Cooperation in the Presence of an Advantaged Outsider*

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Abstract: This paper analyzes how the stability of the tacit cooperation within a fringe of several identical firms is affected by the presence of a more efficient firm which does not take part in their cooperative agreement. The model assumes that the firms of the fringe adopt 'stick and carrot' strategies *à la* Abreu (1986, 1988) to support cooperation, while the outside firm plays its one-period best response function to these strategies, regardless of the history of play. Assuming a linear demand function and constant marginal costs, we then obtain conditions for the cooperation within the fringe to be sustainable and focus on the *most cooperative symmetric punishment* (MCSP) that sustains cooperation. We show that the MCSP is harsher when the number of firms involved in the agreement is relatively large or when their relative cost disadvantage is relatively small. However, both a larger number of firms and a larger cost disadvantage make it more difficult to sustain the cooperation.

Keywords: Repeated Game; Tacit Collusion; Optimal Punishments; Cost Asymmetry, Outsider

JEL Classification: C73; D43; L13

1 Introduction

Since the work of Friedman (1971), the vast majority of the literature on collusion in oligopolistic markets has posited identical firms and that all of them participate in the cartel agreement. A few theoretical works have investigated the relationship between the firms' efficiencies and the collusive behaviour, and have shown that asymmetries in the cost functions hinders, in general, collusion, with the results depending crucially upon the profit sharing rule (e.g., Rothschild 1999, Vasconcelos 2005) and on whether side payments between firms are allowed (see Miklós-Thal 2011). Still, the theory of tacit collusion has maintained the assumption that all the firms participate in the collusive agreement, an assumption which is questionable, especially in a context where the oligopoly is made up of heterogeneous firms. Indeed, a firm with a relatively high competitive advantage may find it more profitable to opt out of an agreement which includes less competitive firms. In fact, this is what is suggested by Vasconcelos's analysis, where the most efficient firm is shown to be the main obstacle to the enforceability of the collusion.[1] In the present article, we will investigate how the stability of tacit cooperation between several identical firms is affected by the presence of a more efficient firm which does not take part in the cooperative agreement.

We employ an oligopoly model in which n symmetric firms, called the *fringe*, plus one cost-advantaged firm, called the *outsider*, play a Cournot game over an infinite horizon. The firms of the fringe adopt two-phase 'stick-and-carrot' punishment schemes *à la* Abreu (1986, 1988) to support their joint-profit maximizing behaviour. I.e., following any deviation, the firms of the fringe conform to a 'stick', or, punishment, phase in which they produce a very high quantity (the punishment output level) for one period (thus resulting in very low market price and profits during that period) to generate a 'carrot' in the possibility of a subsequent return to cooperative behaviour. Deviations from the punishment simply cause it to begin again. As for the outsider, it is assumed to play in every period its best response to the other firms' strategies regardless of the history. We then focus on a subgame perfect equilibrium which supports perfect cooperation within the fringe and non-cooperation between the fringe and the outsider. In this equilibrium, the cooperation among the firms of the fringe makes them act as if they were a single firm, and hence the outcome corresponds to that of a Cournot duopoly game.

As one would expect, there exists an infinity of punishment output levels which support perfect cooperation within the fringe as a subgame perfect outcome. To tackle this problem, we thus

[1] Bae (1987) and Harrington (1991), by investigating the determination of the price of output quotas in heterogeneous cartels, reached the same type of conclusion.

propose a selection criterion that requires equalizing the gains from defection in the cooperative and punishment phases. We then show that this selection criterion singles out what we call the *most cooperative symmetric punishment* (MCSP) in that it most relaxes the incentive compatibility constraints, both in the cooperative and the punishment phases. In other words, the MCSP yields the largest possible range of discount factors for which perfect cooperation within the fringe can be sustained.

The MCSP crucially depends on both the size of the fringe and on the cost asymmetry between the firms of the fringe and the outsider. More specifically, we show that perfect cooperation within the fringe requires harsher punishment output levels when the fringe is relatively large or when the cost asymmetry is relatively small. Intuitively, in those cases, each firm of the fringe has more to gain from cheating on the production cut agreement in the cooperative phase. Harsher punishments are then required to maintain the cooperation.

Next, we characterize the minimum value for the discount factor above which perfect cooperation is sustainable as a subgame perfect outcome. In the context of this paper, this minimum value is interpreted as a measure of the ease of cooperation between the fringe firms. We then show that an increase in the size of the fringe causes perfect cooperation to be less feasible. This result is consistent with the argument made by scholars of collective action, that overcoming the free-rider problem becomes more difficult as the size of the collectivity increases (e.g., Hardin, 1982; Olson, 1982; Sandler 1992). As regards the cost asymmetry between the fringe and the outsider, it only affects the ease of cooperation if the size of the fringe is sufficiently small. If that is the case, a greater cost asymmetry makes cooperation within the fringe increasingly difficult to sustain. The intuition is that the larger the cost disadvantage of the firms of the fringe relative to the outsider, the larger is the one-period loss from the first phase of the punishment strategy. Hence, more weight has to be attached to the future stream of payoffs for the firms to comply with the punishment.

The long history of cartels has produced varied evidence that the strategic interaction with a more efficient outsider makes the stability of the cartel more precarious. In particular, the issue of the strength of cooperation between small producers faced with an advantaged competitor proves especially important for a number of commodity markets, such as cocoa, coffee, natural rubber, and cotton (see, e.g., Gilbert, 1996). For instance, it is widely accepted that the reason for the failure of the International Agreements on Cocoa (ICCAs) and Coffee (ICAs) in the 80s and 90s was the lack of support from the dominant producer in the marketplace, i.e., Ivory Coast for cocoa and Brazil for coffee (see Gayi, 2004). Similarly, observations on the market for lysine have shown that collusion between three Asian-based firms collapsed in the early 90s, precipitating a severe price war, due to the emergence of a large-scale entrant and more efficient competitor, namely the US-based firm Archer Daniels Midland (ADM) (see de Roos, 2004, 2006).

The rest of this paper is organized as follows. In Section 2, we describe the basic model. In Section 3, we explore the infinitely repeated game and provide necessary and sufficient conditions for perfect cooperation within the fringe to be sustained as a subgame perfect outcome through the use of 'stick-and-carrot' strategies *à la* Abreu (1986, 1988). Section 4 derives the MCSP, depending on the size of the fringe and on the cost asymmetry between the firms of the fringe and the outsider. This section not only provides the lowest discount factor for the sustainability of the perfect cooperation, but also investigates the impact of the size of the fringe and of the cost asymmetry on the sustainability of cooperation. Finally, Section 5 offers some concluding comments.

2 The model

2.1 The stage game

We start by specifying the details of the stage game G. There are $n+1$ firms, indexed $i = 0, 1, 2, ..., n$, which produce a homogenous product at constant marginal cost. Firm 0 has a marginal cost normalized to 0, whereas all other firms of the fringe, $i = 1, 2, ...n$, incur a marginal cost $c \geq 0$. Let $q_i \in \mathbb{R}_+$ be the output of firm i, for $i = 0, 1, ..., n$. Then $Q = \sum_{i=0}^{n} q_i$ is the aggregate output. The inverse demand function is given by $p(Q) = \max\{0, 1 - Q\}$, with $c < 1$. Thus, the payoff function of firm 0 is $\pi_0(q_0, q_1, ..., q_n) = p(Q)\, q_0$, while that of firm i, for $i = 1, ..., n$, is $\pi_i(q_0, q_1, ..., q_n) = [p(Q) - c]\, q_i$.

Let $r_i(q_{-i})$, for $i = 0, ..., n$, be the firm i's single-period best reply to the vector of output levels $q_{-i} = (q_0, ..., q_{i-1}, q_{i+1} ..., q_n)$, so that $r_i(q_{-i})$ satisfies $\pi_i(r_i(q_{-i}), q_{-i}) \geq \pi_i(q_i, q_{-i})$ for all $q_{-i} \in \mathbb{R}_+^n$. Thus, we have $r_0(q_{-0}) = \max\left\{0, \left(1 - \sum_{i=1}^{n} q_i\right)/2\right\}$, and $r_i(q_{-i}) = \max\left\{0, \left(1 - q_0 - \sum_{j \neq i} q_j - c\right)/2\right\}$ for $i = 1, ..., n$.

We focus on *symmetric* equilibria in the sense that all firms of the fringe produce the same level of output. For the sake of simplicity, we will write $r_0(q_{-0}) = r_0(x)$ and $r_i(q_{-i}) = r_i(q_0, x)$ if $q_i = x$ for $i = 1, ..., n$. The non-cooperative equilibrium, both within the fringe and between the fringe and the outsider, is characterized by a pair of output levels $\left(q_0^N, q^N\right)$ such that $r_0\left(q^N\right) = q_0^N$ and $r_i\left(q_0^N, q^N\right) = q^N$ for $i = 1, ..., n$. We obtain $\left(q_0^N, q^N\right) = \left(\frac{1+nc}{n+2}, \frac{1-2c}{n+2}\right)$, and so the market clears at price $p^N = (1 + nc)/(n + 2)$.

To guarantee that each firm in the stage game has a positive market share, we make the following assumption.

Assumption 1: $c < 1/2$.

Write π_0^N for $\pi_0\left(q_0^N, q^N, ..., q^N\right)$, the payoff of the outsider, and write π^N for $\pi_i\left(q_0^N, q^N, ..., q^N\right)$, the identical payoff for each firm of the fringe, when there is non-cooperation within the fringe. We have

$$\pi^N = \left[\frac{1-2c}{n+2}\right]^2, \tag{1}$$

and

$$\pi_0^N = \left[\frac{1+nc}{n+2}\right]^2. \tag{2}$$

with $\pi_0^N > \pi^N$.

2.2 Cooperation within the fringe

Assume now that the firms of the fringe collude and jointly choose a common level of output q so as to maximize the sum of their profits. Yet, they continue playing non-cooperatively with the outsider, so that the outcome resembles that of a Cournot duopoly game between the fringe and the outsider. The cooperative output from the viewpoint of the fringe is given by the maximization of $\sum_{i=1}^{n} \pi_i(q_0, q, ..., q) = (p(Q) - c)\, nq$ with respect to q, where $Q = nq + q_0$. The best reply of any firm of the fringe to the output level q_0 produced by the outsider is thus given by the function $R_i(q_0) = \max\{0, (1 - q_0 - c)/2n\}$. The non-cooperative equilibrium between the fringe, which acts as if it were a single firm, and the outsider is thus given by a pair of output levels $\left(q_0^C, q^C\right)$ such that $r_0\left(q^C\right) = q_0^C$ and $R_i\left(q_0^C\right) = q^C$, for $i = 1, ..., n$. Hence, when the firms of the fringe fully cooperate with each other, we have $\left(q_0^C, q^C\right) = \left(\frac{1+c}{3}, \frac{1-2c}{3n}\right)$.

