e^{-x_0} - e^{-x_T} - \frac{v - \alpha\lambda}{\lambda + \bar{u}} \left(e^{-x_0} - e^{-x_1} + x_1 - x_0\right) - \left(1 + e^{-x_T}\right) L = 0, \tag{25}$$

$$\frac{x_T - x_1}{\lambda} + \frac{x_1 - x_0}{\lambda + \bar{u}} - T = 0, \tag{26}$$

that determine the three variables (L, x_1, x_T) as a function of x_0 and T. In order to compute the solution, we can solve the second one for L and the third for x_T and obtain one equation in one unknown for x_1.

We can now compute the agent's payoff under this compensation scheme

$$\begin{aligned}\tilde{W}(x_0, T) &= \int_0^{t_1} \left(1+e^{-x_s}\right)(v - \alpha\lambda - \alpha\bar{u} - v)\,\mathrm{d}s - \int_{t_1}^{T} \left(1+e^{-x_s}\right) v\mathrm{d}s + \left(1+e^{-x_T}\right)(L-k) \\ &= -\int_{x_0}^{x_1} \left(1+e^{-x}\right)\alpha\mathrm{d}x - \int_{x_1}^{x_T} \left(1+e^{-x}\right)\frac{v}{\lambda}\mathrm{d}x + \left(1+e^{-x_T}\right)(L-k),\end{aligned}$$

where (L, x_1, x_T) are the solution to (24)–(26) given (x_0, T). Plugging in the value of L from (23), we can rewrite payoffs as

$$\tilde{W}(x_0, T) = -\int_{x_0}^{x_1} \left(\frac{v-\alpha\lambda}{\lambda+\bar{u}} + e^{-x}\left(\frac{v+\bar{u}\alpha}{\bar{u}+\lambda} - 1\right)\right)\mathrm{d}x - \int_{x_1}^{x_T} \left(\frac{v}{\lambda} + e^{-x}\frac{v-\lambda}{\lambda}\right)\mathrm{d}x - \left(1+e^{-x_T}\right)k.$$

Now fix x_0 and T. We denote by $J(x)$ the payoff under an offer that follows our candidate solution to an agent who holds belief x. This requires solving the system (24)–(26) as a function of the current belief and the residual time. In particular, we have $J(x) = \tilde{W}\left(x, T - \frac{x-x_0}{\lambda+\bar{u}}\right)$ when $x < x_1(x_0, T)$ and $J(x) = \tilde{W}\left(x, T - \frac{x_1-x_0}{\lambda+\bar{u}} - \frac{x-x_1}{\lambda}\right)$ when $x \geq x_1(x_0, T)$.

Finally, we denote by $Y(x)$ the agent's continuation payoff at x under the original scheme. Notice that the bound in (11) ensures that

$$\frac{\lambda+\bar{u}}{1+e^{x_t}} \geq v - \alpha\lambda,$$

for all $t \leq t_1$ and for all T. This means the firm is running a positive flow profit when paying $v - \alpha\lambda$ during full a effort phase, hence effort at t contributes positively to the lump sum L. In other words, the firm does not obtain positive profits when the agent's continuation value is $Y(x)$. The details on how to derive this bound can be found in the working paper.

C.2.2 Original and relaxed programs

Consider the original program $\mathcal{P}$, and rewrite it in terms of the log-likelihood ratios x_t, up to constant terms.

$$W(t, x_t) = \max_{w,u} \int_t^T \left(1+e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - ke^{-x_T}, \tag{27}$$

$$\text{s.t. } u = \arg\max_u \int_t^T \left(1+e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - ke^{-x_T},$$

$$\forall \tau \geq t: \int_\tau^T \left(1+e^{-x_s}\right)(w_s - v - \alpha u_s)\,\mathrm{d}s - ke^{-x_T} \geq W(\tau, x_\tau), \tag{28}$$

$$0 \leq \int_0^T \left(1+e^{-x_t}\right)\left(\frac{\lambda+u_t}{1+e^{x_t}} - w_t\right)\mathrm{d}t. \tag{29}$$

