Supplementary Material

A Note on the Free Disposal Assumption and the Egalitarian Solutions☆

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1. Introduction

In economic theory, *free disposal* is captured by the assumption that goods may be costlessly thrown away. Under the standard hypothesis that preferences are monotone, free disposal implies that the set of feasible utility allocations is *comprehensive*, which means that, whenever it contains an allocation u, it also contains all allocations v satisfying $v \leq u$. For instance, if free disposal fails, an individual consumption set may be bounded below. In classical cooperative game theory with complete information, the possibility to disposing of utility at will is usually taken as an innocuous assumption. The argument is that, when utility is transferable (risk-neutrality in money), if players can agree on the utility allocation u, then they can achieve any utility allocation v satisfying $v \leq u$ by agreeing that each player will "burn" an appropriate amount of money after u has been implemented. In the case of non-transferable utility, the reason for allowing free disposal (in utilities) is that we should not exclude utility allocations from the feasible set, unless we are sure they will not be implemented. If some feasible utility allocation is never chosen, this fact must be a consequence of the rationality of the players. But why would a rational player ever agree to harm himself? The reason is that free disposal increases the strategic possibilities (threats) a player has by allowing him to eliminate some part of the proceeds of the cooperation.

The purpose of this supplementary note is to provide a detailed analysis of the free disposal assumption in the study of some egalitarian-based solution concepts for non-transferable utility games (NTU). In sections 2 and 3, we illustrate why free disposal is a technical assumption of utmost importance for the existence of the egalitarian solutions, and a fortiori, of the Harsanyi NTU value. In section 4, we show that when information is incomplete, incentive constraints restrict what is feasible in a way that makes the analysis of egalitarian solutions more difficult. In particular, we exhibit some examples in which free disposal activities have an important effect on the incentives structure of the game. Finally, section 5 is devoted to establish why the standard free disposal hypothesis does not help for the existence of the Egalitarian value. Instead, we argue that free disposal of virtual utilities allows to solve the aforementioned difficulties.

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2. Egalitarian Solutions for Two-person Bargaining Problems

The free disposal assumption is of particular importance for the analysis of bargaining when negotiations are guided by the principle of *equal gains*. For any two-person bargaining problem, an *egalitarian solution*, as defined by Kalai (1977), will choose the (unique) efficient allocation which grants both players with the same gains (with respect to which each would get in the absence of agreement), i.e., the unique point $u = (u_1, u_2)$ satisfying

(*i*) Efficiency: $u \in \partial V$

(*ii*) Egalitarianism: $u_1 - u_1^0 = u_2 - u_2^0$

where $V \subseteq \mathbb{R}^2$ is the set of feasible utility allocations[1], ∂V denotes the weak Pareto frontier of V and $u^0 = (u_1^0, u_2^0) \in \mathbb{R}^2$ is the allocation both players would get in case of disagreement.

To understand why free disposal is essential, consider the two-person bargaining problem depicted in panel A of figure 1. The set of feasible utility allocations is given by the set which is the locus of all point over the line $\vec{ow}$. Assume that in the absence of agreement each player gets 0, so that the unique feasible allocation equalizing the gains between both players is $(0, 0)$. This allocation is however not efficient. Indeed, the only efficient allocation is the point w.

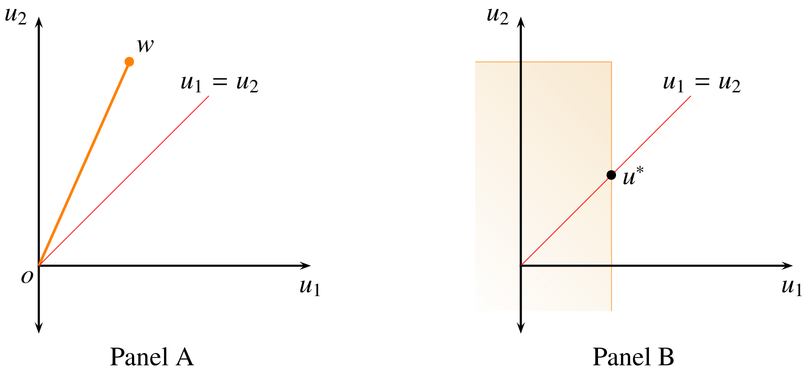


Figure 1

Hence, an egalitarian solution for this example does not exist. This is due to a lack of comprehensiveness. In panel B of figure 1 we depict a bargaining problem in which the set of feasible utility allocations is now the comprehensive hull[2] of the points in the line $\vec{ow}$. This new problem, unlike the former, has an egalitarian solution located at u^*.

Interpersonal comparisons of utility in a bargaining situation can also be made according to the principle of *greatest good*. For any two-person bargaining problem, a *utilitarian solution* selects a feasible allocation maximizing the sum of utilities, i.e., a point $u \in V$ satisfying

(*iii*) Utilitarianism: $u_1 + u_2 = \max_{v \in V} v_1 + v_2$

[1] Notice that whenever the set of feasible decisions for both players is finite, the set V is convex.

[2] The comprehensive hull of a set A is the smallest comprehensive set containing A.