Write π_0^C for $\pi_0\left(q_0^C, q^C, ..., q^C\right)$, the payoff of the outsider, and write π^C for $\pi_i\left(q_0^C, q^C, ..., q^C\right)$, for $i = 1, ..., n$, the identical payoff for each firm of the fringe when there is cooperation within the fringe. The market price equilibrium is $p^C = (a + c)/3$, and then

$$\pi^C = \frac{1}{n}\left(\frac{1-2c}{3}\right)^2, \tag{3}$$

and

$$\pi_0^C = \left(\frac{1+c}{3}\right)^2. \tag{4}$$

We can verify that $\pi^C - \pi^N = n^2 - 5n + 4 \geq 0$, which is positive for $n \geq 4$. Hence, to make the problem interesting, we shall make the following assumption throughout the remainder of our analysis.

Assumption 2. $n \geq 4$.

Moreover, we have that $\pi_0^C - \pi_0^N = (1-2c)(n-1) \geq 0$. In other words, the outsider always benefits from cooperation within the fringe. This is because cooperation within the fringe reduces output levels and increases the market price, which in turn increases the profit of the outsider.

3 The infinitely repeated game

3.1 Preliminaries

The $n+1$ firms play an infinitely repeated game with discounting. Let $G^{\infty}(\delta)$ be the repeated game obtained by repeating G infinitely often, and where $\delta \in (0,1)$ is the discount parameter per period for each player. We assume that the output produced by each firm in each period is perfectly observed by all firms. Let $q_0(t) \in \mathbb{R}_+$ and $q_{-0}(t) = (q_1(t), ..., q_n(t)) \in \mathbb{R}_+^n$ be respectively the output level produced by the outsider and the vector of outputs produced by the n firms of the fringe in period t. Hence, a (finite) history in period $t \geq 1$ is $h(t) = (h_0(t), h_{-0}(t))$, where $h_0(t) = (q_0(1), ..., q_0(t-1))$ and $h_{-0}(t) = (q_{-0}(0), ..., q_{-0}(t-1))$. Let H_t be the set of t-period histories. We further define the initial history to be the null set, $H_0 = \{\emptyset\}$, and H to be the set of all possible publicly observable histories, $H = \overset{\infty}{\underset{t=0}{\cup}} H_t$. A pure strategy for firm i in $G^{\infty}(\delta)$, for $i = 0, 1, 2, ..., n$, is a mapping from the set of all possible histories into the set of output levels, $\sigma_i : H \rightarrow \mathbb{R}_+$.

Let $\sigma_{-0} = (\sigma_1, ..., \sigma_n)$ be the strategy profile of the firms of the fringe. Any strategy profile $\sigma = (\sigma_0, \sigma_{-0})$ generates an output path $\{q_0(\sigma)(t), q_{-0}(\sigma)(t)\}_{t=0}^{\infty}$ defined inductively by $(q_0(\sigma)(0), q_{-0}(\sigma)(0)) = \sigma(\emptyset)$ and $(q_0(\sigma)(t), q_{-0}(\sigma)(t)) = \sigma(t)((q_0(\sigma)(0), q_{-0}(\sigma)(0)), ..., (q_0(\sigma)(t-1), q_{-0}(\sigma)(t-1)))$ for all $t \geq 1$. An outcome path $\{q_0(\sigma)(t), q_{-0}(\sigma)(t)\}_{t=0}^{\infty}$ thus implies an infinite stream of stage-game payoffs $\{\pi_i(q_0(\sigma)(t), q_{-0}(\sigma)(t))\}_{t=0}^{\infty}$ for firms $i = 0, 1, ..., n$. The discounted payoff to firm i from the infinite sequence of stage-game payoffs $\{\pi_i(t)\}_{t=0}^{\infty}$ is given by $\sum_{t=0}^{\infty} \delta^t \pi_i(t)$, so that its payoff in $G^{\infty}(\delta)$ obtained with the strategy profile σ is

$$\pi_i^{\delta}(\sigma) = \sum_{t=0}^{\infty} \delta^t \pi_i(q_0(\sigma)(t), q_{-0}(\sigma)(t)). \tag{5}$$

A strategy profile σ is a Nash equilibrium in $G^{\infty}(\delta)$ if σ_0 is a best response to σ_{-0} and if σ_i, for $i = 1, ..., n$, is a best response to $\sigma_{-0\backslash i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$ and to σ_0. And it is a subgame perfect equilibrium in $G^{\infty}(\delta)$ if after every history $h \in H$, $\sigma|_h$ (i.e., the continuation of σ after h) is a Nash equilibrium in the corresponding subgame. We will restrict attention to stationary subgame perfect equilibria (SSPE), i.e., equilibria in which after any history, a stationary profile of actions is played thereafter, and which also satisfy the additional requirement of symmetry within the fringe, in the sense that all firms of the fringe produce the same level of output at every history.

3.2 The two-phase punishment–cooperation scheme

Throughout the paper, we suppose that the firms of the fringe adopt 'stick-and-carrot' strategies *à la* Abreu (1986, 1988) to support the joint-profit maximizing level of output as a subgame perfect equilibrium. Formally, for any level of output q_0 produced by the outsider in period t, consider two levels of output produced by any firm of the fringe as functions of q_0, i.e., $(\hat{q}(q_0), \tilde{q}(q_0))$, and define a two-phase punishment–cooperation profile $\sigma(\hat{q}(q_0), \tilde{q}(q_0))$ to be 'stick-and-carrot' strategies in which all firms of the fringe produce $\tilde{q}(q_0)$ in the first period and thereafter play $\hat{q}(q_0)$, with any deviation from these strategies causing this prescription to be repeated. Intuitively, $\tilde{q}(q_0)$ is the 'stick', involving a high level of output and $\hat{q}(q_0)$ is the 'carrot', involving a low and cooperative level of output. The punishment specifies a single-period penalty followed by repeated play of the carrot. Deviations from the punishment simply cause it to begin again.

Again, we are concerned with the best subgame perfect equilibrium from the viewpoint of the firms of the fringe and we further assume that the outsider plays in every period its best response to the other firms' strategies regardless of the history, i.e., $\sigma_0(h(t)) = r_0(q_{-0}(t))$ for all $h(t)$. During the cooperative phase within the fringe, we thus pay attention to the levels of output $\hat{q}(q_0) = R_i(q_0^C) = q^C$ for $i = 1, ..., n$ and $r_0(q^C) = q_0^C$. During the punishment phase, we are concerned with the levels of output $\tilde{q}(q_0) = x$ and $r_0(x) = q_0^x$. This strategy profile will be henceforth denoted by $\sigma^*(q^C, \mathrm{x})$, where q^C is the short notation for (q_0^C, q^C) and $\mathrm{x} = (q_0^x, x)$.

The profile $\sigma^*(q^C, x)$ can sustain the cooperation within the fringe as a subgame perfect equilibrium output path if and only if a single period deviation from the strategy (and sticking to it subsequently) after any history is not profitable for any firm of the fringe. There are two kinds of histories to check. The first is that no single deviation has taken place in the previous periods, so that the firms are in a cooperative phase. The second state to check is the one where a firm deviated from the cooperative agreement in the previous period, so that the firms are currently in the punishment phase.

Suppose first that the firms of the fringe are in a cooperative phase, i.e., they all produce q^C and the outsider produces q_0^C, and that firm i considers deviating from q^C. The deviator maximizes $\pi_i(q_0^C, q^C, ..., q_i, ..., q^C) = \max\{0, (1 - q_i - (n-1)q^C - q_0^C - c)q_i\}$ with respect to q_i and hence the optimal deviation output is given by $q^D = (n+1)(1-2c)/6n$.

Let $\pi^D = \pi_i(q_0^C, q^C, ..., q^D, ..., q^C)$ be the optimal deviation profit for each firm i in the fringe. We have

$$\pi^D = \left[\frac{(n+1)(1-2c)}{6n}\right]^2. \tag{6}$$

The deviation payoff given by (6) is decreasing in c. Hence, the greater the cost advantage of the outsider, the lower is the temptation to deviate from the cooperative phase for each firm of the fringe. Moreover, a larger size of the fringe also reduces the deviation payoff.

Now, let $\pi^P(x) = \pi_i(r_0(x), x, ..., x)$ denote the payoff of firm i of the fringe when each fringe firm produces x units of output while the outsider best responds to this level. Let V^P be the present discounted value of the payoffs following a deviation, that is,

$$V^P = \pi^P(x) + \frac{\delta}{1-\delta}\pi^C. \tag{7}$$

No firm of the fringe has an incentive to deviate from $\sigma^*\left(q^C, \mathrm{x}\right)$ in the cooperative phase if and only if

$$\pi^D + \delta V^P \leq \frac{1}{1-\delta}\pi^C. \tag{8}$$

Intuitively, (8) says that the one-shot deviation gain from the cooperative phase plus the discounted payoff of entering the punishment phase next period must not exceed the payoff from continued cooperation. Rearranging this inequality and using (7), we have the following incentive compatibility constraint along the cooperative path

$$\delta \geq \delta^C(x) = \frac{\pi^D - \pi^C}{\pi^C - \pi^P(x)}. \tag{9}$$

Suppose now that one firm deviated from $\sigma^*\left(q^C, \mathrm{x}\right)$ in the previous period, so that the firms of the fringe are in the punishment phase in the current period. Suppose also that firm i considers deviating from x and let $\pi^{DP}(x)$ be the optimal deviation payoff during the punishment phase, i.e., $\pi^{DP}(x) = \pi_i\left(r_0(x), x, ..., q^{DP}(x), ..., x\right)$ where $q^{DP}(x) = \arg\max\limits_{q_i} \pi_i(r_0(x), x, ..., q_i, ..., x)$. No firm of the fringe has an incentive to deviate from $\sigma^*\left(q^C, \mathrm{x}\right)$ during the punishment phase if and only if

$$\pi^{DP}(x) + \delta V^P \leq V^P. \tag{10}$$

Intuitively, the one-shot deviation gain from the punishment phase plus the discounted payoff of staying in the punishment phase next period must not exceed the present value of abiding by the punishment rule (which guarantees a return to the cooperative phase next period). Rearranging this inequality and using (7), we have the following incentive compatibility constraint along the punishment path:

$$\delta \geq \delta^P(x) = \frac{\pi^{DP}(x) - \pi^P(x)}{\pi^C - \pi^P(x)}. \tag{11}$$

Hence, the strategy profile $\sigma^*\left(q^C, \mathrm{x}\right)$ is subgame perfect if and only if

$$\delta \geq \max\left\{\delta^C(x), \delta^P(x)\right\}. \tag{12}$$

We are now ready to determine the set of punishment output levels x for which the strategy profile $\sigma^*\left(q^C, \mathrm{x}\right)$ forms a subgame perfect equilibrium.