We first argue that the non negative profit constraint (29) will be binding. This is immediate if we observe that constraint (28) implies the firm cannot make positive profits on any interval $[t, T]$, $t \geq 0$. If it did, the worker could be poached by a competitor that offers, for example, the same wage plus a signing bonus. We now consider a relaxed problem in which we substitute (28) and (29) with the non positive profit constraint (30).

$$\begin{aligned} W(t, x_t) &= \max_{w,u} \int_0^T \left(1+e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - ke^{-x_T}, \\ \text{s.t. } u &= \arg\max_u \int_0^T \left(1+e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - ke^{-x_T}, \\ 0 &\geq \int_\tau^T \left(1+e^{-x_t}\right)\left(\frac{\lambda + u_t}{1+e^{x_t}} - w_t\right)\mathrm{d}t \text{ for all } \tau \leq T. \end{aligned} \tag{30}$$

We then use the following result to further relax this program.

Lemma C.3 *Let $T > T^*$ and consider our candidate solution described in (24)–(26). If another contract generates a strictly higher surplus $W(0, x_0)$, then it must yield a strictly higher x_T.*

Proof of Lemma C.3. We use the fact that our solution specifies maximal frontloading of effort, given x_T. Notice that we can rewrite the social surplus (which is equal to the agent's payoff at time 0) as

$$-(1+k-\alpha)\,e^{-x_T} - \alpha x_T - \int_0^T \left(1+e^{-x_t}\right)(v - \alpha\lambda)\,\mathrm{d}t + \text{Constant}. \tag{31}$$

Therefore, for a given x_T, surplus is maximized by choosing the highest path for x_t, which is obtained by frontloading effort. Furthermore, (31) is strictly concave in x_T. Because $T > T^*$, we know from Proposition 3.4 that, under any non negative payment function w, the agent works strictly less than the social planner. Since the agent receives the entire surplus, his ex ante payoff is then strictly increasing in x_T. □

We therefore consider the even more relaxed problem $\mathcal{P}'$ which is given by

$$\begin{aligned} &\max_{w,u} \; x_T \\ \text{s.t. } u &= \arg\max_u \int_0^T \left(1+e^{-x_t}\right)(w_t - v - \alpha u_t)\,\mathrm{d}t - ke^{-x_T} \\ 0 &\geq \int_\tau^T \left(1+e^{-x_t}\right)\left(\frac{\lambda + u_t}{1+e^{x_t}} - w_t\right)\mathrm{d}t \text{ for all } \tau \leq T. \end{aligned}$$

In Lemma C.4, whose proof is omitted, we prove that our candidate solves the relaxed program. We also show that the agent's continuation value under the original contract is higher than the value of the best contract offered at a later date, and hence that we have found a solution to the original program $\mathcal{P}$.

Lemma C.4 *Let $T > T^*$. The candidate solution described in (24)–(26) solves the relaxed program $\mathcal{P}'$. Furthermore, under the candidate solution, constraint* (28) *in the original program never binds (except at $t = 0$).*

C.3 Proofs for Subsection 5.1

Proof of Lemma 5.1. Suppose that the equilibrium effort is zero on some open set Ω. Consider the sets $\Omega_{t'} = \{(x,s) : s \in (t',T]\}$ such that the trajectory starting at (x,s) intersects Ω. Suppose that u is not identically zero on Ω_0 and let $\tau = \inf\{t' : \ u = 0 \ \text{ on } \Omega_{t'}\}$. That is, for all $t' < \tau$, there exists $(x,s) \in \Omega_{t'}$ such that $u(x,s) > 0$. Suppose first that we take $(x,\tau) \in \Omega_\tau$. According to the definition of τ and Ω_τ, there exists $(x_k,k) \in \Omega$ such that the trajectory starting at (x,τ) intersects Ω at (x_k,k) and along the path the effort is zero. We can write the payoff

$$V(x,\tau) = \int_x^{x_k} \frac{1+e^{-s}}{1+e^{-x}}\left(\frac{\lambda}{1+e^s} - v\right)\frac{1}{\lambda}\mathrm{d}s + \frac{1+e^{-x_k}}{1+e^{-x}}V(x_k,k),$$

