Dynamics of a ϕ^4 kink in the presence of strong potential fluctuations, dissipation, and boundaries

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We have carried out a number of simulations to study the dynamical behavior of kinks in the ϕ^4 model in the presence of strong fluctuations of its double-well potential. Our work widens the computational and analytical knowledge of this system in four directions. First, we describe in detail a numerical procedure that can be easily generalized to other stochastic, soliton-bearing equations. We demonstrate that it exhibits consistency features never found in previous research on nonlinear stochastic partial differential equations. Second, we fix the range of validity of theoretical approaches based on secular perturbative expansions. We show how this range depends on a combination of noise strength and duration. Third, we numerically study the model beyond the applicability of analytical methods. We compute the main characteristics of kink dynamics in this regime and discuss their stability under this random perturbation. Finally, we introduce dissipation and boundaries in the dynamically disordered model. We establish that the essential consequence of friction action is to soften the noise effects, while boundaries give rise to a critical velocity below which kinks cannot enter the noisy zone.

I. INTRODUCTION

Since the pioneering work of Fermi, Pasta, and Ulam on the anharmonic lattice[1] and the discovery of solitons by Zabusky and Kruskal,[2] an increasing amount of effort has been devoted to the study of nonlinear models,[3] and different nonlinear equations have been related to a large variety of physical phenomena.[4] Among the most widely studied, those of the nonlinear Klein-Gordon (NKG) type have proved themselves very fruitful. The ϕ^4 model belongs to this class, and it was first proposed to describe structural phase transitions by Aubry,[5] and Krumhansl and Schrieffer,[6] and subsequently in a variety of other contexts.[7] As in other nonlinear problems, progress has been achieved in research on this kind of system through the close collaboration of analytical and numerical investigations.[8] In the case of the ϕ^4 model, its nonintegrability severely restricted the number of applicable analytical techniques. Thus, numerical simulations soon turned out to be a very useful (and often the only) tool to build up a nonperturbative picture of this system. Computer work has been concerned both with statistical mechanics[9] and kink dynamics, and kink-antikink interactions,[10] and at the present time the features of the ϕ^4 model are rather well known. However, in real physical systems it becomes necessary to understand how nonlinear excitations interact with external forces, spatial inhomogeneities, heat baths, thermal fluctuations, etc. This is the reason why several different perturbation terms are introduced in the framework of the ϕ^4 model, such as dissipation and external forces[11] or the existence of impurities.[12] Besides, following the rapidly growing attention to the mutual influence between disorder and nonlinearity, random perturbations are being considered in all soliton-bearing models (see Ref. 13 for a review). Aside from physical applications, these problems are interesting on their own, because they lead one to face the hitherto poorly known subject of nonlinear, stochastic partial differential equations.[14,15]

In the ϕ^4 model, some work concerning small-amplitude wave-packet scattering by irregularly spaced arrays of point impurities has been done,[16] but to date most of the studies deal with perturbations fluctuating in time. Certain results have been obtained by Bass, Konotop, and Sinitsyn[17] concerning the so-called collective coordinates of the kink (its center and its speed) using some analytical approach valid for weak noise. The effects of additive and linearly multiplicative noise on the ϕ^4 kinks were numerically studied by Rodríguez-Plaza and Vázquez.[18] Hence, it seems that the natural extension of the previous research would be to consider the fully nonlinear problem, not restricted to weak perturbations but for all noise intensities. We have addressed this question, both numerically (a brief, partial account of our work in the stochastic, but otherwise unperturbed, ϕ^4 equation was presented in Ref. 19) and analytically (we have worked out a secular perturbative approach for weak noises in Ref. 20). This paper is aimed at giving a thorough description of our findings concerning the strongly stochastic ϕ^4 model, including details on the computer simulation and new results on the effect of dissipation and boundaries as external perturbations.

The paper is organized as follows. In Sec. II we review the stochastic ϕ^4 model and some perturbative predictions,[20] deriving them in a simple way for the sake of completeness of exposition. In Sec. III we give details on our numerical procedure and test its accuracy by comparing with exact results and by checking its dependence on the way that averages are done. Section IV is devoted to the propagation of kinks through an infinite fluctuating layer; first, weak noise is studied and its effects on the kink explained by the perturbative theory; and second, we deal with the strong-noise regime and point out its main novel features. Numerical simulations allow us to obtain empirical laws for the mean dispersion and energy evolution of the system. Section V presents results on the same system when dissipation is taken into account. Section VI is concerned with the effects that happen when kinks must cross a boundary between two semi-infinite slabs, from an unperturbed to a perturbed zone. Finally, Section VII contains a summary of our results.

II. WEAK-NOISE REGIME: ADIABATIC APPROACH

The one-dimensional ϕ^4 model[5,6] is a chain of particles of identical mass, each one of them interacting with its two nearest neighbors through harmonic coupling and being under the influence of an on-site, double-well potential (after which the model is named). In the continuum limit, i.e., when the lattice spacing is much less than the wavelength of the excitations propagating along the chain (this is also referred to as *displacive* regime[5,6]), the model dynamics can be described, in dimensionless units, by the following partial differential equation:

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}-\epsilon_0\,(\phi-\phi^3)=0, \tag{1}$$

which is nothing but the famous ϕ^4 field equation. Among its properties (a survey is given in Refs. 21), the most important one is that it has a solitary-wave solution (that henceforth we will term "soliton" or "kink", though it is not a mathematical soliton because it does not have the necessary properties), given by

$$\phi_v(x,t)=\tanh\left(\frac{\gamma}{\sqrt{2}}(x-vt-x_0)\right), \tag{2}$$

where γ is the Lorentz factor $(1-v^2)^{-1/2}$, and the subscript v reflects the fact that, as all values $v\in[0,1]$ are allowed, speed is a free parameter. Moreover, it has two conserved quantities, namely the total energy and the total momentum, whose respective expressions are

$$E=\int_{-\infty}^{\infty}dx\,[\tfrac{1}{2}(\phi_t^2+\phi_x^2)+\tfrac{1}{4}(\phi^2-1)^2], \tag{3}$$

$$P=-\int_{-\infty}^{\infty}dx\,\phi_t\,\phi_x, \tag{4}$$

both equations corresponding to the choice of $\epsilon_0=1$ in Eq. (1), or, so to speak, measuring the energy in units of ϵ_0.[19] If we evaluate these two quantities for a soliton solution ϕ_v, we get

$$E[\phi_v]=\frac{4\gamma}{3\sqrt{2}},\quad P[\phi_v]=\frac{4\gamma v}{3\sqrt{2}}. \tag{5}$$

Another useful quantity is the "center of mass" (also called "center of energy"), defined by

$$X=\frac{1}{E}\int_{-\infty}^{\infty}dx\,x\,[\tfrac{1}{2}(\phi_t^2+\phi_x^2)+\tfrac{1}{4}(\phi^2-1)^2], \tag{6}$$

which equals

$$X[\phi_v]=vt+x_0 \tag{7}$$

when computed for a kink.

Let us now turn to the subject of our work, the stochastic ϕ^4 model. Its name comes from the fact that, among the different parameters in the chain, namely particle masses, harmonic coupling strengths, double-well positions, and barrier heights,[5,6,19] we take these last ones to vary randomly in time. The parameter related to the barrier heights is ϵ_0,[19] and so we choose

$$\begin{aligned}&\epsilon_0\equiv 1+\xi(t),\\&\langle\xi(t)\rangle=0,\\&\langle\xi(t)\xi(t')\rangle=2D\,\delta(t-t'),\end{aligned} \tag{8}$$

the brackets $\langle\cdots\rangle$ standing for averages over the random process realizations. Thus, ϵ_0 becomes a Gaussian white noise with a unit mean and variance $2D$, and we are left with a ϕ^4 model in which the potential fluctuates in time. This kind of perturbation is intended to represent large-spatial-scale changes (large compared to soliton width) of the on-site potential, possibly due to interaction with noisy external fields or fluctuations of the underlying system responsible for the nonlinear potential (think, e.g., of electrons in polyacetylene, which are the origin of the double-well potential for the carbon atoms allowing for a phenomenological ϕ^4 description[22,23]).

Once we have defined the model, we are going to estimate, at least when the stochastic part is weak, its most important consequences on the propagating soliton. Unfortunately, as the ϕ^4 equation is not integrable (nor is its discrete counterpart), we cannot use the perturbation theory based in the inverse spectral transform[24] that has been so successful in dealing with other nonlinear problems. Instead, we will use a technique first proposed by McLaughlin and Scott,[25] and referred to as the adiabatic approach or collective-coordinate formalism. In this approximate treatment, the assumption is made that radiation does not appear and that only the position and the speed of the kink change, becoming functions of time. We are not going to follow the derivation in Ref. 25 but that of Ref. 18 (even one more different derivation was proposed in Ref. 17). The same formulas can be shown[20] to arise as counterterms responsible for the cancellation of secular contributions in the context of a more rigorous, but rather lengthy, perturbative approach. For our present purposes, it is enough, and quite a bit easier, to propose the following ansatz for the solution of the perturbed equation[18]

$$\phi(x,t) = \tanh\psi(x,t),\ \psi(x,t) \equiv \left(\frac{\gamma}{\sqrt{2}}[x - z(t)]\right) \tag{9}$$

that must be supplemented with the constraints

$$\psi_x = \frac{\gamma}{\sqrt{2}},\quad \psi_t = -\frac{\gamma v}{\sqrt{2}}, \tag{10}$$

where the definition $z'(t) = v(t) + x_0'(t)$ has been used to represent, as we already announced above, that now the speed and an additional phase related to the kink center are time dependent. Next, we recall that in the perturbed model, the energy is not constant in time anymore, and the center of mass does not follow the above-mentioned linear law for a kink; instead, these quantities change in time in a precise form that can be easily computed, yielding

$$\frac{dE}{dt} = \xi(t) \int_{-\infty}^{\infty} dx\, \phi_t\, (\phi - \phi^3), \tag{11}$$

$$\frac{dX}{dt} = P - \xi(t) \int_{-\infty}^{\infty} dx\, x\, \phi_t\, (\phi - \phi^3). \tag{12}$$

