Internal Mode Mechanism for Collective Energy Transport in Extended Systems

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(Received 9 May 2003; published 5 December 2003)

We study directed energy transport in homogeneous nonlinear extended systems in the presence of homogeneous ac forces and dissipation. We show that the mechanism responsible for unidirectional motion of topological excitations is the coupling of their internal and translation degrees of freedom. Our results lead to a selection rule for the existence of such motion based on resonances that explain earlier symmetry analysis of this phenomenon. The direction of motion is found to depend both on the initial and the relative phases of the two harmonic drivings, even in the presence of noise.

DOI: 10.1103/PhysRevLett.91.234102 PACS numbers: 05.45.Yv, 02.30.Jr, 05.60.Cd, 63.20.Pw

One intriguing phenomenon that is receiving much attention recently is net directed motion induced by zero average forces. Originally motivated by stochastic models of biomolecular (brownian) motors [1], deterministic ratchetlike systems [2,3] are being intensively studied, chiefly because of their many potential technological applications [4]. Many such models consist of one or two particles on a periodic, asymmetric potential and a periodic force (rocking ratchet [1]). Later, the investigation was generalized to systems with many interacting particles, from noisy soliton-bearing systems [5,6] to other spatially extended (stochastic and deterministic, overdamped and underdamped) systems, both theoretically [7] and from a more applied [8] viewpoint.

Among this class of problems, net transport in *homogeneous* extended systems driven by *homogeneous* ac forces is particularly interesting. A paradigmatic example is the ac driven, damped sine-Gordon (sG) equation:

$$\phi_{tt} - \phi_{xx} + \sin(\phi) = -\beta\phi_t + f(t). \qquad (1)$$

A symmetry analysis, proposed for one-particle systems in [3] and extended to this problem [9,10], indicated that a directed energy current appeared if $f(t)$ broke the symmetry $f(t) = -f(t + T/2)$, T being the period of the external driving. One such choice is $f(t) \equiv \epsilon_1 \sin(\delta t + \delta_0) + \epsilon_2 \sin(m\delta t + \delta_0 + \theta)$ ([9,10] with $\delta_0 = \pi/2$), a case for which numerical simulations of the sG equation confirmed the symmetry analysis results. In what follows, we will refer to δ_0 as the initial phase and to θ as the relative phase. Transport required a nonzero topological charge, implying the existence of sG solitons (kinks) in the system. In this respect, we stress that kink-mediated transport is impossible with only one harmonic for any value of the damping coefficient β [11]. It was argued in [10] that the observed rectification arises from the nonadiabatic excitation of internal kink modes and their interaction with the translational kink motion. This conjecture had no rigorous support; rather, it was based on plots of sG soliton evolution and on the failure of a collective coordinate (CC) approach [12] with one degree of freedom, which assumed that sG solitons behave similar to rigid particles. An attempt to include the width degree of freedom has been recently presented in [13], where it was concluded that the directed energy current vanishes unless the width of the kink, $l(t)$, is a dynamical variable. However, this condition is only a necessary one: $l(t)$ is a dynamical variable in the one-harmonic case but the kink velocity is zero for any value of the damping as already mentioned [11]. Another point not accounted for in [13] is the connection between the internal mode mechanism and the symmetry analysis, which also prohibits motion in other cases where $l(t)$ is a dynamical variable. Therefore, the reasons for the phenomena observed in [9,10] remained largely obscure.

In this Letter, a different CC approach allows us to identify the mechanism through which the width oscillation drives the kink and its relation with the symmetry conditions. Furthermore, our theory predicts, and numerical simulations of Eq. (1) confirm, that the direction of motion depends on the *initial phase* of the driving, even in the presence of additive noise. Our CC theory is based on an ansatz, proposed in [14], for the perturbed kink depending on two CC, $X(t)$ and $l(t)$ (respectively, position and width of the kink). It is not difficult to show [14–16] that the dynamics of these two CC is given by

$$\frac{dP}{dt} = -\beta P - qf(t), \qquad (2)$$

$$\dot{l}^2 - 2l\ddot{l} - 2\beta l\dot{l} = \Omega_R^2 l^2\left[1 + \frac{P^2}{M_0^2}\right] - \frac{1}{\alpha}, \qquad (3)$$

where the momentum $P(t) = M_0 l_0 \dot{X}/l(t)$, $\Omega_R = 1/(\sqrt{\alpha} l_0)$ with $\alpha = \pi^2/12$ is the so-called Rice's frequency, and $M_0 = 8$, $q = 2\pi$, and $l_0 = 1$ are, respectively, the dimensionless kink mass, topological charge, and unperturbed width. Equation (2) can be solved exactly, and in the large time limit ($t \gg \beta^{-1}$) yields