or, rearranging,

$$\left(1+e^{-x}\right)V(x,\tau) = -\left(e^{-x_k} - e^{-x}\right)\left(1 - \frac{v}{\lambda}\right) - \frac{v}{\lambda}(x_k - x) + \left(1+e^{-x_k}\right)V(x_k,k),$$

where $V(x_k,k)$ is differentiable. The Hamilton-Jacobi-Bellman (HJB) equation (a function of (x,τ)) can be derived from

$$\begin{aligned} V(x,\tau) \quad &= \quad \frac{\lambda+\hat{u}}{1+e^x}\mathrm{d}t - v\mathrm{d}t \\ &+ \max_u \left[-\alpha u \mathrm{d}t + \left(1 - \frac{\lambda+u}{1+e^x}\mathrm{d}t + o(\mathrm{d}t)\right)(V(x,\tau) + V_x(x,\tau)(\lambda+u)\mathrm{d}t + V_t(x,\tau)\mathrm{d}t + o(\mathrm{d}t))\right], \end{aligned}$$

which gives, taking limits,

$$0 = \frac{\lambda+\hat{u}}{1+e^x} - v + \max_{u\in[0,\bar{u}]}\left[-\alpha u - \frac{\lambda+u}{1+e^x}V(x,\tau) + V_x(x,\tau)(\lambda+u) + V_t(x,\tau)\right].$$

Therefore, if $u(x,\tau) > 0$,

$$-\frac{V(x,\tau)}{1+e^x} - \alpha + V_x(x,\tau) \geq 0, \text{ or } \left(1+e^{-x}\right)V_x(x,\tau) - e^{-x}V(x,\tau) \geq \alpha\left(1+e^{-x}\right),$$

or finally,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) \geq 0.$$

Notice, however, by direct computation, that, because low effort is exerted from (x,τ) to (x_k,k), for all points (x_s,s) on this trajectory, $s \in (\tau,k)$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x_s}\right)V(x_s,s)\right] - \alpha\left(1+e^{-x_s}\right) = -e^{-x_s}\left(1+\alpha-\frac{v}{\lambda}\right) + \frac{v}{\lambda} - \alpha \leq 0,$$

so that, because $x < x_s$, and $1+\alpha-v/\lambda > 0$,

$$\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right] - \alpha\left(1+e^{-x}\right) < 0,$$

a contradiction to $u(x,\tau) > 0$.

If instead $u(x,\tau)=0$ for all $(x,\tau)\in\Omega_\tau$, then there exists $(x',t')\to(x,\tau)\in\Omega_\tau$, $u(x',t')>0$. Because u is upper semi-continuous, for every $\varepsilon>0$, there exists a neighborhood $\mathcal{N}$ of (x,τ) such that $u<\varepsilon$ on $\mathcal{N}$. Hence

$$\lim_{(x',t')\to(x,\tau)}\frac{\partial}{\partial x}\left[\left(1+e^{x'}\right)V(x',t')\right]-\alpha\left(1+e^{x'}\right)=\frac{\partial}{\partial x}\left[\left(1+e^{-x}\right)V(x,\tau)\right]-\alpha\left(1+e^{-x}\right)<0,$$

a contradiction. □

Proof of Theorem 5.2. We start with (1.). That is, we show that $u(x,t)=\bar{u}$ for $x<\underline{x}_t$ in all equilibria. We first define $\underline{x}$ as the solution to the differential equation

$$(\lambda(1+\alpha)-v+(\lambda+\bar{u})\,\alpha e^{\underline{x}(t)}+\bar{u}-((1+k)(\lambda+\bar{u})-(v+\alpha\bar{u}))e^{-(\lambda+\bar{u})(T-t)})\left(\frac{\underline{x}'(t)}{\lambda+\bar{u}}-1\right)=-\bar{u}, \tag{32}$$

subject to $\underline{x}(T)=x^*$. This defines a strictly increasing function of slope larger than $\lambda+\bar{u}$, for all $t\in(T-t^*,T]$, with $\lim_{t\uparrow t^*}\underline{x}(T-t)=-\infty$.[45] Given some equilibrium, and an initial value (x_t,t), let $u(\tau;x_\tau)$ denote the value at time $\tau\geq t$ along the equilibrium trajectory. For all t, let

