Researcher's Dilemma*

Catherine Bobtcheff[†] Jérôme Bolte[‡] Thomas Mariotti[§]

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Abstract

We propose and analyze a general model of priority races. Researchers privately have breakthroughs and decide how long to let their ideas mature before disclosing them, thereby establishing priority. Two-researcher, symmetric priority races have a unique equilibrium that can be characterized by a differential equation. We study how the shape of the breakthrough distribution and of the returns to maturation affect maturation delays and research quality, both in dynamic and comparative-statics analyses. Making researchers better at discovering new ideas or at developing them has contrasted effects on research quality. Being closer to the technological frontier enhances the value of maturation for researchers, which mitigates the negative impact on research quality of the race for priority. Finally, when researchers differ in their abilities to do creative work or in the technologies they use to develop their ideas, more efficient researchers always let their ideas mature more than their less efficient opponents. Our theoretical results shed light on academic competition, patent races, and innovation quality.

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[†]Toulouse School of Economics (CNRS, LERNA).
[‡]Toulouse School of Economics (GREMAQ).
[§]Toulouse School of Economics (CNRS, GREMAQ, IDEI).

1 Introduction

As emphasized by Merton (1957), one of the primary concerns of scientists is to establish priority of discovery by being first to disclose an advance in knowledge. The recognition for being first depends on the importance attached to the discovery by the scientific community and comes in different forms, from publications and grants to awards or prizes of various prestige.[1] The situation is similar for inventors, who can establish their claim to being first to achieve breakthrough innovation by filing for a patent.

Whereas, as pointed out by Dasgupta and David (1987, 1994), the priority system is essential for granting intellectual property rights to both scientists and inventors, it leaves little room for runner-ups, leading to a highly skewed distributions of rewards. There may valid efficiency reasons for this inequality in rewards, because, as observed by Dasgupta and Maskin (1987) the winning research unit is usually the main contributor to social surplus. Inequality in rewards may also be efficient in inducing certain types of effort, as emphasized in the contest literature.[2] However, this winner-take-all feature increases researchers' fear of preemption and hence their perceived need to publish quickly or to rush patent applications. For instance, according to Stephan (1996), "The probability of being scooped is a constant threat" in academic competition and, as a result, "It is not unknown for scientists to write and submit an article in the same day." The intensity of this race for priority naturally raises the question of the quality of research outputs, which is of great concern given the driving role of research and innovation emphasized by the endogenous-growth literature.[3]

This tension between research quality and preemption risk is relevant in most innovative sectors. Consider first a scientist developing a new theory. Should she publish preliminary results to ensure priority against her potential opponents, for fear that her idea increasingly becomes "in the air"? Or should she wait to present a more mature theory by increasing the amount of evidence in favor of it and answering most likely objections before publishing?[4] Consider next a software company developing a new application. Should it launch it on the market at the earliest opportunity? Or should it wait to verify that the software is completely free of bugs and to develop the interface to make it user-friendly and compatible with other applications? Consider finally a pharmaceutical firm working on a promising new molecule, a breakthrough it keeps secret to avoid duplication by potential competitors. How many tests should it conduct to assess the impact of the molecule and its possible side

[1] See Stephan and Levin (1992) for a discussion of rewards for scientific achievements.

[2] See, for instance, Lazear and Rosen (1981), or Nalebuff and Stiglitz (1983).

[3] See, for instance, Romer (1990) and Aghion and Howitt (1992).

[4] A case in point is the publication by Charles Darwin of his theory of evolution through natural selection. After his attention was drawn in 1856 to a paper by the naturalist Alfred Russel Wallace on the "introduction of new species," Darwin was torn between the desire to produce a complete account of his theory and its applications, and the urgency of publishing a short paper summarizing its main insights. It is only when, upon receiving in 1858 a second parcel from Wallace, Darwin realized that he had been "forestalled" and thus was running the risk of losing priority, that he decided to "publish a sketch of [his] general views in about a dozen pages or so" (Darwin (1887, pages 116–117)). In the end, Darwin's and Wallace's papers were jointly read at the Linnean Society on July 1, 1858 (Desmond and Moore (1991)).

effects before submitting a new drug application? The more tests are run, the higher the effectiveness and safety of the final drug and, therefore, the probability that the submission will be successful, but the higher the risk that a competitor preempts by putting a substitute product on the market.

The objective of this paper is to offer a parsimonious yet flexible theoretical framework allowing one to study the impact of the race for priority on individual research strategies and innovation quality. Our model builds on three stylized features of priority races that can be abstracted from the above examples.

