BOOTSTRAP ASSISTED SPECIFICATION TESTS FOR THE ARFIMA MODEL

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This paper proposes bootstrap assisted specification tests for the autoregressive fractionally integrated moving average model based on the Bartlett T_p-process with estimated parameters whose limiting distribution under the null depends on the estimated model and the estimation method employed. The computation of the asymptotic critical values is not easy if at all possible under these circumstances. To circumvent this problem Delgado, Hidalgo, and Velasco (2005, *Annals of Statistics* 33, 2568–2609) proposed an asymptotically pivotal transformation of the T_p-process with estimated parameters. The aim of this paper is twofold. First, to examine alternative methods based on bootstrap algorithms for estimating the distribution of the test under the null, showing its validity. And second, to study the finite-sample performance of the different alternative procedures via Monte Carlo simulation.

1. INTRODUCTION

A parametric time series linear process is correctly specified when the corresponding innovations of the model are uncorrelated. In this context, Bartlett (1954) introduced two alternative omnibus tests based on estimates of the spectral distribution function. One of these alternatives was based on functionals of the U_p-process, which compares the empirical spectral distribution function with that obtained under the restricted null hypothesis. This procedure resembles the standard empirical process when testing the correct specification of a particular probability distribution function. The second alternative is based on the T_p-process, which is a standardized estimator of the spectral distribution function of the innovations of the model under consideration. Unlike the U_p-process, the T_p-process converges in distribution to the standard Brownian bridge under a simple null without unknown parameters. See, for instance, Anderson (1993). However, when the model depends on a set of unknown parameters, the T_p-process has a limiting distribution that depends on the true parameter values and also on the estimation method employed. One consequence of the latter is that the implementation of the test is difficult if at all possible. To circumvent this drawback, two alternative procedures are discussed and examined in the literature. One of them is based on bootstrap methods, whereas a second one is based on a martingale transformation of the T_p-process in a spirit similar to Khmaladze (1981).

Among the bootstrap methods, Chen and Romano (1999) proposed approximating the distribution of the test statistics based on the U_p-process with estimated parameters using a resample of the residuals. On the other hand, extending an idea of Hidalgo (2003), Hidalgo and Kreiss (2006) proposed a bootstrap test for the T_p-process using a wild resample specifically designed for this problem. Hidalgo and Kreiss (2006) pointed out that, when we allow for long memory dependence, the T_p-process becomes more suitable than the U_p-process for specification testing.

Delgado, Hidalgo, and Velasco (2005) (DHV henceforth), rather than using bootstrap assisted tests, proposed an asymptotically distribution-free transformation of the T_p-process, which entails isolating the martingale component of the process. One advantage of the latter method is that the resulting test has a known and tabulated asymptotic distribution function. The transformation resembles in spirit the cumulative sum (CUSUM) process based on recursive least squares residuals for testing stability of the parameters in a linear regression model, as proposed by Brown, Durbin, and Evans (1975).

The aim of this paper is twofold. On the one hand, we present and examine alternative bootstrap tests, showing their validity for both the T_p-process and its martingale transformation. Following ideas in Hall (1992) and applying the asymptotic expansions for the asymptotic pivotal statistics, as in Götze (1979, 1984), we should expect some size accuracy gains when applying the bootstrap methods to the statistics based on the transformed T_p-process. And second, because several (bootstrap) methods are available to perform valid tests for the null hypothesis, the question of practical interest is which alternative method performs better in finite samples. We explore the latter issue by means of a Monte Carlo experiment.

In addition, we compare the performance of the T_p-process against the Box–Pierce statistic (Box and Pierce, 1970), which is based on the sum of the first m squared sample autocorrelations of the residuals. This is a particular case of the portmanteau tests considered by Hong (1996). The latter proposes a weighted sum of all the squared sample autocorrelations of the residuals, where the weights can be identified by a kernel function with bandwidth parameter of order m^{-1}. In fact, the Box–Pierce statistic corresponds to the choice of the uniform kernel in Hong's approach. When m diverges with n at a suitable rate, the statistic is asymptotically distribution free and is able to detect alternatives converging to the null at the rate $m^{1/4}/n^{1/2}$. However, whereas a relatively large m is required to obtain a good accuracy level for the test (typically $m \simeq n^{1/2}$ is considered a good choice), it is also the case that for a good power performance m needs to be chosen relatively smaller. This is in contrast with the tests put forward in this paper, which are able to detect alternatives at the parametric rate $n^{-1/2}$ without resorting to the choice of a bandwidth or a kernel function.

The reminder of the paper is organized as follows. In the next section, we introduce the basic notation and the testing problem, and we describe the test based on the T_p-process and its asymptotically distribution-free transformation. Section 3 describes the bootstrap tests under two alternative resampling schemes, and it justifies their validity for the T_p-process and its martingale transformation. Section 4 presents a Monte Carlo experiment to shed some light on the performance of the different approaches to test for the null hypothesis. Section 5 gives a series of lemmas that are employed to prove the main results of Section 3 in the last section of the paper.

2. TIME SERIES SPECIFICATION TESTS BASED ON THE T_p-PROCESS

Let f be the spectral density function of a covariance stationary time series process $\{x_t\}_{t\in\mathbb{Z}}$ with mean μ and covariance function given by the relation

$$\text{Cov}(x_t, x_0) = \int_{-\pi}^{\pi} f(\lambda)\cos(\lambda t)\, d\lambda; \qquad t = 0, \pm 1, \pm 2, \ldots.$$

We assume that $\{x_t\}_{t\in\mathbb{Z}}$ admits a Wold representation in terms of a transfer function $\Delta(z) = \sum_{j=0}^{\infty} a_j z^j$. That is,

$$x_t = \mu + \Delta(L)\,\varepsilon_t, \qquad t \in \mathbb{Z}, \tag{2.1}$$

for some sequence $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ satisfying $\mathbb{E}(\varepsilon_t) = 0$ and $\mathbb{E}(\varepsilon_0\varepsilon_t) = \sigma^2\mathbb{I}(t = 0)$, $\mathbb{I}(\cdot)$ denoting the indicator function, and where L is the lag operator. Under (2.1) and denoting $h(\lambda) = \left|\Delta(e^{i\lambda})\right|^2$, $f(\lambda)$ can be factorized as

$$f(\lambda) = \frac{\sigma^2}{2\pi} h(\lambda), \qquad \lambda \in [0, \pi].$$

Statistical inferences on $\{x_t\}_{t\in\mathbb{Z}}$ are usually based on a parametric specification of $\Delta(z)$, $\Delta_\theta(z)$. Among practitioners, the most popular specification is the fractional autoregressive moving average (ARFIMA) model, where

$$\Delta_\theta(z) = \frac{1}{(1-z)^d}\,\Xi(z; \psi, \varphi), \qquad \theta = \left(\psi', d, \varphi'\right)', \tag{2.2}$$

with $\Xi(z; \psi, \varphi) = \Phi_\varphi^{-1}(z)\,\Psi_\psi(z)$ and where $\Psi_\psi(z)$ and $\Phi_\varphi(z)$ are, respectively, the moving average and autoregressive polynomials. The dimensionality of the parameters ψ and φ is, respectively, p_1 and p_2, whereas $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ is known as the long memory parameter. In addition, we shall assume that the parameters ψ and φ are such that $\Psi_\psi(z)$ and $\Phi_\varphi(z)$ have no common roots and they are all lying outside the unit circle. From (2.2) we have that $h_\theta(\lambda) = |\Delta_\theta(e^{i\lambda})|^2$ becomes

$$h_\theta(\lambda) = \frac{1}{\left|1 - e^{i\lambda}\right|^{2d}} \left|\frac{\Psi_\psi\left(e^{i\lambda}\right)}{\Phi_\varphi\left(e^{i\lambda}\right)}\right|^2, \qquad \lambda \in [0, \pi].$$

Denote by $\mathcal{C} = \{\Delta_\theta : \theta \in \Theta\}$ the family of stationary and invertible ARFIMA transfer functions in (2.2), where $\Theta \subset \mathbb{R}^p$ denotes the parameter space. We are interested in testing the hypothesis

$$H_0 : \Delta \in \mathcal{C}$$

with the alternative hypothesis, H_1, being the negation of the null. That is, we are interested in omnibus tests capable of detecting nonparametric alternatives, in the sense that it might not be possible to describe them by a finite number of parameters. It is worth mentioning that, although we shall explicitly focus on the ARFIMA model, this is only the case for notational simplicity and because of its ubiquity in applications. The ARFIMA model was examined by Granger and Joyeux (1980) and Hosking (1981) as a compromise between the stationary autoregressive moving average (ARMA) and the nonstationary autoregressive integrated moving average models, offering greater flexibility to model the long-run dependence by means of a sole extra parameter. Although most economic time series are nonstationary and do require differencing of some sort, it is not necessarily true that after taking first differences, the correct specification of the time series is an ARMA model. In fact, for instance, Robinson (1994b) advocated the use of the Bloomfield (1973) exponential model to describe short-run dynamics. However the latter type of models, which are specified in the frequency domain, require some modification of our bootstrap algorithms; see Section 3 for a reference and some comments. In addition, models that may exhibit long-memory dependence are naturally justified in economics when aggregating cross-sectional observations to construct macro time series (see, e.g., Robinson, 1978; Granger, 1980). For some overviews of the long-memory literature, see, for instance, Beran (1998) or Robinson (1994a).

We can alternatively write H_0 in terms of the spectral density function of $\{\varepsilon_{\theta t}\}_{t\in\mathbb{Z}}$, where

$$\varepsilon_{\theta t} = \Delta_\theta^{-1}(L)(x_t - \mu), \qquad t \in \mathbb{Z}.$$

That is, we can write H_0 as

$$H_0 : \frac{f(\lambda)}{h_{\theta_0}(\lambda)} = \frac{\sigma_0^2}{2\pi}, \qquad \lambda \in [0, \pi],$$

for some $\theta_0 = \left(\psi_0', d_0, \varphi_0'\right)' \in \Theta$. As usual, a subscript 0 in a parameter indicates its true value. Notice that $f(\lambda)/h_\theta(\lambda)$ is the spectral density function of $\{\varepsilon_{\theta t}\}_{t\in\mathbb{Z}}$ and, under H_0, $\varepsilon_{\theta_0 t} = \varepsilon_t$.

The estimator of the spectral distribution function of $\{\varepsilon_{\theta t}\}_{t\in\mathbb{Z}}$,

$$F_\theta(\lambda) = 2\int_0^\lambda \frac{f(\bar{\lambda})}{h_\theta(\bar{\lambda})} d\bar{\lambda}, \qquad \lambda \in [0,\pi],$$

forms the basis for testing H_0. For a generic sequence $\{v_t\}_{t=1}^n$, let us denote its periodogram by

$$I_v(\lambda) := |w_v(\lambda)|^2,$$

where $w_v(\lambda) = (2\pi n)^{-1/2} \sum_{t=1}^n v_t e^{it\lambda}$ is the discrete Fourier transform of the sequence. Then, for a given record of data $\{x_t\}_{t=1}^n$, we estimate F_θ by $F_{\hat{\theta}_n n}$, where $\hat{\theta}_n$ is a $n^{1/2}$-consistent estimator of θ_0 and

$$F_{\theta n}(\lambda) = \frac{2\pi}{\tilde{n}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \frac{I_x(\lambda_j)}{h_\theta(\lambda_j)}, \qquad \lambda \in [0,\pi],$$

with $\lambda_j = 2\pi j/n$ being the Fourier frequencies. Herewith, $\tilde{n} = \lfloor n/2 \rfloor$ with $\lfloor z \rfloor$ being the integer part of z. A natural candidate to estimate the parameters θ_0 is the Whittle estimator defined as

$$\hat{\theta}_n = \arg\min_{\theta\in\Theta} F_{\theta n}(\pi), \tag{2.3}$$

which, under Assumptions A1–A3 stated in Section 3, is known to be $n^{1/2}$-consistent. See, for instance, Velasco and Robinson (2000), among others. Set $\hat{\theta}_n = \left(\hat{\psi}_n', \hat{d}_n, \hat{\varphi}_n'\right)'$. Notice that under H_0, $\sigma_0^2 = F_{\theta_0}(\pi) = \min_{\theta\in\Theta} F_\theta(\pi)$, and so we define $\hat{\sigma}_n^2 = F_{\hat{\theta}_n n}(\pi)$.

We now define the Bartlett's T_p-process as $\alpha_{\hat{\theta}_n n}$, where

$$\alpha_{\theta n}(\lambda) = \tilde{n}^{1/2} \left[\frac{F_{\theta n}(\lambda)}{F_{\theta n}(\pi)} - \frac{\lambda}{\pi}\right], \qquad \lambda \in [0,\pi].$$

Notice that the empirical process $\alpha_{\theta n}$ is a random function with realizations in the functional space $D[0,\pi]$. For a definition see, for instance, Billingsley (1968). Under the null hypothesis and Assumptions A1–A3, DHV showed that

$$\alpha_{\hat{\theta}_n n}(\lambda) = \alpha_{\theta_0 n}(\lambda) - \tilde{n}^{1/2}\left(\hat{\theta}_n - \theta_0\right)' \phi_{\theta_0}(\lambda) + o_p(1), \tag{2.4}$$

where the term $o_p(1)$ is uniform in $\lambda \in [0,\pi]$ and

$$\phi_\theta(\lambda) = \frac{\partial}{\partial\theta} \log h_\theta(\lambda).$$

Contrary to $\alpha_{\theta_0 n}$, $\alpha_{\hat{\theta}_n n}$ does not converge in distribution to the standard Brownian bridge on $[0, \pi]$. In fact, as was shown by DHV, $\alpha_{\hat{\theta}_n n}$ converges to a Gaussian process whose covariance structure depends on the model under the null hypothesis and the specific method employed to estimate the parameters θ_0. More specifically, DHV showed that, under H_0 and suitable regularity conditions, $\alpha_{\hat{\theta}_n n}$ converges to $\alpha_{\theta_0 \infty}$, where

$$\alpha_{\theta\infty}(\lambda) \stackrel{d}{=} B^1(\lambda) - \left(\pi^{-1}\int_0^{\lambda} \phi'_\theta\left(\bar{\lambda}\right) d\bar{\lambda}\right) \Sigma_\theta^{-1}(\pi) \int_0^{\pi} \phi_\theta\left(\bar{\lambda}\right) B^1\left(d\bar{\lambda}\right), \quad \lambda \in [0, \pi],$$

where $\stackrel{d}{=}$ means equal distributions,

$$\Sigma_\theta(\lambda) = \frac{1}{\pi}\int_0^{\lambda} \phi_\theta\left(\bar{\lambda}\right) \phi'_\theta\left(\bar{\lambda}\right) d\bar{\lambda}, \tag{2.5}$$

and B^1 is the standard Brownian bridge in $[0, \pi]$.