$$\tilde{x}(t):=\sup\{x_t:\forall\tau\geq t:u(\tau;x_t)=\bar{u}\text{ in all equilibria}\},$$

with $\tilde{x}(t)=-\infty$ if no such x_t exists. By definition the function $\tilde{x}$ is increasing (in fact, for all $\tau\geq t$, $\tilde{x}(\tau)\geq\tilde{x}(t)+(\lambda+\bar{u})(\tau-t)$), and so it is a.e. differentiable (set $\tilde{x}'(t)=+\infty$ if x jumps at t). Whenever finite, let $s(t)=\tilde{x}'(t)/(\tilde{x}'(t)-\lambda)>0$. Note that, from the transversality condition, $\tilde{x}(T)=x^*$. In an abuse of notation, we also write $\tilde{x}$ for the set function $t\to[\lim_{t'\uparrow t}\tilde{x}(t'),\lim_{t'\downarrow t}\tilde{x}(t')]$.

We first argue that the incentives to exert high effort decrease in x (when varying the value x of an initial condition (x,t) for a trajectory along which effort is exerted throughout). Indeed, recall that high effort is exerted iff

$$\frac{\partial}{\partial x}\left(V(x,t)\left(1+e^{-x}\right)\right)\geq\alpha\left(1+e^{-x}\right). \tag{33}$$

The value $V^H(x,t)$ obtained from always exerting (and being paid for) high effort is given by

$$\begin{aligned}(1+e^{-x})V^H(x,t)&=\int_t^T(1+e^{-x_s})\left[\frac{\lambda+\bar{u}}{1+e^{x_s}}-v-\alpha\bar{u}\right]\mathrm{d}s-k(1+e^{-x_T})\\&=\left(e^{-x}-e^{-x_T}\right)\left(1-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)-(T-t)(v+\alpha\bar{u})-k(1+e^{-x_T})\end{aligned} \tag{34}$$

where $x_T=x+(\lambda+\bar{u})(T-t)$. Therefore, using (33), high effort is exerted if and only if

$$k-\left(1+k-\frac{v+\alpha\bar{u}}{\lambda+\bar{u}}\right)\left(1-e^{-(\lambda+\bar{u})(T-t)}\right)\geq\alpha\left(1+e^{x}\right).$$

[45] The differential equation for $\underline{x}$ can be implicitly solved, which yields

$$\begin{aligned}\ln\frac{k-\alpha}{\alpha}&=(\underline{x}_t+(\lambda+\bar{u})(T-t))+\frac{\bar{u}}{\lambda(1+\alpha)+\bar{u}-v}\ln(k-\alpha)\bar{u}(\lambda+\bar{u})\\&\quad-\frac{\bar{u}}{\lambda(1+\alpha)+\bar{u}-v}\ln\left(\begin{array}{c}e^{(\lambda+\bar{u})(T-t)}(\lambda(1+\alpha)+\bar{u}-v)(\lambda(1+\alpha)-v+\alpha(\lambda+\bar{u})e^{\underline{x}_t})\\-(\lambda(1+\alpha)-v)(\lambda(1+\alpha)+\bar{u}-v+(k-\alpha)(\lambda+\bar{u}))\end{array}\right).\end{aligned}$$

Note that the left-hand side is independent of x, while the right-hand side is increasing in x. Therefore, if high effort is exerted throughout from (x,t) onward, then it is also from (x',t) for all $x' < x$.