Under the reasonable assumptions that V is closed and bounded above, a utilitarian solution always exits. For instance, in our two previous examples (figure 1) the unique utilitarian solution is the allocation w. As evidenced by these examples, egalitarianism and utilitarianism may yield different outcomes. More generally, we cannot expect these two criteria for utility comparisons to coincide. The reason is that both egalitarian and utilitarian solutions violate the scale-covariance axiom[3]. To reconciliate these two principles we must admit a set $\lambda = (\lambda_1, \lambda_2) > 0$ of utility weights in which players make interpersonal utility comparisons. Formally, given a vector λ, an allocation $u \in V$ is a *λ-egalitarian* solution if it satisfies conditions (i) and

(*ii'*) λ-Egalitarianism: $\lambda_1(u_1 - u_1^0) = \lambda_2(u_2 - u_2^0)$

Similarly, an allocation $u \in V$ is a *λ-utilitarian* solution if it satisfies

(*iii'*) λ-Utilitarianism: $\lambda_1 u_1 + \lambda_2 u_2 = \max_{v \in V} \lambda_1 v_1 + \lambda_2 v_2$

These solutions apply the principles of equal-gains and greatest good when players make interpersonal utility comparisons in some λ-weighted utility scale (which does not affect any decision-theoretic property of the utility). A very well known result in the literature asserts that the Nash (1950) bargaining solution is the unique efficient allocation for which there exists utility weights λ such that the allocation is simultaneously both λ-egalitarian and λ-utilitarian (see theorem 1 in Harsanyi (1963) and theorem 8.2 in Myerson (1991)). Hence, for the Nash bargaining solution, utility weights λ are endogenously determined, while for the λ-egalitarian and λ-utilitarian solutions they are externally given. The endogeneity of the utility weights is the result of combining efficiency together with λ-utilitarianism. Observe that conditions (i) and (iii') requieres λ to be a supporting normal vector of ∂V at u. Then, the Nash bargaining solution of a two-person bargaining problem can be constructed according to the following method of *fictitious transfers*:

1. Take a vector $\lambda \in \mathbb{R}^2_+$, $\lambda \neq 0$.
2. The quotients of the coefficients of λ can be regarded as the (local) rates at which players can transfer utility. Consider the game in which λ-weighted utility is transferable.
3. Suppose that the allocation u is such that both players can divide among themselves the maximal transferable weighted-utility worth that they can get (i.e., condition (iii')) in such a way that each player gets the same weighted-utility gain over what he would get in the absence of agreement (i.e., condition (ii')).
4. If u is actually feasible without the fictitious transfers (i.e., condition (i)), then it is a Nash bargaining solution.

Notice that the free disposal assumption is not required for the existence of a Nash bargaining solution[4]. For instance, the Nash bargaining solution of the bargaining problems in figures 1 and 2 is the allocation w. This solution is λ-egalitarian and λ-utilitarian for the weights

[3]According to the scale-covariance axiom, increasing affine transformations of utilities affect the solution in the same way. In particular, this implies that interpersonal comparisons of utility have no decision-theoretic significance. The reader is referred to Myerson (1991, sec. 8.3) for a more detailed discussion on this issue.

[4]The reader familiar with the literature on two-person bargaining problems will remember that a Nash bargaining solution always exists.

$(\lambda_1, \lambda_2) = (w_2, w_1)$. Notice however, that for any vector $\lambda \neq (w_2, w_1)$ a λ-egalitarian solution does not exist unless we take the comprehensive hull of the points in the line $\vec{ow}$ as in panel B of figure 1.

3. Egalitarian Solutions for General NTU Games

For general n-person cooperative games with non-transferable utility, there exist generalizations for the λ-utilitarian and λ-egalitarian solutions. Given a set of positive weights $\lambda = (\lambda_i)_{i\in N}$ in which utilities are interpersonally compared, a (general) λ-egalitarian solution is defined to be the unique allocation $u_N = (u_N^i)_{i\in N}$ for the grand coalition N such that there exist coalitional allocations $(u_S)_{S\subset N}$, with $u_S = (u_S^i)_{i\in S}$, satisfying:

***(i')* Efficiency:** $u_S \in \partial V(S), \quad \forall S \subseteq N$

***(ii")* λ-Egalitarianism:** $\lambda_i(u_S^i - u_{S\setminus j}^i) = \lambda_j(u_S^j - u_{S\setminus i}^j), \quad \forall i, j \in S,\ \forall S \subseteq N$

where for each coalition S, $V(S)$ denotes the set of feasible utility allocations[5] for S (see Harsanyi (1963), and Kalai and Samet (1985)). Conditions (i') and (ii'') naturally extend the definition of a λ-egalitarian solution from two-person bargaining problems to general NTU games. A λ-egalitarian solution is constructed recursively: for each coalition S, given the allocations $(u_{S\setminus i})_{i\in S}$, the allocation u_S is determined by conditions (i') and (ii''). The resulting allocation u_N is the λ-egalitarian solution. Here again, the free disposal assumption is crucial for the existence of a λ-egalitarian solution for every λ (see Kalai and Samet (1985, sec. 5)). To illustrate the difficulties, let us consider the three-person cooperative game (with complete information) presented in the following example.

Example 1. The set of players is $N = \{1, 2, 3\}$ and the sets of feasible joint actions for each coalition are

S	D_S
$\{i\}$	d_i
$\{1,2\}$	$[d_1, d_2]$, d_{12}
$\{1,3\}$	$[d_1, d_3]$
$\{2,3\}$	$[d_2, d_3]$, d_{23}
N	$[d_1, d_2, d_3]$, $[d_{12}, d_3]$, $[d_1, d_{23}]$

Utility functions are defined as follows:

d	(u_1, u_2, u_3)
$[d_1, d_2, d_3]$	$(0, 0, 0)$
$[d_{12}, d_3]$	$(2, 2, 0)$
$[d_1, d_{23}]$	$(0, 1, 1)$

[5] Sets $(V(S))_{S\subseteq N}$ are assumed to be closed and bounded above.