Notice that, under H_0, the Whittle estimator $\hat{\theta}_n$ in (2.3) satisfies the linear expansion

$$\hat{\theta}_n = \theta_0 - \frac{b_{\theta_0 n}}{\bar{u}_{\theta_0 \tilde{n}}} + o_p\left(n^{-1/2}\right), \tag{2.6}$$

where $\bar{v}_m$ denotes, henceforth, the sample mean of any generic sequence $\{v_t\}_{t=1}^m$ and

$$b_{\theta n} = \left(\sum_{j=1}^{\tilde{n}} \phi_\theta\left(\lambda_j\right) \phi'_\theta\left(\lambda_j\right)\right)^{-1} \sum_{j=1}^{\tilde{n}} \phi_\theta\left(\lambda_j\right) u_\theta\left(\lambda_j\right)$$

with $u_\theta\left(\lambda_j\right) = I_x\left(\lambda_j\right)/h_\theta\left(\lambda_j\right)$. Hence, $\alpha_{\hat{\theta}_n n}$ can be asymptotically represented as a CUSUM of least squares residuals. Indeed, combining (2.4) and (2.6), under the null hypothesis H_0 and Assumptions A1–A3 given in Section 3, DHV showed that the T_p-process satisfies the expansion

$$\begin{aligned}\alpha_{\hat{\theta}_n n}(\lambda) &= \frac{1}{\bar{u}_{\theta_0 \tilde{n}} \tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \left\{\left(u_{\theta_0}\left(\lambda_j\right) - \bar{u}_{\theta_0 \tilde{n}}\right) - b'_{\theta_0 n}\left(\phi_{\theta_0}\left(\lambda_j\right) - \bar{\phi}_{\theta_0 \tilde{n}}\right)\right\} \\ &\quad + o_p(1) \\ &= \frac{1}{\bar{u}_{\theta_0 \tilde{n}} \tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \left(u_{\theta_0}\left(\lambda_j\right) - c'_{\theta_0 n} \gamma_{\theta_0}\left(\lambda_j\right)\right) + o_p(1),\end{aligned} \tag{2.7}$$

uniformly in $\lambda \in [0, \pi]$, where $\gamma_\theta\left(\lambda_j\right) = \left(1, \phi'_\theta\left(\lambda_j\right)\right)'$ and

$$c_{\theta n} = \left(a_{\theta n}, b'_{\theta n}\right)' = \left(\bar{u}_{\theta \tilde{n}} - b'_{\theta n} \bar{\phi}_{\theta \tilde{n}}, b'_{\theta n}\right)'$$

are the least squares coefficients of the projection of $\{u_\theta(\lambda_j)\}_{j=1}^{\tilde{n}}$ on $\{\gamma_\theta(\lambda_j)\}_{j=1}^{\tilde{n}}$. Observe that

$$\alpha_{\theta_0 n}(\lambda) = \frac{1}{\bar{u}_{\theta_0 \tilde{n}} \tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \left(u_{\theta_0}(\lambda_j) - \bar{u}_{\theta_0 \tilde{n}}\right) + o_p(1)$$

as $F_{\theta n}(\pi) = 2\pi \bar{u}_{\theta \tilde{n}}$. It is also worth mentioning that in the first equality of (2.7), we have employed the fact that $\bar{\phi}_{\theta \tilde{n}} = o(1)$ because $\int_{-\pi}^{\pi} \phi(\lambda)\, d\lambda = 0$ and Lemma 1 in DHV.

From here, and following ideas of Brown et al. (1975), it is expected that the corresponding CUSUM of recursive residuals will be asymptotically distribution free. In our context, the CUSUM of (forward) recursive residuals is given by $\hat{\alpha}_{\hat{\theta}_n n}(\lambda)$, where

$$\hat{\alpha}_{\theta n}(\lambda) = \frac{1}{\bar{u}_{\theta \tilde{n}} \tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \bar{n}\lambda/\pi \rfloor} \left(u_\theta(\lambda_j) - \gamma'_\theta(\lambda_j)\, \hat{c}_\theta(j)\right), \qquad \textbf{(2.8)}$$

with $\bar{n} = \tilde{n} - p - 1$,

$$\hat{c}_\theta(j) = A_{\theta n}^{-1}(j) \frac{1}{\tilde{n}} \sum_{\ell=j+1}^{\tilde{n}} \gamma_\theta(\lambda_\ell)\, u_\theta(\lambda_\ell),$$

and

$$A_{\theta n}(j) = \frac{1}{\tilde{n}} \sum_{k=j+1}^{\tilde{n}} \gamma_\theta(\lambda_k)\, \gamma'_\theta(\lambda_k),$$

assuming that $A_{\theta_0 n}(\overline{n})$ is nonsingular. In fact, DHV showed that $\hat{\alpha}_{\hat{\theta}_n n}$ converges in distribution to the standard Brownian motion in $[0, \pi]$, denoted by B.

The transformed process $\hat{\alpha}_{\hat{\theta}_n n}$ in (2.8) is related to the martingale transformation of the standard empirical process with estimated parameters proposed by Khmaladze (1981), which has been subsequently extended to other specification testing problems by Koul and Stute (1999), Koenker and Xiao (2002), and Delgado and Stute (2008) among others.

We have then that the test statistics are functionals of the T_p-process $\alpha_{\hat{\theta}_n n}$ or its transformation $\hat{\alpha}_{\hat{\theta}_n n}$. Given a continuous functional on $D[0, \pi]$, $\eta : D[0, \pi] \to R^+$, we have that under H_0, $\eta\left(\alpha_{\hat{\theta}_n n}\right) \to_d \eta\left(\alpha_{\theta_0 \infty}\right)$ and that $\eta\left(\hat{\alpha}_{\hat{\theta}_n n}\right) \to_d \eta(B)$. The most popular functionals are the Kolmogorov–Smirnov $\eta(g) = \sup_{\lambda \in [0,\pi]} |g(\lambda)|$ and the Cramér–von Mises $\eta(g) = \pi^{-1} \int_0^\pi g(\lambda)^2\, d\lambda$. However, as the critical values of $\eta\left(\alpha_{\theta_0 \infty}\right)$ are difficult to tabulate, if at all possible, an alternative approach to the martingale transformation–based tests $\eta\left(\hat{\alpha}_{\hat{\theta}_n n}\right)$ entails using bootstrap assisted algorithms. This is the topic of the next section.

BOOTSTRAP TESTS

The purpose of this section is to provide and justify a bootstrap method for estimating the finite-sample distributions of $\eta\left(\alpha_{\hat{\theta}_n n}\right)$ and $\eta\left(\hat{\alpha}_{\hat{\theta}_n n}\right)$. The motivation to bootstrap the asymptotically pivotal statistic is that, as in many other problems, we can expect that bootstrap methods improve the level of accuracy of the test when they are compared to tests based on the asymptotic critical values. Now consider, for example, $\xi_n = \eta\left(\alpha_{\hat{\theta}_n n}\right)$ and let G_∞ be the asymptotic probability distribution function of ξ_n. Denote by ξ_n^* the bootstrap analogue of ξ_n and denote its bootstrap (conditional) distribution function given the sample $\mathcal{X}_n = \{x_t\}_{t=1}^n$ by G_n^*. We say that the bootstrap is valid if the resampling method employed to compute ξ_n^* satisfies that, under H_0,

$$G_n^* \xrightarrow{P} G_\infty \tag{3.1}$$

at each continuity point of G_∞. Then it is said that ξ_n^* converges in distribution in probability to a random variable ξ_∞ with probability distribution function G_∞, and it is written as $\xi_n^* \xrightarrow{d^*} \xi_\infty$ (in probability). See Giné and Zinn (1990) for some discussion. Moreover, the bootstrap test will be consistent if the bootstrapped p-value converges to zero under the alternative; that is, under H_1,

$$G_n^*\left(\xi_n\right) \xrightarrow{P} 1. \tag{3.2}$$

The resampling method must guarantee that (3.1) and (3.2) are satisfied, which is sometimes referred to as that the bootstrap test is valid to test H_0 in the direction of H_1.

We now describe the bootstrap algorithm. To that end, we denote the coefficients in the series expansion of $(1-z)^{-d}$ by

$$b_j(d) = \frac{\Gamma(j+d)}{\Gamma(d)\,\Gamma(j+1)}, \qquad j = 0, 1, \ldots,$$

where $\Gamma(\cdot)$ is the gamma function.

Step 1. Compute

$$\ddot{\varepsilon}_t = \sum_{j=0}^{t-1} b_j\left(-\hat{d}_n\right) x_{t-j}, \qquad t = 1, \ldots, n.$$

Then, with the initial conditions $\ddot{\varepsilon}_t = \hat{\varepsilon}_t = 0$ for $t \leq 0$, compute

$$\hat{\varepsilon}_t = \ddot{\varepsilon}_t - \sum_{\ell=1}^{p_1} \hat{\varphi}_{n\ell}\,\ddot{\varepsilon}_{t-\ell} - \sum_{q=1}^{p_2} \hat{\psi}_{nq}\,\hat{\varepsilon}_{t-q}.$$

Step 2. For some m large enough, let $\left\{\varepsilon_t^*\right\}_{t=1}^{n+m}$ be a random sample of size $n+m$ from the empirical distribution function of $\left\{\widetilde{\varepsilon}_t\right\}_{t=1}^{n}$, where $\widetilde{\varepsilon}_t = \hat{\varepsilon}_t - n^{-1}\sum_{t=1}^{n}\hat{\varepsilon}_t$, and compute

$$\ddot{\varepsilon}_t^* = \varepsilon_t^* + \sum_{\ell=1}^{p_1}\hat{\varphi}_{n\ell}\ddot{\varepsilon}_{t-\ell}^* + \sum_{q=1}^{p_2}\hat{\psi}_{nq}\varepsilon_{t-q}^*, \qquad t = 1,\ldots,n+m,$$

with initial conditions $\ddot{\varepsilon}_t^* = \varepsilon_t^* = 0$ for $t \le 0$. Next, compute

$$\widetilde{x}_t^* = \ddot{\varepsilon}_t^* + \sum_{j=1}^{t-1} b_j\left(\hat{d}_n\right)\ddot{\varepsilon}_{t-j}^*, \qquad t = 1,\ldots,n+m.$$

Then our bootstrap sample is $\mathcal{X}_n^* = \left\{\widetilde{x}_{t+m}^*\right\}_{t=1}^{n} = \left\{x_t^*\right\}_{t=1}^{n}$.

Remark 1. Notice that we could generate more or less bootstrapped residuals, but at least n. However, it seems convenient to initialize the sample using some additional observations, as it is expected that the effect of the initial conditions on $\left\{\widetilde{x}_t^*\right\}_{t=m+1}^{n+m}$ would not be relevant after choosing m large enough. This is in a spirit similar to when the practitioner simulates an AR(1) model.

Denote by $F_{\theta n}^*$ the bootstrap analogue of $F_{\theta n}$; that is,

$$F_{\theta n}^*(\lambda) = \frac{2\pi}{\tilde{n}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\frac{I_{x^*}\left(\lambda_j\right)}{h_\theta\left(\lambda_j\right)}, \qquad \lambda \in [0,\pi]. \tag{3.3}$$

Step 3. Compute the bootstrap analogue of the Whittle estimate (2.3) as

$$\hat{\theta}_n^* = \hat{\theta}_n - \left(\frac{1}{\tilde{n}}\sum_{j=1}^{\tilde{n}}\phi_{\hat{\theta}_n}\left(\lambda_j\right)\phi_{\hat{\theta}_n}'\left(\lambda_j\right)\right)^{-1}\frac{1}{\hat{\sigma}_n^2}\frac{\partial}{\partial\theta}F_{\hat{\theta}_n n}^*(\pi). \tag{3.4}$$

Then, we compute the bootstrap test as $\hat{\eta}_n^* = \eta\left(\alpha_{\hat{\theta}_n^* n}^*\right)$, where

$$\alpha_{\hat{\theta}_n^* n}^*(\lambda) = \tilde{n}^{1/2}\left[\frac{F_{\hat{\theta}_n^* n}^*(\lambda)}{F_{\hat{\theta}_n^* n}^*(\pi)} - \frac{\lambda}{\pi}\right], \qquad \lambda \in [0,\pi].$$

Remark 2. We can replace our estimator $\hat{\theta}_n^*$ in (3.4) by

$$\hat{\theta}_n^* = \arg\min_{\theta\in\Theta} F_{\theta n}^*(\pi).$$

However, we have preferred to employ (3.4) for computational simplicity; see Shao and Tu (1995, pp. 228, 336).

Remark 3. Because $\mathrm{E}\left[\varepsilon_t^* \middle| \mathcal{X}_n\right] = 0$ and $\mathrm{E}\left[\varepsilon_t^{*2} \middle| \mathcal{X}_n\right] = \hat{\sigma}_n^2$, we have that $\mathrm{E}\left[x_t^* \middle| \mathcal{X}_n\right] = 0$, and the (conditional on $\mathcal{X}_n$) spectral density function of $\left\{x_t^*\right\}_{t=1}^n$ is

$$f_{\hat{\theta}_n n}(\lambda) := \frac{\hat{\sigma}_n^2}{2\pi} h_{\hat{\theta}_n n}(\lambda), \tag{3.5}$$

where $h_{\theta n}(\lambda) = \left|\sum_{\ell=0}^n b_\ell(d) e^{i\ell\lambda}\right|^2 \Xi(\lambda; \psi, \varphi)$ for $\lambda \in [0, \pi]$.

The foregoing bootstrap differs from others in similar problems. In particular it differs from the wild bootstrap proposed by Hidalgo (2003) and improved in our context by Hidalgo and Kreiss (2006). Hidalgo and Kreiss considered a naive resample $\left\{x_t^\dagger\right\}_{t=1}^n$ from the empirical distribution function of $\{x_t\}_{t=1}^n$. Then, using

$$\mathring{F}_{\theta n}^*(\lambda) = \frac{2\pi}{\tilde{n}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \frac{h_{\hat{\theta}_n}(\lambda_j)}{h_\theta(\lambda_j)} I_{x^\dagger}(\lambda_j)$$

instead of $F_{\theta n}^*(\lambda)$ given in (3.3), the bootstrap analogue of $\eta\left(\alpha_{\hat{\theta}_n n}\right)$ is given by $\eta\left(\alpha^*_{\hat{\mathring{\theta}}_n^* n}\right)$, where

$$\alpha^*_{\hat{\mathring{\theta}}_n^* n}(\lambda) = \tilde{n}^{1/2}\left[\frac{\mathring{F}^*_{\hat{\mathring{\theta}}_n^* n}(\lambda)}{\mathring{F}^*_{\hat{\mathring{\theta}}_n^* n}(\pi)} - \frac{\lambda}{\pi}\right], \qquad \lambda \in [0, \pi],$$

with

$$\hat{\mathring{\theta}}_n^* = \arg\min_{\theta \in \Theta} \mathring{F}_{\theta n}^*(\pi)$$

or the analogue of (3.4); that is,

$$\hat{\mathring{\theta}}_n^* = \hat{\theta}_n - \left(\frac{1}{\tilde{n}} \sum_{j=1}^{\tilde{n}} \phi_{\hat{\theta}_n}(\lambda_j) \phi'_{\hat{\theta}_n}(\lambda_j)\right)^{-1} \frac{1}{\hat{\sigma}_n^2} \frac{\partial}{\partial \theta} \mathring{F}^*_{\hat{\theta}_n n}(\pi).$$

The major difference with the bootstrap in steps 1–3 is that in the former we are able to approximate the transfer function $\Delta\left(e^{i\lambda}\right)$, and therefore higher order moments, whereas with the Hidalgo and Kreiss (2006) bootstrap we only approximate its modulus, that is, $\left|\Delta\left(e^{i\lambda}\right)\right|$.

Let us introduce our regularity conditions.

A1. The innovation process $\{\varepsilon_t\}_{t \in \mathbb{Z}}$ satisfies that $\mathbb{E}\left(\varepsilon_t^r \middle| \mathcal{F}_{t-1}\right) = \mu_r$ with μ_r constant ($\mu_1 = 0$ and $\mu_2 = \sigma_0^2$) for $r = 1, \ldots, 4$ and all $t = 0, \pm 1, \ldots$, where $\mathcal{F}_t$ is the sigma algebra generated by $\{\varepsilon_s, s \le t\}$.

A2. $A_{\theta_0 n}(\overline{n})$ is nonsingular for all n large enough.