$$P(t) = -\sqrt{\epsilon}[\, a_1 \sin(\delta t + \delta_0 - \chi_1) + a_2 \sin(m\delta t + \delta_0 + \theta - \chi_2)],$$

where ϵ is merely a rescaling parameter in the perturbation expansion, to be determined later; $\chi_1 = \arctan(\delta/\beta)$, $\chi_2 = \arctan(m\delta/\beta)$, $a_1 = q\epsilon_1/\sqrt{\epsilon(\beta^2+\delta^2)}$, and $a_2 = q\epsilon_2/\sqrt{\epsilon(\beta^2+m^2\delta^2)}$. As we are interested in the damped ($\beta \neq 0$) case and Eq. (3) cannot be solved in that case [15,16], we will study it by a perturbative expansion, $l(t) = l_0 + \epsilon l_1(t) + \epsilon^2 l_2(t) + \cdots$. At order $O(\epsilon)$, we obtain

$$\ddot{l}_1(t) + \beta\dot{l}_1(t) + \Omega_R^2 l_1(t) = -\Omega_R^2 P^2(t) l_0/2\epsilon M_0^2. \tag{4}$$

The key point is that, by substituting the expression of $P(t)$ into (4), we see that the equation for $l_1(t)$ contains harmonics of frequencies 2δ, $2m\delta$, and $(m \pm 1)\delta$; i.e.,

$$\ddot{l}_1(t) + \beta\dot{l}_1(t) + \Omega_R^2 l_1(t) = A_1 + A_2\cos(2\delta t + 2\delta_0 - 2\chi_1) + A_3\cos(2m\delta t + 2\delta_0 + 2\theta - 2\chi_2) + A_4\cos[(m-1)\delta t + \theta - (\chi_2 - \chi_1)] - A_4\cos[(m+1)\delta t + 2\delta_0 + \theta - (\chi_2 + \chi_1)],$$

where $A_1 = -A_2 - A_3$, $A_2 = \Omega_R a_1^2/4\sqrt{\alpha}M_0^2$, $A_3 = \Omega_R a_2^2/4\sqrt{\alpha}M_0^2$, and $A_4 = -\Omega_R a_1 a_2/2\sqrt{\alpha}M_0^2$. After transients elapse, we find

$$l_1(t) = \frac{A_1}{\Omega_R^2} + \frac{A_2\sin(2\delta t + 2\delta_0 - 2\chi_1 + \tilde{\theta}_2)}{\sqrt{(\Omega_R^2 - 4\delta^2)^2 + 4\beta^2\delta^2}} + \frac{A_3\sin(2m\delta t + 2\delta_0 + 2\theta - 2\chi_2 + \tilde{\theta}_{2m})}{\sqrt{(\Omega_R^2 - 4m^2\delta^2)^2 + 4m^2\beta^2\delta^2}} + \frac{A_4\sin[(m-1)\delta t + \theta - (\chi_2 - \chi_1) + \tilde{\theta}_{m-1}]}{\sqrt{[\Omega_R^2 - (m-1)^2\delta^2]^2 + \beta^2(m-1)^2\delta^2}} - \frac{A_4\sin[(m+1)\delta t + 2\delta_0 + \theta - (\chi_2 + \chi_1) + \tilde{\theta}_{m+1}]}{\sqrt{[\Omega_R^2 - (m+1)^2\delta^2]^2 + \beta^2(m+1)^2\delta^2}}, \tag{5}$$

where $\tilde{\theta}_m = \arctan[(\Omega_R^2 - m^2\delta^2)/m\beta\delta]$. A cumbersome but otherwise trivial calculation yields the harmonics contained in $l_2(t)$, collected in Table I.

Next, we need to compute the average velocity over one period $T = 2\pi/\delta$: In the CC approach, we use the definition of the momentum and find

$$\langle \dot{X}(t)\rangle = \frac{1}{T}\int_0^T \frac{P(t)l(t)}{M_0 l_0}\, dt. \tag{6}$$