Fix an equilibrium and a state (x_0, t_0) such that $x_0 + (\lambda + \bar{u})(T - t_0) < x^*$. Note that the equilibrium trajectory must eventually intersect some state $(\tilde{x}_t, t)$. We start again from the formula for the payoff

$$\begin{aligned}\left(1+e^{-x_0}\right) V\left(x_0, t_0\right) &= \int_{t_0}^{t}\left[e^{-x_s}\left(\lambda+u\left(x_s, s\right)\right)-\left(1+e^{-x_s}\right)\left(v+\alpha u\left(x_s, s\right)\right)\right] \mathrm{d} s \\ &\quad +\left(1+e^{-\tilde{x}_t}\right) V^H\left(\tilde{x}_t, t\right).\end{aligned}$$

Let $W(\tilde{x}_t) = V^H(\tilde{x}_t, t)$ (since $\tilde{x}$ is strictly increasing, it is well-defined). Differentiating with respect to x_0, and taking limits as $(x_0, t_0) \to (\tilde{x}_t, t)$, we obtain

$$\begin{aligned}&\lim_{(x_0,t_0)\to(\tilde{x}_t,t)} \frac{\partial\left(1+e^{-x_0}\right) V\left(x_0, t_0\right)}{\partial x_0} \\ =\;& \left[e^{-\tilde{x}_t}\lambda-\left(1+e^{-\tilde{x}_t}\right) v\right] \frac{s\left(\tilde{x}_t\right)-1}{\lambda}+s\left(\tilde{x}_t\right)\left[W'\left(\tilde{x}_t\right)\left(1+e^{-\tilde{x}_t}\right)-W\left(\tilde{x}_t\right) e^{-\tilde{x}_t}\right].\end{aligned}$$

If less than maximal effort can be sustained arbitrarily close to, but before the state $(\tilde{x}_t, t)$ is reached, it must be that this expression is no more than $\alpha\left(1+e^{-\tilde{x}_t}\right)$ in some equilibrium, by (33). Rearranging, this means that

$$\left(1-W(x)+\left(1+e^x\right)\left(W'(x)-\frac{v}{\lambda}\right)\right) s(x)+\left(\frac{v}{\lambda}-\alpha\right) e^x \leq 1+\alpha-\frac{v}{\lambda},$$

for $x = \tilde{x}_t$. Given the explicit formula for W (see (34)), and since $s(\tilde{x}_t) = \tilde{x}'_t / (\tilde{x}'_t - \lambda)$, we can rearrange this to obtain an inequality for $\tilde{x}_t$. The derivative $\tilde{x}'_t$ is smallest, and thus the solution $\tilde{x}_t$ is highest, when this inequality binds for all t. The resulting ordinary differential equation is precisely (32).

Next, we turn to (2.). That is, we show that $u(x,t) = 0$ for $x > \bar{x}_t$ in all equilibria. We define $\bar{x}$ by

$$\bar{x}_t=\ln \left[k-\alpha+\left(\frac{v+\bar{u} \alpha}{\lambda+\bar{u}}-(1+k)\right)\left(1-e^{-(\lambda+\bar{u})(T-t)}\right)\right]-\ln \alpha, \tag{35}$$

which is well-defined as long as $k - \alpha + \left(\frac{v+\bar{u}\alpha}{\lambda+\bar{u}} - (1+k)\right)\left(1 - e^{-(\lambda+\bar{u})(T-t)}\right) > 0$. This defines a minimum time $T - t^*$ mentioned above, which coincides with the asymptote of $\underline{x}$ (as can be seen from (32)). It is immediate to check that $\bar{x}$ is continuous and strictly increasing on $[T - t^*, T]$, with $\lim_{t\uparrow t^*} \bar{x}_{T-t} = -\infty$, $x_T = x^*$, and for all $t \in (T - t^*, T)$, $\bar{x}'_t > \lambda + \bar{u}$.

Let us define $W(x,t) = (1 + e^{-x})V(x,t)$, and re-derive the HJB equation. The payoff can be written as

$$W(x, t)=\left[(\lambda+u(x, t)) e^{-x}-\left(1+e^{-x}\right)(v+\alpha u)\right] \mathrm{d} t+W(x+\mathrm{d} x, t+\mathrm{d} t),$$

which gives

$$0=(\lambda+u(x, t)) e^{-x}-v\left(1+e^{-x}\right)+W_t(x, t)+\lambda W_x(x, t)+\max_{u \in[0, \bar{u}]}\left(W_x(x, t)-\alpha\left(1+e^{-x}\right)\right) u.$$