Let us show that for any vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ such that $\lambda_i = \lambda_j$ for all $i, j \in N$, this game has no egalitarian solution. Here a player can only get 0 by himself. Consider coalition $S = \{1, 2\}$. Since $\lambda_1 = \lambda_2$, the unique allocation meeting conditions (i') and (ii'') is $(u_S^1, u_S^2) = (2, 2)$. Similarly, when looking at coalition $S = \{2, 3\}$, we have that $(u_S^2, u_S^3) = (1, 1)$. Finally, for coalition $S = \{1, 3\}$, $(u_S^1, u_S^3) = (0, 0)$. Then, condition (ii'') for the grand coalition N reduces to

$$\begin{aligned} u_N^1 - u_N^3 &= 1 \\ u_N^2 - u_N^1 &= 1 \end{aligned}$$

It can be easily checked that there is no feasible utility allocation for the grand coalition satisfying the previous two equations. For that, let μ_N be a mechanism for the grand coalition (probability distribution on $\Delta(D_N)$). Notice that $u_N^2 - u_N^1 = 1$ implies that $\mu_N([d_1, d_{23}]) = 1$, thus $\mu_N([d_{12}, d_3]) = 0$. However, equation $u_N^1 - u_N^3 = 1$ implies that $\mu_N([d_{12}, d_3]) = 1$, which is a contradiction. Therefore, this game has no λ-egalitarian solution with respect to such values of λ. As in the case of two-person bargaining problems, this difficulty is due to a lack of comprehensiveness of the feasible set of utility allocations. For instance, we can expand the set of feasible utility allocations for the grand coalition by introducing a new joint decision allowing player 1 to dispose of 1 unit of his utility. Let $\tilde{d}$ such that $\tilde{u}_N = (u_1(\tilde{d}), u_2(\tilde{d}), u_3(\tilde{d})) = (1, 2, 0)$, and consider the new game in which the set of feasible joint decisions for N is $\tilde{D}_N = D_N \cup \tilde{d}$. Notice that decision $\tilde{d}$ is equivalent to implementing the decision $[d_{12}, d_3]$ and then player 1 agrees to dispose of 1 units of his utility. In this new game, unlike the former, there is a λ-egalitarian solution (with $\lambda_i = \lambda_j$ for all $i, j \in N$) given by the allocation $\tilde{u}_N$. □

As for the Nash bargaining solution, one would like to impose an additional utilitarian requirement on the λ-egalitarian solutions in order to deal with the indetermination of the utility weights. Harsanyi (1963) proposed that utility weights were endogenously determined using the following utilitarian criterion: given a vector λ, an allocation u_N for the grand coalition is a (general) *λ-utilitarian* solution if it satisfies

(*iii*") λ-Utilitarianism: $\sum_{i \in N} \lambda_i u_N^i = \max_{v \in V(N)} \sum_{i \in N} \lambda_i v_N^i$

As for the two-person bargaining problem, combining criteria (i') and (iii'') requires that the vector λ be a supporting normal to $\partial V(N)$ at u_N.

A *Harsanyi value* for a general NTU game is obtained by the following procedure:

1. Take a vector $\lambda \in \mathbb{R}_+^N$, $\lambda \neq 0$
2. A λ-egalitarian solution $u_N \in \partial V(N)$ is (recursively) constructed according to (i') and (ii'').
3. If u_N is also λ-utilitarian (i.e., it satisfies (iii'')), then u_N is a Harsanyi NTU value.

Clearly, both the Nash bargaining solution and the Harsanyi NTU value of a two-person bargaining problem coincide. However, unlike the former, the Harsanyi NTU value may fail to exist for general n-person games if the sets $(V(S))_{S \subseteq N}$ are not comprehensive. Two difficulties appear. First, it may happen that, for given utility weights λ, a λ-egalitarian solution does not exist (see example 1). Second, it may also occur that for the values of λ for which a λ-egalitarian solution exists, the solution is not λ-utilitarian. The following example illustrates this latter situation.

Example 2. The set of players is $N = \{1, 2, 3\}$ and the sets of feasible joint actions for each coalition are

S	D_S
$\{i\}$	d_i^0
$\{i, j\}$	$[d_i^0, d_j^0]$, d_{ij}
N	$[d_1^0, d_2^0, d_3^0]$, $[d_{12}, d_3^0]$, $[d_{13}, d_2^0]$, $[d_{23}, d_1^0]$

Utility functions are defined as follows:

d	(u_1, u_2, u_3)
$[d_1^0, d_2^0, d_3^0]$	$(0, 0, 0)$
$[d_{12}, d_3^0]$	$(\frac{1}{3}, \frac{2}{3}, 0)$
$[d_{13}, d_2^0]$	$(\frac{1}{3}, 0, \frac{2}{3})$
$[d_{23}, d_1^0]$	$(0, \frac{1}{2}, \frac{1}{2})$

The sets of feasible utility allocations for all two-person coalitions are as in figure 2.

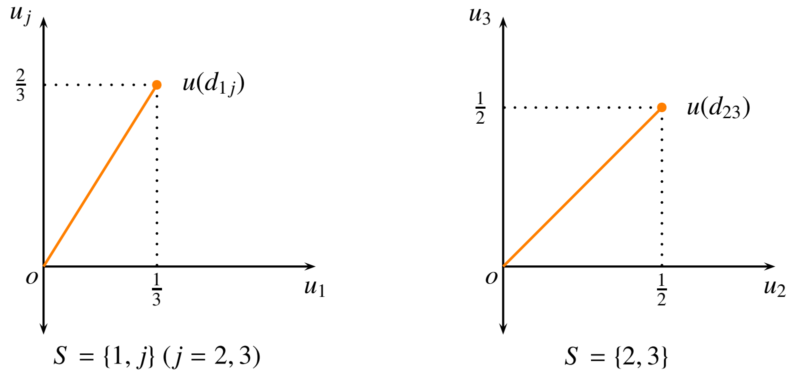


Figure 2: Feasible utility allocations

Let us construct an λ-egalitarian solution for this example. In this game each player can only get zero by himself. Then, for any two-person coalition $S = \{i, j\}$ condition (ii'') reduces to $\lambda_i u_S^i = \lambda_j u_S^j$. Hence, allocations $(u_S)_{S \neq N}$ satisfying (i') and (ii'') can only be constructed if $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (2, 1, 1)$. Given these utility weights, coalitional allocations satisfying (i') and (ii'') are $(u_{\{1,j\}}^1, u_{\{1,j\}}^j) = (\frac{1}{3}, \frac{2}{3})$ (for $j = 2, 3$) and $(u_{\{2,3\}}^2, u_{\{2,3\}}^3) = (\frac{1}{2}, \frac{1}{2})$. Then, condition (ii'') for the grand coalition becomes

$$u_N^2 = u_N^3$$
$$u_N^2 = 2u_N^1 - \tfrac{1}{6}$$

The set of efficient utility allocations for the grand coalition is presented in figure 3.