At $O(\epsilon^0)$, the averages $\langle P(t)\rangle$ and $\langle \dot{X}_0(t)\rangle$ vanish trivially; therefore, net kink motion can arise only in next order. By straightforward calculations from Eqs. (5) and (6), we find for $m = 2$ that, for large enough times,

$$\epsilon\langle \dot{X}_1\rangle = \frac{q^3\Omega_R^2\epsilon_1^2\epsilon_2}{8M_0^3(\beta^2+\delta^2)\sqrt{\beta^2+4\delta^2}}\left(\frac{2\cos[\delta_0 - \theta + (\chi_2 - 2\chi_1) - \tilde{\theta}_1]}{\sqrt{(\Omega_R^2-\delta^2)^2 + \beta^2\delta^2}} - \frac{\cos[\delta_0 - \theta + (\chi_2 - 2\chi_1) + \tilde{\theta}_2]}{\sqrt{(\Omega_R^2 - 4\delta^2)^2 + 4\beta^2\delta^2}}\right). \tag{7}$$

From Eq. (7), we see that for ϵ to be small the prefactor on the right-hand side has to be much smaller than 1. A definite, verifiable prediction from this asymptotic expression is the existence of a nonzero velocity for $m = 2$, with a sinusoidal dependence on δ_0 and θ. This means that the velocity depends on both the initial and the relative phases; indeed, by letting $\delta_0 \equiv \delta t_0$ in Eq. (1) and changing variables to $t' = t + t_0$, it can be immediately seen that an initial phase δ_0 is equivalent to a relative phase $\theta' = \theta - (m-1)\delta_0$ for a kink with its center shifted to $x_0 + Vt_0$. The dependence of the velocity on θ agrees with (and explains) [9,10], whereas the dependence on δ_0 is a totally new result. Nevertheless, these analytical results as well as the numerical simulations we present below strongly support the present conclusion.

For the case $m = 3$, the average velocity is zero at all orders, a result confirmed by direct numerical simulation of the full sG Eq. (1) as we will see below. The reason can be understood by looking at Table I: For $m = 3$, the frequencies of the ac force (or the momentum) are odd harmonics (δ and 3δ), whereas the width of the kink oscillates only with even harmonics ($2n\delta$, $n \in \mathbb{N}$). This leads us to our main conclusion, namely, the mechanism for the appearance of net motion and the corresponding selection rules. Equations (2) and (3) show that the force acts on the kink width through $P^2(t)$, whereas $P(t)$ itself is in turn inversely proportional to $l(t)$. This coupling is the

TABLE I. Harmonic content of the first contributions to the perturbative expansion of $l(t)$.

Harmonic	l_1	l_2
m	2δ, $2m\delta$, $(m \pm 1)\delta$	2δ, 4δ, $2m\delta$, $4m\delta$, $(m \pm 1)\delta$, $2(m \pm 1)\delta$, $(m \pm 3)\delta$, $(3m \pm 1)\delta$
2	δ, 2δ, 3δ, 4δ	δ, 2δ, 3δ, 4δ, 5δ, 6δ, 7δ, 8δ
3	2δ, 4δ, 6δ	2δ, 4δ, 6δ, 8δ, 10δ, 12δ

responsible for the net kink motion but, for it to be actually possible, the harmonic content of the effective force $P^2(t)$ acting on the width degree of freedom must be able to resonate with it. This is evident from Eq. (6), in which the integral is nonzero only if $l(t)$ contains at least one of the harmonics of $P(t)$. It is important to realize that this condition is much more restrictive than that found in [13], where only the necessity of $l(t)$ being a dynamic variable was pointed out. We have just seen that this is indeed necessary, but that additional, crucial resonance conditions have to be fulfilled. Interestingly, our theory shows also that dissipation can change or even revert the kink velocity [see Eq. (7)] in agreement with the numerical results in [9,10]. A more detailed discussion of this point is forthcoming [17].