It is convenient for future reference to observe that the ARFIMA model satisfies Assumptions A2, A3, and A6 of DHV. That is,

1. h is a positive and continuously differentiable function on $(0, \pi]$;
2. $|\partial \log h(\lambda) / \partial\lambda| = O\left(\lambda^{-1}\right)$ as $\lambda \to 0+$;
3. $\phi_{\theta_0}(\lambda)$ is a continuously differentiable function on $(0, \pi]$;
4. $\left\|\partial\phi_{\theta_0}(\lambda) / \partial\lambda\right\| = O(1/\lambda)$ as $\lambda \to 0+$; and for some $0 < \delta < 1$ and all $\lambda \in (0, \pi]$, there exists a $K < \infty$ such that
5. $\sup_{\{\theta:\|\theta-\theta_0\|\leq\iota\}} \|\phi_\theta(\lambda)\| \leq K|\log\lambda|$;

$$\sup_{\{\theta:\|\theta-\theta_0\|\leq\iota/2\}} \frac{1}{\|\theta-\theta_0\|^2}\left|\frac{h_{\theta_0}(\lambda)}{h_\theta(\lambda)} - 1 + \phi'_{\theta_0}(\lambda)(\theta-\theta_0)\right| \leq \frac{K}{\lambda^\delta}\log^2\lambda;$$

   and
6. $\Sigma_{\theta_0} := \Sigma_{\theta_0}(\pi)$ given in (2.5) is positive definite.
7. For some $0 < \iota < 1$ and all $\lambda \in (0, \pi]$, there exists a constant $K < \infty$ such that

$$\sup_{\{\theta:\|\theta-\theta_0\|\leq\iota\}} \frac{1}{\|\theta-\theta_0\|^2}\left\|\phi_\theta(\lambda) - \phi_{\theta_0}(\lambda) - \frac{\partial}{\partial\theta'}\phi_{\theta_0}(\lambda)(\theta-\theta_0)\right\| \leq K|\log\lambda|$$

   and $\phi_\theta(\lambda)/\partial\theta$ satisfies 3–5.

All these properties will be denoted as Assumption A3 in what follows. Denote by O_{p^*}, o_{p^*}, and $\mathbb{E}^*$ the usual stochastic orders of magnitude and expectation, respectively, referred to as the bootstrap law given $\mathcal{X}_n$, $\Pr^*$.

PROPOSITION 1. *Assuming that A1–A3 hold true, under H_0, or under H_1 but assuming that $\tilde{n}^{1/2}\left(\hat{\theta}_n - \theta_1\right) = O_p(1)$ for some $\theta_1 \in \Theta$ and that A2 and A3 hold with θ_0 replaced by θ_1, we have that*

(i) $$\tilde{n}^{1/2}\left(\hat{\theta}^*_n - \hat{\theta}_n\right) = \left(\frac{1}{\tilde{n}}\sum_{j=1}^{\tilde{n}}\phi_{\hat{\theta}_n}(\lambda_j)\phi'_{\hat{\theta}_n}(\lambda_j)\right)^{-1}\frac{2\pi}{\hat{\sigma}^2_n\tilde{n}^{1/2}}\sum_{j=1}^{\tilde{n}}\phi_{\hat{\theta}_n}(\lambda_j) I_{\varepsilon^*}(\lambda_j) + o_{p^*}(1),$$

(ii) $$\tilde{n}^{1/2}\left(\hat{\theta}^*_n - \hat{\theta}_n\right) = O_{p^*}(1).$$

Proof. The proof of this and other results will be given in Section 6. ■

The following theorem provides the consistency of the bootstrap test given in steps 1–3.

THEOREM 1. *If A1–A3 hold, we have that, under H_0,*

$$\alpha^*_{\hat{\theta}^*_n n} \overset{d^*}{\to} \alpha_{\theta_0\infty} \quad \textit{in probability},$$

and under H_1, assuming that $\tilde{n}^{1/2}\left(\hat{\theta}_n - \theta_1\right) = O_p(1)$ for some $\theta_1 \in \Theta$ and that A2 and A3 hold with θ_0 replaced by θ_1,

$$\alpha^*_{\hat{\theta}^*_n n} \xrightarrow{d^*} \alpha_{\theta_1 \infty} \quad \text{in probability.}$$

We now have the following corollary.

COROLLARY 1. *Let η be a continuous mapping in $\mathbb{R}^+$. Under the conditions of Theorem 1 and H_0,*

$$\hat{\eta}^*_n = \eta\left(\alpha^*_{\hat{\theta}^*_n n}\right) \xrightarrow{d^*} \eta\left(\alpha_{\theta_0 \infty}\right) \quad \text{in probability,}$$

and under H_1

$$\hat{\eta}^*_n = \eta\left(\alpha^*_{\hat{\theta}^*_n n}\right) \xrightarrow{d^*} \eta\left(\alpha_{\theta_1 \infty}\right) \quad \text{in probability.}$$

Proof. The proof of the corollary is standard by Theorem 1 and the continuous mapping theorem, and thus it is omitted. ■

Corollary 1 justifies the consistency of the bootstrap test as the previous corollary indicates that under the alternative the power converges to one, that is, $\Pr^*\left\{\hat{\eta}^*_n < \eta\left(\alpha_{\hat{\theta}_n n}\right)\right\} \to 1$, as the distribution of the bootstrap statistic converges to that of $\eta\left(\alpha_{\theta_1 \infty}\right)$ and $\eta\left(\alpha_{\hat{\theta}_n n}\right)$ diverges with n. However because of the difficulty in computing the critical values of the bootstrap distribution, they are approximated by Monte Carlo simulations as accurately as desired, as we now describe. For that purpose, let $\left\{\mathcal{X}^{*(\ell)}_n\right\}_{\ell=1}^c$ be c resamples generated as step 2 and $\left\{\eta^{*(\ell)}_n\right\}_{\ell=1}^c$ their corresponding bootstrap statistics as given in step 3. Then $z^*_{n\varsigma}$, where $\Pr^*\left[\hat{\eta}^*_n \geq z^*_{n\varsigma}\right] = \varsigma$, is approximated by $z^{*c}_{n\varsigma}$ defined from the relation

$$z^{*c}_{n\varsigma} = \inf\left\{z : \frac{1}{c}\sum_{j=1}^{c} \mathbb{I}\left(\eta^{*(j)}_n \geq z\right) \leq \varsigma\right\}.$$

We now describe the bootstrap for the transformation $\hat{\alpha}^*_{\hat{\theta}_n n}$, which only differs from that given in steps 1–3 in the last one. Indeed,

Steps 1 and 2. As before.
Step 3. Compute the bootstrap analog of the Whittle estimate (2.3) as in (3.4). Then, we compute the bootstrap test as $\hat{\eta}^*_n = \eta\left(\hat{\alpha}^*_{\hat{\theta}^*_n n}\right)$, where

$$\hat{\alpha}^*_{\theta n}(\lambda) = \frac{1}{\bar{u}^*_{\theta \tilde{n}} \tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \left(u^*_\theta(\lambda_j) - \hat{c}^{*\prime}_\theta(j)\,\gamma_\theta(\lambda_j)\right), \qquad \lambda \in [0, \pi],$$

with $u^*_\theta\left(\lambda_j\right) = I_{x^*}\left(\lambda_j\right)/h_\theta\left(\lambda_j\right)$ and $u^*_\theta\left(\lambda_j\right) - \hat{c}^{*\prime}_\theta(j)\,\gamma_\theta\left(\lambda_j\right)$ being the recursive residuals in the linear projection of $\left\{u^*_\theta\left(\lambda_j\right)\right\}_{j=1}^{\tilde{n}}$ on $\left\{\gamma_\theta\left(\lambda_j\right)\right\}_{j=1}^{\tilde{n}}$. Recall that $2\pi\,\bar{u}_{\theta\tilde{n}} = F_{\theta n}(\pi)$ so that the bootstrap analogue becomes $2\pi\,\bar{u}^*_{\theta\tilde{n}} = F^*_{\theta n}(\pi)$.

THEOREM 2. *Under the conditions of Proposition 1, under both H_0 and H_1*

$$\hat{\alpha}^*_{\hat{\theta}^*_n n} \xrightarrow{d^*} B \quad \textit{in probability.}$$

Interestingly, unlike in Theorem 1, the limiting distribution under H_0 and the limiting distribution under H_1 are identical. Therefore, power comparisons of resulting tests based on transformed and nontransformed spectral empirical processes are even more involved. As with Theorem 1 we obtain the following corollary.

COROLLARY 2. *Let η be a continuous mapping in $\mathbb{R}^+$. Under the assumptions of Theorem 2, and both H_0 and H_1, we have that*

$$\eta\left(\hat{\alpha}^*_{\hat{\theta}^*_n n}\right) \xrightarrow{d^*} \eta(B) \quad \textit{in probability.}$$

Proof. The proof of the corollary is standard by Theorem 2 and the continuous mapping theorem. ■

The results of Corollary 2 indicate that the conclusions obtained from Corollary 1 apply to this bootstrap-based test also. For models defined in the frequency domain, such as the Bloomfield (1973) exponential model, steps 1 and 2 are more involved. Indeed, to implement the previous bootstrap, we need first to obtain the coefficients of the AR (∞) representation of the Bloomfield (1973) model, which in general have no closed form. So, to avoid this problem, we envisage two procedures. One is based on the use of the wild bootstrap, as we present in the next section. A second method is the route followed by Hidalgo (2009).

4. MONTE CARLO EXPERIMENT

The purpose of this section is to examine and shed some light on the finite-sample performance of the different alternatives or approaches discussed in previous sections to perform valid tests for the null hypothesis H_0. In addition, we are interested in examining whether bootstrap assisted tests for the transformed T_p-process perform better in finite samples than those obtained using the asymptotic critical values. This is motivated from the belief that the bootstrap provides a better approximation to the actual distribution/critical values of $\eta\left(\hat{\alpha}_{\hat{\theta}^*_n n}\right)$ than those obtained from the asymptotic distribution of $\eta(B)$. The latter comes from the observation that the distribution of $\eta(B)$ is pivotal; see Hall (1992). Although a formal proof of this statement could be obtained using arguments in Götze (1979, 1984) for second-order expansions of the Cramér–von Mises criterion, this is beyond the scope of this paper.

In the Monte Carlo experiment, we have considered four alternative specifications: AR (1), MA (1), ARFIMA $(0, d, 0)$, and ARFIMA $(1, d, 0)$ using sample sizes $n = 100$ and $n = 500$ and $D = 50{,}000$ Gaussian Monte Carlo samples. However, to simplify and speed the computations, in this Monte Carlo experiment we have approximated the distribution of G_n^* using the Warp algorithm of Giacomini, Politis, and White (2007). The Warp algorithm permits us to approximate the Monte Carlo distribution of the bootstrap test generating only one additional bootstrap replication for each Monte Carlo sample, $X_{n,b}^{*(1)}$, $b = 1, \ldots, D$. Then the bootstrap critical values are obtained via the empirical distribution function of the D Monte Carlo samples employed in the experiment in the usual way. The significance level employed has been $\alpha = 0.05$.

As we mentioned at the end of the previous section, we shall also employ the "wild bootstrap" approach defined as

$$\frac{1}{\bar{u}_{\hat{\theta}_n \tilde{n}} \tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \left(u_{\hat{\theta}_n j} - \gamma'_{\hat{\theta}_n} \left(\lambda_j\right) \hat{c}_{\hat{\theta}_n} \left(\lambda_j\right) \right) V_j,$$

instead of $\hat{\alpha}^*_{\hat{\theta}^*_n n}$, where $\left\{ V_j \right\}_{j=1}^{\tilde{n}}$ is a sequence of zero mean independent identically distributed random variables with variance one and mutually independent of $\{\varepsilon_t\}_{t\in\mathbb{Z}}$.

We only present the results for the Cramér–von Mises type statistics

$$C_{\hat{\theta}_n n} = \frac{1}{n} \sum_{j=1}^{\tilde{n}} \alpha^2_{\hat{\theta}_n n} \left(\lambda_j\right) \quad \text{and} \quad \hat{C}_{\hat{\theta}_n n} = \frac{1}{n} \sum_{j=1}^{\tilde{n}} \hat{\alpha}^2_{\hat{\theta}_n n} \left(\lambda_j\right),$$

as the performance for the Kolmogorov–Smirnov functional was very similar. More specifically, in the tables that follow we present the results for the following test statistics:

1. asymptotic test based on $\hat{C}_{\hat{\theta}_n n}$, denoted $\hat{C}_n$;
2. naive bootstrap version of $\hat{C}_{\hat{\theta}_n n}$, denoted $\hat{C}_n^*$;
3. naive bootstrap version of $C_{\hat{\theta}_n n}$, denoted C_n^*;
4. wild bootstrap version of $\hat{C}_{\hat{\theta}_n n}$, denoted $\hat{C}_n^{**}$;
5. Hidalgo–Kreiss bootstrap version of $C_{\hat{\theta}_n n}$, denoted C_n^{**}.

We also report results for the Ljung and Box (1978) test,

$$\hat{Q}_{m,n} = n\,(n+2) \sum_{j=1}^{m} \frac{\hat{\rho}^2_{\hat{\theta}_n n}(j)}{n-j},$$

based on critical values obtained from an asymptotic χ^2_{m-p} distribution, where $\hat{\rho}_{\hat{\theta}_n n}(j)$ is the jth residual sample autocorrelations, and its bootstrap analogue

$\hat{Q}^*_{m,n}$ using the naive bootstrap procedure described in Section 3. We report the proportion of rejections choosing $m = n^{1/2}$, which is the common choice for the asymptotic test, and both $m = n^{1/2}$ and $m = 3$ for the bootstrap version, the latter being expected to perform much better in terms of power.

Tables 1 and 2 present the proportion of rejections under the null hypothesis for sample sizes of $n = 100$ and $n = 500$, respectively. In the first three blocks of these tables we consider single parameter models. The performance of the popular Box–Ljung test depends very much on the choice of the smoothing parameter m, and, as is well known, its size accuracy is appropriate when we choose $m = n^{1/2}$ for any specification considered. Interestingly, the bootstrap version of the Box–Ljung test does not perform much better than the asymptotic counterpart when $m = n^{1/2}$. The accuracy level for the bootstrap tests based on the transformed T_p-process is excellent using either the naive or wild bootstrap methods, even for the smallest sample size with any model and parameter combinations, such as MA(1) with $\theta_0 = 0.8$. However, the wild bootstrap performs slightly worse than the naive one. The latter confirms our comment that as the wild bootstrap only approximates $|\Delta(\lambda)|$ rather than $\Delta(\lambda)$, as the naive bootstrap does, then the finite-sample performance of the former may be worse than that of the naive bootstrap tests. Also, it is interesting to observe that the bootstrap assisted test for the transformed T_p-process shows better finite-sample performance than that using the asymptotic critical values. This appears to corroborate the comments made earlier with regard to bootstrapping pivotal tests.

The second three blocks in Tables 1 and 2 consider the two parameter model ARFIMA$(1, d, 0)$ for all previous value combinations. For the smaller sample size, $n = 100$, we observe that only the wild bootstrap is able to report good size for all cases, whereas for $n = 500$ the naive bootstrap provides accurate size for two parameter models only if applied to pivotal statistics. The Hidalgo–Kreiss test produces similar results to the naive bootstrap test C^*_n, whereas the asymptotic tests perform worse than their bootstrap counterparts.

We now investigate power properties for the previous tests. Tables 3 and 4 report the proportion of rejections when testing two null specifications with data generated according to the different models under the alternative. In particular we test the null of an AR(1) model against MA(1) and I(d) data and the null of MA(1) or $I(d)$ model specifications against AR(1) data. We confirm the well-known property of the Box–Ljung test exhibiting better power behavior in general by choosing m small. The T_p-process tests have better power in general, but the differences among alternative versions are small, and no one is always superior to the other. The most relevant information that we extract from these tables is that there is not a clear advantage in terms of power by using the bootstrap algorithm for the T_p-process and its transformation. However, it appears that, in general, both naive and wild bootstrap tests based on the transformation of the T_p-process have slightly less power than asymptotic tests, though the latter ones tend to control the size less accurately.