Condition (i') is equivalent to $u_N^1 + u_N^2 + u_N^2 = 1$ and $u_N \in V(N)$. Then, the unique λ-egalitarian solution is the allocation[6] $(u_N^1, u_N^2, u_N^3) = (\frac{8}{30}, \frac{11}{30}, \frac{11}{30})$. On the other hand, given the vector λ, we have that

$$\sum_{i \in N} \lambda_i u_N^i = \tfrac{19}{15} < \tfrac{4}{3} = \max_{v \in V(N)} \sum_{i \in N} \lambda_i v_N^i$$

[6]This allocation is implemented by a mechanism $\mu_N([d_{12}, d_3^0]) = \mu_N([d_{13}, d_2^0]) = 1 - \mu_N([d_{23}, d_1^0]) = \frac{2}{5}$

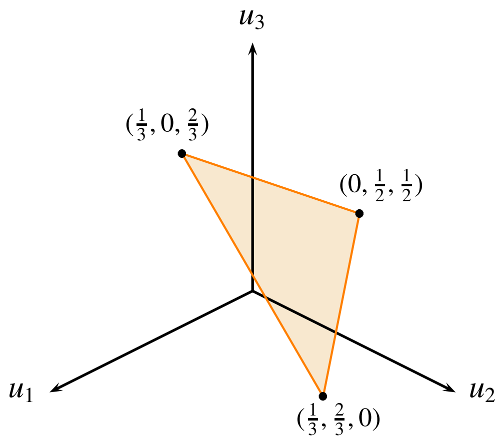


Figure 3: $\partial V(N)$

Hence, u_N is not λ-utilitarian. Since a λ-egalitarian solution does not exist for any other vector λ, we conclude that this game has no Harsanyi NTU value. The difficulty in this example can alternatively be understood as follows: condition (iii'') requires that the vector of utility weights λ be a supporting normal to $\partial V(N)$. Then, the natural candidate is $\lambda = (1, 1, 1)$, since it supports $\partial V(N)$ at any efficient allocation. However, a λ-egalitarian solution cannot be constructed for these utility weights.

Let us proceed as in example 1. We expand the set of feasible utility allocations for coalitions $\{1, j\}$ (with $j = 2, 3$) by introducing a new joint decision allowing player j to dispose of $2/3$ units of his utility. Let $\tilde{d}_j$ (with $j = 2, 3$) be such that $(u_1(\tilde{d}_j), d_j(\tilde{d}_j)) = (\frac{1}{3}, 0)$, and consider the new game in which $\tilde{D}_{\{1,j\}} = D_{\{1,j\}} \cup \tilde{d}_j$ for $j = 2, 3$. The superadditivity assumption implies that the set of joint decisions for the grand coalition is now

$$\tilde{D}_N = D_N \cup \{[\tilde{d}_2, d_3^0], [\tilde{d}_3, d_2^0]\}$$

By orthogonality of coalitions, utility functions are extended as follows:

d	(u_1, u_2, u_3)
$[d_1^0, d_2^0, d_3^0]$	$(0, 0, 0)$
$[d_{12}, d_3^0]$	$(\frac{1}{3}, \frac{2}{3}, 0)$
$[d_{13}, d_2^0]$	$(\frac{1}{3}, 0, \frac{2}{3})$
$[d_{23}, d_1^0]$	$(0, \frac{1}{2}, \frac{1}{2})$
$[\tilde{d}_2, d_3^0]$	$(\frac{1}{3}, 0, 0)$
$[\tilde{d}_3, d_2^0]$	$(\frac{1}{3}, 0, 0)$

Observe that decision $[\tilde{d}_j, d^0_{N\setminus\{1,j\}}]$ (for $j = 2, 3$) is equivalent to implement the decision $[d_{1j}, d^0_{N\setminus\{1,j\}}]$ and then player j agrees to dispose $\frac{2}{3}$ units of his utility. The set of (weakly) efficient utility allocations of the expanded game is represented in figure 4. Notice that adding new feasible decisions to $(D_S)_{S\subseteq N}$ may change not only the Pareto frontier of $V(S)$, but also that of $V(N)$. This is so because of our assumption of superadditivity. Fortunately, when information is complete, adding (strongly) Pareto dominated decisions (i.e., allowing for partial free disposal, as in our example) can be done while keeping unaffected the original Pareto frontier of

$V(N)$. Of course this may add linear leveled segments[7] to $\partial V(N)$. For instance, in our example, introducing decisions $(\tilde{d}_j)_{j=2,3}$ enlarges the weak Pareto frontier of $V(N)$ by including strong Pareto dominated outcomes. The rest of the original Pareto frontier remains unchanged. Thus, utility weights $\lambda = (1, 1, 1)$ continue to be the natural scale factors for a Harsanyi NTU value.

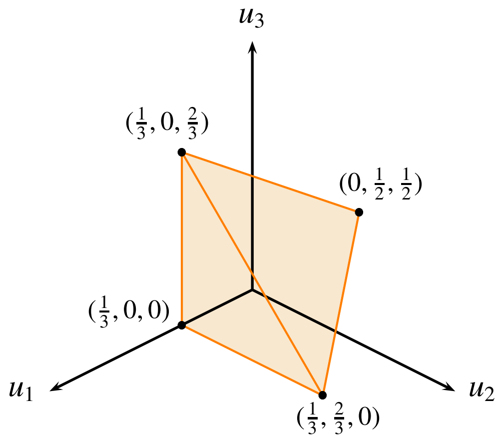


Figure 4: Efficient allocations in the expanded game

Let us show that in the expanded game there exists a Harsanyi NTU value. Fix the utility weights $\lambda = (1, 1, 1)$. Now we construct a λ-egalitarian solution for these weights. Since nothing has changed for coalition $\{2, 3\}$ in the expanded game, the unique allocations satisfying (i') and (ii'') for this coalition is $(u^2_{\{2,3\}}, u^3_{\{2,3\}}) = (\frac{1}{2}, \frac{1}{2})$, as in the original game. Consider now a coalition $\{1, j\}$ with $j = 2, 3$, for which the set of feasible allocations is represented in figure 5.