In the punishment phase, the outsider optimally responds to the punishment output x, thereby producing

$$r_0(x)=\begin{cases}\frac{1-nx}{2} \text{ if } x<\tilde{x},\\ \\ 0 \text{ otherwise,}\end{cases} \tag{13}$$

where $\tilde{x}=1/n$ corresponds to the threshold value of the punishment output level above which the market price turns out to be nil for all $x\geq\tilde{x}$, thereby driving the outsider out of business during the punishment period. Each firm within the fringe then obtains

$$\pi^P(x)=\begin{cases}x\left[\frac{1-2c-nx}{2}\right] \text{ if } x<\tilde{x},\\ \\ -cx \text{ otherwise.}\end{cases} \tag{14}$$

The punishment payoff function changes when $x\geq\tilde{x}$ because (again) the market price becomes zero, so that the firms of the fringe produce and give the good for free. Clearly, in that case, the profit of any firm of the fringe is negative. However, even if the market price is positive (i.e., when $x<\tilde{x}$), the punishment profit may still be negative due to dumping, with a market price which falls short of c. This is the case for $x\geq(1-2c)/n$. One can also observe that $\pi^P(x)$ is decreasing in x for any $x\geq(1-2c)/2n$. The idea is that raising the punishment output floods the market, which exerts a downward pressure on the market price. Nevertheless, the loss entailed by the punishment should be recouped by reverting back to cooperative behaviour.

We can now determine the optimal deviation payoff for the firm which defects from the punishment phase. Anticipating the outsider's best reply to x, the payoff function for the deviator i is given by

$$\pi_i\left(r_0(x), x, ..., q_i, ..., x\right)=\begin{cases}\max\{0,[1-q_i-(n-1)x-\frac{1-nx}{2}-cq_i]\} \text{ if } x<\tilde{x},\\ \\ \max\left\{0,[1-q_i-(n-1)x-c]\,q_i\right\} \text{ otherwise.}\end{cases} \tag{15}$$

Note that the firm which deviates from the punishment phase can always choose not to produce (with a payoff equal to 0) if it cannot get positive profits. Denote by $\hat{x}=(1-2c)/(n-2)$ and $\hat{x}'=(1-c)/(n-1)$ two peculiar punishment outputs levels. As shown below, they are the minimum punishment threats required to drive the deviator out of business depending on whether the market price is positive or nil on the punishment path.

Maximizing the above payoff function with respect to q_i, we obtain the optimal deviation output during the punishment phase, i.e.,

$$q^{DP}(x)=\begin{cases}\frac{1-2c-x(n-2)}{4} \text{ if } \begin{cases}\text{either } c\geq\frac{1}{n} \text{ and } x<\hat{x},\\ \text{or } c<\frac{1}{n} \text{ and } x<\tilde{x},\end{cases}\\ \\ \frac{1-c-x(n-1)}{2} \text{ if } c<\frac{1}{n} \text{ and } \tilde{x}\leq x<\hat{x}',\\ \\ 0 \text{ otherwise.}\end{cases} \tag{16}$$

When $x<\tilde{x}$, the market price is positive and hence the outsider is better off producing $r_0(x)>0$, as one can see from (13). In this situation, the deviator also produces strictly positive quantities whenever $x<\hat{x}$. When $c<\frac{1}{n}$, we have $\tilde{x}<\hat{x}$, so that $x<\hat{x}$ whenever $x<\tilde{x}$. If however $c\geq\frac{1}{n}$, we have $\hat{x}\leq\tilde{x}$ and hence the relevant constraint is indeed given by $\hat{x}$.

When the punishment output is relatively large, i.e., when $x\geq\tilde{x}$, the market price is equal to 0 and hence the best response of the outsider is to stay out of the market, i.e., $r_0(x)=0$. However, it might possible that the behaviour of the deviator, by producing a lower level of output than the agreed punishment output level, gives rise to a positive market price, so that the optimal deviation output is positive. For this situation to happens, one must have for $x\in[\tilde{x},\hat{x}')$, which necessarily implies $\tilde{x}<\hat{x}'$ and $c<\frac{1}{n}$. In all other cases, i.e., $c\geq\frac{1}{n}$ and $x\geq\hat{x}$ or $c<\frac{1}{n}$ and $x\geq\hat{x}'$, we have $q^{DP}(x)=0$.

Substituting (16) into (15) yields the optimal deviation payoff function

$$\pi^{DP}(x)=\begin{cases}\left[\frac{1-2c-x(n-2)}{4}\right]^2 \text{ if } \begin{cases}\text{either } c\geq\frac{1}{n} \text{ and } x<\hat{x},\\ \text{or } c<\frac{1}{n} \text{ and } x<\tilde{x},\end{cases}\\ \\ \left[\frac{1-c-x(n-1)}{2}\right]^2 \text{ if } c<\frac{1}{n} \text{ and } \tilde{x}\leq x<\hat{x}',\\ \\ 0 \text{ otherwise.}\end{cases} \tag{17}$$

$\pi^{DP}(x)$ is decreasing in c whenever the market price induced by a deviation from the punishment path is positive. In other words, the higher the cost advantage of the outsider, the lower is the incentive to deviate from the punishment phase for any firm of the fringe even though the outsider is driven out of business, which happens when $c<\frac{1}{n}$ and $\tilde{x}\leq x<\hat{x}'$, arranging a market price equal to 0 in the absence of a deviation.

We now state the following lemma, which will prove useful.[2]

Lemma 1. *Let $\bar{x}=2\gamma/3n$ be the highest solution to $\pi^C-\pi^P(x)=0$, with $\gamma=1-2c$. A necessary condition for the strategy profile $\sigma^*\left(q^C,\mathrm{x}\right)$ to be a* SSPE *is then that $x\geq\bar{x}$.*

The punishment output level must be sufficiently large for the strategy profile $\sigma^*\left(q^C,\mathrm{x}\right)$ to be a subgame perfect equilibrium. Punishments which are too small strengthen the incentives to

[2] All the proofs are given in the Appendix.

deviate either from the cooperative phase by raising $\pi^P(x)$, or from the punishment phase by raising $\pi^{DP}(x)$.

4 The most cooperative symmetric punishment

We now determine the strongest credible punishment in the sense that $x \in [\overline{x}, +\infty)$ is chosen to minimize the continuation value following a deviation given by (7) subject to the incentive constraints (8) and (10). This requires that the incentive constraint along the punishment phase (10) holds with equality, or equivalently that $\delta = \delta^P(x)$. Suppose it is not. Then the punishment output level x can be raised so as to decrease both $\pi^{DP}(x)$ and the continuation valuation V^P until (10) holds with strict equality. Indeed, since $\pi^P(x)$ is decreasing in x for any $x \geq \gamma/2n$, V^P is also decreasing in x on $[\overline{x}, +\infty)$, since $\gamma/2n < \bar{x}$. Furthermore, decreasing V^P makes the incentive constraint along the cooperative path, given by (8), more likely to be satisfied, because π^C and π^D do not depend on x. Note that when $\pi^{DP}(x) = 0$, the harshest punishment $V^P = 0$, i.e., the punishment level such that the losses incurred by the firm during the punishment phase are exactly recouped by the cooperative profits in the following periods, can be sustained since the incentive constraint (10) holds as an equality.[3]

Typically, there are multiple punishment levels such that $\delta = \delta^P(x)$ and such that the incentive constraint in the cooperative phase (8) holds. To deal with this multiplicity problem, we use the following selection criterion. Observing that $\delta^C(x)$ is decreasing in x, we further impose that (8) holds with strict equality, i.e., $\delta = \delta^C(x)$, so that we focus on the strongest credible punishment levels satisfying $\delta^C(x) = \delta^P(x)$. From (14) and (17), both payoff functions $\pi^P(x)$ and $\pi^{DP}(x)$ are not everywhere differentiable on $[\overline{x}, +\infty)$, and moreover $\pi^{DP}(x)$ may be non-monotonic on this set. Hence, the equation $\delta^C(x) = \delta^P(x)$ may admit several solutions. Therefore, we single out the highest value of x, say x^*, satisfying this equality. In turn, we ensure that $\delta^P(x)$ is strictly increasing in x on $[x^*, +\infty)$. Hence, x^* corresponds to the (strongest credible) punishment level which implies the largest possible range of discount factors for which cooperation within the fringe can be enforced as an SSPE. We call this punishment level the *most cooperative symmetric punishment* (MCSP). In other words, the MCSP is the punishment output level which most relaxes the incentive compatibility constraints both in the cooperative and the punishment phases.

First, using (8) and (10), equation $\delta^C(x) = \delta^P(x)$ becomes $\left[\pi^{DP}(x) - \pi^P(x)\right] = \left[\pi^D - \pi^C\right]$,

[3] When the most severe punishment can be sustained, Abreu (1986) has shown that the 'stick and carrot' strategy is optimal in the set of symmetric stationary strategies.

which reduces to

$$\pi^{DP}(x) - \pi^{P}(x) = \left[\frac{\gamma(n-1)}{6n}\right]^2. \quad (18)$$

As a result, our selection criterion requires that the net gain from deviating from the cooperative path is equal to that from deviating from the punishment path. Building on the previous analysis, we will distinguish in what follows between two cases: first, $c \geq \frac{1}{n}$, and second, $c < \frac{1}{n}$.

4.1 Case 1: $c \geq \frac{1}{n}$, a high competitive disadvantage of the fringe

Recall that $c \geq \frac{1}{n}$ is equivalent to $\hat{x} \leq \tilde{x}$, which means that the best deviation profit from the punishment path can be equal to 0 (for $x \geq \hat{x}$) even though the market price is positive (for $x < \tilde{x}$).

When $c \geq 1/n$, there are three types of punishment regimes, depending on the severity of the punishment output level relative to $\hat{x}$ and $\tilde{x}$. When the punishment output level is higher than $\tilde{x}$, the outsider is driven out of business (see (13)), since the market price falls down to 0, while the deviator cannot do better than cutting its production level to 0 (see (16)). The MCSP which solves (18) in $[\tilde{x}, +\infty)$ will be denoted by x_1 whenever it exists. If not, we turn to less severe punishment levels inside $[\hat{x}, \tilde{x})$. In this case, the market price is positive and the outsider is better off remaining active in the market. But again, in this case, the best deviation from the punishment path is to cut the production level to 0. The MCSP which solves (18) in $[\hat{x}, \tilde{x})$ will be denoted by x_1' whenever it exists. Otherwise, given Lemma 1, we focus on less severe subgame perfect punishments within the interval $[\bar{x}, \hat{x})$. In this case, the market price is still positive and both the outsider and the deviator are active in the market. The MCSP, if it exists, which solves (18) in $[\bar{x}, \hat{x})$ will be denoted by x_1''.

Proposition 1. *Assume that $c \geq \frac{1}{n}$. Furthermore, let $f(n)$ be the lowest value of c which satisfies $x_1 - \tilde{x} = 0$. Then, the strategy profile $\sigma^*\left(q^C, \mathrm{x}\right)$ admits a unique* MCSP *given by*: (i) $x_1 \geq \tilde{x}$ *if* $n \geq 9$ *and* $c \leq f(n)$; (ii) $x_1' \in [\hat{x}, \tilde{x})$ *if* $n \geq 9$ *and* $c > f(n)$; (iii) $x_1'' \in [\bar{x}, \hat{x})$ *if* $n < 9$.