First, having a breakthrough is not in itself sufficient to deliver an accomplished piece of scientific work, a marketable product, or a valuable innovation. That is, one of the determinants of the final quality of new ideas is the time spent maturing them. This shifts the focus of the analysis to the tradeoff faced by researchers between letting their ideas mature optimally and risking being preempted by their opponents.

Second, breakthroughs are privately observed by researchers, if only because they have an incentive to keep their research agendas secret to avoid being imitated and let their ideas mature in relative safety. A consequence of this is that, to some extent, a researcher developing a new idea works in the dark: she does not know whether she has an active opponent until it is too late and she realizes she has been preempted.

Third, the context in which new ideas are first discovered and then developed changes over time. Among the relevant factors are the by-products of economic growth itself, such as technological progress, human-capital accumulation, improved access to information or innovations in related fields, and more efficient institutions. As a result, there is an inherent nonstationarity in the innovation process.

In line with these broad features, we propose to model priority races as Bayesian games in which researchers first exogenously have breakthroughs that they each privately observe, and then choose how long to let their ideas mature before disclosing them. Depending on the interpretation of the model, disclosure may consist in submitting a paper to a scientific journal, filing for a patent, or putting a new product on the market. In analogy with an auction, our model is thus fully specified by two functions.

The first is the distribution of researchers' breakthrough times, which describes how good they are at discovering new ideas. The rate at which breakthroughs occur can vary over time. Such variations may reflect the evolution of technology or human capital, the growth of the research community, or exogenous fashion trends that dictate whether a given area of research becomes "hotter" or "colder" over time.[5]

The second is the payoff from being first, which describes the returns to maturation. These returns may depend on technological factors, such as how good researchers are at

[5]Building on Kuhn's (1962) terminology, Stephan (1996) notes that, in academic competition, the former scenario is particularly likely "in the case of "normal" science where the accumulated knowledge and focus necessary for the next scientific breakthrough is "in the air"," while the latter scenario may arise because "Scientists [can choose to] minimize the threat of being scooped by choosing to work on problems that fall outside the mainstream of "normal science" or by working in the "backwaters" of research."

developing new ideas. But they may also be affected by the institutional context in which research takes place, such as the efficiency of scientific journals and patent offices, or the steepness of the incentives faced by researchers. Our model is flexible enough to allow for technological progress or institutional change.

Our main theoretical contribution is that, in the two-researcher, symmetric case, the priority race admits a unique equilibrium that is characterized by a differential equation subject to a specific boundary condition. This equation reflects the tension between the rate at which a researcher's payoff from being first grows as her idea matures and the rate at which her opponent has breakthroughs. It is used throughout the paper to derive the main economic implications of the model.

The evolution of the breakthrough rate is key to the qualitative features of equilibrium. When breakthroughs become more frequent, researchers are under increasing competitive pressure and have decreasing incentives to wait and let their ideas mature. Thus ideas developed later on are less elaborated than ideas developed earlier on. This points at a new kind of Schumpeterian effect: by making researchers better at discovering new ideas over time, growth may have a detrimental effect on innovation quality by intensifying the race for priority that takes place between them. This leads scientists to publish "quick-and-dirty" papers or inventors to prematurely patent their innovations. Hence, whereas a broad prediction of endogenous-growth models is that higher innovativity has a positive impact on innovation and in turn on growth,[6] the reverse prediction holds in our model because of increased preemption risk. An important countervailing effect on the quality of research output is that researchers, because of growth-induced technological progress or institutional change, may become better at developing their ideas or may face steeper incentives. Then researchers' marginal gain of letting their ideas mature relative to the potential loss if they are preempted is higher. In contrast with an increase in the breakthrough rate, such increases in research projects' growth potential reduce researchers' fear of preemption and slow down the maturation process.

This tension between the prospection phase, in which researchers have breakthroughs, and the development phase, in which researchers let their breakthroughs mature, raises the question of the quality of research outputs. If the development technology does not change over time, a decline in maturation delays is associated to a decline in quality. This increasing relationship need no longer hold if the development technology improves over time. For instance, better access to relevant information or improved technology is likely to reduce the time necessary to bring an idea to the same level of development; as a result, hours spent developing a late breakthrough are likely to be more productive than those spent developing an early breakthrough. To formalize this idea, we develop an explicit model of research quality in which researcher's returns to maturation depend on when they have breakthroughs. This allows us, in particular, to study the impact of the distance to the technological frontier on the quality of research output. We show that, the more researchers

[6] See, for instance, Aghion and Howitt (1998, 2005).

can benefit from ongoing technological progress while developing their breakthroughs, the higher the quality of their output, even though they may end up devoting less time to this task. This reflects that being closer to the frontier enhances the value of maturation, which mitigates the negative impact of the race for priority.