As we already know (see (33)), effort is maximum or minimum depending on $W_x(x,t) \lesseqgtr \alpha(1 + e^{-x})$. Let us rewrite the previous equation as

$$\begin{aligned}& v-\alpha \lambda-W_t(x, t) \\ =\;& ((1+\alpha) \lambda-v+u(x, t)) e^{-x}+\lambda\left(W_x(x, t)-\alpha\left(1+e^{-x}\right)\right)+\left(W_x(x, t)-\alpha\left(1+e^{-x}\right)\right)^{+} \bar{u}.\end{aligned}$$

Given W_x, W_t is maximized when effort $u(x,t)$ is minimized: the lower $u(x,t)$, the higher $W_t(x,t)$, and hence the lower $W(x,t-\mathrm{d}t) = W(x,t) - W_t(x,t)\,\mathrm{d}t$. Note also that, along any equilibrium trajectory, no effort is never strictly optimal (by (iv)). Hence, $W_x(x,t) \geq \alpha(1+e^{-x})$, and therefore, again $u(x,t)$ (or $W(x,t-\mathrm{d}t)$) is minimized when $W_x(x,t) = \alpha(1+e^{-x})$: to minimize $u(x,t)$, while preserving incentives to exert effort, it is best to be indifferent whenever possible.

Hence, integrating over the equilibrium trajectory starting at (x,t),

$$\begin{aligned}&(v-\alpha\lambda)(T-t) + k\left(1+e^{-x_T}\right) + W(x,t)\\ =\ &\int_t^T u(x_s,s)\,e^{-x_s}\mathrm{d}s + \int_t^T \left[((1+\alpha)\lambda - v)\,e^{-x_s} + (\lambda+\bar{u})\left(W_x(x_s,s) - \alpha\left(1+e^{-x_s}\right)\right)^+\right]\mathrm{d}s.\end{aligned}$$

We shall construct an equilibrium in which $W_x(x_s,s) = \alpha(1+e^{-x_s})$ for all $x > \underline{x}_t$. Hence, this equilibrium minimizes

$$\int_t^T u(x_s,s)\,e^{-x_s}\mathrm{d}s,$$

along the trajectory, and since this is true from any point of the trajectory onward, it also minimizes $u(x_s,s)$, $s \in [t,T]$; the resulting $u(x,t)$ will be shown to be increasing in x, and equal to $\bar{u}$ at $\bar{x}_t$.

Let us construct this interior effort equilibrium. Integrating (33) over any domain with non-empty interior, we obtain that

$$(1+e^x)V(x,t) = e^x(\alpha x + c(t)) - \alpha, \tag{36}$$

for some function $c(t)$. Because the trajectories starting at (x,t) must cross $\underline{x}$ (whose slope is larger than $\lambda+\bar{u}$), value matching must hold at the boundary, which means that

$$(1+e^{\underline{x}_t})V^H(\underline{x}_t,t) = e^{\underline{x}_t}(\alpha\underline{x}_t + c(t)) - \alpha,$$

which gives $c(t)$ (for $t \geq T - t^*$). From (36), we then back out $V(x,t)$. The HJB equation then reduces to

$$v - \alpha\lambda = \frac{\lambda + u(x,t)}{1+e^x} + V_t(x,t),$$

which can now be solved for $u(x,t)$. That is, the effort is given by

$$\begin{aligned}\lambda + u(x,t) &= (1+e^x)(v-\alpha\lambda) - \frac{\partial}{\partial t}\left[(1+e^x)V(x,t)\right]\\ &= (1+e^x)(v-\alpha\lambda) - e^x c'(t).\end{aligned}$$

It follows from simple algebra (c' is detailed below) that $u(x,t)$ is increasing in x. Therefore, the upper end $\bar{x}_t$ cannot exceed the solution to

$$\lambda + \bar{u} = \left(1+e^{\bar{x}}\right)(v-\alpha\lambda) - e^{\bar{x}}c'(t),$$

and so we can solve for

$$e^{\bar{x}} = \frac{\lambda(1+\alpha) - v + \bar{u}}{v - \alpha\lambda - c'(t)}.$$