The MCSP, i.e., the punishment output level which most relaxes the incentive compatibility constraints on both the cooperative and punishment paths, depends on both the number of firms within the fringe and on their competitive disadvantage relative to the outsider. First, it must be remembered that the two incentive constraints depend on the harshness of the punishment. The higher the punishment output level, the lower is the profit of any firm of the fringe on the punishment path (i.e., $\partial\pi^P(x)/\partial x < 0$). This makes the incentive constraint for cooperation more likely to be satisfied (i.e., $\partial\delta^C(x)/\partial x < 0$). However, a harsher punishment also raises the temptation to deviate from the punishment path, which tightens the incentive constraint on this path. We indeed show, in the proofs of Proposition 1 and 2, that $\partial \pi^P(x)/\partial x > 0$ for any punishment output level above the one which makes the two incentive constraints binding.

Now observe that the lower the size of the fringe, the lower is the gain from cheating on the cooperative path, given by $\pi^D - \pi^C$. Intuitively, with a small number of firms within the fringe, each firm has a relatively large market so that the potential gain of deviating by producing beyond the production cut agreement is relatively low. Therefore, when the size of the fringe is relatively low (i.e., $n < 9$) a relatively low punishment output level $x_1'' \in [\bar{x}, \hat{x})$ is sufficient to sustain the cooperation among the firms of the fringe and to satisfy the two incentive compatibility constraints. Point A in Figure 1 shows a parameter configuration (n, c) for which the MCSP is x_1''. Treating the size of the fringe as a continuous variable, we show in the Appendix that the threshold value of n below which the MCSP is given by x_1'' is precisely equal to $\tilde{n} = \left(9 + \sqrt{73}\right)/2 \simeq 8.77$.

As the number of firms increases to a certain point (i.e., $n \geq 9$) the higher temptation to deviate from cooperation requires harsher punishments. However, in this case, the interval within which the punishment output levels can fall depends on the competitive disadvantage of the firms of the fringe. More specifically, consider now the dotted arrow in Figure 1 which depicts from point A, and for a given level of c, an increase in the size of the fringe above 9. Cooperation then requires a punishment output level x_1' harsher than x_1'', as is the case at point B, provided that $c > f(n)$. In this case, the best deviation profit from the punishment path is equal to 0, but the outsider still makes a positive profit since the market price remains positive.

Now consider that the cost disadvantage of the firms of the fringe, for a given size n, decreases, as depicted by the bold arrow starting from point B in Figure 1. If c becomes smaller than $f(n)$, cooperation requires a punishment output level x_1 harsher than x_1', as is the case at point C. An explanation of this is that the benefit from cheating on the cooperative path is decreasing in c. The larger the competitive disadvantage of the firms, the lower is the benefit from cheating on the production cut agreement. Conversely, the lower the cost competitiveness of the firms, the higher is the incentive to defect from the punishment path because of the high level of production prescribed by the punishment scheme. Therefore, when c is relatively large (i.e., $c > f(n)$), a relatively low punishment output level (i.e., $x_1' \in [\hat{x}, \tilde{x})$) is sufficient to sustain the cooperation, thus relaxing the incentive constraint on the punishment path. Finally, a relatively low cost disadvantage (i.e., $c \leq f(n)$) together with a large size of the fringe requires the highest punishment level (i.e., $x_1 \geq \tilde{x}$). Note again that in this case, the market price falls to 0, so that the best deviation profit from the punishment phase for any firm of the fringe as well the profit of the outsider are both equal to 0.

4.2 Case 2: $c < \frac{1}{n}$, a low competitive disadvantage of the fringe

Recall that $c < \frac{1}{n}$ is equivalent to $\tilde{x} < \hat{x}'$, which means that the best deviation profit from the punishment path can be positive (for $x < \hat{x}'$) even though the market price is equal to 0 in the absence of a deviation (for $x \geq \tilde{x}$).

As in the previous case, when $c < \frac{1}{n}$, there are three types of punishment regimes, depending on the severity of the punishment output level relative to $\hat{x}'$ and $\tilde{x}$. First, when the punishment output level is higher than $\hat{x}'$, then both the market price and the best deviation profit from the punishment path are equal to 0. Furthermore, in this case, the profit of a firm which abides by the punishment rule is the same as that prevailing when $c \geq 1/n$ and $x \geq \tilde{x}$. Therefore, the MCSP which solves (18) in $[\hat{x}', +\infty)$ must still be given by x_1, provided it exists. If not, we turn to less severe punishment output levels within the interval $[\tilde{x}, \hat{x}')$. In this case, the market price is still equal to 0 if all firms of the fringe abide by the punishment rule. If however a firm decides to deviate from the punishment path, it produces a level of output lower than the agreed punishment level. This causes a positive market price so that the deviator makes positive profits, as shown by (17). Provided it exists, the MCSP within the interval $[\tilde{x}, \hat{x}')$ will be denoted by x_2'. If it does not exist, then we look for a solution to (18) in the interval $[\bar{x}, \tilde{x})$. In this case, the profit of a firm in the punishment phase and that of the deviator are identical to those obtained in the previous case, where $c \geq 1/n$ and $[\bar{x}, \hat{x})$ (since now $\tilde{x} < \hat{x}$). Therefore, if it exists, the MCSP which solves (18) in $[\bar{x}, \tilde{x})$ must be given by x_1''.

Proposition 2. *Assume that $c < \frac{1}{n}$. Furthermore, let $h(n)$ be the lowest value of c which satisfies $x_1 - \hat{x}' = 0$, and $g(n)$ the highest value of c which satisfies $x_2' - \tilde{x} = 0$. Then, the strategy profile $\sigma^* \left(q^C, \mathrm{x}\right)$ admits a unique* MCSP *given by* (i) *$x_1 \geq \hat{x}'$ if $n \geq 9$ or if $n < 9$ and $c \leq h(n)$;* (ii) *$x_2' \in [\tilde{x}, \hat{x}')$ if $n \in \{7, 8\}$ and $c \in (h(n), g(n)]$;* (iii) *$x_1'' \in [\bar{x}, \tilde{x})$ if $n \in \{7, 8\}$ and $c > g(n)$ or if $n < 7$ and $c > h(n) > g(n)$.*

The characterization of the MCSP when $c < \frac{1}{n}$ is slightly more complicated than when $c \geq \frac{1}{n}$. One can observe that there are two sets of parameters under which the MCSP with $c < \frac{1}{n}$ is exactly the same as when $c \geq \frac{1}{n}$. When the size of the fringe is relatively small (i.e., $n < 9$), the lowest MCSP is still given by x_1'', but now there is a lower bound to the competitive disadvantage of the firms (i.e., $c > g(n)$ if $n = \{7, 8\}$ or $c > h(n)$ if $n < 7$). If this additional constraint is not satisfied (i.e., when $c \leq h(n)$), then the MCSP is given by x_1, which was obtained in the previous case (i.e., $c \geq \frac{1}{n}$) for a relatively low disadvantage cost (i.e., $c \leq f(n)$ provided $c \geq \frac{1}{n}$) and a relatively large size of the fringe (i.e., $n \geq 9$). In the current size, this last condition (i.e., $n \geq 9$) also guarantees that x_1 is the

MCSP independently of the disadvantage cost. Finally, there is now a new punishment level given by x_2', which holds only for intermediate values of the size of the fringe (i.e., $n \in \{7, 8\}$) and of the disadvantage cost (i.e., $c \in (h(n), g(n)]$.

The driving forces behind the results in Proposition 2 are the same as those underlying Proposition 1. Again, this can be explained with the help of a figure. Point A in Figure 2 represents a parameter configuration for which $n < 9$ and c is relatively large (i.e., $c \geq \max\{h(n), g(n)\}$). As the incentive to deviate in the cooperative phase is relatively low, cooperation within the fringe can be supported by the MCSP which involves the least severe punishment output level x_1''. An increase in the size of the fringe, for a given level of c, raises the temptation to deviate from the cooperative path, which may require a punishment output level x_2' harsher than x_1''. This is illustrated by the dotted arrow from point A to point B in Figure 2. Observe that the parameter configuration for which x_2' is the MCSP is quite limited (i.e., $n \in \{7, 8\}$ and $h(n) < c \leq g(n)$[4]). This restricted parameter area corresponds to the situation where a firm which deviates from the punishment path causes a positive market price, by producing less than the agreed punishment level, thus obtaining positive profits.

Now assume that the cost disadvantage of the firms of the fringe, for a given size n, decreases below $h(n)$, as depicted by the bold arrow starting from point B in Figure 2. Just as for an increase in n, a greater cost competitiveness increases the temptation to deviate from the cooperative path. This thus requires a punishment output level x_1 harsher than both x_2' and x_1'', as shown by point C in Figure 2. When $n \geq 9$, the MCSP is also given by x_1 independently of c because in that case the assumption that $c < 1/n$ necessarily implies $c \leq h(n) < 1/n$. This parameter area corresponds to a situation where the best deviation profit from the punishment path is equal to 0 (just as the market price is in the punishment path). To summarize, a relatively large size of the fringe and a relatively small disadvantage cost require greater punishment output levels than in the reverse situation.

Combining Propositions 1 and 2, one can conclude that the MCSP is characterized by the harshest punishment output level x_1 independently of n, when the cost disadvantage of the firms of the fringe is low enough, i.e., when $c \leq h(n) \leq 1/n \leq f(n)$). In particular, this is the case when the fringe is as efficient as the outsider, i.e., when $c = 0$. However, we cannot state that the MCSP is given by x_1 independently of c, if n is large enough (or that it is given by the lowest punishment output level x_1'' independently of c if n is low enough). This is a clear illustration of the fact that

[4] Using the expression of $h(n)$ given by (A9) and that of $g(n)$ given by (A16) in the Appendix, we have that c must lie in the interval $\left(\left[\sqrt{73} - 7\right]/2\sqrt{73}; 1/11\right] \simeq (0.0904; 0.0909]$ when $n = 7$, and $\left(\left[\sqrt{919} - 24\right]/2\sqrt{919}; 2/19\right] \simeq (0.1042; 0.1052]$ when $n = 8$.

the existence of different punishment regimes relies on the cost asymmetry between the fringe and the outsider.

4.3 The minimum discount factor

We can now determine the minimum discount factor such that the two incentive compatibility constraints are binding. That is, we substitute the MCSP which solves (18) into (9) or (11). From the analysis of the previous section, we have that the MCSP can take four different expressions, depending on c, n, and on the interval within which the punishment output level x can fall.

We start by considering Case 1. If $n \geq 9$ and $c \leq f(n)$, then the MCSP is x_1, given by (A1) in the Appendix. This also implies $\pi^P(x_1) = -cx_1$ since $x_1 \geq \tilde{x}$. Substituting $\pi^P(x_1)$, π^C given by (3), and π^D given by (6) into $\delta^C(x)$ given by (9), we obtain

$$\delta_1(n) = \frac{(n-1)^2}{(n+1)^2}. \tag{19}$$

Suppose now that $n \geq 9$ and $c > f(n)$, so that the MCSP is x_1', given by (A5) in the Appendix. This also implies $\pi^P(x_1') = x_1'[(\gamma - nx_1')/2]$, since $x_1' \in [\hat{x}, \tilde{x})$. Substituting $\pi^P(x_1')$, π^C, and π^D into (9), we obtain again $\delta_1(n)$ given by (19). This is not surprising, since $\pi^{DP}(x_1) = \pi^{DP}(x_1') = 0$, and thus (18) yields that $-\pi^P(x_1) = -\pi^P(x_1') = \pi^D - \pi^C$. It follows that $\delta^C(x_1) = \delta^C(x_1') = \left[\pi^D - \pi^C\right]/\pi^D$.