Whereas our basic model focuses on symmetric priority races, research contests feature in practice important asymmetries (Stephan (1996)). First, researchers differ in their abilities to do creative work; that is, in our terminology, more innovative researchers have more frequent breakthroughs. Second, researchers differ in their motivations or in the technologies they use to develop their ideas; that is, in our terminology, researchers may engage in projects with more or less growth potential. We show that a common feature of these two dimensions of heterogeneity is that more efficient researchers always let their ideas mature more than their less efficient opponents. Thus, for instance, speed of discovery and quality of research output are positively correlated. An important difference, however, is that widening the gap between more and less innovative researchers increases the risk of preemption and thus tends to deteriorate the overall quality of research outputs; by contrast, increasing the growth potential of some researchers' projects, for instance by subjecting them to steeper incentive schemes or by rewarding them more for long-term performance, mitigates the impact of preemption risk and thus tends to enhance the overall quality of research outputs.

Related Literature A first way to model competition among researchers is to represent it as a contest, in line with Lazear and Rosen (1981), Nalebuff and Stiglitz (1983), or Dasgupta and Maskin (1987). In these models, each contestant makes an ex-ante effort or project-choice decision, which stochastically influences the quality of her individual output. The winner is the contestant who offers the highest-quality output. Lazear and Rosen (1981) provide conditions under which a well-chosen prize for winning the contest induces first-best effort choices, while the compensation for runner-ups is set so as to induce participation to the contest. In a model of research-portfolio choices, Dasgupta and Maskin (1987) show that researchers may select projects having an expected yield higher than the expected yields of the projects in an efficient portfolio, reflecting that each researcher is induced to spend more than is socially efficient so as to give herself a better chance of winning the contest. Besides being explicitly dynamic, our model of quality is different in that, instead of making an ex-ante decision, each researcher can condition her maturation decision on her privately-observed breakthrough time. This leads to the reverse prediction that, to ensure priority, researchers may be lead to downgrade the quality of their output.

It is also interesting to contrast our assumptions and results to those of the literature on R&D races with step-by-step innovation (Harris and Vickers (1987), Aghion, Harris, and Vickers (1997), and Aghion, Harris, Howitt, and Vickers (2001)). In these models, researchers control the intensity with which they have breakthroughs, but not the value of a breakthrough; in particular, there is no role for maturation. Correspondingly, the patent race, or any of its steps, is over as soon as a breakthrough occurs, an event which is assumed to be publicly observable. Our model is dual in that breakthroughs are exogenous, but researchers can enhance the quality of their ideas by choosing how long to let them mature; this, in turn, is made possible by the fact that breakthroughs are privately observed. As a result, the impact of an increase in competition is different. Indeed, in R&D races with step-by-step innovation, an increase in product-market competition fosters innovation by increasing the incremental value of innovation, an "escape-competition" effect. In our model, an increase in the competitive pressure, as measured by the breakthrough rate, not only increases innovativity, but also the disclosure of innovations. But the interpretation is different because, for a fixed development technology, shorter maturation delays lead to lower-quality research outputs. This suggests, in particular, that one should be cautious in interpreting an increase in the patent rate as a positive indicator of R&D productivity, as it may only reflect an intensification of the race for priority.

From a theoretical viewpoint, this paper belongs to the literature on preemption games, that is, timing games with a first-mover advantage. In a seminal paper on the strategic adoption of new technology, Fudenberg and Tirole (1985) show in a complete-information setting that there always exists a subgame-perfect equilibrium in which firms' payoffs are equalized and rents are fully dissipated. This does not arise in our setting because wasteful competition is alleviated by the asymmetry of information between players. As a result, there is a genuine tradeoff between the gains from letting one's project mature and the risk of preemption. Weeds (2002), extending Fudenberg and Tirole's (1985) analysis, studies a real-option model of R&D competition in which firms invest into research projects. Once the decision to invest has been made by a firm, a breakthrough occurs randomly at a time that is drawn from an exponential distribution. The first firm that has a breakthrough immediately wins the R&D race, as in R&D races with step-by-step innovation. By contrast, breakthrough times in our model are the researchers' private information and our analysis focuses on the endogenous wedge between the occurrence of a breakthrough and the disclosure of the corresponding research.