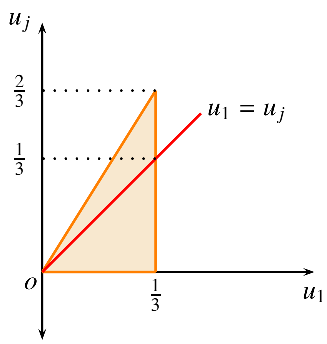


Figure 5: Feasible allocations for $S = \{1, j\}$

An allocation $u_{\{1,j\}}$ satisfies (i') and (ii'') if and only if $(u^1_{\{1,j\}}, u^j_{\{1,j\}}) = (\frac{1}{3}, \frac{1}{3})$. This allocation is equivalent to implement the decision d_{1j} and then player j agrees to dispose $1/3$ units of his utility. Thus, condition (ii'') for the grand coalition becomes

$$u^2_N = u^3_N$$

$$u^2_N = u^1_N + \frac{1}{6}$$

[7] A linear segment $L \subset \partial V(S)$ is leveled if there exist $x, y \in L$ such that $x \leq y$ and $x \neq y$.

Straightforward algebra shows that the previous two equations cannot be satisfied by an allocation lying on the part of the weak Pareto frontier consisting of strong Pareto dominated outcomes. Then, condition (i') requires that $u_N^1 + u_N^2 + u_N^2 = 1$ and $u_N \in V(N)$. The unique λ-egalitarian solution is the allocation[8] $(u_N^1, u_N^2, u_N^3) = (\frac{4}{18}, \frac{7}{18}, \frac{7}{18})$. On the other hand, given the vector λ, we have that

$$\sum_{i \in N} \lambda_i u_N^i = 1 = \max_{v \in V(N)} \sum_{i \in N} \lambda_i v_N^i$$

Hence, condition (iii'') is satisfied and u_N is a Harsanyi NTU value. □

Examples 1 and 2 exhibit why the free disposal assumption is particularly important for guaranteeing the existence of the egalitarian solutions and *a fortiori* of the Harsanyi NTU value. Example 2 also shows why free disposal is usually taken as an innocuous assumption: adding strongly Pareto dominated decisions can be done while keeping unaffected the original Pareto frontier of $V(N)$. As a consequence of this, when information is complete, cooperative games can reasonably be described as a collection $(V(S))_{S \subseteq N}$ of fully comprehensive feasible utility sets. Full comprehensiveness guarantees that a λ-egalitarian solution always exists for any vector of utility weights λ. This characteristic (coalitional) representation suppresses any explicit mention of the decisions generating the utilities. Although implicitly, we assume that a utility allocation u_S is feasible for coalition S if the players in S together have a joint feasible strategy that enables them to allocate u_S (this may include the implementation of a correlated strategy or a joint decision, burning utility or even transferring utility).

It is worth emphasizing that free disposal, apart from its significance as a strategic option, is a technical assumption that has been widely integrated in the study of NTU games. Other hypothesis of this kind include the *non-levelness* assumption[9] which guarantees that only strictly positive weights can emerge in a value allocation. The analysis of general NTU games is difficult due to the arbitrariness of the outcomes that the players can achieve. Free disposal serves as a very weak form of "transferability" that makes the study of these games more tractable.

4. Free Disposal in the Presence of Incentive Constraints

When information is incomplete, the free disposal assumption must be carefully considered. Take for instance the case of a pure exchange economy with differential information in which consumers enter in coalitional contracts at the interim stage (i.e., once every agent has private information). We can assume free disposal by allowing mechanisms to throw away goods (before the reallocation of goods). When information is complete, this disposability assumption implies a comprehensive characteristic representation of the economy. However, when information is differential, even if goods are fully disposable, incentive constraints may lead to feasible (interim) utility sets that are not comprehensive. This will be illustrated in example 3 (see also de Clippel (2012)). The way incentive constraints restrict what is feasible makes free disposal and comprehensiveness two different notions. Difficulties of this kind suggest that a cooperative game cannot be simply described as a collection of comprehensive utility sets, one for every

[8] This allocation is implemented by a mechanism $\mu_N([d_{12}, d_3^0]) = \mu_N([d_{13}, d_2^0]) = \mu_N([d_{23}, d_1^0]) = \frac{1}{3}$

[9] A game is *non-leveled* if for all $S \subseteq N$, $V(S)$ is non-leveled, that is, for all $u_S, v_S \in \partial V(S)$, $u_S \geq v_S$ implies that $u_S = v_S$.

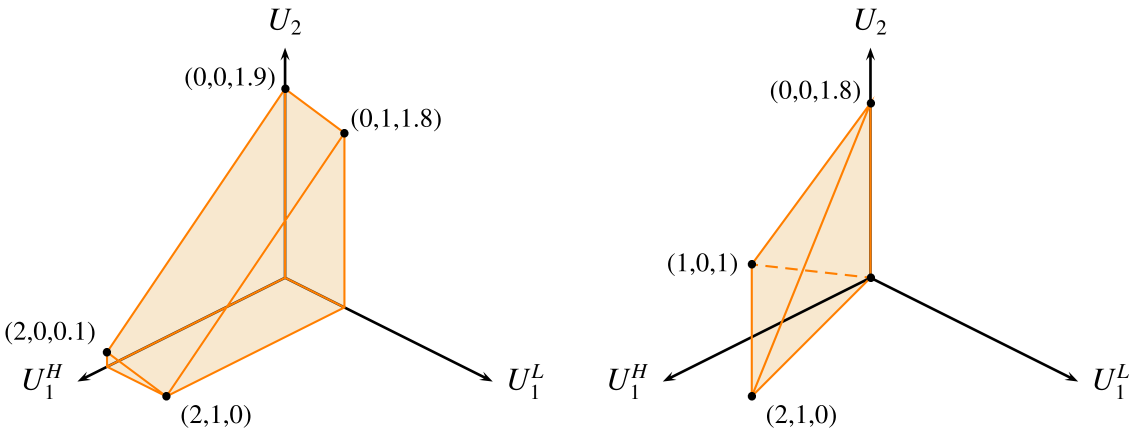
coalition on each possible information state. In particular, we require to keep track of the incentives constraints and its effects on feasibility. Indeed, the fact that a utility vector is feasible at some type profile does not allow us to determine what would be the utility an individual would get by reporting a different type. Thus, strategies or decisions should be explicitly included in the structure of the game.