Now if $n < 9$, the MCSP is x_1'', given by (A6) in the Appendix. We also have $\pi^P(x) = x_1''[(\gamma - nx_1'')/2]$, since $x_1'' < \hat{x} \leq \tilde{x}$. Substituting $\pi^P(x_1'')$, π^C, and π^D into (9), we obtain

$$\delta_1'(n) = \frac{(n-1)(n+2)^2}{24n(n-2)}, \tag{20}$$

which is lower than 1 for any $4 \leq n < 9$.

We now consider Case 2. If $n \geq 9$ or $n < 9$ and $c \leq h(n) < 1/n$, the MCSP is given by x_1 and hence $\pi(x_1) = -cx_1$ since $x_1 \geq \tilde{x}$. Hence, the threshold value of the discount factor is the same as when $n \geq 9$ and $1/n \leq c \leq f(n)$, namely, $\delta_1(n)$ given by (19). If $n \in \{7, 8\}$ and $c > g(n)$ or if $n < 7$ and $c > h(n) > g(n)$, the MCSP is given by x_1'', and hence $\pi^P(x_1'') = x_1''[(\gamma - nx_1'')/2]$, since $x_1'' < \tilde{x}$. It follows that the threshold value of the discount factor has the same expression as that obtained when $n \leq 9$ and $c \geq 1/n$, namely, $\delta_1'(n)$ is given by (20).

Finally, suppose that $n \in \{7, 8\}$ and $h(n) < c \leq g(n)$ so that the MCSP is x_2' given by (A13) in the Appendix. We then have $\pi(x_2') = -cx_2'$, since $x_2' \geq \tilde{x}$. Now, substituting $\pi^P(x_2')$, π^C, and π^C

into (9), we obtain

$$\delta_2'(n,c) = \frac{\gamma^2 (n-1)^4}{4n\left[\gamma^2 - n\left[c\left(17c+1\right)+2\right] + n^2\left[5c(1-c)+1\right] + 3c\sqrt{\gamma^2 (n-1)^4 - 36n^2 c\left[(n-1)-cn\right]}\right]}. \tag{21}$$

The next proposition summarizes these results.

Proposition 3. *The minimal threshold for the discount factor above which the strategy profile* $\sigma^*\left(q^C, \mathrm{x}\right)$ *forms an* SSPE *is given by*

(i) $\delta_1(n)$ *if* (ia) $n \geq 9$; *or* $n < 9$ *and* $c \leq h(n) < 1/n$;

(ii) $\delta_1'(n)$ *if* (ia) $n < 9$ *and* $c \geq 1/n$; *or* $n \in \{7, 8\}$ *and* $h(n) < g(n) < c < 1/n$; *or* $n < 7$ *and* $g(n) < h(n) < c < 1/n$;

(iii) $\delta_2'(n,c)$ *if* $n \in \{7, 8\}$ *and* $h\left(n\right) < c \leq g(n) < 1/n$.

Clearly, $\delta_1(n)$ is increasing in n and approaches 1 as the size of the fringe goes to infinity. One can also verify that $\delta_1'(n)$ is increasing in n for $n \geq 5$.[5] Furthermore, since $\delta_1'(8)$ is strictly lower than $\delta_1(9)$, one can conclude that when $c \geq 1/n$, an increase in the size of the fringe generally makes cooperation more difficult to sustain (except if the fringe increases from 4 to 5 firms). For $c \geq 1/n$, the evolution of the minimum discount factor as a function of the size of the fringe is depicted in Figure 3. It is discontinuous because the MCSP is itself discontinuous at $\tilde{n} = \left(9 + \sqrt{73}\right)/2 \simeq 8.77$. For $c < 1/n$, there is a (restricted) parameter configuration, i.e., $n \in \{7, 8\}$ and $c \in (h(n), g(n)]$, which gives rise to a third expression for the minimum discount factor, viz., $\delta_2'(n,c)$. We show in the Appendix, Section 6.4, that $\delta_1'(6) < \delta_2'(7,c) < \delta_2'(8,c) < \delta_1(9)$, as depicted in Figure 4. In other words, when $c < 1/n$, we still obtain that the cooperation within the fringe is more difficult to sustain as the size of the fringe increases.

Keeping the size of the fringe n constant, we can also evaluate the impact of a change in c on both the MCSP and the sustainability of cooperation within the fringe. First, observe that both $\delta_1(n)$ and $\delta_1'(n)$ are independent of the cost disadvantage of the fringe. In other words, equalizing the net benefits from deviating from the cooperative and punishment paths may yield a punishment threat (i.e., the MCSP x_1, x_1', or x_1'', depending on the parameter configuration), which makes the sustainability of the cooperation within the fringe immune to its disadvantage cost. Yet, whether the relevant minimal discount factor is given by $\delta_1(n)$ or $\delta_1'(n)$ depends on the relation between

[5] The sign of the derivative of $\delta_1'(n)$ with respect to n is the same as the sign of $n^3 - 6n^2 + 6n - 4$. Using Mathematica, one obtains that it admits a unique real root given by $\bar{n} \simeq 4.95$, under (above) which the polynomial is negative (positive).

c and n. Furthermore, still depending on the parameter pair (c, n), the minimum discount factor might also be given by $\delta_2'(n, c)$, which is a function of c.

If $n < 7$, the lowest discount factor is $\delta_1(n)$ when $c \leq h(n)$, but it is given by $\delta_1'(n)$ when $c > h(n)$. And one can easily verify that $\delta_1'(n) > \delta_1(n)$ for any $n < 7$. If $n \in \{7, 8\}$, then the lowest discount factor is $\delta_1(n)$ when $c \leq h(n)$, but is $\delta_2'(n, c)$ when $c \in (h(n), g(n)]$, or $\delta_1'(n)$ when $c > g(n)$. We show in the Appendix, Section 6.4, that $\delta_2'(n, c)$ is increasing in c and furthermore that $\delta_1(n) < \delta_2'(n, c) < \delta_1'(n)$ for $n \in \{7, 8\}$ and $c \in (h(n), g(n)]$. Finally, if $n \geq 9$, then the lowest discount factor is given by $\delta_1(n)$ independently of c. In other words, cooperation within the fringe is more difficult to sustain as the cost asymmetry increases, provided the size of the fringe is lower than 9 firms. When the size of the fringe is larger than 9 firms, then the cost disadvantage of the firms (relative to the outsider) has no effect on the difficulty in sustaining the cooperation within the fringe. Yet, in this case, the MCSP depends on c, since it is given by x_1 for $c \leq f(n)$ and by $x_1' < x_1$ for $c > f(n)$.

What intuition can we now provide about the effects of changes in the size and the competitive disadvantage of the fringe on the minimum discount factor?

As regards the impact of the size of the fringe, the results illustrated in Figures 3 and 4 are in line with the traditional literature on tacit collusion in symmetric oligopoly games. It becomes harder to collude with more firms because of the greater incentive to deviate from the cooperative agreement. Indeed, in a similar model, i.e., in a Cournot market with linear demand and constant marginal costs, but with all firms being identical and participating in the collusive agreement, the critical discount factor is increasing in the number of firms involved in the agreement whether firms use Nash-reversion strategies (see Vives, 1999, p. 307) or stick-and-carrot strategies (see Motta, 2004, p. 171). A general explanation still available here is that a larger number of firms has the effect of decreasing the individual collusive profit, thereby increasing the net gain from deviating in the cooperative phase, captured by $\pi^D - \pi^C$. One important difference from the traditional literature is that the deviating firm gains less in our setup, since the resulting outcome following the deviation is an (asymmetric) triopoly. Moreover, the MCSP requires equalizing the gains from deviating in the cooperative and the punishment phases. When the increase in the number of firms within the fringe leads to a more severe punishment regime, the gain from cheating in the punishment phase reflected by $\pi^{DP}(x) - \pi^P(x)$ rises. This partly offsets the increase in $\pi^D - \pi^C$. However, a harsher punishment output level also raises the loss due to the punishment in the cooperative phase (i.e., $\pi^C - \pi^P(x)$), which causes a decline of $\delta^C(x)$. This effect contradicts the overall result that an increase in n at the MCSP makes cooperation less likely. We can thus conclude that the direct, positive effect on the minimum discount factor caused by an increase in the size of the fringe dominates the induced negative effect associated with a consequent switch to a more severe punishment regime.

The same reasoning helps separate the intricate effects of changes in the competitive disadvantage of the fringe on the minimum discount factor. As previously seen, the overall result is that an increase in c does not make it easier to sustain the cooperation. This is far from intuitive, since a decrease in the competitiveness of the fringe leads to a reduction in the net gain from deviating in the cooperative phase, which, on the contrary, strengthens the incentive to cooperate. Nevertheless, the equalization of deviation gains required by the MCSP implies a reduction in the net gain from deviating in the punishment phase. This can be achieved by a softer punishment regime, which in turn reduces the one-period loss caused by punishment in the cooperative phase, thereby dampening the incentive to cooperate. As a result, the latter effect dominates the direct, negative impact on the minimum discount factor of the lower gain from cheating in the cooperative phase.

5 Conclusion

In this paper, we explored the ability of several identical firms to maintain perfect cooperation in a quantity-setting supergame in the presence of a low-cost firm which does not take part in the cooperative agreement. The less competitive firms, collectively referred to as the fringe, are assumed to adopt two-phase punishment schemes in the style of Abreu (1986, 1988) to sustain the joint-profit maximizing outcome, while the outsider is assumed to play its one-period best response to the fringe firms' strategies in every period. We focused on the maximal punishment regime that can be enforced, referred to as the MCSP, and then determined the minimal threshold value for the discount factor above which perfect cooperation can be sustained as a (stationary) subgame perfect equilibrium.

An important insight to be gained from this analysis is that the firms' ability to sustain the cooperation in the presence of a more competitive and non-cooperative firm depends crucially, and in a quite complex way, on the number of firms involved in the agreement and on their cost disadvantage relative to the non-participating firm. Indeed, the MCSP and the corresponding minimal discount factor result from the interplay between the firms' incentives to deviate from the cooperative agreement, on the one hand, and on their incentives to deviate from the punishment phase, on the other. The overall insight is that the MCSP requires less severe punishments when the fringe either is at a higher competitive disadvantage, or is smaller. At lower levels of the MCSP, the minimum of patience needed to cooperate is higher than if firms were at a lower competitive disadvantage. Hence, cooperation proves more difficult although deviation is less beneficial. By contrast, when the fringe is smaller, cooperation is easier, essentially because deviation is less beneficial. This positive effect on cooperation more than offsets the negative effect due to the lax punishment.