We can therefore insert our ansatz (9) in Eqs. (11) and (12), and so obtain the derivative of E and X with time for such a variable-speed soliton. On the other hand, we can make the calculation just the other way round, because these two quantities have a precise value for a kink that depends only on its speed, which is given in (5) and (7). We can then compute their evolution in time, taking derivatives in those formulas assuming that v and x_0 are functions of t. Both procedures should lead to the same result; hence, imposing their consistency, we get the following equations for the kink speed and center, respectively:

$$v'(t) = 0, \tag{13}$$

$$z'(t) = v(t) - \tfrac{1}{2}v(t)\,[1 - v^2(t)]\,\xi(t). \tag{14}$$

The first outcome of this calculation is that, according to Eq. (13), the perturbation does not affect the evolution of the parameter v in time [note that $z'(t)$ now plays the role of the velocity of the nonlinear wave]. Variations of v with time can occur in second-order perturbation theory. That is not so with respect to the kink center position, which is indeed affected by noise. Its mean value and variance (also called dispersion) can be, after some algebraic calculation (either directly averaging or through a Fokker-Planck equation[17]), written down as

$$\langle z(t)\rangle = v_0 t, \tag{15}$$

$$\sigma_z^2(t) \equiv \langle z^2(t)\rangle - \langle z(t)\rangle^2 = \frac{D}{2} v_0^2\,(1 - v_0^2)\,t, \tag{16}$$

where the initial conditions $z(0) = 0$ and $v(0) = v_0$ have been chosen, without loss of generality. Thus, in the framework of the adiabatic approach, we find that the effect of the noise is mainly the introduction of an uncertainty in the position of the kink, given by its standard deviation $\sigma_z(t)$. It is predicted to grow as the square root of time, as if the kink, in its own reference frame, mimicked the Brownian behavior of a pointlike particle. On the other hand, the coefficient in front of t in Eq. (16) becomes maximum at $v_0 = 1/\sqrt{2}$, and vanishes if either $v_0 = 0$ or $v_0 = 1$. This is the first time we have found the noise effects to be speed dependent, though up to now this dependence has been rather simple. Numerical simulations will not bear this out, neither will the effects be so easily understandable in the strong-noise regime.

Had we added a dissipative contribution to our system, that is, a term $-\alpha\phi_t$ in the right-hand side (rhs) of Eq. (1), the same procedure as that described above would have led us to

$$v'(t) = -\alpha\gamma v(t), \tag{17}$$

while the behavior of $z(t)$ would be unchanged and remain governed by Eq. (14). We can now solve Eq. (17) for $v(t)$, and we find

$$v(t) = v_0\left[v_0^2 + (1 - v_0^2)\exp(2\alpha t)\right]^{-1/2}. \tag{18}$$

This purely dissipative expression for the speed can be then inserted in Eq. (14), allowing us to write an integral expression for $z(t)$, from which its statistical properties can be straightforwardly obtained in the following form:

$$\langle z(t)\rangle = z_0 + \frac{1}{2\alpha}\ln\left[\left(\frac{1+v_0}{1-v_0}\right)\frac{\left[v_0^2 + (1-v_0^2)\exp(2\alpha t)\right]^{1/2} - v_0}{\left[v_0^2 + (1-v_0^2)\exp(2\alpha t)\right]^{1/2} + v_0}\right], \tag{19}$$

$$\sigma_z^2(t) = \frac{Dv_0^2}{8\alpha}\left(2 - v_0^2 - \frac{v_0^2 + (1-v_0^2)2\exp(2\alpha t)}{\left[v_0^2 + (1-v_0^2)\exp(2\alpha t)\right]^2}\right). \tag{20}$$

It is worth commenting on some points in these expressions. First, it is easy to check that if there is no dissipation the formulas (15) and (16) can be recovered taking the limit $\alpha \to 0$ in Eqs. (19) and (20); therefore, both sets of equations are consistent. On the other hand, if $t \to 0$, $\langle z(t)\rangle \to z_0$ and $\sigma_z^2(t) \to 0$, as one should expect. More interestingly, both formulas go to a finite limit when $t \to \infty$, namely,

$$\langle z(t)\rangle \stackrel{t\to\infty}{\longrightarrow} z_0 + \frac{1}{2\alpha}\ln\left(\frac{1+v_0}{1-v_0}\right), \tag{21}$$

$$\sigma_z^2(t) \stackrel{t\to\infty}{\longrightarrow} \frac{Dv_0^2}{8\alpha}(2-v_0^2), \tag{22}$$

implying that at large times, if our hypothesis is correct (i.e., radiation is absent), the kink is finally stopped at the point given by Eq. (21) and its dispersion does not grow anymore due to the action of dissipation. Notice that when v_0 is small, the final position of the kink is near z_0, while if $v_0 \simeq 1$ the limit $\langle z(\infty)\rangle$ is very far away from the initial position. This makes sense, as does the fact that the final values of $\langle z(t)\rangle$ and $\sigma_z(t)$ are inversely proportional to α. So, we see that the adiabatic approach give reasonable results, a condition it first must meet if we expect to get something out of it. In the remainder of the paper our results will come from numerical simulations of the system, and, comparing them to formulas (19) and (20), we will see what the range of validity of this approach is, or, in other words, what we can consider as weak or strong noises.

III. COMPUTER SIMULATION DETAILS

A. Numerical scheme and parameters

Our procedure is a generalization of the Strauss-Vázquez scheme for NKG equations,[26] and it is given by

$$\frac{\phi_j^{n+1}-2\phi_j^n+\phi_j^{n-1}}{\Delta t^2} - \frac{\phi_{j+1}^n-2\phi_j^n+\phi_{j-1}^n}{\Delta x^2} + \alpha\frac{\phi_j^{n+1}-\phi_j^{n-1}}{2\Delta t} + \tfrac{1}{4}\left(1+\xi^n\right)\frac{[(\phi_j^{n+1})^2-1]^2-[(\phi_j^{n-1})^2-1]^2}{\phi_j^{n+1}-\phi_j^{n-1}} = 0, \tag{23}$$

where $\phi_j^n \equiv \phi(j\Delta x, n\Delta t)$, Δx and Δt are, respectively, the spatial and temporal steps, and ξ^n is the discretization of $\xi(n\Delta t)$. Provided the noise is interpreted in the Stratonovich sense,[27] ξ^n is obtained from a pseudo-random-number Gaussian generator with variance $D_{\rm num} \equiv D(\Delta t)^{-1}$,[18] D being the analytical coefficient in Eq. (8). Similar schemes apply straightforwardly for any NKG system[26] (see, for instance, the sine-Gordon version in Ref. 28), their key feature being the discretization of the potential derivative term.

A very noticeable property of the scheme (23) is that, when there is neither dissipation nor perturbation, its stability and convergence can be proved[29] using its conservative character, i.e., using the fact that it has a discrete analog of the energy whose expression is

$$E^n \equiv \Delta x \sum_{j=-\infty}^{+\infty} \frac{1}{2}\left(\frac{\phi_j^n-\phi_j^{n-1}}{\Delta t}\right)^2 + \frac{1}{2}\left(\frac{\phi_j^n-\phi_{j-1}^n}{\Delta x}\right)\left(\frac{\phi_j^{n-1}-\phi_{j-1}^{n-1}}{\Delta x}\right) + \tfrac{1}{8}\{[(\phi_j^n)^2-1]^2+[(\phi_j^{n-1})^2-1]^2\}, \tag{24}$$

which is nothing but a discretization of Eq.(3) and which remains invariant in the (numerical) discrete time evolution. However, to our knowledge, rigorous results on stability and convergence of stochastic schemes are unfortunately very few and they do not apply to (24) when noise is present. We will test its accuracy and get a practical "proof" of consistency below. Let us now conclude this subsection with a summary of the parameters employed in our simulations.

(i) $\Delta x = 2\Delta t = 0.05$: The choice of Δx is much smaller than soliton widths (~ 4, in dimensionless units) and avoids discreteness effects,[30] because we are interested in the displacive limit of the model and, as a consequence, in Eq. (1). Were we to study the relationship between fluctuations and discreteness, Δx would be such that the soliton width would cover a few lattice spacings.

(ii) Number of spatial points: 400 ($x \in [-10,10]$).

(iii) Boundary conditions: 200 additional points on each side, not under the influence of noise, to avoid interference with reflections from the boundaries. At their edge fixed boundary conditions [$\phi_t(-L,t) = \phi_t(L,t) = 0$] were used. Some runs, repeated with other boundary conditions, led to the same results.

(iv) Initial data: a kink with different velocities in the interval $0 < v < 0.99$. Kinks with higher speeds are too narrow and show discreteness effects.[30]

(v) Noises in the interval $10^{-3} < 2D < 1$.

B. Accuracy of averages and consistency of the scheme

It goes without saying that the first point one must check in this kind of stochastic problem is the way in which averages are computed. We did it by generating several realizations of what is now a stochastic process,

$\phi(x,t)$; that is to say, we computed $\phi(x,t)$ for different sequences of (pseudo)random numbers ξ^n and averaged over this set. The point about this is, what is the appropiate number of realizations to average over which? In previous related works[18,28] mean values and variances were calculated using 15 and 30 realizations. Very recently the simulations in Ref. 28 were repeated[31] but, there, 2000 trajectories were employed for averaging, and no essential difference between the main results of both works could be appreciated. So, we again work with 30 realizations as our standard ensemble for averages (each one took two CPU minutes to be computed on an IBM 3090-150, and that is why we preferred not to go further), and repeated some runnings with 60 realizations, getting a nice agreement between both procedures. This is shown in Fig. 1, the mean value and dispersion of the center of a kink standing as examples. So, we can at least trust that the statistical behavior of the stochastic kink comes from its random nature and not from poor-statistics errors.

The proper way to ensure the reliability of any numerical simulation is to check whether exact known properties of the system are verified in the simulation. Unfortunately, as we already mentioned, the knowledge of stochastic, nonlinear partial differential equations is rather poor, and not too many of their features have been rigorously established. However, very recently, the combined use of geometrical techniques and the Hamiltonian formulation of the nonlinear Klein-Gordon system

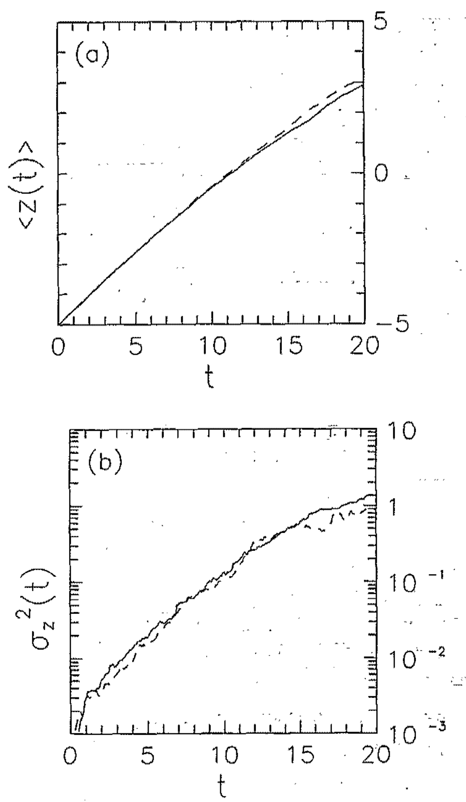

FIG. 1. (a) Mean value $\langle z(t)\rangle$, and (b) mean dispersion $\sigma_z^2(t)$ of the center of a kink propagating freely (i.e, without dissipation, $\alpha = 0$) along an unbounded stochastic zone (all other figures correspond to this propagation in the infinite random layer unless otherwise stated). Solid line: averages over 60 realizations. Dashed line: averages over 30 realizations. The initial speed is $v_0 = 0.5$, and the initial center position is $z_0 = -5$. The noise strength was $2D = 0.1$.