Note that, from totally differentiating the equation that defines $c(t)$,

$$\begin{aligned} c'(t) &= \underline{x}'(t)\, e^{-\underline{x}(t)} \left[\left(W'(\underline{x}(t)) - \alpha\right)\left(e^{\underline{x}(t)} + 1\right) - W(\underline{x}(t)) \right] \\ &= v - \alpha\lambda + e^{-\underline{x}(t)} \left(v - (1+\alpha)\lambda\right), \end{aligned}$$

where we recall that $\underline{x}$ is the lower boundary below which effort must be maximal, and $W(\underline{x}) = V^H(\underline{x}_t, t)$. This gives

$$e^{\bar{x}} = e^{\underline{x}} \frac{\lambda(1+\alpha) - v + \bar{u}}{\lambda(1+\alpha) - v}, \text{ or } e^{\underline{x}} = \frac{\lambda(1+\alpha) - v}{\lambda(1+\alpha) - v + \bar{u}} e^{\bar{x}}.$$

Because (32) is a differential equation characterizing $\underline{x}$, we may substitute for $\bar{x}$ from the last equation to obtain a differential equation characterizing $\bar{x}$, namely

$$\begin{aligned} &\frac{\bar{u}}{1 - \frac{\bar{x}'(t)}{\lambda + \bar{u}}} + \left((1+k)(\lambda + \bar{u}) - (v + \alpha\bar{u})\right) e^{-(\lambda+\bar{u})(T-t)} \\ = \;& \lambda(1+\alpha) + \bar{u} - v + \frac{\alpha(\lambda+\bar{u})(\lambda(1+\alpha) - v)}{\lambda(1+\alpha) - v + \bar{u}} e^{\bar{x}}, \end{aligned}$$

with boundary condition $\bar{x}(T) = x^*$. It is simplest to plug in the formula given by (35) and verify that it is indeed the solution of this differential equation.

Finally, we prove (3.). The same procedure applies to both, so let us consider $\bar{\sigma}$, the strategy that exerts high effort as long as $x < \bar{x}_t$, (and no effort above). We shall do so by "verification." Given our closed-form expression for $V^H(x,t)$ (see (34)), we immediately verify that it satisfies the (33) constraint for all $x \leq \bar{x}_t$ (remarkably, $\bar{x}_t$ is *precisely* the boundary at which the constraint binds; it is strictly satisfied at $\underline{x}_t$, when considering $\underline{\sigma}$). Because this function $V^H(x,t)$ is differentiable in the set $\{(x,t) : x < \bar{x}_t\}$, and satisfies the HJB equation, as well as the boundary condition $V^H(x,T) = 0$, it is a solution to the optimal control problem in this region (remember that the set $\{(x,t) : x < \bar{x}_t\}$ cannot be left under any feasible strategy, so that no further boundary condition needs to be verified). We can now consider the optimal control problem with exit region $\Omega := \{(x,t) : x = \bar{x}_t\} \cup \{(x,t) : t = T\}$ and salvage value $V^H(\bar{x}_t, t)$ or 0, depending on the exit point. Again, the strategy of exerting no effort satisfies the HJB equation, gives a differentiable value on $\mathbb{R} \times [0,T] \setminus \Omega$, and satisfies the boundary conditions. Therefore, it is a solution to the optimal control problem. □

Comparative statics for the boundary: We focus on the equilibrium that involves the largest effort region.

Proposition C.5 *The boundary of the maximal effort equilibrium $\underline{p}(t)$ is non-increasing in k and v and non-decreasing in α and λ.*

This proposition follows directly by differentiating expression (35) for the frontier $\bar{x}(t)$.

The effect of the maximum effort level $\bar{u}$ is ambiguous. Also, one might wonder whether increasing the penalty k increases welfare, for some parameters, as it helps resolve the commitment problem. Unlike in the non-observable case, this turns out never to occur, at least in the maximum-effort equilibrium: increasing the penalty decreases welfare, though it increases total effort. The proof is in Appendix C.3. Similarly, increasing v increases effort (in the maximum-effort equilibrium), though it decreases the worker's payoff.