A theoretical exercise such as this may have some derived practical value. For instance, it is well known that governments of the United States, the European Union, China, and India widely subsidize their cotton farmers despite the World Trade Organisation's ruling some of these subsidies illegal[6]. Clearly, subsidies provide their recipients with a noticeable competitive advantage relative to the rival producers in some of the poorest regions of the world, especially in West Africa, and finally push down the market price. Our analysis suggests that tacit cooperation based on 'stick and carrot' strategies may be a useful retaliatory device in the hands of farmers who find themselves at a lower competitive disadvantage because they are not subsidized.

In the model presented, we admittedly made the most simple assumptions about cost and demand. Nonetheless, the derivations proved rather complicated, albeit feasible, and the intricate features of our results may carry over to more general functional forms. One interesting extension would be to assume that the outsider has a strategic behaviour more sophisticated than just playing non-cooperatively in each period of the game. For example, there might be attempts to cooperate between the outsider and a firm cheating on the fringe agreement. Alternatively, since cooperation within the fringe benefits the outsider, one could imagine that the latter takes part in the punishment.

6 Appendix A

6.1 Proof of Lemma 1

For condition (9) to be satisfied, we must have $\pi^C - \pi^P(x) \geq 0$, otherwise each firm of the fringe would have an incentive to deviate from q^C during the cooperative phase so as to enter in the punishment phase and get higher profits. Clearly, when $x > \tilde{x}$, $\pi^C - \pi^P(x)$ is positive since $\pi(x) = -cx < 0$. When $x \leq \tilde{x}$, we have $\pi^C - \pi^P(x) = (1/18n)[3nx - 2\gamma][3nx - \gamma]$. Hence, $\pi^C - \pi^P(x) = 0$ has two solutions: $\underline{x} = \gamma/3n$ and $\bar{x} = 2\gamma/3n$. Furthermore, the second derivative of $\pi^C - \pi^P(x)$ with respect to x is positive, which implies that this function has a global minimum. It follows that $\pi^C - \pi^P(x)$ is positive if and only if $x \in [0, \underline{x}] \cup [\bar{x}, +\infty)$, otherwise $\pi^P(x) \geq \pi^C$. This last case might be possible because $\pi^P(x)$ internalizes the best reply function of the outsider. Hence, the cartel could act as a Stackelberg leader by choosing x so as to maximize $\pi^P(x)$, in which case we would have $\pi^P(x) \geq \pi^C$. But in that situation, the strategy profile $\sigma^*\left(q^C, \mathrm{x}\right)$ is not subgame perfect because any firm within the fringe would have an incentive to

[6] see **http://www.guardian.co.uk/global-development/poverty-matters/2011/may/24/american-cotton-subsidies-illegal-obama-must-act, http://www.guardian.co.uk/environment/2010/nov/15/cotton-subsidies-west-africa, http://www.washingtonpost.com/wp-dyn/content/article/2010/06/02/AR2010060204228.html**

deviate from the cooperative phase in order to get $\pi^P(x)$ instead of π^C. To summarize, the incentive compatibility constraint (9) in the cooperative phase can be satisfied if and only if $x \in [0, \underline{x}] \cup [\bar{x}, +\infty)$.

We now show that, for any $x \in [0, \underline{x}]$, the incentive compatibility constraint (11) in the punishment phase cannot be satisfied. Indeed, to ensure subgame perfection, one must also have $\pi^{DP}(x) \leq \pi^C$. Suppose this inequality does not hold. Then any firm would have an incentive to deviate in the cooperative phase to earn $\pi^D \geq \pi^C$. This would be worthwhile because in the subsequent period, i.e., when punishment begins, the cheater would deviate again to get a greater profit level than that obtained along the cooperative path, thereby triggering the same defection scenario forever. Formally, $\pi^{DP}(x) \geq \pi^C$ implies $\delta^P(x) \geq 1$, so that (11) cannot be satisfied. Since $\underline{x} < \tilde{x}$, we have $x \leq \tilde{x}$ for any $x \in [0, \underline{x}]$, and hence $\pi^{DP}(x) = [(\gamma - x(n-2))/4]^2$. This function is decreasing on $[0, \gamma/(n-2)]$, and then increasing for any $x \geq \gamma/(n-2)$. Since $\underline{x} < \gamma/(n-2)$, $\pi^{DP}(x)$ is decreasing on $[0, \underline{x}]$ and reaches a minimum in $\underline{x}$ on $[0, \underline{x}]$. Evaluating at $\underline{x}$ the profit of the deviator along the punishment path, one obtains $\pi^{DP}(\underline{x}) = [\gamma(n+1)/6n]^2$, which is greater than π^C (The inequality $\pi^{DP}(\underline{x}) \geq \pi^C$ reduces to $(n-1)^2 \geq 0$). It follows that the incentive compatibility constraint (11) cannot be satisfied for $x \in [0, \underline{x}]$. Therefore, a necessary condition for the two incentive compatibility constraints to be simultaneously satisfied is that $x \in [0, \underline{x}] \cup [\bar{x}, +\infty)$.

6.2 Proof of Proposition 1

When $c \geq 1/n$, $\pi^{DP}(x) = 0$ for any $x \geq \hat{x}$ as shown by (17). However, the payoff function $\pi^P(x)$ during the punishment phase depends on whether $x \in [\hat{x}, \tilde{x})$ or $x \geq \tilde{x}$ as shown by (14). Suppose first that $x \geq \tilde{x}$ which implies $\pi^P(x) = -cx$. In that case, (18) has only one solution:

$$x_1 = \frac{1}{c}\left[\frac{\gamma(n-1)}{6n}\right]^2. \tag{A1}$$

The punishment output x_1 is relevant if $x_1 \geq \tilde{x}$. The equation $x_1 - \tilde{x}$ is quadratic in c and therefore $x_1 - \tilde{x} = 0$ has two roots, viz.,

$$f(n) = \frac{n(n+7) + 1 - 3\sqrt{n(n+2)(2n+1)}}{2(n-1)^2}. \tag{A2}$$

and

$$F(n) = \frac{n(n+7) + 1 + 3\sqrt{n(n+2)(2n+1)}}{2(n-1)^2}. \tag{A3}$$

Furthermore, the second derivative of $x_1 - \tilde{x}$ with respect to c is positive, which implies that this function has a global minimum in c. Therefore, we have $x_1 \geq \tilde{x}$ for any $c \notin (f(n), F(n))$. First, recall that assumption 1 states that $c < 1/2$, which guarantees that any firm has a positive market share in the stage game. Second, one can easily verify that $f(n) \leq 1/2$, this inequality being equivalent to $(n-1)^2 \geq 0$, and that $F(n) > 1/2$. It follows that one can have $x_1 \geq \tilde{x}$ only for $c \leq f(n)$.

Since we are in the case $c \geq 1/n$, one must also verify that $f(n) \geq 1/n$. We have

$$f(n) - 1/n = \frac{n^3 + 5n^2 + 5n - 2 - 3n\sqrt{n\,(n+2)\,(2n+1)}}{2n\,(n-1)^2}. \tag{A4}$$

The numerator of this expression is positive if $\left[n^3 + 5n^2 + 5n - 2\right]^2 \geq 3n^3\,(n+2)\,(2n+1)$. This inequality can be equivalently rewritten as $(n-1)^2(n+2)(n+1)(n^2 - 9n + 2) \geq 0$, which is verified for any $n \geq \tilde{n}$ with $\tilde{n} = \left(9 + \sqrt{73}\right)/2 \simeq 8.77$.

Therefore, if $n \geq 9$ and $1/n \leq c \leq f(n)$, then x_1 (greater than $\tilde{x}$) solves (18). Finally, to ensure that x_1 is the MCSP, we need to verify that $\delta^P(x)$ is increasing in x for any $x \geq x_1 \geq \tilde{x}$. Again, for any $x \geq \hat{x}$, we have $\pi^{DP}(x) = 0$, implying that $\delta^P(x) = -\pi^P(x)/[\pi^C - \pi^P(x)]$. Since $\pi^P(x)$ is always decreasing in x, clearly, $\delta^P(x)$ is increasing in x whenever $x \geq \hat{x}$. Since $x_1 \geq \tilde{x} \geq \hat{x}$, x_1 given by (A1) is indeed the MCSP when $n \geq 9$ and $1/n \leq c \leq f(n)$.

If $c > f(n)$, then $x_1 < \tilde{x}$, implying that x_1 is not the MCSP. Then, if (18) admits a solution, it must be lower than $\tilde{x}$. We now investigate whether there exists a solution within the interval $[\hat{x}, \tilde{x})$. When $x \in [\hat{x}, \tilde{x})$, we have $\pi^P(x) = x\,[(\gamma - nx)/2]$ and still $\pi^{DP}(x) = 0$. In this case, (18) has two roots. The lower root is $\gamma\left[3n - \sqrt{n\,(n+2)\,(2n+1)}\right]/6n^2$, which is negative for any $n \geq 4$. The upper root is

$$x_1' = \frac{\gamma\left[3n + \sqrt{n\,(n+2)\,(2n+1)}\right]}{6n^2}. \tag{A5}$$

The punishment output x_1' is relevant only if $\hat{x} \leq x_1' < \tilde{x}$. The inequality $x_1' < \tilde{x}$ reduces to $c > f(n)$, while the inequality $x_1' \geq \hat{x}$ reduces to $(n+1)(n^2 - 9n + 2) \geq 0$, which is equivalent to the inequality $f(n) \geq 1/n$ or $n \geq 9$. Therefore, if $n \geq 9$ and $c > f(n) > 1/n$, then x_1' solves (18) on $[\hat{x}, \tilde{x})$. Furthermore, this punishment level is the MCSP because, again, $\delta^P(x)$ is increasing in x whenever $x \geq \hat{x}$.

Finally, suppose that $n < 9$, implying that $c \geq 1/n > f(n)$. In this case, neither x_1 nor x_1' can be the MCSP. If (18) admits a solution, then it must be lower than $\hat{x}$, implying $\pi^{DP}(x) \geq 0$ and $V^P \geq 0$. In this case, we have $\pi^P(x) = x\,[(\gamma - nx)/2]$ and $\pi^{DP}(x) = [(\gamma - x(n-2))/4]^2$. Equation (18) has two roots, namely $\gamma/3n$ and $\gamma(5n-2)/3n\,(n+2)$. The lower root does not satisfy Lemma 1 and hence the relevant solution is the upper root (satisfying Lemma 1), i.e.

$$x_1'' = \frac{\gamma(5n-2)}{3n\,(n+2)}. \tag{A6}$$

One can check that $x_1'' < \hat{x}$ reduces to $-n^2 + 9n - 2 > 0$, which implies that $n < 9$. Finally, to ensure that x_1'' is the MCSP, we need to verify that $\delta^P(x) = \left[\pi^{DP}(x) - \pi^P(x)\right]/[\pi^C - \pi^P(x)]$ is also increasing in x on $[x_1'', \hat{x})$. When $x < \hat{x}$, we have

$$\delta^P(x, n, \gamma) = \frac{9n\,[\gamma - x(n+2)]^2}{8\,(\gamma - 3nx)\,(2\gamma - 3nx)}. \tag{A7}$$

Calculating the derivative of this expression, we have

$$\frac{\partial \delta^P(x,n,\gamma)}{\partial x} = \frac{9\gamma n\left[\gamma - x(n+2)\right]\left[\gamma(5n-8) - 9nx(n-2)\right]}{8(\gamma-3nx)^2(2\gamma-3nx)^2}. \tag{A8}$$

This derivative is positive whenever $x \geq \gamma/(n+2) > \bar{x} > \gamma(5n-8)/9n(n-2)$. Since $x_1'' > \gamma/(n+2)$, we have that $\delta^P(x)$ is increasing in x on $[x_1'', \hat{x})$. Since $\delta^P(x)$ is also increasing in x for any $x > \hat{x}$, x_1'' is the MCSP for any $n < 9$.