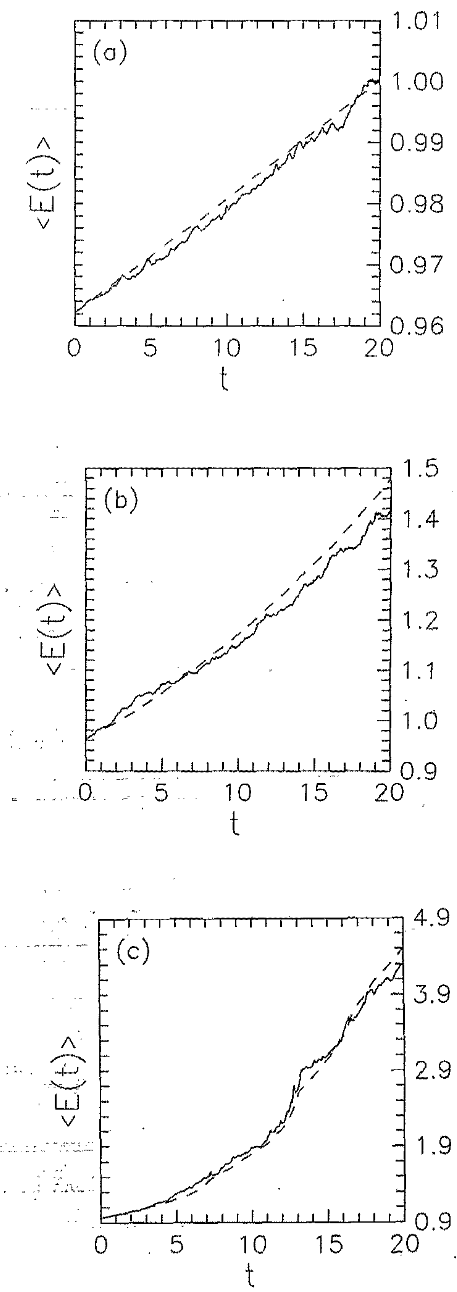

FIG. 2. Mean total energy for an initial kink with $v_0 = 0.2$, $\alpha = 0$ and noise of strength (a) $2D = 0.01$, (b) $2D = 0.1$, and (c) $2D = 0.25$. Solid line: energy as directly computed from the simulation using its definition, Eq. (24). Dashed line: energy obtained from PMR formula, Eq. (26). Notice that scales are not the same in each plot because of the largely different energy increments.

allowed Parrondo, Mañas, and de la Rubia[32] to obtain some exact results for this kind of system when perturbed not only with white noise but also with colored noise. The most important outcome of that work, concerning ours, is that it was shown that the mean total energy, as defined in Eq. (3), of the system *exactly* verifies an integro-differential equation that can be written down explicitly. The proof of that formula, henceforth called the PMR formula, can be found in Ref. 32, and we only quote it:

$$\frac{d\langle E\rangle}{dt} = -\alpha \int_{-\infty}^{\infty} dx\, \langle|\phi_t|^2\rangle + 2D \int_{-\infty}^{\infty} dx\, \langle(-\phi+\phi^3)^2\rangle. \quad (25)$$

We must emphasize that the PMR formula is exact and does not involve any approximation at all. Hence, it provides an excellent test for our simulations. In all of them, we checked the discretization of this law, given by

$$\frac{\langle E^n\rangle - \langle E^{n-1}\rangle}{\Delta t} = \sum_{j=-N/2}^{N/2} \left[-\alpha\Delta x \left\langle \left(\frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} \right)^2 \right\rangle + \Delta x \langle [-(\phi_j^n) + (\phi_j^n)^3]^2 \rangle \right], \quad (26)$$

where N is the spatial grid size and n is the time step at which the energy is to be calculated. In all computed cases, i.e., for all different choices of the initial kink speed and noise strength, we find a fairly good agreement between the energy as obtained via the discretization of its definition (24) and via the PMR formula (26). As an illustration, three different calculations are shown in Fig. 2. Though energy increasing differs by an order of magnitude when noise strength is changed by the same amount, the coincidence of direct and PMR results is excellent for all values. This applies also to weaker ($2D = 10^{-3}$) or stronger ($2D = 1$, up to the time the kink integrity is preserved, see below) noises, the final energy values being enormously different. Throughout the paper, for further demonstration, energy plots show both PMR and direct computations, and they always compare well to each other [Figs. 3(f), 4(f), 5(f), 6, and 10(f)]. We will come back to this question when describing the problem of boundaries below, where a generalization of (25) still applies. Let us state for the moment in the above case the agreement is once more very satisfactory.

We feel that this verification, if it is by no means a rigorous proof neither of its stability nor of its convergence, it is indeed a very remarkable consistency feature: the discrete scheme verifies an analog of the properties of the underlying stochastic continuous system. The related conservative character of the ordinary Strauss-Vázquez scheme[26] is the very basis for the proof of all its properties.[29] It might be possible that this could also take place in the stochastic case, but we have not been able to obtain such a demonstration; this is an interesting issue that remains an open problem. Anyway, the satisfactory checking we have done, namely, the practical invariance of the computation under a different number of realizations (which implies that our statistics is sufficient) and the existence of a discrete PMR analog (which implies good properties for our discretization) are enough to firmly establish our simulations; furthermore, if this is considered along with the agreement to perturbative results in the weak-noise regime (see below) we can say that a good basis for our numerical approach to the stochastic ϕ^4 model has been set. To finish, we would like to stress that this procedure is suitable to study any NKG problem. Besides, since the very root of its favorable properties seems to be the energy conservation of the unperturbed scheme that is transferred to the perturbed one as the verification of the discrete PMR formula, the method could be also generalized to other conservative problems for which similar schemes can be found.

IV. NONDISSIPATIVE KINKS IN THE INFINITE CHAIN

Let us now move to the main topic of the paper, namely, the numerical simulations of the model. After comparing the results to the above-described adiabatic approach, we decided that it would be more convenient to split this section into two parts: the first dealing with weak noises, values around $2D \simeq 0.01$ or below, and the second devoted to strong noises, $2D \simeq 0.1$ or above. Before entering the results, we must comment on the magnitudes we computed in the simulations. In principle, there is a large number of measurable quantities that one can think of because we are dealing with a system of infinitely many degrees of freedom. Among these quantities, we chose to monitor, of course, the mean kink shape and the mean energy density, and, aside from them, the collective coordinates of the kink (while they remain as proper quantities to describe the kink motion, i.e., if the kink is not severely destorted or even distroyed). In addition, since we always checked the verification of the PMR formula, we also followed in detail the energy evolution. To quantify (at least approximately) the agreement between the perturbative approach and the numerical results, we fitted power-law behaviors to our computed $\sigma_z^2(t)$, of the form $\sigma_z^2(t) \simeq \sigma_0\, t^\delta$ [remember that the adiabatic prediction for the behavior of $z(t)$ is that it must be uniform motion in the absence of dissipation and this is checked directly]. With respect to the energy, we found that the most accurate functional form to describe it was exponential, $E(t) = E_0 \exp(t/t_e)$. This behavior cannot be predicted from the PMR formula, which only ensures that the energy will increase. The results of these fittings are summarized for both regimes of noise in Table I, and we comment on them in full detail below.

A. Weak-noise regime

In this regime we have mainly studied two values of noise, $2D = 0.001$ and $2D = 0.01$. For the former value, the effects of noise were too small to cause visible effects on the soliton, and the soliton was indeed able to propagate through the perturbed zone, showing no essential differences with respect to a usual ϕ^4 kink in its shape or its energy. However, the center dispersion shows the signature of the noise effect, as can be seen from the exponents in Table I, but, as the prefactor σ_0 is much less than unity, this variance behavior has very little physical relevance; i.e., everything happens almost as if noise were absent. The center motion has a mean evolution identical to that of an unperturbed kink, and the energy remains practically constant, its very small amount of increasing being exponential with a tiny exponent (see Table I).

All these features are present in a more appreciable fashion in the simulations for $2D = 0.01$, the noise value for which we show a detailed set of plots in Fig. 3. It can be seen that the kink is not altered too much by noise. Its shape remains practically the same (which implies the absence of radiation and verification of the adiabatic hypothesis); its energy density does not spread out; the center follows a line with quasiconstant velocity, as is predicted by Eq. (15); and the dispersion grows more or less with t, also in good agreement with the perturbative Eq. (16). Finally, with respect to energy, we see that the PMR formula is again verified, and that the energy evolution in time is, as we mentioned above, exponential. Hence, the conclusion one can draw from simulations is that in this regime the adiabatic formalism[20] is a good description for kink dynamics. However, the threshold between weak and strong perturbation is not so sharp and it depends on the kink initial speed, as will become clear in Sec. IV B.

B. Strong-noise regime

As we have just seen, noises around $2D = 0.01$ or less can be considered weak. Let us now discuss what happens when noise is increased up to $2D = 0.1$. We have studied this value in great detail because it exhibits the

TABLE I. Parameters characterizing the kink behavior as obtained from the simulations for nondissipative kinks in the infinite stochastic model. All the entries are defined in the beginning of Sec. IV. The noise scaled exponent is defined as the energy exponent t_e^{-1} divided by the noise strength, a quantity that is around 0.2 except for relativistic kinks.