□

Proof of Proposition 5.3 (1.) The equation defining the full effort frontier in the unobservable case $x_2(t)$ is given by

$$(k-\alpha)\, e^{-x_2-(\lambda+u)(T-t)} - \alpha - \int_{x_2}^{x_2+(\lambda+u)(T-t)} e^{-x} \left(\frac{1}{1+e^x} - \frac{v-\alpha\lambda}{\lambda+\bar{u}} \right) \mathrm{d}x. \tag{37}$$

Plug the expression for $\bar{x}(t)$ given by (35) into (37) and notice that (37) cannot be equal to zero unless $\bar{x}(t) = x^*$ and $t = T$, or $\bar{x}(t) \to -\infty$. Therefore, the two frontiers cannot cross before the deadline T, but they have the same vertical asymptote.

Now suppose that $\phi'(x^* \mid \bar{u}) > 0$ so that the frontier $x_2(t)$ goes through (T, x^*). Consider the slopes of $x_2(t)$ and $\bar{x}(t)$ evaluated at (T, x^*). We obtain

$$[\bar{x}'(t) - x_2'(t)]_{t=T} \propto (\bar{u}+\lambda)(k-\alpha) > 0,$$

so the unobservable frontier lies above the observable one for all t.

Next, suppose $\phi'(x^* \mid \bar{u}) < 0$, so there is no mixing at x^* and the frontier $x_2(t)$ does not go through (T, x^*). In this case, we still know the two cannot cross, and we also know a point on $x_2(t)$ is the pre-image of (T, x^*) under full effort. Since we also know the slope $\bar{x}'(t) > \lambda + \bar{u}$, we again conclude that the unobservable frontier $x_2(t)$ lies above $\bar{x}(t)$.

Finally, consider the equation defining the no effort frontier $x_3(t)$,

$$(k-\alpha)\, e^{-x_3-\lambda(T-t)} - \alpha - \int_{x_3}^{x_3+\lambda(T-t)} e^{-x} \left(\frac{1}{1+e^x} - \frac{v-\alpha\lambda}{\lambda} \right) \mathrm{d}x = 0. \tag{38}$$

Totally differentiating with respect to t shows that $x_3'(t) < \lambda$ (might be negative). Therefore, the no effort region does not intersect the full effort region defined by $\bar{x}(t)$ in the observable case.

(2.) To compare the effort regions in the unobservable case and the full effort region in the social optimum, consider the planner's frontier $x_P(t)$, which is given by

$$x_P(t) = \ln \Big((1+k-v/\lambda) e^{-\lambda(T-t)} - (\alpha - v/\lambda) \Big) - \ln \alpha.$$

The slope of the planner's frontier is given by

$$x_P'(t) = \lambda \frac{(1+k-v/\lambda)e^{-\lambda(T-t)}}{(1+k-v/\lambda)e^{-\lambda(T-t)} + v/\lambda - \alpha} \in [0, \lambda].$$

In the equilibrium with unobservable effort, all effort ceases above the frontier $x_3(t)$ defined in (38) above, which has the following slope

$$x_3'(t) = \lambda \frac{\Big(\big(1+e^{x_3+\lambda(T-t)}\big)^{-1} + k - v/\lambda \Big) e^{-\lambda(T-t)}}{\Big(\big(1+e^{x_3+\lambda(T-t)}\big)^{-1} + k - v/\lambda \Big) e^{-\lambda(T-t)} + v/\lambda - \alpha - (1+e^{x_3})^{-1}}.$$

We also know $x_3(T) = x^*$ and $x_P(T) = \ln((1+k-\alpha)/\alpha) > x^*$. Now suppose towards a contradiction that the two frontiers crossed at a point (t, x). Plugging in the expression for $x_P(t)$ in both slopes, we obtain

$$x_3'(t) = \left(1 + \frac{v/\lambda - \alpha - s(t)}{(1+k-v/\lambda+(1-s(t)))e^{-\lambda(T-t)}}\right)^{-1} > \left(1 + \frac{v/\lambda - \alpha}{(1+k-v/\lambda)e^{-\lambda(T-t)}}\right)^{-1} = x_P'(t),$$

with

$$s(t) = 1/\left(1 + e^{x_P(t)}\right) \in [0,1],$$

meaning the unobservable frontier would have to cross from below, a contradiction. □