6.3 Proof of Proposition 2

When $c < 1/n$ and $x \geq \hat{x}'$, we have $\pi^{DP}(x) = 0$ and $\pi^P(x) = -cx$ since $\hat{x}' > \tilde{x}$. Again, (18) has one solution, given by (A1). One must verify that $x_1 \geq \hat{x}'$. The equation $x_1 - \hat{x}' = 0$ is quadratic in c and hence has two roots, viz.,

$$h(n) = \frac{\sqrt{(n-1)^3 + 9n^2} - 3n}{2\sqrt{(n-1)^3 + 9n^2}}, \tag{A9}$$

and

$$H(n) = \frac{\sqrt{(n-1)^3 + 9n^2} + 3n}{2\sqrt{(n-1)^3 + 9n^2}}. \tag{A10}$$

Furthermore, the second derivative of $x_1 - \hat{x}'$ with respect to c is positive, which implies that this function has a global minimum in c. Therefore, we have $x_1 \geq \hat{x}'$ for any $c \notin (h(n), H(n))$. Again, one can observe that $H(n) > 1/2$, while that $h(n) < 1/2$. Since one must have $c < 1/2$, one can have $x_1 \geq \hat{x}'$ only for $c \leq h(n)$.

Since we are in the case $c < 1/n$, we now evaluate the difference between $h(n)$ and $1/n$. We have

$$h(n) - 1/n = \frac{(n-2)\sqrt{(n-1)^3 + 9n^2} - 3n^2}{2n\sqrt{(n-1)^3 + 9n^2}} \tag{A11}$$

The numerator of this expression is positive if $(n-2)^2[(n-1)^3 + 9n^2] \geq 9n^4$. This inequality can be equivalently rewritten as $(n-1)^2(n+2)\left[n^2 - 9n + 2\right] \geq 0$, which is verified for any $n \geq \tilde{n}$, where $\tilde{n} = \left(9 + \sqrt{73}\right)/2$. We then have $h(n) \geq 1/n$ for any $n \geq 9$, while the inequality is reversed for any $n < 9$. Hence, when $c < 1/n$, x_1 solves (18) if $n \geq 9$ or if $n < 9$ and $c \leq h(n) < 1/n$. Finally, when $x \geq \hat{x}'$, we have $\delta^P(x) = -\pi^P(x)/[\pi^C - \pi^P(x)]$. Since $\pi^P(x)$ is always decreasing in x, $\delta^P(x)$ is increasing for any $x \geq \hat{x}'$. Therefore, when $c < \tilde{x}$ and $n \geq 9$ or $n < 9$ and $c \leq h(n) < 1/n$, x_1 given by (A1) is the MCSP.

Suppose now that $n < 9$ and $h(n) < c < 1/n$, so that there does not exist a punishment output level higher than $\hat{x}'$ satisfying (18). We then look for a punishment level satisfying (18) lower than $\hat{x}'$. When $x < \hat{x}'$, the profit functions $\pi^P(x)$ and $\pi^{DP}(x)$ depend on whether $x \in [\tilde{x}, \hat{x}')$ or $x < \tilde{x}$, as shown by (14) and (17). Assume first that $x \in [\tilde{x}, \hat{x}')$, implying $\pi^P(x) = -cx$ and $\pi^{DP}(x) = [(1-c-x(n-1))/2]^2$. Equation (18) then admits two roots, given by

$$\varkappa_2' = \frac{3n\left[(n-1) - c(n+1)\right] - \sqrt{\gamma^2 (n-1)^4 - 36n^2 c\left[(n-1) - cn\right]}}{3n(n-1)^2} \tag{A12}$$

and

$$x_2' = \frac{3n\left[(n-1) - (n+1)\right] + \sqrt{\gamma^2 (n-1)^4 - 36n^2 c\left[(n-1) - cn\right]}}{3n(n-1)^2}, \tag{A13}$$

where $\gamma = (1-2c)$. We now show that $\varkappa_2'$ given by (A12) does not satisfy Lemma 1, i.e., that $\underline{x} \geq \varkappa_2'$. The equation $\underline{x} - \varkappa_2'$ is quadratic in c and has two roots, given by

$$c(n) = \frac{\left(n^2 + 2n - 2\right) - n\sqrt{n^2 + 6n + 33}}{(n^2 + 16n - 4)} \tag{A14}$$

and

$$C(n) = \frac{\left(n^2 + 2n - 2\right) + n\sqrt{n^2 + 6n + 33}}{(n^2 + 16n - 4)}. \tag{A15}$$

The lower root $c(n)$ is negative. Furthermore, the second derivative of $\underline{x} - \varkappa_2'$ with respect to c is negative, which implies that this function has a global maximum in c. Therefore $\underline{x} - \varkappa_2'$ is positive (i.e., $\underline{x} \geq \varkappa_2'$) for any $c \in [0, C(n)]$. Since we are in the case $c < 1/n$, a sufficient condition to have $\underline{x} \geq \varkappa_2'$ is that $1/n < C(n)$. A sufficient condition for this last inequality to be satisfied is that $1/n < \left(n^2 + 2n - 2\right)/(n^2 + 16n - 4)$ or that $n\left[n(n+1) - 18\right] + 4 > 0$, which is indeed verified for any $n \geq 4$. It follows that $\varkappa_2'$ does not satisfy Lemma 1.

The punishment output x_2' is relevant if $\tilde{x} \leq x_2' < \hat{x}'$. The inequality $x_2' < \hat{x}'$ is equivalent to $c > h(n)$. Furthermore, $x_2' - \tilde{x}$ is quadratic in c and hence $x_2' - \tilde{x} = 0$ has two roots, given by -1 and

$$g(n) = \frac{n-4}{5n-2}. \tag{A16}$$

Since the second derivative of $x_2' - \tilde{x}$ with respect to c is negative, $x_2' - \tilde{x}$ has a global maximum and hence $x_2' \geq \tilde{x}$ only for $c \leq g(n)$.[7]

Recall that we are now in the case $n < 9$ and $h(n) < c < 1/n$. First, one can observe that the sign of $g(n) - 1/n$ is the same as the sign of (again) $n^2 - 9n + 2$, which is negative for $n < 9$ (thus implying

[7] With this constraint, we can now verify that the term under the radical in the numerator of (A13), i.e., $(1-2c)^2 (n-1)^4 - 36n^2 c\left[(n-1) - cn\right]$, is positive for any $c \leq g(n)$. The derivative of this term with respect to c is $-4(1-2c)^2(n-1)^4 - 36n^2\left[(n-1) - 2cn\right]$, which is negative since $c < 1/n < (n-1)/2n$ for any $n \geq 4$. Now replacing c by $g(n)$ in the term under the radical in (A13), we obtain $\left[3(n^3 - 8n^2 + 3n - 2)/(5n-2)\right]^2$, which is strictly positive. This implies that the term under the radical in the numerator of (A13) is strictly positive for any $c \leq g(n)$.

$g(n) < 1/n$). Second, we must verify that $g(n) > h(n)$. We have

$$g\left(n\right)-h(n)=\frac{3\left[n(5n-2)-(n+2)\sqrt{\left(n-1\right)^{3}+9n^{2}}\right]}{2(5n-2)\left[\sqrt{\left(n-1\right)^{3}+9n^{2}}\right]}. \tag{A17}$$

The numerator of this expression is positive if $-n^5+15n^4-51n^3-31n^2-8n+4 \geq 0$ or if $n^3(-n^2+15n-51) \geq 31n^2+8n-4$. The right-hand term of this inequality is always positive, while the left-hand term is positive only if $n \in \left[\left(15-\sqrt{21}\right)/2, \left(15+\sqrt{21}\right)/2\right] \simeq [5,21; 9,79]$. Therefore, a necessary condition for $g\left(n\right)-h(n)$ to be positive is that $n = \{6,7,8\}$ since we are now assuming that $n < 9$. However, one can easily verify that the above inequality is not satisfied for $n = 6$, but only for $n = 7$ or $n = 8$.[8] Hence, x_2' solves (18) only for $n \in \{7,8\}$ and $h(n) < c \leq g(n) < 1/n$.

Finally, to ensure that x_2' is the MCSP, we need to verify that $\delta^P\left(x\right) = \left[\pi^{DP}\left(x\right)-\pi^{P}\left(x\right)\right]/[\pi^C - \pi^P\left(x\right)]$ is also increasing in x on $[x_2', \hat{x}')$. When $x < \hat{x}'$, we have

$$\delta^{P}\left(x,n,c\right)=\frac{9n\left[4cx+\left[(1-c)-x(n-1)\right]^{2}\right]}{4\left[9nxc+(1-2c)^{2}\right]}. \tag{A18}$$

Calculating the derivative of this expression with respect to x, we have

$$\frac{\partial\delta^{P}\left(x,n,c\right)}{\partial x}=\frac{9n\left\{x(n-1)^{2}\left[9nxc+2(1-2c)^{2}\right]-(1+c)\left[c^{2}(n-8)-c\left(3n-8\right)+2(n-1)\right]\right\}}{4\left[9nxc+(1-2c)^{2}\right]^{2}}. \tag{A19}$$

We need to show that the numerator of this expression is positive for any $x \geq x_2'$. Denote by $\Psi(x,n,c)$ the term in $\{.\}$ in the numerator of this expression. Calculating the derivative of $\Psi(x,n,c)$ with respect to x, we obtain $\partial\Psi(x,n,c)/\partial x = 18n(n-1)^2[9nxc+(1-2c)^2]$, which is positive, implying that $\Psi(x,n,c)$ is increasing in x. Therefore, a sufficient condition for the numerator of $\partial\delta^P\left(x,n,c\right)/\partial x$ to be positive for any $x \geq x_2'$ is that $\Psi(x,n,c)\left|_{x=x_2'}\right. \geq 0$. Unfortunately, one cannot obtain the sign of $\Psi(x,n,c)\left|_{x=x_2'}\right.$ independently of n and c. Therefore, we also calculate the derivative of $\Psi(x,n,c)$ with respect to c. We obtain $\partial\Psi(x,n,c)/\partial c = x(n-1)^2[9nx+8\left(1-2c\right)]+(n-6)+24c^2+nc(4-3c)$, which is always positive since $n \in \{7,8\}$ and $c < 1/n$. Therefore, a necessary and sufficient condition for $\partial\delta^P\left(x,n,c\right)/\partial x$ to be positive for any $x \geq x_2'$ and $c \in (h(n), g(n)]$ is that $\Psi(x,n,c)\left|_{x=x_2',c=h(n)}\right. \geq 0$ when $n = 7$ and $n = 8$. When $n = 7$, we have $\Psi(x,n,c)\left|_{x=x_2',c=h(n),n=7}\right. = 6174\sqrt{73}(\sqrt{73}-7)/5329 \simeq 15.28 > 0$, while when $n = 8$, we have $\Psi(x,n,c)\left|_{x=x_2',c=h(n),n=8}\right. = 82944\sqrt{919}(\sqrt{919}-24)/844561 \simeq 18.80 > 0$. It follows that $\delta^P\left(x,n,c\right)$ given by (A18) is increasing in x for $x \geq x_2'$, and hence x_2' is the MCSP when $n \in \{7,8\}$ and $c \in (h(n), g(n)]$.