Noise strength	Initial speed	Energy fit $\xi \equiv t_e^{-1}$	Energy fit E_0	Noise scaled exponent	Dispersion fit δ	Dispersion fit σ_0
1.0×10^{-3}	0.00	1.678×10^{-4}	0.9432	0.1678	[a]	[a]
	0.05	1.885×10^{-4}	0.9438	0.1885	1.212	3.718×10^{-7}
	0.10	2.299×10^{-4}	0.9475	0.2299	1.023	2.396×10^{-6}
	0.20	1.953×10^{-4}	0.9621	0.1953	1.165	6.564×10^{-6}
5.0×10^{-3}	0.20	1.059×10^{-3}	0.9615	0.2118	1.161	3.770×10^{-5}
1.0×10^{-2}	0.00	1.741×10^{-3}	0.9468	0.1741	[a]	[a]
	0.05	2.547×10^{-3}	0.9418	0.2547	1.534	2.746×10^{-6}
	0.10	2.143×10^{-3}	0.9458	0.2143	1.107	2.410×10^{-5}
	0.20	1.932×10^{-3}	0.9611	0.1932	1.144	7.389×10^{-5}
2.5×10^{-2}	0.20	4.606×10^{-3}	0.9663	0.1842	1.483	1.169×10^{-4}
5.0×10^{-2}	0.20	1.042×10^{-2}	0.9599	0.2084	1.880	2.179×10^{-4}
1.0×10^{-1}	0.00	1.936×10^{-2}	0.9596	0.1936	[a]	[a]
	0.025	1.462×10^{-2}	0.9601	0.1462	1.757	1.144×10^{-5}
	0.05	1.718×10^{-2}	0.9387	0.1718	2.380	1.815×10^{-5}
	0.10	2.436×10^{-2}	0.9278	0.2436	2.087	1.080×10^{-4}
	0.20	1.883×10^{-2}	0.9656	0.1883	[b]	[b]
	0.50	1.845×10^{-2}	1.0711	0.1845	1.975	1.527×10^{-3}
	0.80	7.704×10^{-3}	1.5651	0.0770	1.446	1.364×10^{-3}
	0.99	3.790×10^{-4}	6.6438	0.0038	1.161	7.845×10^{-6}
2.5×10^{-1}	0.20	8.327×10^{-2}	0.8617	0.3331	2.213	1.368×10^{-3}

[a]The center of the kink remained at rest within an error of 10^{-11}, and therefore the dispersion results were meaningless.
[b]No available data.

most interesting characteristics. First, we will deal with slow speeds, as that of Fig. 4. The perturbative prediction ceases to be valid, and the reason becomes evident from Figs. 4(a) and 4(b). The kink, forced by the noise, emits radiation (*phonons* in condensed-matter terminology, sometimes also called *mesons* in field-theoretical jargon) mainly backwards. In addition, Fig. 4(a) shows also an important contribution of the localized mode that distorts the kink structure (see the description in terms of modes in Ref. 20). What is happening can be explained in the following way: notice that, at the beginning, when the noise is switched on, the only part of the kink that

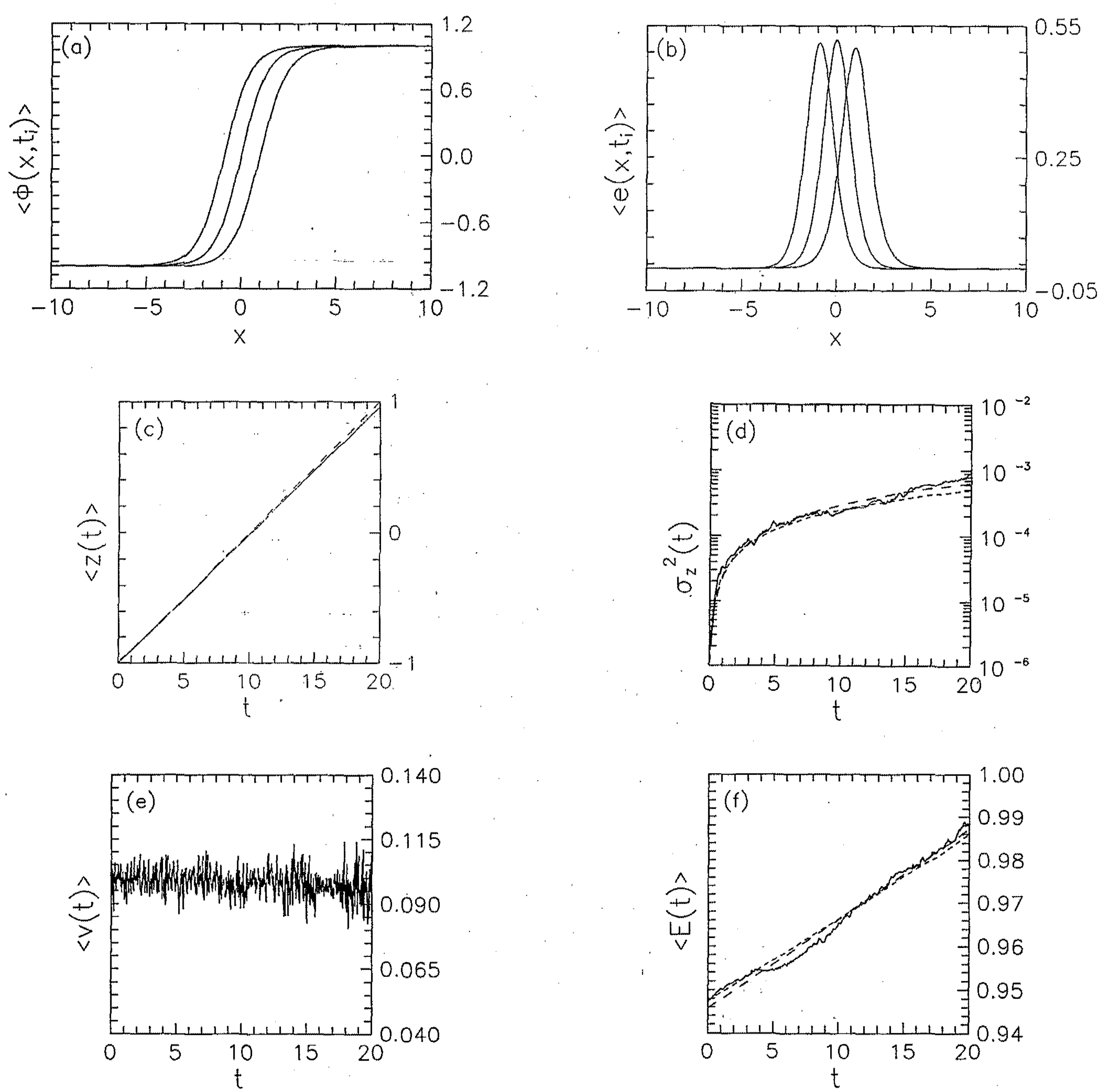

FIG. 3. A typical outcome of a simulation for weak noise. Initial speed is $v_0 = 0.1$ rightwards, initial center position is $z_0 = -1$, and noise strength is $2D = 0.01$. $\alpha = 0$. (a) Mean kink shape at three different time instants, $t = 1, 10, 20$. (b) Mean energy density at the same time instants. (c) Mean center position. The line is numerical; the dashed line is the adiabatic approach prediction of uniform motion. (d) Mean center dispersion. The solid line is numerical; the small-dashed line is the adiabatic approach prediction $\sigma_z^2(t) = 2.475 \times 10^{-5}\, t$, and the large-dashed line is a power-law fit, given by $\sigma_z^2(t) = 2.30 \times 10^{-5}\, t^{1.107}$. (e) Mean kink speed. The adiabatic prediction is $v(t) = v_0 = 0.1$. (f) Mean total energy. The solid line is numerical, small-dashed line is the PMR formula, and the large-dashed line is an exponential fit, given by $E(t) = 0.9458 \exp(0.002\,14\, t)$.

it affects is its *structure* or *kernel*, that is, the part in which the chain particles are sensitively away from the minima of the potential $[\phi(x,t) = \pm 1]$; particles in the *wings* (also called *tails*) are not influenced at all [see Eq. (1)]. Hence, particles in the chain begin to suffer the noise action when the kink structure passes over them. After that moment they go on experiencing the perturbation effects and continue to emit small-amplitude linear waves, a motion that is maintained by noise action (see Ref. 20 for a description of the early stages of this phenomenon, when it still can be described by a linear approximation), this constituting the radiation visible in Fig. 4. Moreover, the center is slowed down and it does not reach the position that it should according to the adiabatic approach [Fig. 4(c)]; and the dispersion, after an initial transient in which it more or less follows the perturbative Brownian-like law (up to a time around $t = 5$ for this $2D = 0.1$ noise), crosses over to a more rapid