Table 1. Proportion of rejections under H_0, $n = 100$

θ_0	$\hat{C}_n$	$\hat{C}_n^*$	C_n^*	$\hat{C}_n^{**}$	C_n^{**}	$\widetilde{Q}_{3,n}^*$	$\widetilde{Q}_{n^{1/2},n}^*$	$\widetilde{Q}_{n^{1/2},n}$
				AR (1)				
−0.8	5.29	5.14	3.39	4.65	4.47	4.01	4.33	5.27
−0.5	5.70	5.14	3.25	4.63	4.26	4.59	4.57	4.82
0.0	7.24	5.03	3.93	5.00	4.26	5.05	4.77	4.57
0.5	9.30	5.10	4.55	5.41	4.43	5.02	4.86	4.77
0.8	11.66	5.17	4.68	5.27	4.44	4.75	4.68	4.70
				MA (1)				
−0.8	4.73	4.75	2.27	3.96	4.61	3.94	4.25	6.75
−0.5	5.57	4.90	2.86	4.27	4.50	4.37	4.38	4.83
0.0	7.32	4.95	3.70	4.85	4.23	4.60	4.62	4.45
0.5	9.74	4.96	4.30	5.38	3.27	4.37	4.38	4.57
0.8	16.58	7.82	0.07	5.71	3.22	0.04	0.81	10.49
				$I(d)$				
0.0	8.41	4.86	3.59	4.67	2.84	4.84	4.86	4.83
0.2	8.38	4.93	3.62	4.66	2.81	4.80	4.80	4.82
0.4	8.46	5.05	3.65	4.73	2.81	4.77	4.82	4.77
				ARFIMA $(1, d_0 = 0.0, 0)$				
−0.8	8.54	5.41	1.20	3.53	2.16	3.75	4.12	3.27
−0.5	11.77	5.20	1.30	4.36	2.22	4.06	4.30	2.78
0.0	17.41	3.33	0.08	4.94	0.89	0.69	1.14	2.40
0.5	17.02	0.06	0.00	4.28	1.45	0.00	0.00	4.17
0.8	17.69	0.99	0.00	4.90	0.80	0.00	0.00	3.42
				ARFIMA $(1, d_0 = 0.2, 0)$				
−0.8	8.57	5.50	1.26	3.56	2.11	3.83	4.14	3.19
0.5	11.85	5.21	1.28	4.33	2.21	4.15	4.29	2.76
0.0	17.32	3.40	0.16	4.87	1.01	0.88	1.36	2.36
0.5	17.08	0.07	0.00	4.29	1.37	0.00	0.00	4.15
0.8	17.89	1.01	0.00	4.95	0.87	0.00	0.00	3.50
				ARFIMA $(1, d_0 = 0.4, 0)$				
−0.8	8.60	5.49	1.27	3.71	2.08	3.77	4.09	3.10
−0.5	11.77	5.18	1.29	4.38	2.24	4.16	4.33	2.63
0.0	17.45	3.55	0.21	4.98	1.04	1.05	1.43	2.33
0.5	17.08	0.08	0.00	4.31	1.34	0.00	0.00	4.03
0.8	17.31	0.84	0.00	4.82	0.54	0.00	0.00	3.98

5. TECHNICAL LEMMAS

We shall first introduce some notation used in this and the next sections. We shall denote $\zeta_\theta(\lambda) = \zeta(\lambda;\theta) : (0,\pi] \times \Theta \to \mathbb{R}^p$ a function satisfying the same conditions of ϕ_θ in A3 and abbreviate $\zeta_{\theta_0}(\lambda)$ by $\zeta(\lambda)$ and $\zeta_{\hat{\theta}_n}(\lambda)$ by $\hat{\zeta}(\lambda)$. Also,

TABLE 2. Proportion of rejections under H_0, $n = 500$

θ_0	$\hat{C}_n$	$\hat{C}_n^*$	C_n^*	$\hat{C}_n^{**}$	C_n^{**}	$\widetilde{Q}_{3,n}^*$	$\widetilde{Q}_{n^{1/2},n}^*$	$\widetilde{Q}_{n^{1/2},n}$
				AR (1)				
−0.8	5.32	5.16	4.63	5.18	5.19	4.73	4.85	5.24
−0.5	5.43	5.16	4.67	5.11	5.13	5.10	4.99	5.11
0.0	5.46	4.99	4.87	4.96	4.84	5.20	5.04	5.08
0.5	5.92	5.07	5.08	5.16	5.04	5.26	5.10	5.17
0.8	6.42	5.11	5.05	5.25	5.07	5.09	5.04	5.22
				MA (1)				
−0.8	5.17	5.14	4.43	4.85	4.91	5.00	4.65	5.51
−0.5	5.37	5.14	4.60	5.00	4.99	5.01	4.81	5.08
0.0	5.49	4.92	4.78	5.03	4.86	5.06	4.83	5.07
0.5	5.96	5.13	5.09	5.26	4.92	5.07	4.78	4.83
0.8	6.45	5.04	4.98	5.03	5.36	4.99	4.74	5.14
				$I(d)$				
0.0	5.81	5.09	4.74	4.97	4.35	5.20	4.93	5.25
0.2	5.81	5.13	4.77	5.01	4.24	5.16	4.94	4.93
0.4	5.85	5.14	4.73	5.13	4.32	5.23	5.20	5.20
				ARFIMA (1, $d_0 = 0.0$, 0)				
−0.8	5.84	5.27	3.60	4.73	4.03	4.82	4.53	3.49
−0.5	6.79	5.42	3.81	5.57	4.01	5.23	4.51	3.45
0.0	8.21	4.70	4.43	5.14	4.54	4.88	4.50	3.30
0.5	7.25	4.79	3.07	4.43	4.00	3.04	2.72	3.69
0.8	8.39	5.17	1.66	4.49	2.10	3.59	3.28	4.13
				ARFIMA (1, $d_0 = 0.2$, 0)				
−0.8	5.77	5.12	3.60	4.78	4.10	4.72	4.46	3.47
−0.5	6.79	5.44	3.66	5.62	3.96	5.23	4.48	3.42
0.0	8.19	4.89	4.37	5.33	4.52	4.73	4.53	3.25
0.5	7.23	4.75	2.99	4.47	3.81	3.12	2.72	3.73
0.8	8.58	5.44	1.48	4.47	1.48	3.60	3.22	4.20
				ARFIMA (1, $d_0 = 0.4$, 0)				
−0.8	5.85	5.15	3.85	4.70	4.08	4.87	4.43	3.45
−0.5	6.89	5.54	3.93	5.61	4.09	5.36	4.35	3.37
0.0	8.14	4.41	4.47	5.02	3.84	5.07	4.53	3.35
0.5	7.20	4.69	2.88	4.71	3.24	3.28	2.72	3.80
0.8	8.20	5.24	1.80	4.96	0.31	4.69	3.25	4.45

for a generic function $g(\lambda)$, we shall abbreviate $g(\lambda_j)$ by g_j. We shall abbreviate $b_p(d_0)$ and $b_p(\hat{d}_n)$, respectively, by b_p and $\hat{b}_p$. Finally, henceforth, $z^{(k)}$ denotes the kth element of a $p \times 1$ vector z and K a finite positive constant.

TABLE 3. Proportion of rejections under H_1, $n = 100$

θ_0	$\hat{C}_n$	$\hat{C}_n^*$	C_n^*	$\hat{C}_n^{**}$	C_n^{**}	$\widetilde{Q}_{3,n}^*$	$\widetilde{Q}_{n^{1/2},n}^*$	$\widetilde{Q}_{n^{1/2},n}$
				H_0 : AR (1) vs. H_1 : MA (1)				
−0.8	99.56	98.58	97.35	86.89	97.00	96.15	71.97	71.42
−0.5	62.18	48.77	49.45	33.97	47.99	44.91	23.93	23.58
0.2	8.39	6.28	5.03	6.12	6.75	6.86	5.76	5.70
0.5	43.45	39.71	41.38	33.15	46.11	45.23	24.35	25.05
0.8	86.07	84.26	93.42	67.57	92.25	95.62	70.56	71.73
				H_0 : MA (1) vs. H_1 : AR (1)				
−0.8	99.92	99.79	99.68	98.86	99.50	99.77	98.72	98.92
0.5	66.13	56.80	54.95	56.48	51.29	50.25	30.60	30.07
0.2	8.52	6.59	4.31	6.19	5.78	4.93	4.68	4.59
0.5	48.80	46.50	49.32	27.83	55.65	44.88	27.35	27.85
0.8	99.10	99.09	99.38	41.21	99.62	99.51	97.92	98.18
				H_0 : I (d) vs. H_1 : AR (1)				
0.2	13.48	7.25	10.24	4.48	7.96	4.93	9.20	9.29
0.5	22.83	13.79	24.91	5.49	20.99	22.52	19.34	19.42
0.8	12.15	6.34	11.06	3.08	8.28	10.88	12.83	12.93
				H_0 : AR (1) vs. H_1 : I (d)				
0.1	9.50	6.85	5.58	6.67	5.64	5.64	5.10	4.94
0.2	15.47	11.40	10.46	10.62	10.46	8.90	7.75	7.42
0.3	23.39	17.33	17.09	15.95	17.09	13.77	12.09	11.80
0.4	30.67	22.90	23.98	21.53	23.98	17.54	15.59	15.35

LEMMA 1. *Let* $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. *Then*

(i) $$\left| \left| \sum_{p=0}^{\infty} b_p (d) e^{ip\lambda} \right|^2 - \left| \sum_{p=0}^{n} b_p (d) e^{ip\lambda} \right|^2 \right| = O\left(\frac{1}{\lambda^{1+d} n^{1-d}} \right) \quad \text{if } \frac{1}{2}\lambda_1 < \lambda \le \pi,$$

(ii) $$\left| \sum_{p=0}^{n} b_p (d) e^{ip\lambda} \right|^2 = O\left(n^{2d} \right) \quad \text{if } 0 \le \lambda < \frac{1}{2}\lambda_1.$$

Proof. We first show part (i). After standard algebra, the left side of the equality in (i) is

$$\left| \sum_{p=n+1}^{\infty} b_p (d) e^{ip\lambda} \right|^2 + 2\mathrm{Re} \left(\sum_{p=n+1}^{\infty} b_p (d) e^{ip\lambda} \right) \left(\sum_{p=0}^{n} b_p (d) e^{-ip\lambda} \right), \quad \textbf{(5.1)}$$

where Re (z) denotes the real part of a complex number z. Because $\left| \sum_{p=0}^{n} e^{ip\lambda} \right| < K\lambda^{-1}$ for $\lambda > 0$, we obtain that by summation by parts, monotonicity of $b_p (d)$,

TABLE 4. Proportion of rejections under H_1, $n = 500$

θ_0	$\hat{C}_n$	$\hat{C}_n^*$	C_n^*	$\hat{C}_n^{**}$	C_n^{**}	$\widetilde{Q}_{3,n}^*$	$\widetilde{Q}_{n^{1/2},n}^*$	$\widetilde{Q}_{n^{1/2},n}$
				H_0 : AR (1) vs. H_1 : MA (1)				
−0.8	100	100	100	100	100	100	100	100
−0.5	99.87	99.85	99.85	99.59	99.61	99.61	81.98	82.57
0.2	13.29	12.29	13.43	12.40	12.14	12.14	6.73	7.14
0.5	98.67	98.64	99.77	97.95	97.95	99.63	82.64	83.39
0.8	100	100	100	100	100	100	100	100
				H_0 : MA (1) vs. H_1 : AR (1)				
−0.8	100	100	100	100	100	100	100	100
−0.5	99.77	99.75	99.79	99.71	99.40	99.69	91.18	91.37
0.2	13.11	12.45	12.58	11.95	13.69	10.33	6.41	6.57
0.5	99.23	99.22	99.83	99.78	99.67	99.64	90.21	90.53
0.8	100	100	100	100	100	100	100	100
				H_0 : I (d) vs. H_1 : AR (1)				
0.2	40.00	37.29	58.32	29.50	54.03	41.42	21.85	22.16
0.5	79.88	77.41	99.20	60.26	96.87	86.99	7.44	54.96
0.8	38.91	35.54	83.55	22.65	81.03	48.18	54.77	5.23
				H_0 : AR (1) vs. H_1 : I (d)				
0.1	18.41	17.28	20.04	16.23	20.01	17.49	12.42	12.60
0.2	53.83	51.88	60.72	45.17	59.21	54.19	45.05	45.21
0.3	84.11	82.94	87.93	71.64	86.68	81.55	78.31	78.58
0.4	95.18	94.99	96.60	86.26	96.18	91.20	92.41	92.28

and because $b_p(d) = O\left(p^{d-1}\right)$, the first term of (5.1) is bounded by

$$K\frac{1}{\lambda^2}\left|\sum_{p=n+1}^{\infty}\left|b_p(d) - b_{p+1}(d)\right|\right|^2 = O\left(\frac{1}{\lambda^2 n^{2-2d}}\right).$$

Proceeding similarly, the second term of (5.1) is bounded in modulus by

$$O\left(\frac{1}{\lambda n^{1-d}}\left|\sum_{p=0}^{n} b_p(d)\, e^{ip\lambda}\right|\right) = O\left(\frac{1}{\lambda^2 n^{2-2d}} + \frac{1}{\lambda^{1+d} n^{1-d}}\right)$$

because

$$\left|\sum_{p=0}^{n} b_p(d)\, e^{ip\lambda}\right| \le \left|\sum_{p=0}^{\infty} b_p(d)\, e^{ip\lambda}\right| + \left|\sum_{p=n+1}^{\infty} b_p(d)\, e^{ip\lambda}\right|$$

and, by definition,

$$\left|\sum_{p=0}^{\infty} b_p(d)\, e^{ip\lambda}\right| \le K\lambda^{-d}.$$

This concludes the proof of part (i).

Next we show part (ii). When $d \geq 0$ the proof is obvious because $b_p(d) = O\left(p^{d-1}\right)$. On the other hand, when $d < 0$, the proof follows using the last two displayed inequalities. ■

LEMMA 2. *Assuming A1–A3,*

$$\text{(i)} \quad \left|\sum_{p=1}^{n} b_p e^{ip\lambda}\right|^2 - \left|\sum_{p=1}^{n} \hat{b}_p e^{ip\lambda}\right|^2 = O_p\left(\frac{\left|1-e^{i\lambda}\right|^{-2d_0}\log\left|1-e^{i\lambda}\right|}{n^{1/2}} + \frac{n^{d_0-1}}{\lambda^{1+d_0}}\right) \quad \text{if } \frac{1}{2}\lambda_1 < \lambda \leq \pi,$$

$$\text{(ii)} \quad \left|\sum_{p=1}^{n} b_p e^{ip\lambda}\right|^2 - \left|\sum_{p=1}^{n} \hat{b}_p e^{ip\lambda}\right|^2 = O_p\left(n^{d_0-1/2}\right) \quad \text{if } 0 < \lambda \leq \frac{1}{2}\lambda_1,$$

where the orders of magnitude are uniformly in λ.