[8] Using Mathematica, one finds that the relevant roots which solve $g(n)-f(n)=0$ are given by $\hat{n} = \left[6+(378-3\sqrt{1137}])^{1/3}+(378+3\sqrt{1137})^{1/3}\right]/3 \simeq 6,78$ and $\tilde{n} = \left(9+\sqrt{73}\right)/2 \simeq 8.77$.

Suppose now that $n \in \{7,8\}$ and $h(n) < g(n) < c < 1/n$. Then, neither x_1 nor x_2' can be the MCSP. If (18) admits a solution, then it must be lower than $\tilde{x}$, implying $\pi^{DP}(x) \geq 0$ and $V^P \geq 0$. In this case, we have $\pi^P(x) = x[(\gamma - nx)/2]$ and $\pi^{DP}(x) = [(\gamma - x(n-2))/4]^2$. As previously shown, (18) admits one solution satisfying Lemma 1 which is given by (A6) provided that $x_1'' \leq \tilde{x}$. One can easily check that this last constraint is equivalent to $c > g(n)$. Now, again assuming that $c > h(n)$, suppose that $n < 7$ (implying $1/n > h(n) > g(n)$). In that case, we also have that neither x_1 nor x_2' can be the MCSP. Moreover, the inequality $c > g(n)$ is necessarily satisfied since $c > h(n)$. Thus, when $n \leq 7$ and $h(n) < c < 1/n$, the solution to (18) is also given by x_1''. Finally, to ensure that x_1'' is the MCSP, we need to verify that $\delta^P(x) = \left[\pi^{DP}(x) - \pi^P(x)\right]/[\pi^C - \pi^P(x)]$ is also increasing in x on $[x_1'', \tilde{x})$. For any $x < \tilde{x}$, $\delta^P(x)$ is still given by (A7), which has been shown to be increasing in x for any $x \geq \gamma/(n+2)$. Since $x_1'' > \gamma/(n+2)$, x_1'' is the MCSP for $n \in \{7,8\}$ and $g(n) < c < 1/n$ or $n < 7$ and $h(n) < c < 1/n$.

6.4 The minimum discount factor $\delta_2'(n,c)$

We show here that $\delta_2'(n,c)$ is strictly lower than 1 and that it is increasing in c for $h(n) < c \leq g(n) < 1/n$ and $n = \{7,8\}$. When $n = 7$, we have

$$\delta_2'(7,c) = \frac{18\gamma^2}{7\left[\lambda_1(c) + c\sqrt{\lambda_2(c)}\right]}, \tag{A20}$$

where $\lambda_1(c) = 2 + 13c - 20c^2$ and $\lambda_2(c) = 36 - 438c + 487c^2$. We have $\lambda_1'(c) = 13 - 40c$, which is positive for any $c < 1/7$. Hence, $\lambda_1(c)$ is increasing in c, so that a sufficient condition to have $\lambda_1(c) > 0$ for any $c \in (h(7), g(7)]$ is that $\lambda_1(0) = 2 > 0$. Similarly, we have $\lambda_2'(c) = -438 + 974c$, which is negative for any $n < 1/7$. Hence, $\lambda_2(c)$ is decreasing in c and reaches a minimum at $c = g(7) = 1/11$. We have $\lambda_2(1/11) = 25/121 > 0$, implying that $\lambda_2(c)$ is positive for any $c \in (h(7), g(7)]$. Furthermore, the derivative of $\delta_2'(7,c)$ with respect to c is given by

$$\frac{\partial \delta_2'(7,c)}{\partial c} = \frac{-18\gamma\left[\lambda_3(c) + 7(3-2c)\sqrt{\lambda_2(c)}\right]}{7\sqrt{\lambda_2(c)}\left[\lambda_1(c) + c\sqrt{\lambda_2(c)}\right]^2}, \tag{A21}$$

where $\lambda_3(c) = 36 - 585c + 536c^2$. We have $\lambda_3'(c) = -585 + 1072c$, which is negative for any $c < 1/7$. Hence, $\lambda_3(c)$ is decreasing in c, and furthermore is negative at $c = h(7) = \left[\sqrt{73} - 7\right]/2\sqrt{73}$, implying that $\lambda_3(c)$ is negative for any $c \in (h(7), g(7)]$. We then need to evaluate the sign of $\Psi(c) = \lambda_3(c) + 7(3-2c)\sqrt{\lambda_2(c)}$. We have $\Psi'(c) < 0$ since $\lambda_2'(c) < 0$ and $\lambda_3'(c) < 0$. Evaluating $\Psi(c)$ at $c = h(7)$, we have $\Psi(c) = 0$, and hence $\Psi(c)$ is negative for any $c \in (h(7), g(7)]$. This implies that $\delta_2'(7,c)$ is increasing in c and that it reaches a maximum at $c = g(7) = 1/11$. We have $\delta_2'(7,c)\big|_{c=g(7)} = 729/1295 \simeq 0.5629$, and hence $\delta_2'(7,c) < 1$ for any $c \in (h(7), g(7)]$. Furthermore, $\delta_2'(7,c)$ reaches a minimum at $c = h(7)$ and we have that $\delta_2'(7,c)\big|_{c=h(7)} = \delta_1(7) = 9/16 = 0.5625$. Since we also have $\delta_2'(7,c)\big|_{c=g(7)} < \delta_1'(7) = 81/41 \simeq 0.5786$, we can conclude that $\delta_1(7) < \delta_2'(7,c) < \delta_1'(7)$ for any $c \in (h(7), g(7)]$.

When $n = 8$, we have

$$\delta_2'(8,c) = \frac{2401\gamma^2}{32\left[\lambda_4(c) + 3c\sqrt{\lambda_5(c)}\right]}, \tag{A22}$$

where $\lambda_4(c) = 49+308c-452c^2$ and $\lambda_5(c) = 2401-25732c+28036c^2$. We have $\lambda_4'(c) = 308-904c$, which is positive for any $c < 1/8$. Hence, $\lambda_4(c)$ is increasing in c, so that a sufficient condition to have $\lambda_4(c) > 0$ for any $c \in (h(8), g(8)]$ is that $\lambda_4(0) = 49 > 0$. Similarly, we have $\lambda_5'(c) = -25732 + 56072c$, which is negative for any $n < 1/7$. Hence, $\lambda_5(c)$ is decreasing in c, and reaches a minimum at $c = g(8) = 2/19$. We have $\lambda_5(2/19) = 1089/361 > 0$, implying that $\lambda_5(c)$ is positive for any $c \in (h(8), g(8)]$. Furthermore, the derivative of $\delta_2'(8,c)$ with respect to c is given by

$$\frac{\partial\delta_2'(8,c)}{\partial c} = \frac{-7203\gamma\left[\lambda_6(c) + 24\,(7-4c)\sqrt{\lambda_5(c)}\right]}{32\sqrt{\lambda_5(c)}\left[\lambda_4(c) + 3c\sqrt{\lambda_5(c)}\right]^2}, \tag{A23}$$

where $\lambda_6(c) = 2401 - 33796c + 30340c^2$. We have $\lambda_6'(c) = -33796 + 60680c$, which is negative for any $c < 1/8$. Hence, $\lambda_6(c)$ is decreasing in c, and furthermore is negative at $c = h(8) = \left[\sqrt{919} - 24\right]/2\sqrt{919}$, implying that $\lambda_6(c)$ is negative for any $c \in (h(8), g(8)]$. We then need to evaluate the sign of $\Lambda(c) = \lambda_6(c) + 24\,(7-4c)\sqrt{\lambda_5(c)}$. We have $\Lambda'(c) < 0$ since $\lambda_5'(c) < 0$ and $\lambda_6'(c) < 0$. Evaluating $\Lambda(c)$ at $c = h(8)$, we have $\Lambda(c) = 0$, and hence $\Lambda(c)$ is negative for any $c \in (h(8), g(8)]$. This implies that $\delta_2'(8,c)$ is increasing in c and that it reaches a maximum at $c = g(8) = 2/19$. We have $\delta_2'(8,c)\big|_{c=g(8)} = 175/288$, and hence $\delta_2'(8,c) < 1$ for any $c \in (h(8), g(8)]$. Furthermore, $\delta_2'(8,c)$ reaches a minimum at $c = h(8)$, and we have that $\delta_2'(8,c)\big|_{c=h(8)} = \delta_1(8) = 49/81 \simeq 0.6049$. Since we also have $\delta_2'(8,c)\big|_{c=g(8)} = \delta_1'(8) = 175/188 \simeq 0.6076$, we can conclude that $\delta_1(8) < \delta_2'(8,c) < \delta_1'(8)$ for any $c \in (h(8), g(8)]$.

Finally, we compare $\delta_2'(7,c)$, $\delta_2'(8,c)$, $\delta_1'(6)$ and $\delta_1(9)$. We have

$$\delta_2'(8,c) - \delta_2'(7,c) = \frac{\gamma\left[\lambda_7(c) + 16807c\sqrt{\lambda_2(c)} + 1728c\sqrt{\lambda_5(c)}\right]}{224\left[\lambda_1(c) + c\sqrt{\lambda_2(c)}\right]\left[\lambda_4(c) + 3c\sqrt{\lambda_5(c)}\right]}, \tag{A24}$$

where $\lambda_7(c) = 5390 + 41083c - 75788c^2$. We have $\lambda_7'(c) = 41083 - 151576c$, which is positive for any $c < 1/7$. Hence, $\lambda_7(c)$ is increasing in c and furthermore $\lambda_7(0) > 0$. It follows that $\delta_2'(8,c) - \delta_2'(7,c) > 0$ for any $c \in (h(7), g(7)] \subset (h(8), g(8)]$. Furthermore, the maximum of $\delta_2'(8,c)$ is (again) $\delta_2'(8,c)\big|_{c=g(8)} = 175/288 \simeq 0.6076$, which is lower than both $\delta_1(9) = (4/5)^2 = 0.64$. Finally, the minimum of $\delta_2'(7,c)$ is (again) $\delta_2'(7,c)\big|_{c=h(7)} = 9/16 = 0.5625$, which is greater than $\delta_1'(6) = 5/9 \simeq 0.5556$. We then have $\delta_1'(6) < \delta_2'(7,c) < \delta_2'(8,c) < \delta_1(9)$.

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