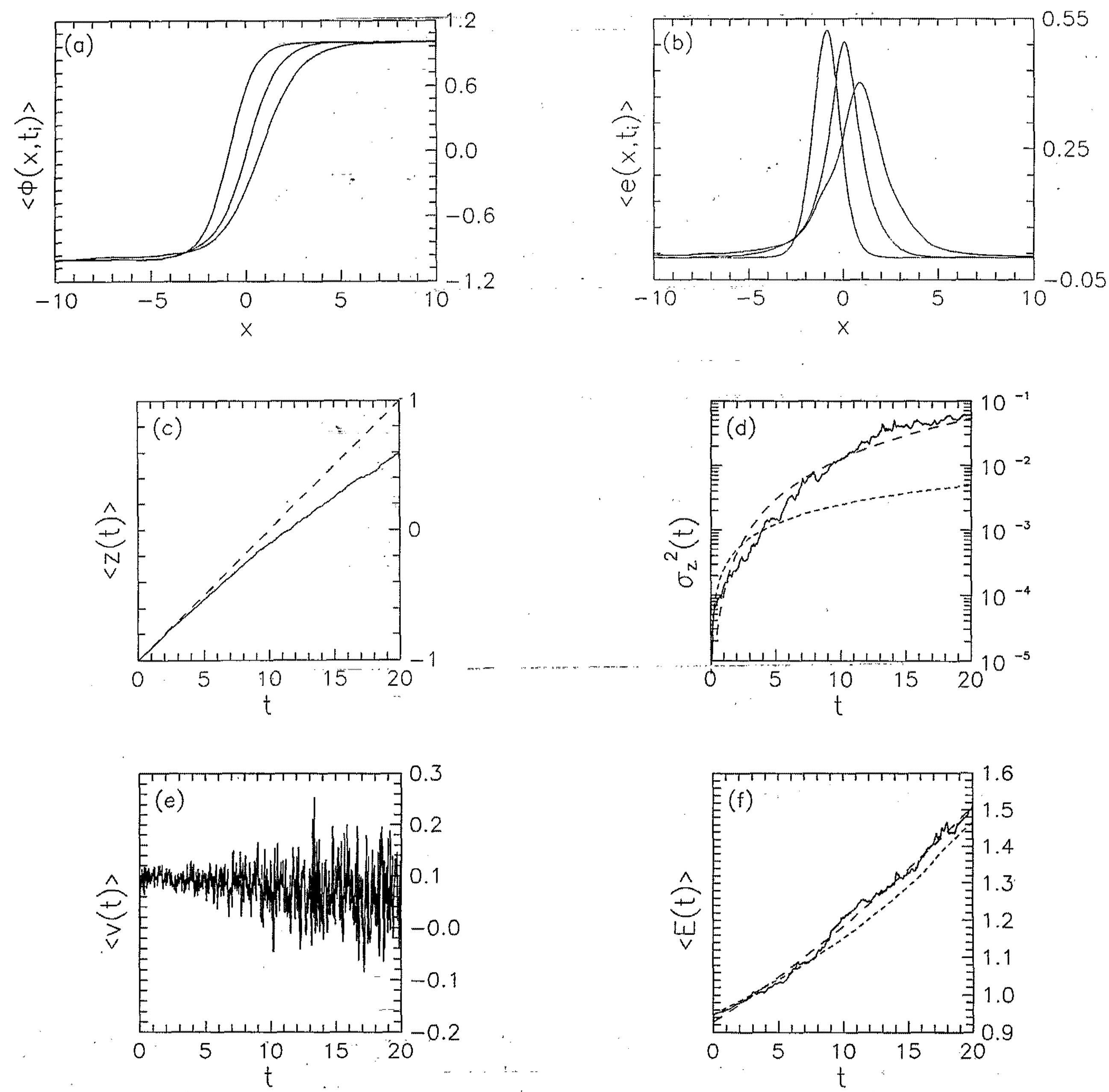

FIG. 4. As in Fig. 3, but for stronger noise. The initial speed is $v_0 = 0.1$ rightwards, the initial center position is $z_0 = -1$, and the noise strength is $2D = 0.1$. $\alpha = 0$. In (d) the adiabatic prediction is now $\sigma_z^2(t) = 2.475 \times 10^{-4}\, t$, and the fitting is $\sigma_z^2(t) = 1.03 \times 10^{-4}\, t^{2.087}$. In (f), the fitting is $E(t) = 0.927\,81 \exp(0.0244\, t)$.

growth [see Fig. 4(d)].The energy exponential behavior is characterized by an exponent of an order of magnitude higher than in the previous case of weak noise. Its spatial distribution is also severely changed, and we were able to detect in the simulations that more than the amount of energy injected in the chain by the noise flows outwards from the structure in radiation form. This energy redistribution could be responsible for the global slowing down of the soliton. We must add that kinks at rest present the same characteristics as slow kinks, except for the fact that the stochastic field ϕ remains symmetric around the kink center and radiation goes in both directions at the same ratio (this is rather reasonable because there is nothing in our system that breaks the symmetry under translations for a stopped kink).

Nevertheless, it has also turned out that the impor-

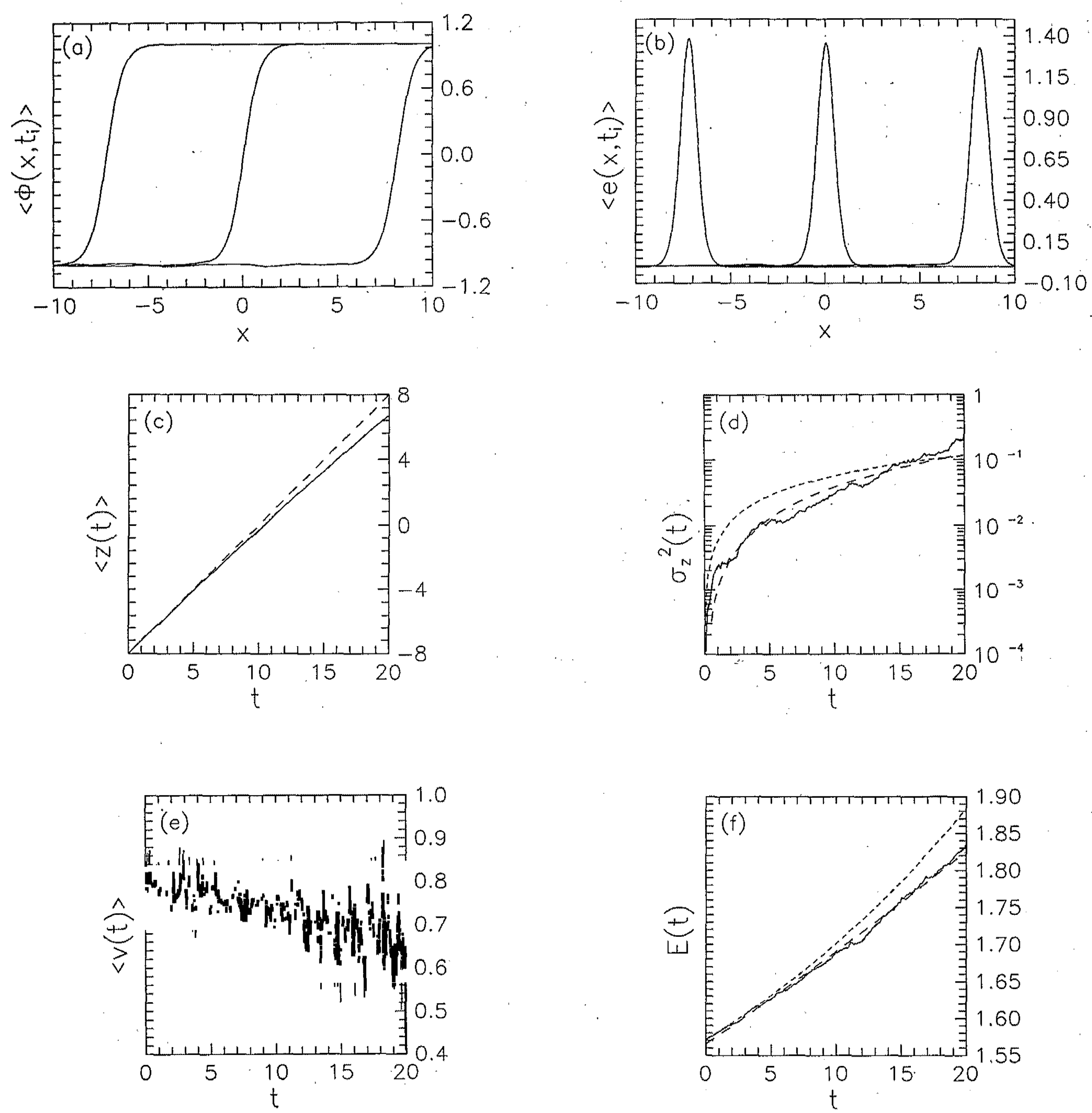

FIG. 5. As in Fig. 3, but for higher speed. The initial speed is $v_0 = 0.8$ rightwards, the initial center position is $z_0 = -8$, the noise strength is $2D = 0.1$. In (d) the adiabatic prediction is now $\sigma_z^2(t) = 5.76\times10^{-3}t$, and the fitting is $\sigma_z^2(t) = 9.85\times10^{-4}t^{1.596}$. In (f) the fitting is $E(t) = 1.5651\ \exp(0.0077\ t)$.

tance of these effects and the separation from the adiabatic formulas become less and less important if we increase the speed of the propagating kink. An example is shown in Fig. 5. Loosely speaking, we can say that the results of these experiments are clearly closer to those of weak noise (Fig. 3) than the ones of the $v_0 = 0.2$ kink (Fig. 4). Kinks with an even higher initial speed show more clearly this fact (see entries for $v_0 = 0.99$ in Table I). We are forced to conclude that the suitability of the collective-coordinate treatment to describe the kink motion depends not only on the noise strength but also on the initial kink speed. This must not be confused with the dependance of $\sigma_z^2(t)$ on v_0, which we already commented on; rather, it is related to the radiation emission power that is also a function of the speed.[20] As relativistic kinks radiate much less than slow kinks, the adiabatic approach remains valid for higher noises.

If we further increase the noise up to values around $2D = 1$, we get a non-numerical, intrinsic blowup behavior before the end of the integration time $t = 20$ is reached. The kink structure resists for a while, but finally some particles get too close to each other and they are subsequently repelled due to their harmonic coupling; this and similar catastrophic processes lead to a final destruction of the soliton and of the very structure of the system. It is actually remarkable that the energy grows in a moderately rapid fashion, but when it is two or three times its initial value the increasing becomes suddenly faster (see Fig. 6), the PMR formula still being consistent, up to a final blowup. This crossover might be related to the presence in the chain of enough energy to create a new kink-antikink pair that could accelerate the chain destabilization.

Another interesting outcome of the simulations concerns kink stability under these potential fluctuations. We have found that, if noise is switched off before a destruction, nonreturn point is reached (like the one described in the preceding paragraph), the kink rearranges itself by emitting more radiation and recovers its structure, suffering a global phase shift (see Fig. 7) and speed diminution. This happens even when the kink is severely deformed due to the internal mode and to phonons. The stabilization process seems to occur via dissipation of the energy contained in the internal mode into phonons and subsequent propagation of these and the noise-generated ones far away from the kernel. Moreover, the parameters of the (so to speak) "new" kink are those of the old at the time when the noise is switched off. This means that the propagation of the rearranged kink starts at the center position the perturbed kink had when the noise was turned off, and it takes place from that moment at the velocity it had, which is less than the initial one. This evidence is what allows us to state that kinks are stable against this perturbation if it does not continue for too much time, the exact time depending on the noise strength. Roughly we can say that the destabilization time $t_{\rm ins}$ verifies $2Dt_{\rm ins} \simeq 10$.

The main results in this section are summarized in Table I and Figs. 8 and 9. Let us consider first Fig. 8; there, the deviation from the adiabatic prediction is seen to be directly proportional to the noise strength, and it ranges from a relative deviation of 1% when $2D = 0.001$

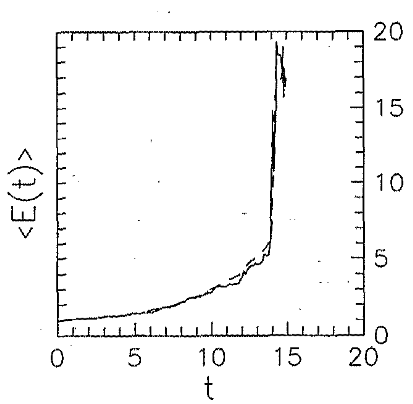

FIG. 6. Mean energy of a kink with $v_0 = 0.2$, $\alpha = 0$ and noise $2D = 0.5$. The crossover to a faster growth regime takes place at $E \simeq 3E_0$. A blowup happens before the end of the run. The dashed line is obtained from the PMR formula.