Example 3. Let us consider the following two-person bargaining problem with incomplete information. Player 1 may be one of two possible types, H or L. Types are chosen according to the probability distribution $p(H) = 1 - p(L) = 0.9$. Player 1's type is private information. Suppose that there are three possible decisions, $D = \{(d_i)_{i=0,1,2}\}$, and the two individuals' payoffs (u_1, u_2) depend on the decision and types as follows:

(u_1, u_2)	L	H
d_0	$(0, 0)$	$(0, 0)$
d_1	$(-1, 2)$	$(0, 2)$
d_2	$(1, 0)$	$(2, 0)$

In case of disagreement, both players have (type-independent) reservation utilities normalized to zero. Let us start assuming that when the final agreement is implemented player 2 can costlessly verify the true type of player 1. This implies that 1's type is public information at the implementation stage and incentive constraints are not required. Then, a mechanism is feasible if it is (interim) individually rational. The set of feasible interim utility allocations under the verifiability assumption is depicted in panel A of figure 6. Given a point d in some Euclidean space $\mathbb{R}^m$, and a set $V \subseteq \mathbb{R}^m$, we say that the set V is *d-comprehensive* if $d \leq v \leq u$ and $u \in V$ implies that $v \in V$. According to this definition, the set of feasible allocations in panel A is 0-comprehensive.

Let us comme back to our original set-up in which information is not verifiable and feasibility is constrained by individual rationality and incentive compatibility. The set of feasible utility allocations in this case is illustrated in panel B. When comparing panels A and B, it is clear that incentive compatibility restricts feasibility. In particular, it reduces interim efficiency and wipes out 0-comprehensiveness. □



Panel A: Feasible allocations under verifiable types

Panel B: Feasible allocations under incentive constraints

Figure 6

Example 4 puts in evidence one of the issues that must be addressed for an appropriate generalization of the principle of equal gains to bargaining situations with incomplete information. For instance, de Clippel (2012) introduces an egalitarian solution for social choice problems in which an uninformed mediator acts as a mechanism designer. His solution straightforwardly generalizes conditions (*i*) and (*ii*) to social choice problems with incomplete information. Assume that reservation utilities are type-independent and normalized to zero. A mechanism μ (for the grand coalition) satisfies the *interim egalitarian criterion* if

(*i"*) Efficiency: μ is interim incentive efficient.

(*ii"'*) Egalitarianism: $U_i(\mu \mid t_i) = U_j(\mu \mid t_j), \quad \forall i, j \in N, \forall t \in T$

That is, an interim utility allocation is an egalitarian solution in de Clippel's terms if it is incentive efficient and all individuals experience the same interim expected utility whatever the true information state might be. It can be easily seen from panel B of figure 6 that in the bilateral bargaining problem of example 4 there is no feasible mechanism passing the interim egalitarian criterion[10]. Here again, the difficulty comes from the fact that the feasible utility set is not 0-comprehensive, thus making the intersection between the line $U_1^H = U_1^L = U_2$ and the weak Pareto frontier empty. Then, one is tempted to proceed as in examples 1 and 2, and expand the feasible utility set by introducing a new decision allowing player 1 to dispose of his utility (in state H). Let $\tilde{d}$ be such that $u_1(\tilde{d}, H) = 0$, $u_1(\tilde{d}, L) = 1$ and $u_2(\tilde{d}, H) = u_2(\tilde{d}, L) = 0$. Decision $\tilde{d}$ is equivalent to implement decision d_2 in both states but then player 1 agrees to dispose of 2 units of his utility in state H. Now consider the expanded problem with decisions set $\tilde{D} = D \cup \tilde{d}$. The set of feasible utility allocations is depicted in figure 7.

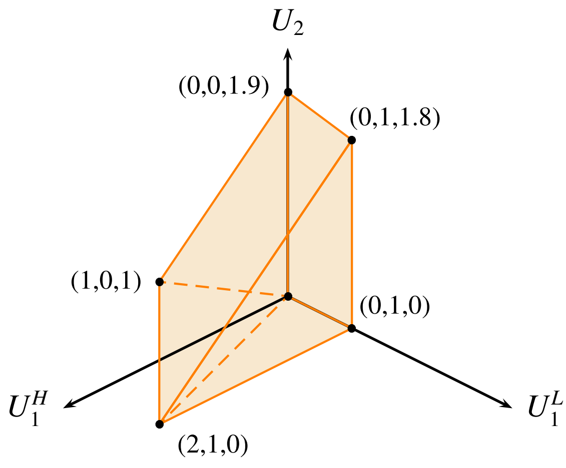


Figure 7

As required, in the expanded problem the utility allocation $(U_1^H, U_1^L, U_2) = (\frac{19}{20}, \frac{19}{20}, \frac{19}{20})$ meets the egalitarian criterion. However, unlike the complete information case, here the game has

[10]In de Clippel's (2012) words, the game in example 4 is not *simple*. Because not every mechanism design problem is simple, only a partial axiomatic characterization of the interim egalitarian criterion can be provided in terms of a weak monotonicity property.

substantially changed after $\tilde{d}$ was introduced. Permitting free disposal for type H of player 1 facilitated the fulfillment of incentive constraints, thus allowing both players to achieve higher interim utilities with respect to the original problem[11]. In particular, any incentive-efficient allocation in the expanded game is ex-post efficient, which is not the case in the original game (see figures 6 and 7). This implies, for instance, that the normal vector to the Pareto frontier of the original bargaining problem is no longer the same as in the expanded problem.