Proof. We begin with part (i). By Lemma 1 and because the $n^{1/2}$-consistency of the Whittle estimator in (2.3) implies that $n^{\hat{d}_n - d_0} - 1 = o_p(1)$, the left side of the equality in (i) is

$$\left|\sum_{p=1}^{\infty} b_p e^{ip\lambda}\right|^2 - \left|\sum_{p=1}^{\infty} \hat{b}_p e^{ip\lambda}\right|^2 + O_p\left(\frac{n^{d_0-1}}{\lambda^{1+d_0}}\right).$$

Now the conclusion follows because the first and second terms of the last displayed expression are, respectively, $\left|1-e^{i\lambda}\right|^{-2d_0}$ and $\left|1-e^{i\lambda}\right|^{-2\hat{d}_n}$ and because the Whittle estimator is $n^{1/2}$-consistent. Part (ii) follows immediately proceeding as in the proof of Lemma 1, after observing that $\left|b_p - \hat{b}_p\right| = O_p\left(n^{-1/2}\right) b_p \log p$. ■

LEMMA 3. *Let ζ_θ be such that for all $\lambda > 0$ and θ, $\|\zeta_\theta(\lambda)\| + \|\partial\zeta_\theta(\lambda)/\partial\theta\| \leq K\left|\log\lambda\right|^{\ell}$, $\ell \geq 1$, and $\|\partial\zeta_\theta(\lambda)/\partial\lambda\| + \left\|\partial^2\zeta_\theta(\lambda)/\partial\theta\partial\lambda\right\| \leq K\lambda^{-1}\left|\log\lambda\right|^{\ell-1}$ and for some $0 \leq \delta \leq 1$,*

$$\sup_{\|\theta-\theta_0\|\leq K/n^{1/2}} \|\theta-\theta_0\|^{-2}\left|\zeta_\theta^{(k)}(\lambda) - \zeta_{\theta_0}^{(k)}(\lambda) - \frac{\partial}{\partial\theta'}\zeta_{\theta_0}^{(k)}(\lambda)(\theta-\theta_0)\right| \leq \lambda^{-\delta}.$$

Then, for $k = 1, \ldots, p$,

$$\sup_{\lambda\in[0,\pi]}\left\|\frac{1}{\tilde{n}}\sum_{j=1}^{\lfloor\tilde{n}\lambda/\pi\rfloor}\hat{\zeta}_j^{(k)} - \frac{1}{\pi}\int_0^\lambda \zeta_{\theta_0}^{(k)}(v)\,dv\right\| = O_p\left(\frac{\log^\ell n}{n}\right) + O_p\left(\frac{1}{n^{1/2}}\right) \times \int_0^\lambda \frac{\partial\zeta_{\theta_0}^{(k)}(v)}{\partial\theta}dv.$$

Proof. By Lemma 1 of DHV, it suffices to show that

$$\sup_{\lambda\in[0,\pi]}\left\|\frac{1}{\tilde{n}}\sum_{j=1}^{\lfloor\tilde{n}\lambda/\pi\rfloor}\left\{\hat{\zeta}_j^{(k)} - \zeta_{\theta_0 j}^{(k)}\right\}\right\| = O_p\left(\frac{1}{n^{1/2}}\right)\int_0^\lambda \frac{\partial\zeta_{\theta_0}^{(k)}(v)}{\partial\theta}dv.$$

But the latter holds true because $\left|\hat{\zeta}_j^{(k)} - \zeta_{\theta_0 j}^{(k)} - (\partial/\partial\theta')\zeta_{\theta_0 j}^{(k)}\left(\hat{\theta}_n - \theta_0\right)\right| = O\left(n^{-1}\lambda_j^{-\delta}\right)$ and as in Lemma 1 of DHV, we conclude that $\tilde{n}^{-1}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \partial\zeta_{\theta_0 j}^{(k)}/\partial\theta$ $-\pi^{-1}\int_0^\lambda \left(\partial\zeta_{\theta_0}^{(k)}(\nu)/\partial\theta\right) d\nu = O\left(n^{-1}\log^\ell n\right)$. ■

Denote $\upsilon^*(\lambda) = h_{\hat{\theta}_n n}^{-1/2}(\lambda)\, w_{x^*}(\lambda)$, $\overline{a}$ the conjugate of the complex number a, and $\varsigma(j) = \min\left(j^{-1}, j^{d_0-1}\right)\log j$. Recall that $w_{x^*}(\lambda)$ is the discrete Fourier transform of $\left\{x_t^*\right\}_{t=1}^n$.

LEMMA 4. *Assume A1–A3.*

(i) For $\ell \leq k \leq j \leq \tilde{n}$ with $\ell^{-1} = o(1)$,

$$\mathbb{E}^*\left(\upsilon_j^*\overline{\upsilon}_k^*\right) - \frac{\hat{\sigma}_n^2}{2\pi}\mathbb{I}(j=k) = O_p(\varsigma(j)), \qquad \mathbb{E}^*\left(\upsilon_j^*\upsilon_k^*\right) = O_p(\varsigma(k)). \tag{5.2}$$

(ii) For fixed positive integers $k \leq j$, $\mathrm{E}^*\left(\upsilon_j^*\upsilon_k^*\right) = O_p(\varsigma(k))$.

Proof. We shall begin with part (i). We shall prove only the case $j = k$, because the case $j \neq k$ can be similarly handled. From the definition of $h_{\hat{\theta}_n n}^{-1}(\lambda_j)$ in (3.5), the left side of (5.2) is $\hat{\sigma}_n^2 h_{\hat{\theta}_n n}^{-1}(\lambda_j)$ times

$$\int_{-\pi}^{\pi}\left(h_{\hat{\theta}_n n}(\lambda) - h_{\hat{\theta}_n n}(\lambda_j)\right)\mathcal{F}(\lambda - \lambda_j)\, d\lambda, \tag{5.3}$$

where $\mathcal{F}(\lambda) = (2\pi n)^{-1}|D(\lambda)|^2$ denotes the Fèjer's kernel, with $D(\lambda) = \sum_{t=1}^n \exp\{it\lambda\}$. So, it suffices to show that (5.3) is $O_p\left(\lambda_j^{-2d_0}\varsigma(j)\right)$, which proceeds similarly as in Theorem 2 of Robinson (1995a). Indeed, we first notice that (5.3) is

$$\left(\left\{\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right\} + \left\{\int_{-\delta}^{-1/2\lambda_j} + \int_{2\lambda_j}^{\delta}\right\} + \int_{1/2\lambda_j}^{2\lambda_j} + \int_{-1/2\lambda_j}^{1/2\lambda_j}\right) \times \left(h_{\hat{\theta}_n n}(\lambda) - h_{\hat{\theta}_n n}(\lambda_j)\right)\mathcal{F}(\lambda - \lambda_j)\, d\lambda. \tag{5.4}$$

Because $\mathcal{F}(\lambda) \leq K\lambda^{-2}n^{-1}$ for$\lambda > 0$, $\Xi(\lambda;\psi,\varphi)$ is twice continuously differentiable, and Lemma 2 implies that $h_{\hat{\theta}_n n}(\lambda) = h_{\theta_0 n}(\lambda)\left(1 + o_p(1)\right)$, we obtain that the contribution of $\left\{\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right\}$ into (5.4) is bounded by

$$Kn^{-1}\int_{\delta}^{\pi}\left(h_{\theta_0 n}(\lambda) + h_{\theta_0 n}(\lambda_j)\right)\left(1 + o_p(1)\right) d\lambda = O_p\left(j^{-1}\lambda_j^{-2d_0}\right)$$

proceeding as in Lemma 1 and Theorem 2, part (a), in Robinson (1995a).

Next, because $\mathcal{F}(\lambda) \leq K\lambda^{-2}n^{-1}$ for $\lambda > 0$, the contribution due to $\left\{\int_{-\delta}^{-1/2\lambda_j} + \int_{2\lambda_j}^{\delta}\right\}$ into (5.4) is bounded by

$$Kn^{-1}\left\{\int_{-\delta}^{-1/2\lambda_j} + \int_{2\lambda_j}^{\delta}\right\}\left|h_{\hat{\theta}_n n}(\lambda) - h_{\hat{\theta}_n n}(\lambda_j)\right|\left|\lambda - \lambda_j\right|^{-2} d\lambda$$
$$= Kn^{-1}\left\{\int_{-\delta}^{-1/2\lambda_j} + \int_{2\lambda_j}^{\delta}\right\}\left|h_{\theta_0}(\lambda) - h_{\theta_0}(\lambda_j)\right|\left|\lambda - \lambda_j\right|^{-2} d\lambda$$
$$+ O_p\left(\frac{n^{2d_0}}{j^{2+d_0}}\right)$$

by Lemmas 1 and 2, because $\Xi(\lambda; \psi, \varphi)$ is twice continuously differentiable and $\hat{\theta}_n - \theta_0 = O_p(n^{-1/2})$ implies that $n^{\hat{d}_n - d_0} - 1 = o_p(1)$. Now proceed as with the proof of Theorem 2 of Robinson (1995a) to conclude that the right side of the last displayed equality is $O_p\left(\lambda_j^{-2d_0} j^{d_0 - 1}\right)$. Next the contribution in (5.4) due to $\int_{1/2\lambda_j}^{2\lambda_j}$ is bounded by

$$K \sup_{1/2\lambda_j \leq \lambda \leq 2\lambda_j}\left|\frac{\partial h_{\hat{\theta}_n n}(\lambda)}{\partial \theta}\right| \int_{\frac{1}{2}\lambda_j}^{2\lambda_j}\left|\lambda - \lambda_j\right| \mathcal{F}(\lambda - \lambda_j)\, d\lambda. \quad \textbf{(5.5)}$$

But by (3.5) and because $\Xi(\lambda; \psi, \varphi)$ is twice continuously differentiable and Lemma 2, $\left|\partial h_{\hat{\theta}_n n}(\lambda)/\partial\theta\right| = \left|\sum_{p=1}^{n} p b_p e^{ip\lambda}\right|\left|\sum_{p=1}^{n} b_p e^{ip\lambda}\right|(1 + o_p(1))$, which by summation by parts is $O\left(\lambda^{-d_0}\left(\lambda^{-1}\sum_{p=1}^{n}\left|p b_p - (p+1) b_{p+1}\right| + n^{d_0}\right)\right)$. Then, because by Robinson (1995a), $\int_0^{2\lambda_j}\left|D(\lambda - \lambda_j)\right| d\lambda = O(\log j)$ and $|\lambda D(\lambda)| < K$, we conclude that (5.5) is $O_p\left(\max\left(j^{d_0 - 1} + j^{-1}\right)\lambda_j^{-2d_0}\log j\right)$.

Finally, the contribution due to $\int_{-1/2\lambda_j}^{1/2\lambda_j}$ into (5.4) is bounded by

$$O\left(\frac{n}{j^2}\right)\left\{\int_0^{1/2\lambda_1} + \int_{1/2\lambda_1}^{1/2\lambda_j}\right\} h_{\hat{\theta}_n n}(\lambda)\, d\lambda + O_p\left(\frac{\lambda_j^{-2d_0}}{j}\right) = O_p\left(\frac{\lambda_j^{-2d_0}}{j}\right)$$

by Lemma 1, because $\int_{1/2\lambda_1}^{1/2\lambda_j} h_{\hat{\theta}_n}(\lambda)\, d\lambda = O_p\left(\lambda_j^{1-2\hat{d}_n}\right)$ and $n^{2\hat{d}_n - 2d_0} - 1 = o_p(1)$. This completes the proof of part (i).

The proof of part (ii) proceeds similarly, and thus it is omitted. ■

LEMMA 5. *Assuming A1–A3, as $n \to \infty$, for $1 \leq r < s \leq \tilde{n}$, $k = 1, \ldots, p$,*

$$\mathbb{E}^*\left|\sum_{j=r}^{s}\zeta_j^{(k)} v_j^*\left(\overline{v}_j^* - \overline{w}_{\varepsilon^* j}^*\right)\right|^2$$
$$= O_p\left(\log^2(n)\sum_{j=r}^{s}\left\{\frac{\log(n)}{j^{1-d_0}} + \sum_{\ell=r}^{s}\left(\frac{\log^2(n)}{\ell^{2-2d_0}} + \frac{\ell^{d_0 - 1/2}}{j^{1-d_0}}\right)\right\}\right).$$

Proof. The proof follows in the same way as that of expression (4.8) of Robinson (1995b, pp. 1648–1651), using Robinson's Lemma 3 but using our Lemma 4 instead of Theorems 1 and 2 of Robinson (1995a) where appropriate. Notice that the proof of Lemma 4 indicates that the terms $O_p(\varsigma(j))$ are uniformly in j in the sense that the maximum in j of the left side normalized by $\varsigma(j)$ is $O_p(1)$. ∎

Define

$$\varrho_n^{*\zeta}(\lambda) := \frac{1}{\tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \hat{\zeta}_j \left(I_{\varepsilon^* j} - \frac{\hat{\sigma}_n^2}{2\pi} \right),$$

$$\widetilde{\varrho}_n^{*\zeta}(\lambda) := \frac{1}{\tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \hat{\zeta}_j \left(\frac{I_{x^* j}}{h_{\hat{\theta}_n n}(\lambda_j)} - \frac{\hat{\sigma}_n^2}{2\pi} \right).$$

LEMMA 6. *Under A1–A3, for some* $0 < \delta < 1/6$, $\mathbb{E}^* \sup_{\lambda \in [0,\pi]} \left\| \widetilde{\varrho}_n^{*\zeta}(\lambda) - \varrho_n^{*\zeta}(\lambda) \right\| = O_p\left(n^{-\delta}\right)$.

Proof. The proof proceeds in the same way as that of Lemma 4 of DHV, and thus it is omitted. ∎

For $\mu, \lambda \in [0,\pi]$, let $\hat{c}_s(\mu,\lambda) = \frac{2}{n\tilde{n}^{1/2}} \sum_{p=\lfloor \tilde{n}\mu/\pi \rfloor + 1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \hat{\zeta}_p \cos\left(s\lambda_p\right)$ and $\Upsilon_g(\mu,\lambda) = \pi^{-1} \int_\mu^\lambda g(v)\, dv$.

LEMMA 7. *For* $0 \le \mu < \lambda \le v \le \pi$, *as* $n \to \infty$,

$$\sum_{t=1}^{n-1} \sum_{s=1}^{n-t} \hat{c}_s(\mu,\lambda)\, \hat{c}_s'(\mu,v) = \left(\Upsilon_{\zeta\zeta'}(\mu,\lambda) - \Upsilon_\zeta(\mu,\lambda)\, \Upsilon_\zeta'(\mu,v) \right) \left(1 + o_p(1)\right).$$

Proof. We omit the proof as it proceeds in the same way as that of Lemma 5 of DHV. ∎

Let us introduce the following notation. For $0 \le \lambda_1 < \lambda_2 \le \pi$,

$$\mathcal{E}_{1,n}^*(\lambda_1,\lambda_2) := \left(\frac{1}{\tilde{n}} \sum_{p=\lfloor \tilde{n}\lambda_1/\pi \rfloor + 1}^{\lfloor \tilde{n}\lambda_2/\pi \rfloor} \hat{\zeta}_p \right) \left(\frac{\tilde{n}^{1/2}}{n} \sum_{t=1}^{n} \left(\varepsilon_t^{*2} - \hat{\sigma}_n^2 \right) \right), \qquad \textbf{(5.6)}$$

$$\mathcal{E}_{2,n}^*(\lambda_1,\lambda_2) := \sum_{t=2}^{n} \varepsilon_t^* \sum_{s=1}^{t-1} \varepsilon_s^* \hat{c}_{t-s}(\lambda_1,\lambda_2), \qquad \textbf{(5.7)}$$

and $H_n(\lambda_1,\lambda_2)$ denotes a generic sequence of $O_p(1)$ random variables.