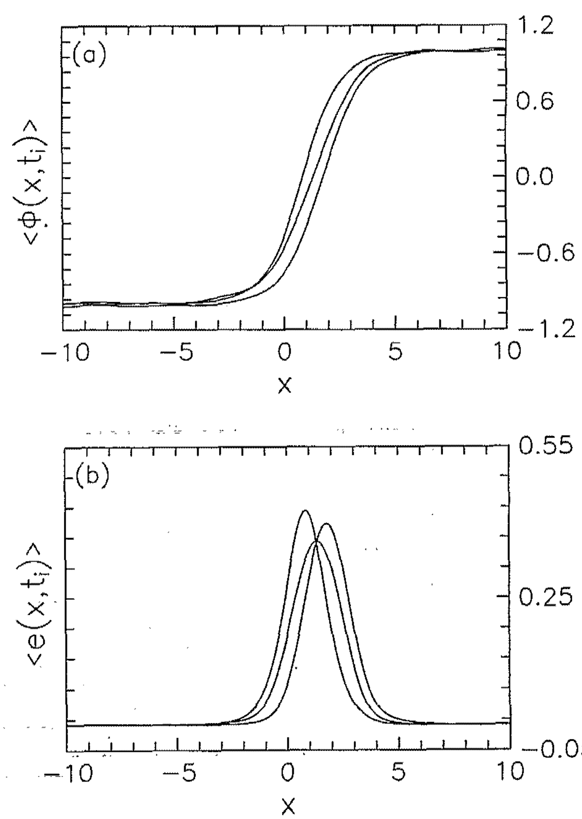

FIG. 7. (a) Mean kink shape and (b) mean energy density of the kink simulation shown in Fig. 4, when at $t = 20$ noise is switched off and the kink is allowed to evolve freely ($\alpha = 0$). $t = 20, 30, 40$ are shown in both plots.

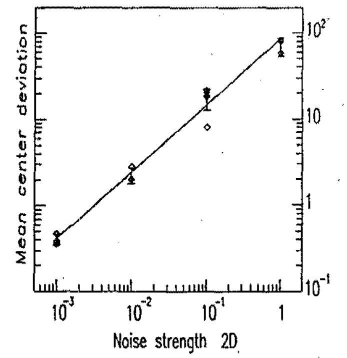

FIG. 8. Mean percent of relative center deviation from the adiabatic prediction in nondissipative simulations. We plot $\Delta x \equiv 100[\langle x_{\rm adiab}(t=20)\rangle - \langle x_{\rm num}(t=20)\rangle]/\langle x_{\rm adiab}(t=20)\rangle$ vs noise strength $2D$, except for $2D = 1$, where this was computed when energy was twice its initial value (a small interval later the kink suffered a blowup). The line is a power-law fitting, given by $\Delta x = 88.97\ 2D^{0.78}$. Error bars show the sample standard deviation when we have several values (coming from different speeds) for the same noise strength.

to a relative deviation of 100% when $2D = 1$. Second, the dispersion law exponent, as is shown in Table I, grows more or less linearly with noise strength, going from laws of the form t to laws of the form t^2. This "anomalous diffusion" might be understood as an extended structure effect. Finally, energy grows exponentially in time with an exponent that scales well with noise strength (Fig. 9); as we have already reported, this behavior takes place up to a crossover instability time that is proportional to $(2D)^{-1}$. This must be related to some stochastic resonance process that we have discussed in Ref. 20; the problem there is that we were not able to get this estimation for destabilization times, because our description

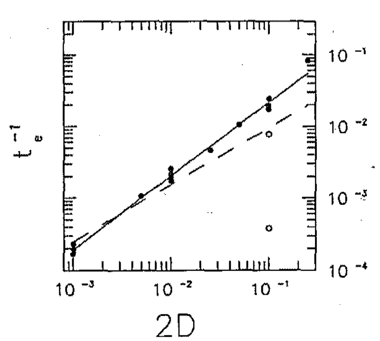

FIG. 9. Dependence of the energy exponent t_e^{-1} on the noise strength in nondissipative simulations. The dashed line is a linear fit of all points, of slope ≈ 0.8. The solid line fits all but the two empty ones (upper, $v_0 = 0.8$; lower, $v_0 = 0.99$) and its slope is ≈ 1.022. Points at the same noise strength were obtained for several initial speeds.

is only valid when radiation is beginning to be generated. As a general remark, all these features are speed dependent and decrease for relativistic kinks, the perturbative results being then valid for stronger (even an order of magnitude) noises.

V. DISSIPATIVE KINKS IN THE INFINITE CHAIN

If our model must have something to do, even as a simple phenomenological model, with some real problems, we must consider an aspect that is always present in any physical system, namely, friction or dissipation coming from a number of different causes that can stop or disturb the otherwise eternal (in the pure model) propagation of solitons. Therefore, we must include a dissipative term to account for energy losses due to several different reasons, such as $-\alpha\phi_t$, in the rhs of our Eq. (1) and study the new effects arising from it. This is what we aim for in this section, by repeating the above-described simulations for the same system under the action of a such a friction. After some trial runs, in which we decided on a proper value for this viscous coefficient α, such that neither was the kink pinned too fast (preventing us from getting a clear idea of transient behaviors) nor was it sensitively affected, we chose the standard value $\alpha = 0.1$. Then we repeated simulations on every range of speed and noise. What we found, in a few words, was that in this case perturbative predictions remain valid for stronger noises than in the nondissipative problem. Under a noise of strength $2D = 0.1$ kinks behave essentially as predicted by formulas (19) and (20), with very small deviation, as is shown in Fig. 10, where a remarkable analytical-numerical agreement is evident. This agreement comprises all computed aspects, from the absence of radiation (the adiabatic hypothesis once more) to the speed decreasing. One thing that changes is that the energy behavior is not exponential anymore as a consequence of the balance between energy injection of noise and energy dissipation through friction, and it seems likely that it finally tends to a constant stationary value; if this is so, it does in a time later than the one that we reached in our computations. This can be mathematically treated if noise is small enough,[20] and it can be shown that a threshold appears for the parametrical excitation of the linear modes to happen in this stochastic, dissipative system. Our simulations show that this is the case even if noise is not so small, leading to a decrease in the radiation generation power that allows for an adiabatic description. Let us insist upon the result that, apart from this consideration concerning energy and from the dissipative slowing down, everything happens as in the nondissipative model for weak noise, and the same discussion applies here. It is not unreasonable to suppose that the kink finally is completely stopped and then it undergoes the same kind of process as that of a nondissipative one, except that now the radiation growth is limited by the nonzero α. All our simulations support the idea that dissipation is a factor that softens the noise effect in such a way that the chain tends to an equilibrium state, with kinks pinned and a steady flow of radiation coming out of them (the emission power is perturbatively predicted to be constant[20]).

It seems certain that the reason why perturbation treatment works well in this case is that we remain, for stronger noise, in a situation in which our main assumption holds: that all the effect of noise is concentrated in changing the kink center and speed, and its shape is not altered. The physical explanation seems very direct to us: when noise is beginning to create radiation, dissipation begins to act, equilibrating this process and inducing a progressive diminution of the amplitude of these phonons. Then the energy cannot easily flow outwards from the kink structure and all noise effects become con-

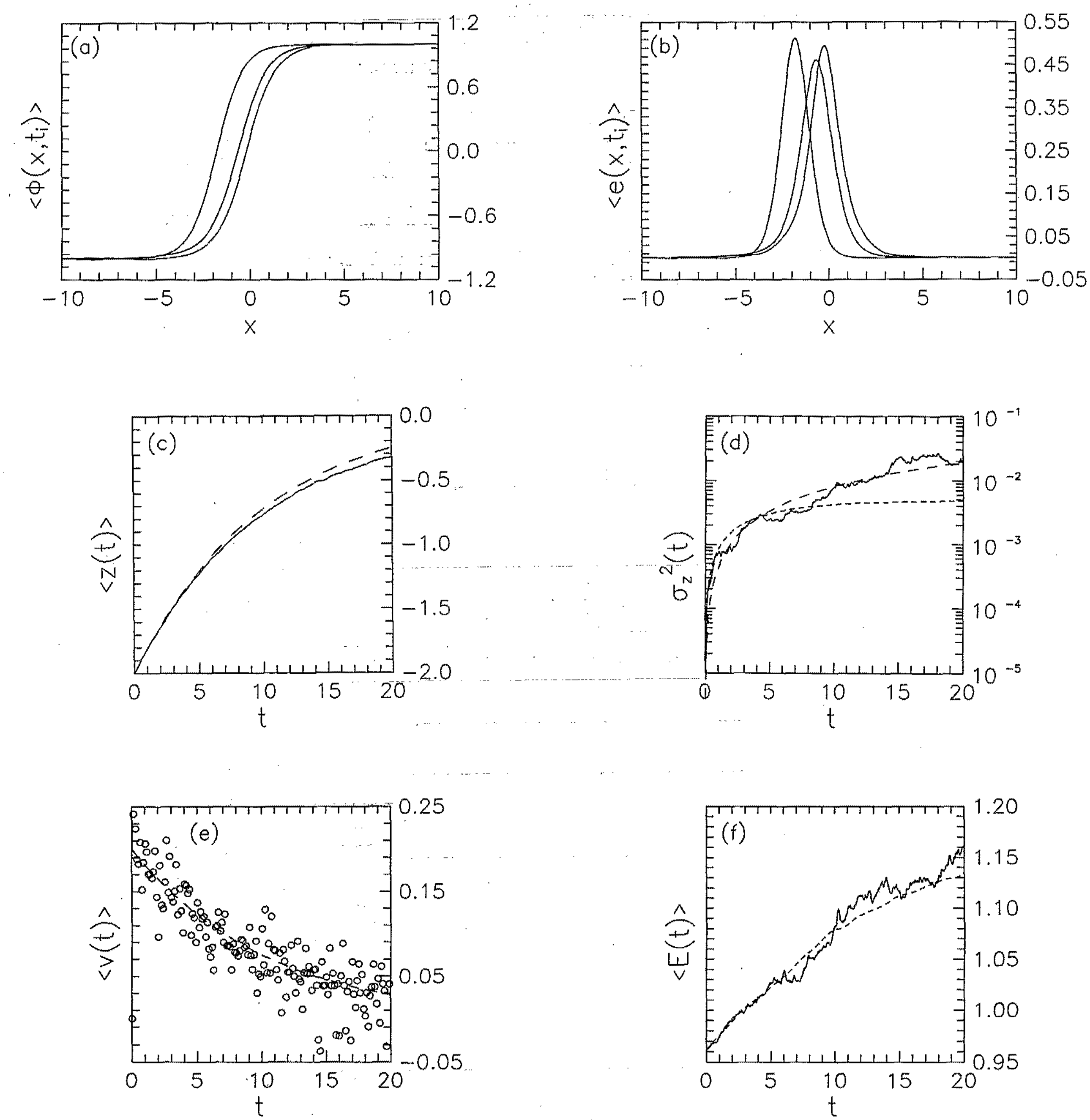

FIG. 10. As in Fig. 3, but with dissipation ($\alpha = 0.1$). The initial speed is $v_0 = 0.2$ rightwards, the initial center position is $z_0 = -2$, and the noise strength is $2D = 0.1$. In (c) we show the adiabatic prediction by a dashed line, obtained from Eq.(19). In (d) the adiabatic prediction comes from Eq. (20); the fitting is $\sigma_z^2(t) = 4.43 \times 10^{-3}\, t^{1.258}$. In (e) the points are instantaneous speeds; we do not join them with a line to allow one to see the adiabatic prediction; instead, one point for each five time steps is shown. In (f) there is no exponential behavior of the energy.