These predictions from the CC approximation must be confirmed by a numerical solution of the full partial differential Eq. (1). We do this by using the Strauss-Vázquez scheme [18], on systems of length $L = 100$, 1000, with steps $\Delta t = 0.01$, $\Delta x = 0.1$, free boundary conditions, and a kink at rest as an initial condition. Instead of the perturbative expressions (which are only qualitatively correct unless $\epsilon \ll 1$), to assess the validity of our theory we numerically integrate Eq. (3) and the equation for the velocity obtained from the expression of $P(t)$ [Eq. (2)] with a fourth-order Runge-Kutta method. Our main results are shown in Figs. 1–3; they fully confirm the accuracy, even quantitative, of our approach. Figure 1 exhibits clearly the sinusoidal dependence of the velocity as a function of the initial phase. The dependence on θ is also seen as a simple shift when changing from $\theta = 0$ to $\theta = \pi/2$. The agreement with the CC results is perfect. As a further check of the robustness of this dependence, following [9] we have simulated Eq. (1) with an additional additive Gaussian white noise term with variance D. While one could, in principle, think that this noise would suppress the initial phase dependence, Fig. 2 shows that the opposite is the case: The noise enhances the dependence on the initial phase, increasing the maximum values of the velocity while keeping the same general sinusoidal dependence and the location of the zeros. It is tempting to conclude from this plot that the noise, at least if it is not very large ($D \ll 1$), assists the process of energy transfer between the width and the translation degrees of freedom, activating it. Finally, Fig. 3 makes it clear that our main result, namely the interpretation of the physics of the problem, is indeed true, by showing the harmonic content of $l(t)$ for $m = 2$ and 3. In this case, the agreement between our CC theory and the full numerical simulation of Eq. (1) is indeed impressive, and validates firmly our resonance criterion for net kink motion. It is important to stress that the present theory does not apply to the net motion found for $m = 3$ in [10]. We have confirmed their result in our simulations, which allowed us to realize that this is an altogether different phenomenon: First, it appears only above a (moderately large, $\epsilon_i \gtrsim 0.4$) threshold amplitude, and, second, it is induced by the *kink wings*, which are highly distorted in the process yielding the CC picture inappropriate (even kink-antikink pairs are created).

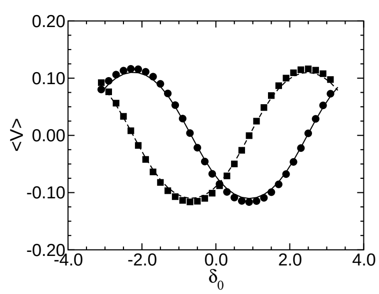

FIG. 1. Dependence of the kink velocity on the initial phase. Parameters are $\epsilon_1 = \epsilon_2 = 0.2$, $\beta = 0.05$, $\delta = 0.1$. Relative phase $\theta = \pi/2$: solid line, CC theory; filled circles, simulation results. Relative phase $\theta = 0$: dashed line, CC theory; squares, simulation results.

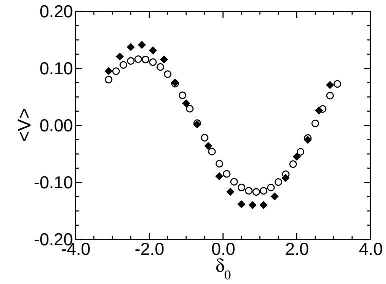

FIG. 2. Dependence of the kink velocity on the initial phase for relative phase $\theta = \pi/2$ in the deterministic ($D = 0$, empty circles) and the stochastic ($D = 0.03$, diamonds) cases. Other parameters are as in Fig. 1.

In conclusion, we have found that the symmetry conditions set forth in [9,10] have their physical origin in the mechanism of the directed motion: the indirect action of the force through the coupling of the translational and width degrees of freedom. To make net motion possible, this indirect driving has to resonate with the available frequencies for the width. This interpretation does not contradict the nonexistence of internal modes in sG kinks, shown in [16], because external forces can induce, via excitation of certain phonons, behavior similar to the one expected from an intrinsic internal mode [19,20]. The fact that a force with only one harmonic would not drive the damped sG kink [11], and that two harmonics are needed to simultaneously excite the width oscillations and induce net motion, fits nicely in this picture. On the other hand, this point raises the question as to the generality of our results, in view of the fact that most kink-bearing systems do have internal modes. To answer this question, we have studied the same problem in the framework of the ϕ^4 model, reaching the same conclusions [17]: Indeed, the intrinsic internal mode of ϕ^4 kinks

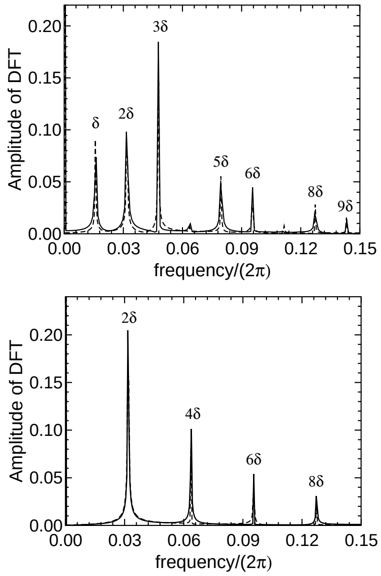

FIG. 3. Discrete Fourier transform of the kink width. Upper panel: $m = 2$; lower panel: $m = 3$. Solid line: amplitude measured in simulations. Dashed line: numerical integration of the CC equations. Parameters are as in Fig. 1 for relative phase $\theta = \pi/2$ and initial phase $\delta_0 = -2.5$.

makes the phenomenon even more noticeable, making us confident on the wide applicability of this work.