LEMMA 8. *Let* $0 \le \lambda_1 < \lambda < \lambda_2 \le \pi$. *Then, assuming A1–A3, for* $k = 1, \ldots, p$ *and some* $\beta > 0$ *and* $0 \le \delta < 1$,

$$\mathbb{E}^*\left(\left|\mathcal{E}_{j,n}^{*(k)}(\lambda_1,\lambda)\right|^{\beta}\left|\mathcal{E}_{j,n}^{*(k)}(\lambda,\lambda_2)\right|^{\beta}\right)=H_n(\lambda_1,\lambda_2)(\lambda_2-\lambda_1)^{2-\delta},\quad j=1,2. \tag{5.8}$$

Proof. We begin with $j=1$. By Lemma 3 and because we can take $\tilde{n}^{-1}\leq(\lambda_2-\lambda_1)$,

$$\left|\frac{1}{\tilde{n}}\sum_{p=\lfloor\tilde{n}\lambda_1/\pi\rfloor+1}^{\lfloor\tilde{n}\lambda_2/\pi\rfloor}\hat{\zeta}_p^{(k)}-\frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\zeta^{(k)}(\nu)\,d\nu\right|=H_n(\lambda_1,\lambda_2)(\lambda_2-\lambda_1)^{1-\delta/2}$$

because $\left|\int_\lambda^\mu(\partial/\partial\theta)\zeta^{(k)}(\nu)\,d\nu\right|\leq K|\lambda-\mu|^{1-\delta/2}$. Next, because by standard arguments we have that

$$\mathbb{E}^*\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^{*\ell}=\frac{1}{n}\sum_{t=1}^{n}\hat{\varepsilon}_t^{\ell}\xrightarrow{P}\mathbb{E}\varepsilon_t^{\ell}, \tag{5.9}$$

for $\ell\leq 4$, we can conclude that $\mathbb{E}^*\left(\sum_{t=1}^{n}\left(\varepsilon_t^{*2}-\hat{\sigma}_n^2\right)\right)^2=O_p(n)$. So, using that $(\lambda_2-\lambda)(\lambda-\lambda_1)\leq(\lambda_2-\lambda_1)^2$ and then the Cauchy–Schwarz inequality, $\mathbb{E}^*\left(\left|\mathcal{E}_{1,n}^{*(k)}(\lambda_1,\lambda)\right|\left|\mathcal{E}_{1,n}^{*(k)}(\lambda,\lambda_2)\right|\right)=H_n(\lambda_1,\lambda_2)(\lambda_2-\lambda_1)^{2-\delta}$.

To complete the proof, we examine that (5.8) holds for $j=2$. Now

$$\mathbb{E}^*\left(\mathcal{E}_{2,n}^{*(k)}(\lambda_1,\lambda_2)\right)^4=16\prod_{j-1}^{4}\sum_{1\leq s_j<t_j\leq n}\hat{c}_{t_j-s_j}^{(k)}(\lambda_1,\lambda_2)\,\mathbb{E}^*\left(\varepsilon_{t_1}^*\varepsilon_{s_1}^*\ldots\varepsilon_{t_4}^*\varepsilon_{s_4}^*\right).$$

Because the number of equal indexes in the set $\{t_1,s_1,\ldots,t_4,s_4\}$ does not exceed 4, it follows that $\left|\mathbb{E}^*\left(\varepsilon_{t_1}^*\varepsilon_{s_1}^*\ldots\varepsilon_{t_4}^*\varepsilon_{s_4}^*\right)\right|=O_p(1)$ by (5.9). Also, $\left|\mathbb{E}^*\left(\varepsilon_{t_1}^*\varepsilon_{s_1}^*\ldots\varepsilon_{t_4}^*\varepsilon_{s_4}^*\right)\right|\neq 0$ can only hold if any t_j,s_j are repeated in $\{t_1,s_1,\ldots,t_4,s_4\}$ at least twice. Hence by the Cauchy–Schwarz inequality, the left side of the last displayed equality is

$$H_n(\lambda_1,\lambda_2)\prod_{j=1}^{4}\left(\sum_{1\leq s_j<t_j\leq n}\left(\hat{c}_{t_j-s_j}^{(k)}(\lambda_1,\lambda_2)\right)^2\right)^{1/2}=H_n(\lambda_1,\lambda_2)(\lambda_2-\lambda_1)^{2-\delta}$$

by Lemma 7 and $\left|\int_{\lambda_1}^{\lambda_2}\left(\zeta^{(k)}(\nu)\right)^2d\nu\right|\leq K(\lambda_2-\lambda_1)^{1-\delta/2}$. We now conclude the proof by choosing $\beta=2$ in (5.8) and the Cauchy–Schwarz inequality. ■

Let $R_n^{(1)*}(\lambda)=\frac{2\pi}{\tilde{n}^{1/2}}\sum_{j=1}^{\lfloor\tilde{n}\lambda/\pi\rfloor}\hat{\zeta}_j J_p^*$, $R_n^{(2)*}(\lambda)=\frac{2\pi}{\tilde{n}^{1/2}}\sum_{j=\lfloor\tilde{n}\lambda/\pi\rfloor+1}^{\tilde{n}}\hat{\zeta}_j J_p^*$, where

$$J_p^*=I_{\varepsilon^*p}-\frac{\hat{\sigma}_n^2}{2\pi}. \tag{5.10}$$

LEMMA 9. *Assume A1–A3. Then, for some* $0\leq\delta<1$ *and* $\beta>0$*, we have that for all* $0<\lambda_1<\lambda<\lambda_2\leq\pi$ *and* $j=1,2$, *(i)* $\mathbb{E}^*\left(\left\|R_n^{(j)*}(\lambda_2)-R_n^{(j)*}(\lambda)\right\|^{\beta}\left\|R_n^{(j)*}(\lambda)-R_n^{(j)*}(\lambda_1)\right\|^{\beta}\right)$

$$= H_n(\lambda_1,\lambda_2)(\lambda_2-\lambda_1)^{2-\delta}. \qquad \textbf{(5.11)}$$

(ii) $R_n^{(j)*}(\lambda) \overset{d^*}{\to} \mathcal{N}\left(0, 4\pi^2\sigma^4 V^{(j)}(\lambda)\right)$ *in probability, (5.11)*

where $V^{(1)}(\lambda) = \Upsilon_{\zeta\zeta'}(0,\lambda) + \kappa_\varepsilon \Upsilon_\zeta(0,\lambda)\Upsilon_\zeta'(0,\lambda)$, $V^{(2)}(\lambda) = \Upsilon_{\zeta\zeta'}(\lambda,\pi) + \kappa_\varepsilon \Upsilon_\zeta(\lambda,\pi)\Upsilon_\zeta'(\lambda,\pi)\ldots\kappa_\varepsilon$ *is the fourth cumulant of* ε_t.

Proof. We begin with (i). The proof follows directly from Lemma 8 after we notice that $R_n^{(2)*}(\lambda) - R_n^{(2)*}(\lambda_2) = \mathcal{E}_{1,n}^*(\lambda,\lambda_2) + \mathcal{E}_{2,n}^*(\lambda,\lambda_2)$. Regarding part (ii), the proof proceeds very similarly to that of Propositions 6.4 and 6.5 of Hidalgo and Kreiss (2006), and thus it is omitted. ■

LEMMA 10. *Under A1–A3, for some* $\delta > 0$,

$$\mathbb{E}^* \sup_{\lambda\in[0,\pi]}\left\|\frac{1}{\tilde n^{1/2}}\sum_{j=1}^{\lfloor \tilde n\lambda/\pi\rfloor}\hat\zeta_j\left(\frac{I_{x^*j}}{h_{\hat\theta_n j}}-\frac{I_{x^*j}}{h_{\hat\theta_n n}(\lambda_j)}\right)\right\| = O_p\left(n^{-\delta}\right).$$

Proof. By standard algebra, the left side is bounded by

$$\mathbb{E}^* \sup_{\lambda\in[0,\pi]}\left\|\frac{1}{\tilde n^{1/2}}\sum_{j=1}^{\lfloor \tilde n\lambda/\pi\rfloor}\hat\zeta_j\xi_{\hat\theta_n j}\left(\frac{I_{x^*j}}{h_{\hat\theta_n n}(\lambda_j)}-I_{\varepsilon^*j}\right)\right\|$$
$$+\mathbb{E}^* \sup_{\lambda\in[0,\pi]}\left\|\frac{1}{\tilde n^{1/2}}\sum_{j=1}^{\lfloor \tilde n\lambda/\pi\rfloor}\hat\zeta_j\xi_{\hat\theta_n j}\left(I_{\varepsilon^*j}-\frac{\hat\sigma_n^2}{2\pi}\right)\right\| + \frac{\hat\sigma_n^2}{2\pi}\sup_{\lambda\in[0,\pi]}\left\|\frac{1}{\tilde n^{1/2}}\sum_{j=1}^{\lfloor \tilde n\lambda/\pi\rfloor}\hat\zeta_j\xi_{\hat\theta_n j}\right\|,$$

where $\xi_{\theta j} = h_{\theta j}^{-1}\left(h_{\theta n}(\lambda_j) - h_{\theta j}\right)$. The first term of the last displayed expression is $O_p\left(n^{-\delta}\right)$ by Lemma 6 as $\hat\zeta_j\xi_{\hat\theta_n j}$ satisfies the same conditions of ζ, whereas the second term is $O_p\left(n^{-\delta}\right)$ by Lemma 9 and because Lemma 1 implies that

$$h_{\hat\theta_n n}(\lambda_j) - h_{\hat\theta_n}(\lambda_j) = O_p\left(\lambda_j^{-2d_0} j^{d_0-1}\right) \qquad \textbf{(5.12)}$$

as $n^{\hat d_n - d_0} - 1 = o_p(1)$ and that for ARFIMA models $K^{-1}\lambda_j^{-2d} < h_{\theta j} < K\lambda_j^{-2d}$. The latter will also imply that the third term of the last displayed expression is bounded by $K\hat\sigma_n^2\tilde n^{-1/2}\sum_{j=1}^{\tilde n}\left\|\hat\zeta_j j^{d_0-1}\right\| = O_p\left(n^{d_0-1/2}\right)$ using (5.9) and Lemma 3. Now choose $2d_0 - 1 = 2\delta$ to conclude as $d_0 < \frac{1}{2}$. ■

Define

$$\varkappa_j^* = \frac{I_{x^*j}}{h_{\hat\theta_n^* j}} - I_{\varepsilon^*j}, \qquad j = 1, \ldots, \tilde n. \qquad \textbf{(5.13)}$$

LEMMA 11. *Assume A1–A3. Then, uniformly in* $\lambda \in [0,\pi]$, *for some* $0 < \delta < \frac{1}{2}$,

$$\frac{2\pi}{\tilde{n}^{1/2}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\hat{\zeta}_j \varkappa_j^* = -\left(\frac{\hat{\sigma}_n^2}{\tilde{n}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\hat{\zeta}_j \phi'_{\hat{\theta}_n j}\right)\tilde{n}^{1/2}\left(\hat{\theta}_n^* - \hat{\theta}_n\right) + O_{p^*}\left(n^{-\delta}\right). \quad \textbf{(5.14)}$$

Proof. The left side of (5.14) is

$$\begin{aligned}
&\frac{2\pi}{\tilde{n}^{1/2}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\hat{\zeta}_j \frac{I_{x^* j}}{h_{\hat{\theta}_n j}}\left[\frac{h_{\hat{\theta}_n j}}{h_{\hat{\theta}_n^* j}} - 1 + \phi'_{\hat{\theta}_n j}\left(\hat{\theta}_n^* - \hat{\theta}_n\right)\right] \\
&\quad + \frac{2\pi}{\tilde{n}^{1/2}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\hat{\zeta}_j\left(\frac{I_{x^* j}}{h_{\hat{\theta}_n n}\left(\lambda_j\right)} - I_{\varepsilon^* j}\right) \qquad \textbf{(5.15)} \\
&\quad + \frac{2\pi}{\tilde{n}^{1/2}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\hat{\zeta}_j \xi_{\hat{\theta}_n j}\frac{I_{x^* j}}{h_{\hat{\theta}_n n}\left(\lambda_j\right)} - \frac{2\pi}{\tilde{n}^{1/2}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\hat{\zeta}_j \phi'_{\hat{\theta}_n j}\frac{I_{x^* j}}{h_{\hat{\theta}_n j}}\left(\hat{\theta}_n^* - \hat{\theta}_n\right)
\end{aligned}$$

with $\xi_{\theta j}$ as defined in the proof of Lemma 10.

First, A3 implies that, uniformly in $\lambda \in [0, \pi]$, the norm of the first term of (5.15) is bounded by

$$K\tilde{n}^{1/2}\left\|\hat{\theta}_n^* - \hat{\theta}_n\right\|^2 \frac{1}{\tilde{n}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\left|\log^2 \lambda_j\right|\left\|\hat{\zeta}_j\right\|\frac{I_{x^* j}}{h_{\hat{\theta}_n j}} = O_{p^*}\left(n^{-\delta}\right), \qquad \textbf{(5.16)}$$

because by Proposition 1, $\hat{\theta}_n^* - \hat{\theta}_n = O_{p^*}\left(n^{-1/2}\right)$, and hence we can take $\iota = Kn^{-1/2}$ in A3 so that $\lambda_j^{-\iota} < K$, and also because by Markov's inequality and Lemmas 6, 9, and 10,

$$\sup_{\lambda\in[0,\pi]}\left|\frac{1}{\tilde{n}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\left|\log^2 \lambda_j\right|\left\|\hat{\zeta}_j\right\|\left(\frac{I_{x^* j}}{h_{\hat{\theta}_n j}} - \frac{\hat{\sigma}_n^2}{2\pi}\right)\right| = O_{p^*}\left(n^{-\delta}\right).$$

Notice that $\zeta(\lambda)\log^2\lambda$ satisfies the same properties as $\zeta(\lambda)$.

The second term of (5.15) is $O_{p^*}\left(n^{-\delta}\right)$ by Lemma 6 and Markov's inequality, whereas Lemmas 6 and 9 imply that the third term is

$$\frac{\hat{\sigma}_n^2}{\tilde{n}^{1/2}}\sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor}\hat{\zeta}_j \xi_{\hat{\theta}_n j} + O_{p^*}\left(n^{-\delta/2}\right) = O_{p^*}\left(n^{-\delta}\right)$$

using (5.9) and (5.12). Next, proceeding similarly as in (5.16), because $\hat{\zeta}(\lambda)\,\phi'_{\hat{\theta}_n}(\lambda)$ satisfies the same conditions of $\hat{\zeta}(\lambda)\,|\log\lambda|$, the last term of (5.15) is the first term on the right of (5.14) plus $O_{p^*}\left(n^{-\delta}\right)$ by Lemmas 6, 9, and 10. This concludes the proof of the lemma. ■

LEMMA 12. *Assuming A1–A3, for any* $0 \le \upsilon < (1-\delta)/4$, *with* $0 \le \delta < 1$, *we have that for all* $k = 1, \ldots, p$,

$$\mathbb{E}^*\left(\frac{\mathcal{E}_{1,n}^{*(k)}(\lambda_1, \pi)}{(\pi - \lambda_1)^{\upsilon}} - \frac{\mathcal{E}_{1,n}^{*(k)}(\lambda_2, \pi)}{(\pi - \lambda_2)^{\upsilon}}\right)^2 = H_n(\lambda_1, \lambda_2)(\lambda_2 - \lambda_1)^{2-\delta-2\upsilon}, \qquad \textbf{(5.17)}$$

$$\mathbb{E}^* \left(\frac{\mathcal{E}_{2,n}^{*(k)}(\lambda_1, \pi)}{(\pi - \lambda_1)^{\upsilon}} - \frac{\mathcal{E}_{2,n}^{*(k)}(\lambda_2, \pi)}{(\pi - \lambda_2)^{\upsilon}} \right)^4 = H_n(\lambda_1, \lambda_2)(\lambda_2 - \lambda_1)^{2-\delta-4\upsilon} \tag{5.18}$$

for all $0 < \lambda_1 < \lambda_2 < \pi$ *and where* $\mathcal{E}_{1,n}^{*(k)}(\lambda_1, \lambda_2)$ *and* $\mathcal{E}_{2,n}^{*(k)}(\lambda_1, \lambda_2)$ *are given in (5.6) and (5.7), respectively.*

Proof. The proof proceeds in the same way as that of Lemma 9 of DHV, but instead of using their Lemmas 6 and 7, we use Lemmas 8 and 9. ■

In what follows we shall abbreviate $\gamma'_{\theta q} A_{\theta n}^{-1}(q)$ by $\Im_{\theta n}(q)$ and recall the notation introduced in (5.10) and (5.13).