fined there, hidden by the more visible ones produced by friction. If we were to introduce even stronger noises, then the kink would be finally destroyed unless we simultaneously increased dissipation, but if done so, the kink would rapidly stop and we would be back in the nondissipative situation, with energy growing rapidly in the structure and inducing its blowup. Note that in the case of sufficiently strong dissipation, we actually deal with a damped ϕ^4 system, analogous to the damped sine-Gordon one. For the analytical description of this prob-

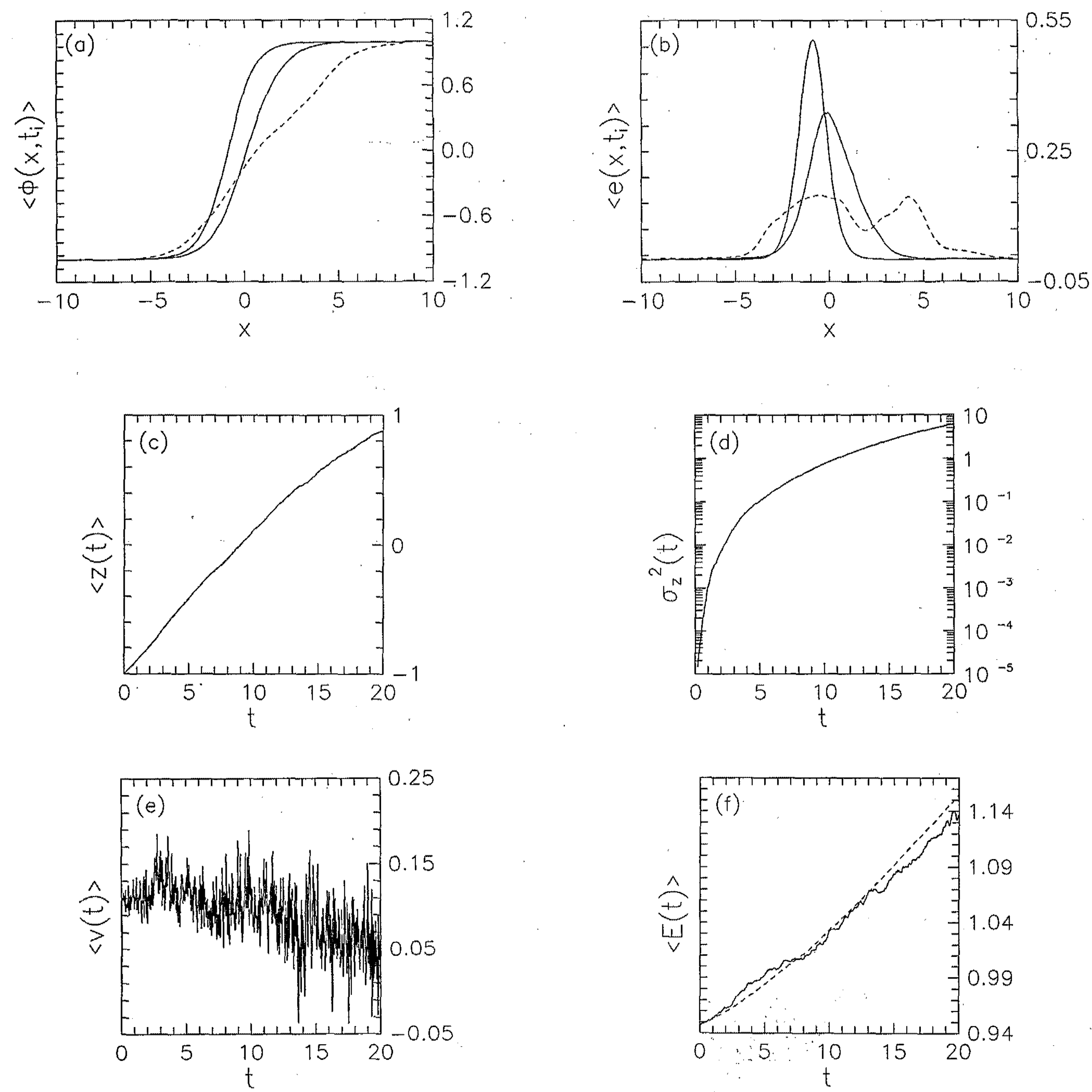

FIG. 11. Incidence of a kink from the left on a noise zone, in $x \in [0, 10]$. The initial speed is $v_0 = 0.1$ rightwards, the initial center position is $z_0 = -1$, the noise strength is $2D = 0.1$, and no dissipation is present, $\alpha = 0$. Adiabatic predictions are not available now. (a) Mean kink shape at $t = 1, 10, 20$ (the last one is dashed to distinguish it from the others). (b) Mean energy density at the same times. (c) Mean center position. (d) Mean center dispersion. (e) Mean center speed. (f) Mean total energy. The solid line is numerical; the dashed line is the modified PMR formula, Eq. (27).

lem, one would have to make use of a singular perturbation theory developed by Kaup and Osman[33] for the sine-Gordon problem (see also Ref. 13).

VI. BOUNDARY EFFECTS ON NONDISSIPATIVE KINKS

We next move to the remaining problem, namely, the scattering of the kink by a boundary between an unperturbed and a stochastic zone. In our previous calculations, both analytical and numerical, we have supposed that the kink is propagating in a fluctuating medium that is infinitely extended. This would correspond to a situation in which the soliton is in a medium initially unperturbed and suddenly the perturbation (some random external field, say) is switched on, affecting the whole system. It seems sensible to think also that this can affect not the entire chain but only a piece of it (this is much more so if we think that this perturbation can be due to localized external sources), and hence we must analyze the problem of whether the kink is able to propagate across such a boundary. The simulations are carried out

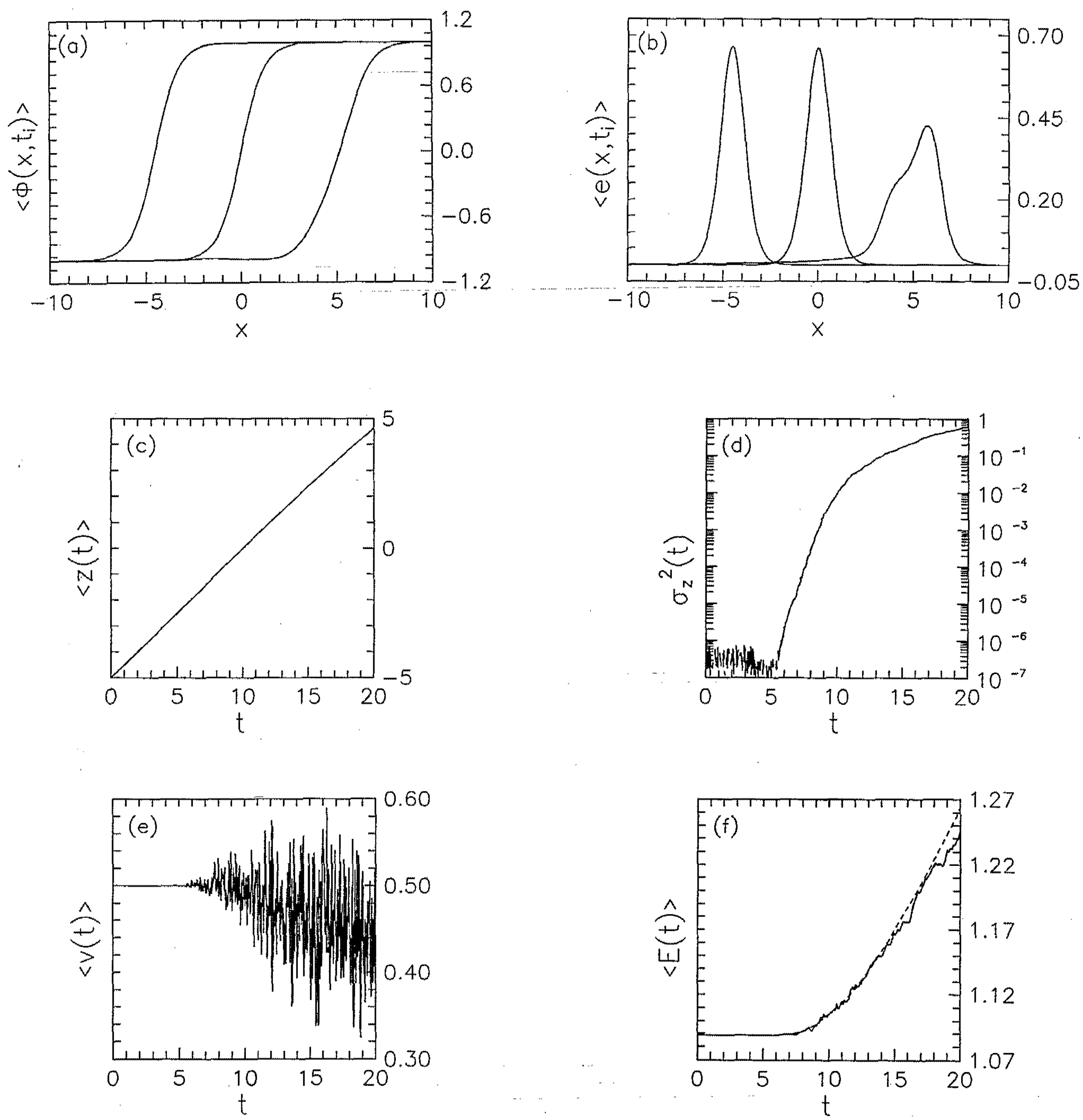

FIG. 12. As in Fig. 11 but for higher speed. The initial speed is $v_0 = 0.5$, the initial center position is $z_0 = -5$, and the noise strength is $2D = 0.1$.