Another important conclusion is the dependence of the velocity on the initial phase δ_0, not mentioned in earlier work [9,10]. We note that this dependence allows much more flexibility in controlling the kink velocity, providing an alternative to the use of the relative phase suggested earlier. On the other hand, this may have important consequences for applications as a way of separating, e.g., fluxons in long Josephson junctions [8]. Interestingly, such superconducting devices provide the best possible laboratory to verify our results. This experimental confirmation is crucial in order to ascertain their applicability. Given the accuracy with which the sG equation describes long Josephson junctions, and the fact that an external force such as the one proposed in this and earlier works [9,10] is easy to implement, we hope that the corresponding measurements will soon be carried out. A conclusive, positive verification of our theory would yield the picture we provide here very useful in that and related contexts.

This work has been supported by the Ministerio de Ciencia y Tecnología of Spain through Grants No. BFM2001-3878-C02 (N. R. Q.), No. BFM2000-0006, and No. BFM2003-07749-C05-01 (A. S.), by the Junta de Andalucía under Project No. FQM-0207, by DAAD (Germany) A0231253/Ref. 314, and by the International Research Training Group "Nonequilibrium Phenomena and Phase Transitions in Complex Systems."

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[1] P. Reimann, Phys. Rep. **361**, 57 (2002).
[2] P. Jung *et al.*, Phys. Rev. Lett. **76**, 3436 (1997); T. E. Dialynas *et al.*, Phys. Rev. E **56**, 3976 (1997); J. L. Mateos, Phys. Rev. Lett. **84**, 258 (2000).
[3] S. Flach *et al.*, Phys. Rev. Lett. **84**, 2358 (2000).
[4] Special issue on *Ratchets and Brownian Motors: Basics, Experiments and Applications*, edited by H. Linke [Appl. Phys. A **75** (2002)].
[5] F. Marchesoni, Phys. Rev. Lett. **77**, 2364 (1996).
[6] A.V. Savin *et al.*, Phys. Lett. A **229**, 279 (1997); Phys. Rev. E **56**, 2457 (1997).
[7] P. Reimann *et al.*, Europhys. Lett. **45**, 545 (1999); C. Van den Broeck *et al.*, Ann. Phys. (Leipzig) **9**, 713 (2000); J. Buceta *et al.*, Phys. Rev. E **61**, 6287 (2000); G. Costantini and F. Marchesoni, Phys. Rev. Lett. **87**, 114102 (2001); Z. Zheng *et al.*, Phys. Rev. Lett. **89**, 154102 (2002); M. Salerno and N. R. Quintero, Phys. Rev. E **65**, 025602 (2002); G. Costantini *et al.*, Phys. Rev. E **65**, 051103 (2002); L. M. Floría *et al.*, Europhys. Lett. **60**, 174 (2002).
[8] G. Carapella and G. Costabile, Phys. Rev. Lett. **87**, 077002 (2001); E. Goldobin *et al.*, Phys. Rev. E **63**, 031111 (2001); F. Falo *et al.*, in [4], p. 263.
[9] S. Flach *et al.*, Phys. Rev. Lett. **88**, 184101 (2002).
[10] M. Salerno and Y. Zolotaryuk, Phys. Rev. E **65**, 056603 (2002).
[11] N. R. Quintero and A. Sánchez, Phys. Lett. A **247**, 161 (1998); Eur. Phys. J. B **6**, 133 (1998).
[12] A. Sánchez and A. R. Bishop, SIAM Rev. **40**, 579 (1998).
[13] C. R. Willis and M. Farzaneh, arXiv:cond-mat/0212125.
[14] M. Salerno and A. C. Scott, Phys. Rev. B **26**, 2474 (1982); M. J. Rice, Phys. Rev. B **28**, 3587 (1983).
[15] N. R. Quintero *et al.*, Phys. Rev. Lett. **84**, 871 (2000).
[16] N. R. Quintero *et al.*, Phys. Rev. E **62**, R60 (2000).
[17] L. Morales-Molina, N. R. Quintero, F. G. Mertens, and A. Sánchez (unpublished).
[18] W. A. Strauss and L. Vázquez, J. Comput. Phys. **28**, 271 (1978).
[19] N. R. Quintero and P. G. Kevrekidis, Phys. Rev. E **64**, 056608 (2001).
[20] J. A. González *et al.*, Phys. Rev. E **65**, 065601 (2002); Chaos, Solitons & Fractals (to be published).