Example 3 has shown that free disposal may have an important effect on the incentives structure of the game. The following example will enhance our understanding of this phenomenon.

Example 4. Consider a two-person bargaining problem with incomplete information in which player 1 may be of two possible types, H or L. Types are chosen according to the probability distribution $p(H) = 1 - p(L) = p > 0$. Player 1's type is private information. The decision set is $D = \{d_H, d_L\}$. Individuals' payoffs depend on decisions and types as follows:

(u_1, u_2)	L	H
d_H	$(u_1 + m, 0)$	$(u_1 + M, u_2)$
d_L	(u_1, u_2)	$(u_1, 0)$

where $0 < m < M$ and $u_1, u_2 > 0$. Player 1 prefers the action d_H in both states while player 2 prefers the decision that matches the true state. Hence, a mechanism is incentive compatible if and only if it is non-revealing, that is, $\mu_H = \mu_L$ where μ_t denotes the probability of decision d_H in state t.

Now we introduce a new decision allowing player 1 to dispose of his utility in both states. Let $\tilde{d}$ be such that $u_1(\tilde{d}, H) = u_1(\tilde{d}, L) = 0$ and $u_2(\tilde{d}, H) = u_2(\tilde{d}, L) = u_2$. Decision $\tilde{d}$ is equivalent to implement d_H (resp. d_L) in state H (resp. L) and then player 1 agrees to dispose $u_1 + M$ (resp. u_1) units of his utility. Given $r \in (0, 1)$, consider the mechanism μ^r defined by

$$\mu^r_H = \begin{cases} d_H, & \text{with probability } 1 - r; \\ \tilde{d}, & \text{with probability } r. \end{cases} \quad , \qquad \mu^r_L = d_L$$

This mechanism is equivalent to implement d_H (resp. d_L) in state H (resp. L) and then player 1 agrees to dispose $r(u_1 + M)$ units of his utility in state H. Furthermore, μ^r is incentive compatible in the expanded game if and only if

$$\frac{u_1}{u_1 + M} \leq 1 - r \leq \frac{u_1}{u_1 + m}$$

Since $0 < m < M$, then r can be chosen such that μ^r is incentive compatible. Moreover, if m is close enough to zero, r can be arbitrarily small, so that the mechanism μ^r is almost-fully-revealing and player 1 decreases his utility only by an arbitrary small amount. Then, free disposal enlarges the set of incentive feasible mechanisms: recall that non-revealing mechanisms are always incentive compatible. □

[11] This example shares some features in common with an exchange economy with differential information proposed by Forges, Mertens and Vohra (2002, sec. 2.5).

5. Free Disposal of Virtual Utility

The method of fictitious transfers of weighted utility has proven to be a very useful approach to generalize (almost) any solution concept for transferable utility (TU) games to general NTU games. In particular, the Harsanyi NTU value extends simultaneously both the Nash bargaining solution and Shapley's (1953) TU value to general NTU games. Unfortunately, the method of fictitious transfers does not immediately extend to games with incomplete information. The difficulty is that allowing players to transfer utilities at the interim stage may affect the incentives structure of the game in the same way free disposal does (see examples 3 and 4). Let us illustrate this fact by means of the following example.

Example 5. Consider again the game analyzed in example 3. Because of the presence of incentive constraints, both players are constrained in their abilities to share the total gains from cooperation, so that they face a lack of transferability. In fact, the mechanism that gives the entire surplus to player 2 in both states is not incentive compatible, i.e., the interim allocation $(U_1^H, U_1^L, U_2) = (0, 0, 1.9)$ is not incentive compatible (see panel B of figure 6). Assume now that player 1 is given the additional option to transfer the proceeds of cooperation to player 2. For that, let $\tilde{d}$ be such that $u_1(\tilde{d}, H) = u_1(\tilde{d}, L) = 0$, $u_2(\tilde{d}, H) = 2$ and $u_2(\tilde{d}, L) = 1$, and consider the bargaining problem where $\tilde{D} = D \cup \tilde{d}$. Decision $\tilde{d}$ is equivalent to implement d_2 and then player 1 transfer his total utility to player 2 in both states. Then, the set of feasible (i.e., interim individually rational and incentive compatible) utility allocations is as in figure 8.

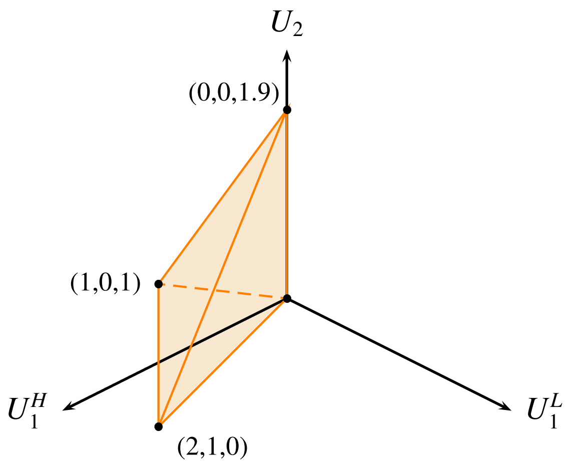


Figure 8: Feasible allocations with transferable utility

As for the free disposal, allowing bounded utility transfers made easy the fulfillment of incentive constraints, thus permitting both players to attain higher payoffs with respect to the original bargaining situation. Indeed, unlike the original game, when utility is transferable any incentive-efficient allocation is ex-post efficient (see figures 6 and 8). □

Aware of this difficulty, Myerson (1984) developed the virtual utility approach for generalizing the method of fictitious transfers to cooperative games with incomplete information (see section 2.2 of the paper). The advantage of his approach is that, for any interim utility allocation on the

Pareto frontier of the grand coalition, it associates a set of additional decisions that extends the original game in such a way that players can make bounded virtual utility transfers conditionally on every state, while leaving the original utility allocation efficient in the expanded problem, exactly as in the case of complete information (see also lemma 4 in de Clippel (2012)). Then, we may consider the fictitious game in which each player's payoff is represented in the virtual utility scales, players' types are verifiable, so that there are no incentive constraints, and virtual payoffs are transferable among the players (conditionally on every state).