as the preceding ones, but now noise acts only on the 200 particles at positions [0, 10]. The kink is situated initially outside just before its structure enters the zone (remember that wings are not affected at all), and the rest of the simulation parameters remain the same. The PMR formula remains still *exactly* valid in the following modified form:

$$\frac{d\langle E\rangle}{dt} = -\alpha \int_{-\infty}^{\infty} dx\, \langle|\phi_t|^2\rangle + 2D \int_{-\infty}^{\infty} dx\, \chi_{[a,b]}(x)\, \langle(-\phi+\phi^3)^2\rangle, \qquad (27)$$

where $[a, b]$ stands for the interval where the noise acts and $\chi_{[a,b]}(x)$ is the characteristic function of that interval (that equals 1 inside it and vanishes outside of it). With this generalization, at least our consistency test remains useful, and of course we have checked its verification. It is another consistency result for our scheme that the discretization of Eq.(27) is quite well verified, as can be seen in Figs. 11(f) and 12(f). In this last one, the initial plateau is due to the fact that the kink kernel had not entered the noise zone up to time $t = 7$, more or less, and in this situation the kink is not influenced by effects of noise.

Concerning simulations, we have found that there are two possible behaviors depending on the ratio of noise strength to initial speed. These two cases are shown in Figs. 11 and 12, and, in a few words, we can classify them by saying that the soliton gets pinned at the boundary if its speed is not too high, and otherwise it easily overcomes the boundary and the situation is then equal to that of evolution in an infinite noisy zone. This is also indicated in Fig.13 where the evolution of the energy at the left and at the right of the boundary is shown, while if the kink is a slow one a great part of its energy is not able to cross the boundary and recoils [Fig. 13(a)], the fast soliton crosses, losing practically no energy [Fig. 13(b)], and closely resembles the behavior of kinks in the infinite stochastic chain after that point. The "critical" speed (or range of speeds, it does not have to be a sharp value necessarily) depends on how strong the noise is; the exact determination of a kind of "phase diagram" for this problem would require a lot of computer time, which is beyond the scope of this work.

The novel feature of this aspect of our simulations takes place only when we consider slow kinks because, otherwise, after a short transient (the time the soliton takes to cross the boundary) and a little distortion then originates, the situation is the same as in the propagation along an infinite noisy layer (see Fig. 12). When the kink slowly reaches the edge of the perturbed zone, it is rapidly pinned to it. A great part of the energy is not allowed to enter the noisy slab, and thereafter the kink has no means of propagating undistortedly along it. Subsequently, this energy begins to reflect backwards, not as phonons but as a strain of the whole chain [see Fig. 11(a)], due to the action of the harmonic coupling that still connects the parts of the kink inside and outside the noise, leading to a rather unstable situation. The final result of the process is not clear in our computations, and much longer ones would be necessary in order to arrive at a definite conclusion. This problem is also difficult analytically: the adiabatic equations are very complicated and one can only get a few results[20] for weak noise, namely, that kinks get pinned if their speed is not much greater than zero. Nothing is known about the radiation generation power of this process; the only hint of this is that our simulations strongly suggest that the leading role is being played by the internal mode. Further work is needed to clarify completely the boundary problem. Let us close the discussion of boundary effects with a comment on the combined action of boundaries and dissipation. When both are present, simulations show the result that one can expect from all the preceding descriptions, i.e., the threshold velocity below which kinks are pinned is increased, due to the fact that dissipation contributes to slowing them enough as to allow for their capture by the boundary. Except for this variation, anything new happens seemingly because phonon modes, the ones that are most affected by dissipation, do not appear to be responsible for this pinning phenomenon.

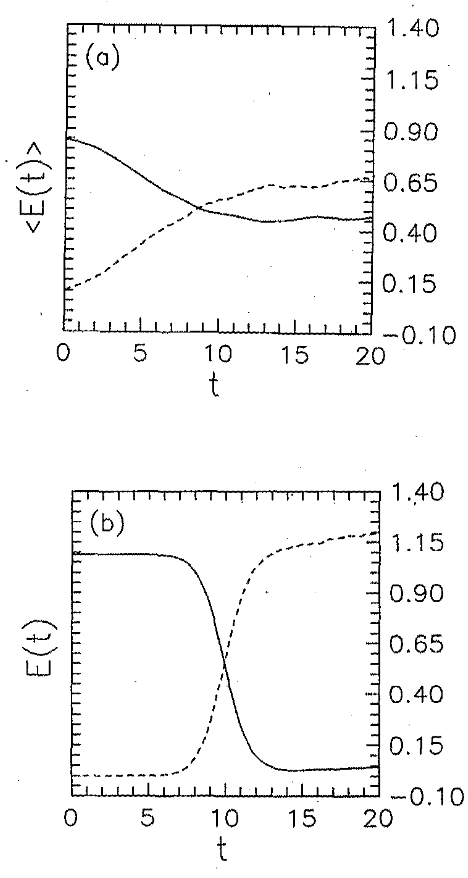

FIG. 13. Energy contained in the zone to the left and to the right of the noise boundary. (a) The initial speed is $v = 0.2$. (b) the initial speed is $v = 0.5$. The solid line is E_{left}, the dashed line is E_{right}. $\alpha = 0$.

VII. CONCLUSION

We have studied the problem of nonlinear excitations propagating along a ϕ^4 chain in which the on-site potential randomly fluctuates in time to a large spatial extent. First, we have settled the basis of our numerical procedure, proving that the number of realizations over which we average is not crucially important, and showing that a discrete analog of the PMR (Ref. 32) formula, which is exact in the continuum limit, is verified by our simulations in all cases. This is important because, to our knowledge, such a property has not been found previously in numerical simulation of stochastic differential equations; besides, as the scheme can be generalized to a variety of problems, this provides a procedure to treat them beyond the weak-perturbation regime. Thus ensured of the reliability of our computations, we have seen that kinks are stable if perturbed for a time not exceeding a critical value that depends on the strength of the stochastic term; if the perturbation time is greater, the model reaches an unstable situation that crosses over to a rapid evolution regime leading to a blowup end. We have shown also that this stability is real by tracking the evolution of kinks after the noise is switched off, in order to see whether the distortions of their shape, which are induced by the perturbation, are permanent or not. Our simulations demonstrate that they are able to recover their usual appearance, the only permanent effects being the diminution of the speed and a phase shift of the center position.

With respect to noise regimes, we have established that when $2D$ is less than or equal to 0.01, the noise can be thought of as weak, and the adiabatic approximate treatment is able to account for the main features of soliton propagation. For these perturbation values, the collective coordinates are enough to describe soliton motion, and the growing of the dispersion of the center happens as expected. Hence in this regime it is possible to account even for the radiation generation and the internal mode evolution through a secular perturbative expansion.[20] This is not so anymore when noise is around $2D \simeq 0.1$; the appearance of phonons propagating along the chain limits the validity of our perturbative predictions. The speed begins to behave clearly as a stochastic process and does not remain constant, but shows a trend of decreasing. The kink center consequently slows down and its dispersion grows faster than linearly with time, as fast as time squared. A more accurate analytical approach, which takes the radiation properly into account, i.e., nonlinearly, is then needed. Notice that the linear treatment of Ref. 20 is clearly not valid when radiation amplitude is not small; however, the stochastic parametrical resonance phenomena there described could be at the root of the crossover between normal and anomalous diffusion.

One thing that was revealed by the simulation and that was not suspected in principle is the dependence of the results on the speed of the soliton. In all situations we have found that fast kinks are quite a bit more resistent to noise action than slow ones, and that the range in which the collective coordinate treatment holds extends to stronger values of noise as the soliton speed approaches 1. It seems as though the perturbative prediction that radiation is small for relativistic kinks[20] extended its validity to the strong noise regime, and, besides, the difference in radiation emission power can be as dramatic as to allow relativistic kinks to propagate when noise is an order of magnitude over the value that causes severe distortions to slow solitons. Simulations have allowed us to fix numerical values for all these phenomena in the entire range of noises.

Nevertheless, the most novel and unexpected property of these systems is perhaps the behavior of the energy in the chain. Aside from the PMR prediction, which makes no explicit predictions on what is the precise function governing the energy evolution, we have found that it is always exponential when there is no dissipation, with an exponent that depends roughly linearly on the noise strength. This is the reason why kinks are stable only for a certain period: the monotonic increasing of the energy content of the chain with time always must lead to unstable configurations and blowups. The existence of such an empirical law is very interesting: let us recall that the blowup regime is seen to start when energy is around twice the initial value. Then, it is enough to compute how much time is needed to reach this point by means of our *a priori* knowledge of the energy increasing ratio to get a semiquantitative prediction on the stability time for each noise strength. As far as we know, there are no previous results with similar predictive power concerning soliton stability for this kind of system.

We have also tried to introduce more physical content in our model, studying the influence of dissipation and boundaries. Our simulations show that due to the damping of phonons by dissipation, the collective coordinates are valid for a description of soliton evolution under noises an order of magnitude above those of nondissipative systems. Solitons present an essentially dissipative character that hides the stochastic behavior, which is only appreciated in the dispersion of the center and an initial growing of energy up to some saturation value. The reason for this behavior must be the inhibition of radiation emission due to friction that concentrates the noise effects on the kink collective coordinates. On the other hand, when boundaries between pure and perturbed parts of the chain exist, there is a critical velocity for each value of noise under which solitons cannot propagate into the perturbed zone, and a distinctly nonzero part of the energy remains in the first part of the chain. Over this velocity, the kink rapidly enters the zone and after that it evolves as in an infinite noisy layer. This topic would require a larger amount of computation in order to define, not only the value of this critical speed, but also if it is a sharp value or not. This could be also the starting point for also introducing a dependence of the noise on the spatial coordinate or, at least, more than one of these inhomogeneities. The study and comprehension of systems disordered both in time and space and the joint effect of this disorder and nonlinearity are still poorly understood, and much more theoretical and numerical effort must be devoted to them. We hope that this work can serve as a basis for a deeper insight into wave propagation in these nonlinear disordered media.

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