LEMMA 13. *Assuming A1–A3, for all* $\varepsilon > 0$,

$$\lim_{\lambda_0 \to \pi} \overline{\lim_{n \to \infty}} \Pr{}^* \left\{ \sup_{\lambda_0 \le \lambda \le \pi} \left| \frac{1}{\widetilde{n}} \sum_{k=\lfloor \overline{n}\lambda_0/\pi \rfloor + 1}^{\lfloor \overline{n}\lambda/\pi \rfloor} \frac{\Im_{\hat{\theta}_n n}(k)}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \left(\varkappa_j^* + \jmath_j^* \right) \right| > \varepsilon \right\} = 0. \tag{5.19}$$

Proof. Take $\lambda_0 > \pi/2$ without loss of generality. First we observe that

$$\sup_{\lambda_0 \le \lambda \le \pi} \left| \frac{1}{\widetilde{n}} \sum_{k=\lfloor \overline{n}\lambda_0/\pi \rfloor + 1}^{\lfloor \overline{n}\lambda/\pi \rfloor} \frac{\Im_{\hat{\theta}_n n}(k)}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \left(\varkappa_j^* + \jmath_j^* \right) \right| \tag{5.20}$$

$$\le \frac{K}{\widetilde{n}} \sum_{k=\lfloor \overline{n}\lambda_0/\pi \rfloor + 1}^{\overline{n}} \left\| \Im_{\hat{\theta}_n n}(k) \right\| \left(1 - \frac{k}{\widetilde{n}} \right)^{\delta/2}$$
$$\times \left\{ \sup_{\lfloor \overline{n}\lambda_0/\pi \rfloor \le k \le \overline{n}} \left\| \frac{\left(1 - \frac{k}{\widetilde{n}}\right)^{-\delta/2}}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \varkappa_j^* \right\| + \sup_{\lfloor \overline{n}\lambda_0/\pi \rfloor \le k \le \overline{n}} \left\| \frac{\left(1 - \frac{k}{\widetilde{n}}\right)^{-\delta/2}}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \jmath_j^* \right\| \right\},$$

for any $0 < \delta < 1$. The first factor on the right of (5.20) is bounded by

$$K \left| \frac{1}{\widetilde{n}} \sum_{k=\lfloor \overline{n}\lambda_0/\pi \rfloor + 1}^{\overline{n}} \left\| \gamma_{\hat{\theta}_n k} \right\| \left(1 - \frac{k}{\widetilde{n}} \right)^{\frac{\delta}{2} - 1} \right| = O_p \left(|\pi - \lambda_0|^{\frac{\delta}{2}} \right),$$

using that

$$\left\| A_{\hat{\theta}_n n}^{-1}(k) \right\| \le K \left(1 - \frac{k}{\widetilde{n}} \right)^{-1}, \tag{5.21}$$

because $\left\| A_{\theta_0}(\lambda) \right\| \geq K^{-1}(\pi - \lambda)$ by A3 and Lemma 3 imply that $\sup_{\lfloor \widetilde{n}\lambda_0/\pi \rfloor \leq k \leq \widetilde{n}} \left\| A_{\hat{\theta}_n n}(k) - A_{\theta_0}\left(\lfloor k\pi/\widetilde{n} \rfloor\right) \right\| = O_p\left(n^{-1/2}\right)$.

Next, by Lemma 12, the second term inside the braces on the right of (5.20) is $O_p(1)$ for $\delta > 0$ small enough, whereas Lemma 11 and Proposition 1 imply that the first term is bounded by

$$O_p\left(\sup_{\lfloor \widetilde{n}\lambda_0/\pi \rfloor \leq k \leq \widetilde{n}} \left\{ \left\| \frac{\left(1 - \frac{k}{\widetilde{n}}\right)^{-\delta/2}}{\widetilde{n}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \phi'_{\hat{\theta}_n j} \right\| + \frac{\left(1 - \frac{k}{\widetilde{n}}\right)^{-\delta/2}}{n^{\delta}} \right\} \right)$$

$$= O_p\left(|\pi - \lambda_0|^{\delta/2} \right)$$

because $n^{-1} \leq \widetilde{n}^{-1} \leq \inf_{\lfloor \widetilde{n}\lambda_0/\pi \rfloor \leq k \leq \widetilde{n}} \left(1 - k/\widetilde{n}\right)$ and $0 < \delta < 1$ and because of an obvious extension of Lemma 3 with $\hat{\zeta}(\lambda) = \gamma_{\hat{\theta}_n}(\lambda) \phi'_{\hat{\theta}_n}(\lambda)$ there. So, (5.20) is $O_{p^*}\left(|\pi - \lambda_0|^{\delta/2}\right)$, which implies that (5.19) holds true because $\delta > 0$. ■

LEMMA 14. *Assuming A1–A3,*

$$\sup_{\lambda \in [0,\pi]} \left\| \frac{1}{\widetilde{n}^{1/2}} \sum_{j=\lfloor \widetilde{n}\lambda/\pi \rfloor + 1}^{\widetilde{n}} \left(\phi_{\hat{\theta}_n^* j} - \phi_{\hat{\theta}_n j} \right) \left(\varkappa_j^* + \jmath_j^* \right) \right\| = O_{p^*}\left(\frac{\log n}{n^{1/2}} \right). \quad \textbf{(5.22)}$$

Proof. The expression inside the norm on the left of (5.22) is

$$\frac{1}{\widetilde{n}^{1/2}} \sum_{j=\lfloor \widetilde{n}\lambda/\pi \rfloor + 1}^{\widetilde{n}} \frac{\partial}{\partial \theta} \phi_{\hat{\theta}_n j} \varkappa_j^* \left(\hat{\theta}_n^* - \hat{\theta}_n \right) + \frac{1}{\widetilde{n}^{1/2}} \sum_{j=\lfloor \widetilde{n}\lambda/\pi \rfloor + 1}^{\widetilde{n}} \frac{\partial}{\partial \theta} \phi_{\hat{\theta}_n j} \jmath_j^* \left(\hat{\theta}_n^* - \hat{\theta}_n \right)$$

$$+ \frac{1}{\widetilde{n}^{1/2}} \sum_{j=\lfloor \widetilde{n}\lambda/\pi \rfloor + 1}^{\widetilde{n}} \left(\phi_{\hat{\theta}_n^* j} - \phi_{\hat{\theta}_n j} - \frac{\partial}{\partial \theta} \phi_{\hat{\theta}_n j} \left(\hat{\theta}_n^* - \hat{\theta}_n \right) \right) \left(\varkappa_j^* + \jmath_j^* \right). \quad \textbf{(5.23)}$$

By A3 and then noting that $|a - b| \leq (a - b) + 2b$ for $a > 0$ and $b > 0$, the norm of the third term of (5.23) is bounded by

$$K \frac{\left\| \hat{\theta}_n^* - \hat{\theta}_n \right\|^2}{\widetilde{n}^{1/2}} \left\{ \sum_{j=1}^{\widetilde{n}} \left| \log\left(\lambda_j\right) \right| \left(\varkappa_j^* + \jmath_j^* \right) + \frac{\hat{\sigma}_n^2}{\pi} \sum_{j=1}^{\widetilde{n}} \left| \log \lambda_j \right| \right\} = O_{p^*}\left(\frac{\log n}{n^{1/2}} \right)$$

by Proposition 1 and then using Lemmas 11 and 9 with $\hat{\zeta}(\lambda) = |\log \lambda|$, and Lemma 3, respectively. So, uniformly in λ the third term of (5.23) is $o_{p^*}(1)$. Likewise, the first term of (5.23) is $O_{p^*}\left(n^{-1/2}\right)$ uniformly in λ using Lemma 11 with $\hat{\zeta}(\lambda) = (\partial/\partial\theta)\phi_{\hat{\theta}_n}(\lambda)$ there and Proposition 1. Observe that $(\partial/\partial\theta)\phi_{\hat{\theta}_n}(\lambda)$ satisfies the same conditions of $\zeta(\lambda)$ in Lemma 10 by A3. Finally, the second term of (5.23) is $O_{p^*}\left(n^{-1/2}\right)$ by Lemma 9 with $\hat{\zeta}(\lambda) = (\partial/\partial\theta)\phi_{\hat{\theta}_n}(\lambda)$ there. ■

LEMMA 15. *Assuming A1–A3, for all* $\varepsilon > 0$,

$$\lim_{\lambda_0 \to \pi} \overline{\lim_{n\to\infty}} \Pr{}^* \left\{ \sup_{\lambda_0 \le \lambda \le \pi} \left| \frac{1}{\widetilde{n}} \sum_{k=\lfloor \widetilde{n}\lambda_0/\pi \rfloor + 1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \frac{\Im_{\hat{\theta}_n^* n}(k)}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n^* j} \left(\varkappa_j^* + \jmath_j^* \right) \right| > \varepsilon \right\} = 0. \tag{5.24}$$

Proof. Notice that Proposition 1 implies that it suffices to show (5.24) in the set $\left\{ \left\| \hat{\theta}_n^* - \hat{\theta}_n \right\| < K n^{-1/2} m_n^{-1} \right\}$, where $m_n + m_n^{-1} n^{-1/2} \to 0$. On the other hand, Lemma 11 implies that, uniformly in k,

$$\frac{1}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n^* j} \varkappa_j^* = \left(\frac{\hat{\sigma}_n^2}{\widetilde{n}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \phi'_{\hat{\theta}_n j} \right) \widetilde{n}^{1/2} \left(\hat{\theta}_n - \hat{\theta}_n^* \right) + O_{p^*}\left(n^{-\delta/2} \right),$$

$$\frac{1}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n^* j} \jmath_j^* = \frac{1}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \jmath_j^* + O_{p^*}\left(n^{-\delta/2} \right) \tag{5.25}$$

proceeding as in the proof of (5.22) but with $\varkappa_j^* + \jmath_j^*$ replaced by $\jmath_j^*$ there. Observe that we can take $\lambda_0 > \pi/2$. Next, uniformly in k, A3 implies that

$$\sup_{\lfloor \widetilde{n}\lambda_0/\pi \rfloor \le k \le \widetilde{n}} \left\| A_{\hat{\theta}_n^* n}(k) - A_{\hat{\theta}_n n}(k) \right\| = (\pi - \lambda_0)\, O_{p^*}\left(\left\| \hat{\theta}_n^* - \hat{\theta}_n \right\| \right),$$

which together with (5.21) implies that $\left\| A^{-1}_{\hat{\theta}_n^* n}(k) \right\| = O_p\left(\left(1 - \frac{k}{\widetilde{n}}\right)^{-1} \right)$.

So, we have that for some $0 < \delta < \frac{1}{2}$,

$$\sup_{\lambda_0 \le \lambda \le \pi} \left\| \frac{1}{\widetilde{n}} \sum_{k=\lfloor \widetilde{n}\lambda_0/\pi \rfloor + 1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \frac{\Im_{\hat{\theta}_n^* n}(k)}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n^* j} \left(\varkappa_j^* + \jmath_j^* \right) \right\| \tag{5.26}$$

$$= O_{p^*}(1) \sup_{\lambda_0 \le \lambda \le \pi} \left| \frac{1}{\widetilde{n}} \sum_{k=\lfloor \widetilde{n}\lambda_0/\pi \rfloor + 1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \left\| \gamma_{\hat{\theta}_n k} \right\| \left(1 - \frac{k}{\widetilde{n}} \right)^{-1+\delta/2} \right|$$
$$\times \left\{ \sup_{\lfloor \widetilde{n}\lambda_0/\pi \rfloor \le k \le \widetilde{n}} \left\| \left(1 - \frac{k}{\widetilde{n}} \right)^{-\delta/2} \frac{1}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \jmath_j^* \right\| + O_{p^*}\left(|\pi - \lambda_0|^{\delta/2} \right) \right\}$$

by (5.25) and because $n^{-1} \le \widetilde{n}^{-1} \le \inf_{\lfloor \widetilde{n}\lambda_0/\pi \rfloor \le k \le \widetilde{n}} \left(1 - k/\widetilde{n} \right)$. But Lemma 12 implies that $\sup_{\lfloor \widetilde{n}\lambda_0/\pi \rfloor \le k \le \widetilde{n}} \left\| \left(1 - k/\widetilde{n} \right)^{-\delta/2} \widetilde{n}^{-1/2} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \jmath_j^* \right\| = O_{p^*}(1)$, and Lemma 3 implies that

$$\sup_{\lambda_0 \le \lambda \le \pi} \frac{1}{\widetilde{n}} \sum_{k=\lfloor \widetilde{n}\lambda_0/\pi \rfloor + 1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \left\| \gamma_{\hat{\theta}_n k} \right\| \left(1 - \frac{k}{\widetilde{n}} \right)^{-1+(\delta/2)} = O_p\left(|\pi - \lambda_0|^{\delta/2} \right),$$

and hence the left side of (5.26) is $O_p\left(|\pi - \lambda_0|^{\delta/2} \right)$. Now, we conclude that (5.24) holds true because $\delta > 0$. ■

6. PROOFS

6.1. Proof of Proposition 1. The proof of part (i) follows by Lemma 6. Next part (ii). First we notice that $\sum_{j=1}^{\tilde{n}} \phi_{\hat{\theta}_n j} = O(\log n)$ by Lemma 3 because $\int_0^{\pi} \phi_\theta(\lambda)\, d\lambda = 0$ for all θ and A3 implies that $\phi_{\hat{\theta}_n}(\lambda)$ satisfies the same conditions of $\hat{\zeta}(\lambda)$ in Lemma 3. So, $\sum_{j=1}^{\tilde{n}} \phi_{\hat{\theta}_n j} I_{\varepsilon^* j} = \sum_{j=1}^{\tilde{n}} \phi_{\hat{\theta}_n j} \left(I_{\varepsilon^* j} - \hat{\sigma}_n^2/2\pi\right) + o_{p^*}\left(\tilde{n}^{1/2}\right)$. From here we conclude the proof using Lemma 9. ■

6.2. Proof of Theorem 1. From Lemma 11, we have that

$$F^*_{\hat{\theta}^*_n n}(\lambda) = \frac{2\pi}{\tilde{n}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} I_{\varepsilon^* j} - \left(\frac{\hat{\sigma}_n^2}{\tilde{n}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \phi'_{\hat{\theta}_n j}\right) \left(\hat{\theta}^*_n - \hat{\theta}_n\right) + o_{p^*}(1).$$

On the other hand, because $2\pi \sum_{j=1}^{\tilde{n}} I_{\varepsilon^* j} = \sum_{t=1}^{n} \varepsilon_t^{*2} = \sum_{t=1}^{n} \hat{\varepsilon}_t^2 + O_{p^*}\left(n^{1/2}\right)$ using (5.9) and because by Lemma 3, $\sum_{j=1}^{[\tilde{n}\lambda/\pi]} \phi_{\hat{\theta}_n j} = O_p(\log n)$, standard algebra implies that

$$\alpha^*_{\hat{\theta}^*_n n}(\lambda) = \frac{2\pi}{\hat{\sigma}_n^2 \tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} J_j^* - \left(\frac{\hat{\sigma}_n^2}{\tilde{n}} \sum_{j=1}^{\lfloor \tilde{n}\lambda/\pi \rfloor} \phi'_{\hat{\theta}_n j}\right) \tilde{n}^{1/2} \left(\hat{\theta}^*_n - \hat{\theta}_n\right) + o_{p^*}(1).$$