For the virtual game, we would like to identify mechanisms that are efficient and equitable in any well defined sense. In our theory, virtual efficiency requires the total virtual utility worth of a coalition to be maximal (virtual ex-post efficiency). Because virtual payoffs are transferable, this condition is a counterpart of (i'). On the other hand, equity requires, according to our egalitarian criterion (see proposition 1 in the paper), expected average marginal contributions to be balanced. This last condition is an analogous version of (ii''). Thus, considerations of equity and efficiency in the virtual game lead to our optimal threat criterion (see definition 3 in the paper). Because types are verifiable in the virtual game, optimal egalitarian threats are not required to be incentive compatible.

Since the definition of an optimal egalitarian threat generalizes conditions (i') and (ii''), it is not surprising that all difficulties identified in section 2 of this note are inherited by our optimal threat criterion. In particular, for given virtual utility scales, problem (3.7) (in the paper) may not be feasible. It may also happen that the virtual utility scales for which the problem (3.7) has a solution, do not support any incentive efficient allocation which is equitable in the virtual game. Then, for a general existence result of our egalitarian value to be proven, feasibility of problem (3.7) must be guaranteed for given virtual utility scales, exactly as in the case of complete information. A sufficient condition for this is that the feasible virtual utility sets be comprehensive on each state, i.e., that virtual utility be disposable conditionally on every state (see section 4.2). Unfortunately, virtual free disposal cannot be accommodated within our model by introducing decisions in each D_S allowing players to discard utility. The reason for this was discussed in section 3 of this complementary note: adding new decisions may change not only the incentives structure, but also the efficient frontier of the game.

In order to examine this issue in depth, recall that virtual scales are defined by a vector of utility weights λ together with a vector of Lagrange multipliers α for the incentives constraints of the primal problem. As in the case of complete information, λ is a supporting normal to the interim efficient frontier. Fix (λ, α) and assume that you can expand the set of feasible virtual utility allocations for a coalition S by introducing a new decision $\tilde{d}_S$ allowing some player $j \in S$ to dispose some amount of his virtual utility in some state $\bar{t} \in T$. Let $\tilde{v} = (\tilde{v}_i(t))_{i \in N, t \in T}$ be the vector of virtual utilities associated to $\tilde{d}_S$. Now, we need to translate virtual payoffs $\tilde{v}$ into real utilities. Consider the linear function $\upsilon^{\lambda,\alpha} : \mathbb{R}^{N \times T} \to \mathbb{R}^{N \times T}$ that maps any profile of (ex-post) utilities to its associated virtual utilities:

$$(\upsilon^{\lambda,\alpha}(u))_i(t) = \frac{1}{p(t_i)} \left[\left(\lambda_i(t_i) + \sum_{\tau_i \in T_i} \alpha_i(\tau_i \mid t_i) \right) u_i(t) - \sum_{\tau_i \in T_i} \alpha_i(t_i \mid \tau_i) u_i(\tau_i, t_{-i}) \right],$$

for each $u \in \mathbb{R}^{N \times T}$, each $t \in T$ and $i \in N$. Then, in order to determine the real utilities associated to $\tilde{v}$ we need the map $\upsilon^{\lambda,\alpha}(\cdot)$ to be invertible, which is only known to be possible whenever λ

is strictly positive[12]. However, due to the incentive constraints, we cannot prevent λ to vanish in some (but not all) of its components. Assume for simplicity that $\lambda > 0$ so that we can associate real utilities $\tilde{u} = (\tilde{u}_i(t))_{i \in N, t \in T}$ to decision $\tilde{d}_S$. Now consider the expanded game in which $\tilde{D}_S = D_S \cup \tilde{d}_S$. The superadditivity assumption implies that the set of joint decisions for the grand coalition in the expanded game is $\tilde{D}_N = D_N \cup D_{N \setminus S} \times \tilde{d}_S$. Orthogonal coalitions together with $\tilde{u}$ expand the definition of $u_i(\cdot)$ to the enlarged game. As it was illustrated in examples 3 to 5, adding decisions to D_N may alter the incentives inside N so that the vector of dual variables α may change after introducing $\tilde{d}_S$. Also, the efficient frontier can be modified so that the utility weights λ are not longer the same. Besides the fact that the expanded game may have substantially changed with respect to the original game, it may also occur that, for the new scales (λ, α), the virtual utilities $(v^{\lambda,\alpha}(\tilde{u}))_i(\bar{t})$ associated to $\tilde{d}_S$ in the expanded game do not correspond any more to a disposal activity for player $j \in S$ in state $\bar{t}$.

We solve the preceding dilemma by considering a class of mechanisms that allow players to agree to unilaterally decrease their individual virtual utility levels by any arbitrary amount, conditional on every state. Free disposal of virtual utility is a weak linear activity that can be embedded into virtual utility transfers. Thus, we think that there is very little loss of generality in assuming that virtual payoffs are freely disposable in the fictitious game. For the grand coalition N, any incentive efficient mechanism is ex-post efficient in terms of the virtual utility scales. Then, permitting or not free disposal of virtual payoffs for N can be done without loss of generality. Furthermore, our solution concept is based on the idea that payoffs are granted by the grand coalition, and therefore coalitional agreements for all coalitions $S \neq N$ are only important as an expression of the power structure of the game. Hence, the possibility to "burn" virtual utilities is mainly used for providing the players with additional threats during the bargaining process within the grand coalition.

6. References

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[12]This is a consequence of lemma 3 in Clippel (2012).