The proof now follows by Proposition 1 and Lemma 9 after we observe that

$$\tilde{n}^{1/2}\left(\hat{\theta}^*_n - \hat{\theta}_n\right) = \left(\frac{1}{\tilde{n}} \sum_{j=1}^{\tilde{n}} \phi_{\hat{\theta}_n j} \phi'_{\hat{\theta}_n j}\right)^{-1} \frac{1}{\tilde{n}^{1/2}} \sum_{j=1}^{\tilde{n}} \phi'_{\hat{\theta}_n j} J_j^* + o_{p^*}(1),$$

where $0 < \tilde{n}^{-1} \sum_{j=1}^{\tilde{n}} \phi_{\hat{\theta}_n j} \phi'_{\hat{\theta}_n j} = O_p(1)$ and (5.9) and Lemma 3 imply, respectively, $\hat{\sigma}_n^2 \to_P \sigma^2$ and $\tilde{n}^{-1} \sum_{j=1}^{[\tilde{n}\lambda/\pi]} \phi_{\hat{\theta}_n j} \to_P \pi^{-1} \int_0^{\lambda} \phi_{\theta_0}(v)\, dv$ under H_0 and $\to_P \pi^{-1} \int_0^{\lambda} \phi_{\theta_1}(v)\, dv$ under H_1. ■

6.3. Proof of Theorem 2. We shall only consider H_0 in the proofs. For H_1, we just replace θ_0 by θ_1 in the expressions and in A2 and A3. The proof is done in two steps. Step (a) shows that, in probability,

$$\check{\alpha}^*_{\hat{\theta}_n n}(\lambda) = \frac{2\pi}{F_n^*(\pi)} \frac{1}{\tilde{n}^{1/2}} \sum_{j=1}^{\lfloor \overline{n}\lambda/\pi \rfloor} \left\{I_{\varepsilon^* j} - \gamma'_{\hat{\theta}_n j} b_n^*(j)\right\} \overset{d^*}{\Rightarrow} \frac{1}{\pi^{1/2}} B_\pi(\lambda),$$

where $b_n^*(j) = A^{-1}_{\hat{\theta}_n n}(j)\, \tilde{n}^{-1} \sum_{k=j+1}^{\tilde{n}} \gamma'_{\hat{\theta}_n k} I_{\varepsilon^* k}$, whereas step (b) will show that

$$\sup_{\lambda \in [0,\pi]} \left|\hat{\alpha}^*_{\hat{\theta}^*_n n}(\lambda) - \check{\alpha}^*_{\hat{\theta}_n n}(\lambda)\right| = o_{p^*}(1).$$

We begin with step (a). Using $F_n^*(\pi) = \hat{\sigma}_n^2 + o_{p^*}(1)$ and recalling that $\Im_{\theta n}(j) = \gamma_{\theta j}' A_{\theta n}^{-1}(j)$, we obtain that uniformly in $\lambda \in [0, \pi]$,

$$\check{\alpha}_{\hat{\theta}_n n}^*(\lambda) = \frac{2\pi}{\hat{\sigma}_n^2} \frac{1}{\widetilde{n}^{1/2}} \sum_{j=1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} J_j^* - \frac{2\pi}{\hat{\sigma}_n^2} \hat{\Lambda}_n^*(\lambda) + o_{p^*}(1), \qquad \textbf{(6.1)}$$

where $\hat{\Lambda}_n^*(\lambda) = \widetilde{n}^{-1} \sum_{j=1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \Im_{\hat{\theta}_n n}(j) \left(\frac{1}{\widetilde{n}^{1/2}} \sum_{k=j+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n k} J_k^* \right)$.

Suppose, to be shown later, that the convergence in $[0, \lambda_0]$ holds true for any $0 < \lambda_0 < \pi$. Then, because $B(\cdot)$ and the limit of the process $\widetilde{n}^{-1/2} \sum_{j=1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} J_j^*$ are continuous in $[0, \pi]$, Theorem 4.2 of Billingsley (1968) implies that it suffices to show that for all $\varepsilon > 0$,

$$\lim_{\lambda_0 \to \pi} \overline{\lim_{n \to \infty}} \Pr{}^* \left\{ \sup_{\lambda_0 \le \lambda \le \pi} \left| \frac{1}{\widetilde{n}} \sum_{j=\lfloor \widetilde{n}\lambda_0/\pi \rfloor + 1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \frac{\Im_{\hat{\theta}_n n}(j)}{\widetilde{n}^{1/2}} \sum_{k=j+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n k} J_k^* \right| > \varepsilon \right\} = 0.$$

But the latter holds true by Lemma 13; cf. the second term on the right of (5.20).

So, to complete the proof of step (a) we need to show that, for any $0 < \lambda_0 < \pi$, the first term on the right of (6.1) converges in bootstrap distribution to $\pi^{-1/2} B(\lambda)$ in $[0, \lambda_0]$ in probability. Fidi's convergence follows by Lemma 9 part (ii) after we note that

$$\hat{\Lambda}_n^*(\lambda) = \frac{1}{\widetilde{n}^{1/2}} \sum_{k=1}^{\widetilde{n}} \left(\frac{1}{\widetilde{n}} \sum_{j=1}^{k \wedge \lfloor \widetilde{n}\lambda/\pi \rfloor} \Im_{\hat{\theta}_n n}(j) \right) \gamma_{\hat{\theta}_n k} J_k^*$$

and $\left(\widetilde{n}^{-1} \sum_{j=1}^{k \wedge \lfloor \widetilde{n}\lambda/\pi \rfloor} \Im_{\hat{\theta}_n n}(j) \right) \gamma_{\hat{\theta}_n k}$ satisfies the same conditions of Lemma 9 for $\hat{\zeta}(\lambda)$. Then, we are left to prove tightness. Because $\widetilde{n}^{-1/2} \sum_{j=1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} J_j^*$ is tight by Lemma 8 (see also Lemma 9(i)), all we need to show is that $\hat{\Lambda}_n^*(\lambda)$ is tight.

By Theorem 15.6 of Billingsley (1968), it suffices to show that

$$\mathbb{E}^* \left(\left| \hat{\Lambda}_n^*(\vartheta) - \hat{\Lambda}_n^*(\mu) \right| \left| \hat{\Lambda}_n^*(\lambda) - \hat{\Lambda}_n^*(\vartheta) \right| \right) = H_n(\lambda, \mu) |\lambda - \mu|^{2\delta}$$

for all $0 \le \mu < \vartheta < \lambda \le \pi$ and some $\delta > \frac{1}{2}$. Observe that we can take $\widetilde{n}^{-1} < |\lambda - \mu|$ because otherwise the last equality is trivial. Because $(\lambda - \vartheta)(\vartheta - \mu) < (\lambda - \mu)^2$, by the Cauchy–Schwarz inequality, it suffices to show that the last displayed equality holds for $\mathbb{E}^* \left| \hat{\Lambda}_n^*(\lambda) - \hat{\Lambda}_n^*(\mu) \right|^2$ which is

$$\begin{aligned} &\frac{1}{\widetilde{n}^3} \sum_{j,k=\lfloor \widetilde{n}\mu/\pi \rfloor + 1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \Im_{\hat{\theta}_n n}(j) \left\{ \sum_{\ell_1=j+1}^{\widetilde{n}} \sum_{\ell_2=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n \ell_1} \gamma_{\hat{\theta}_n \ell_2}' \mathbb{E}^* \left(J_{\ell_1}^* J_{\ell_2}^* \right) \right\} \Im_{\hat{\theta}_n n}'(k) \\ &\quad = O_p(1) \frac{1}{\widetilde{n}^2} \sum_{j,k=\lfloor \widetilde{n}\mu/\pi \rfloor + 1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \left\| \Im_{\hat{\theta}_n n}(j) \right\| \left\| \Im_{\hat{\theta}_n n}(k) \right\| \\ &\quad = O_p(1) \left(\left\| \widetilde{\Im}(\lambda, \mu) \right\|^2 + |\lambda - \mu|^{2\delta} \right), \end{aligned}$$

where $\widetilde{\Im}(\lambda,\mu) := \pi^{-1}\int_{\mu}^{\lambda}\Im_{\theta_0}(\nu)\,d\nu$ and $\Big\|\widetilde{n}^{-1}\sum_{j=1+\lfloor\overline{n}\mu/\pi\rfloor}^{\lfloor\overline{n}\lambda/\pi\rfloor}\Im_{\hat{\theta}_n n}(j) - \widetilde{\Im}(\lambda,\mu)\Big\| = O_p\left(|\lambda-\mu|^{\delta}\right)$ by Lemma 3 for some $\delta > \frac{1}{2}$. From here we conclude the proof of part (a) by Theorem 15.6 of Billingsley (1968), because $\widetilde{\Im}(\lambda,0)$ is a monotonic, continuous, and nondecreasing function such that $\Big|\widetilde{\Im}(\lambda,0) - \widetilde{\Im}(\mu,0)\Big| = |\lambda-\mu|^{\delta}$.

To show step (b), by definitions of $\hat{\alpha}^{*}_{\hat{\theta}^{*}_n n}(\lambda)$ and $\check{\alpha}^{*}_{\hat{\theta}_n n}(\lambda)$, it suffices to show that

$$\left|\frac{1}{\widetilde{n}^{1/2}}\sum_{k=1}^{\lfloor\overline{n}\lambda/\pi\rfloor}\left\{\varkappa^{*}_{k} - \Im_{\hat{\theta}_n n}(k)\frac{1}{\widetilde{n}}\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}_n j}\varkappa^{*}_{j}\right\}\right| \tag{6.2}$$

and

$$\frac{1}{F_{\hat{\theta}^{*}_n n}(\pi)}\left(\frac{1}{\widetilde{n}}\sum_{k=1}^{\lfloor\overline{n}\lambda/\pi\rfloor}\Im_{\hat{\theta}_n n}(k)\frac{1}{\widetilde{n}^{1/2}}\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}_n j}\left(\frac{I_{x^{*}j}}{h_{\hat{\theta}^{*}_n j}} - \frac{F_{\hat{\theta}^{*}_n n}(\pi)}{2\pi}\right)\right)$$
$$-\frac{1}{F_{\hat{\theta}^{*}_n n}(\pi)}\left(\frac{1}{\widetilde{n}}\sum_{k=1}^{\lfloor\overline{n}\lambda/\pi\rfloor}\Im_{\hat{\theta}^{*}_n n}(k)\frac{1}{\widetilde{n}^{1/2}}\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}^{*}_n j}\left(\frac{I_{x^{*}j}}{h_{\hat{\theta}^{*}_n j}} - \frac{F_{\hat{\theta}^{*}_n n}(\pi)}{2\pi}\right)\right) \tag{6.3}$$

are $o_{p^{*}}(1)$ uniformly in $\lambda \in [0,\pi]$. Indeed, expression (6.2) is $o_{p^{*}}(1)$ uniformly in $\lambda \in [0,\pi]$ by Lemma 11 and because

$$\phi'_{\hat{\theta}_n k} - \Im_{\hat{\theta}_n n}(k)\,\widetilde{n}^{-1}\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}_n j}\phi_{\hat{\theta}_n j} = 0.$$

Next we examine (6.3). Because by Lemma 11 and proceeding as in the proof of Theorem 1, $F_{\hat{\theta}^{*}_n n}(\pi) - \hat{\sigma}^{2}_{n} = O_{p^{*}}\left(n^{-1/2}\right)$, it suffices to show that

$$\frac{1}{\widetilde{n}}\sum_{k=1}^{\lfloor\overline{n}\lambda/\pi\rfloor}\left\{\frac{\Im_{\hat{\theta}_n n}(k)}{\widetilde{n}^{1/2}}\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}_n j}\left(\varkappa^{*}_{j} + \jmath^{*}_{j}\right) - \frac{\Im_{\hat{\theta}^{*}_n n}(k)}{\widetilde{n}^{1/2}}\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}^{*}_n j}\left(\varkappa^{*}_{j} + \jmath^{*}_{j}\right)\right\} \tag{6.4}$$

converges to zero uniformly in $\lambda \in [0,\pi]$, after observing that for all $\lambda \in [0,\pi]$,

$$\sum_{k=1}^{\lfloor\overline{n}\lambda/\pi\rfloor}\left\{\Im_{\hat{\theta}^{*}_n n}(k)\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}^{*}_n j} - \Im_{\hat{\theta}_n n}(k)\sum_{j=k+1}^{\widetilde{n}}\gamma_{\hat{\theta}_n j}\right\} = 0.$$

First, we observe that Lemmas 11 and 13 imply that it suffices to show the uniform convergence in $\lambda \in [0, \lambda_0]$ for any $\lambda_0 < \pi$. But (6.4) is equal to

$$\frac{1}{\widetilde{n}} \sum_{k=1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \Im_{\hat{\theta}_n^* n}(k) \frac{1}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \left(\gamma_{\hat{\theta}_n j} - \gamma_{\hat{\theta}_n^* j} \right) \left(\varkappa_j^* + \jmath_j^* \right), \tag{6.5}$$

$$+ \frac{1}{\widetilde{n}} \sum_{k=1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \left(\Im_{\hat{\theta}_n n}(k) - \Im_{\hat{\theta}_n^* n}(k) \right) \frac{1}{\widetilde{n}^{1/2}} \sum_{j=k+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \left(\varkappa_j^* + \jmath_j^* \right). \tag{6.6}$$

So, the theorem follows if (6.5) and (6.6) are $o_{p^*}(1)$ uniformly in $\lambda \in [0, \lambda_0]$.

To that end, we first observe that proceeding as in DHV but using Lemma 3 instead of their Lemma 1, we have that

$$\sup_{\lambda \in [0,\pi]} \frac{1}{\widetilde{n}} \sum_{j=1}^{\lfloor \widetilde{n}\lambda/\pi \rfloor} \left\| \phi_{\hat{\theta}_n j} - \phi_{\hat{\theta}_n^* j} \right\| = o_{p^*}(1), \tag{6.7}$$

$$\sup_{\lambda \in [0,\lambda_0]} \left\| A_{\hat{\theta}_n n}^{-1}(\lambda) - A_{\hat{\theta}_n}^{-1}(\lambda) \right\| = o_p(1), \tag{6.8}$$

$$\sup_{\lambda \in [0,\lambda_0]} \left\| A_{\hat{\theta}_n^* n}^{-1}(\lambda) - A_{\hat{\theta}_n n}^{-1}(\lambda) \right\| = o_{p^*}(1). \tag{6.9}$$

Next, uniformly in $\lambda \in [0, \lambda_0]$, (6.5) is $o_p(1)$ by Lemma 14 and using (6.7)–(6.9) after observing that $\left(\gamma'_{\hat{\theta}_n j} - \gamma'_{\hat{\theta}_n^* j} \right) = \left(0, \phi'_{\hat{\theta}_n j} - \phi'_{\hat{\theta}_n^* j} \right)$. Next that (6.6) is also $o_p(1)$, uniformly in $\lambda \in [0, \lambda_0]$, follows by (6.7) and (6.9) and because $\sup_{\lambda \in [0,\pi]} \left| \widetilde{n}^{-1/2} \sum_{j=[\widetilde{n}\lambda/\pi]+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \left(\varkappa_j^* + \jmath_j^* \right) \right| = O_{p^*}(1)$ by Lemmas 7 and 8 with $\hat{\zeta}(\lambda) = \gamma_{\hat{\theta}_n}(\lambda)$ there and observing the results in Proposition 1. Also, recall that, by Lemma 3, $\widetilde{n}^{-1} \sum_{j=[\widetilde{n}\lambda/\pi]+1}^{\widetilde{n}} \gamma_{\hat{\theta}_n j} \phi'_{\hat{\theta}_n j} \to P \int_\lambda^\pi \gamma_{\theta_0}(\nu) \phi'_{\theta_0}(\nu)\, d\nu$. ■

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