The idea that players in preemption games may face uncertainty about whether they have active competitors has first been introduced by Hendricks (1992), who extends Fudenberg and Tirole's (1985) analysis to the case where it is determined at the outset of the game whether firms are innovators or imitators, in which case they cannot move first. Innovators have an incentive to build a reputation for being imitators, which alleviates rent dissipation; they reveal their information gradually by playing according to a mixed strategy. Bobtcheff and Mariotti (2012) consider a setting closer to the one developed in the present paper, in which players randomly and secretly come into play; they show that all equilibria give rise to the same distribution for each player's moving time. However, in their model, as in Fudenberg and Tirole (1985) and Hendricks (1992), the payoff a player derives from being first only depends on calendar time and not, as in our model, on the time elapsed since she had a breakthrough; as a result, a player who comes into play late is not at a disadvantage relative to one who came into play earlier on but did not make a move in the meanwhile.

By contrast, we consider situations in which ideas take time to mature, which is a more appropriate assumption in the case of priority races.

Closely related to this paper, Hopenhayn and Squintani (2014) consider a sequential model of R&D races in which research builds on previously patented products. In a given race, firms have breakthroughs at a constant common rate, and decide when to disclose them and file for a patent. A constant proportion of firms is randomly selected to participate in each race. Therefore, a firm that joins in a race does not know whether it will take part in the subsequent ones. There exists a symmetric equilibrium in which firms wait a constant amount of time following a breakthrough. This allows for detailed comparative-statics analyses and for an explicit comparison between the equilibrium outcome and the social optimum. Weaker patents may be socially beneficial, as they reduce competition and lead firms to postpone patenting. Our model is simpler in that we focus on a single race. However, by allowing for arbitrary breakthrough distributions and time-dependent payoff functions, we can study how researchers' strategies evolve in nonstationary environments in which their ability to discover and develop new ideas changes over time. Another distinctive feature of our analysis is to study the impact of heterogeneity among researchers.

The paper is organized as follows. Section 2 describes the model. Section 3 provides our main characterization result. Section 4 draws the economic implications of the model. Section 5 extends the analysis to a class of asymmetric priority races. Section 6 concludes.

2 The Model

2.1 The General Framework

Time is continuous and indexed by $t \geq 0$. There are two symmetric players, a and b. In what follows, i refers to an arbitrary player and j to her opponent. Player i comes into play at some random time $\tilde{\tau}^i$ at which she has a breakthrough. Calendar time is common knowledge; time zero can be interpreted as the date of a common knowledge event, such as a pioneering discovery, that in turn enables players to have breakthroughs.[7]

2.1.1 Actions and Payoffs

Each player i has a single opportunity to make a move. Making a move consists for player i in disclosing her research, thereby establishing priority. As in Bobtcheff and Mariotti (2012) and Hopenhayn and Squintani (2014), this must occur at some time $t^i \geq \tilde{\tau}^i$, for a maturation delay $t^i - \tilde{\tau}^i$. We say that player i preempts player j if $t^i < t^j$. To capture the winner-take-all feature of priority races emphasized by Merton (1957), we assume that, in that case, player j's payoff is zero.

Player i's payoff from preempting player j first depends on her maturation delay, reflecting

[7]Hence players' clocks are synchronized, unlike in Abreu and Brunnermeier (2003), Brunnermeier and Morgan (2010), or Barbos (2015).

that a development period is necessary to bring a breakthrough to fruition. It may also depend on her breakthrough time, reflecting that the development technology is susceptible to change over time. Formally, consider player i with breakthrough time τ^i. Then the present value, evaluated at time τ^i, of her preempting player j at time t^i is $L(t^i - \tau^i, \tau^i)$.[8] This present value is lowered to $\alpha L(t^i - \tau^i, \tau^i)$, for some $\alpha \in [0, 1]$, if players i and j simultaneously make a move at time t^i.

We first describe how payoffs vary with maturation delays for a fixed breakthrough time.[9]

Assumption 1 *The function L is continuous over $[0, \infty) \times [0, \infty)$ and thrice continuously differentiable over $(0, \infty) \times (0, \infty)$. For each τ, there exists $M(\tau) > 0$ such that*

$$
\begin{aligned}
&L(0, \tau) = 0 \text{ and } L(m, \tau) > 0 \text{ if } m > 0, \\
&L_1(m, \tau) > 0 \text{ if } M(\tau) > m > 0 \text{ and } L_1(m, \tau) < 0 \text{ if } m > M(\tau), \\
&L_{11}(M(\tau), \tau) < 0.
\end{aligned}
$$

Hence some maturation time is required for a time-τ breakthrough to have positive value,[10] whereas maturing it more than $M(\tau)$ is counterproductive. We refer to $M(\tau)$ as to the stand-alone maturation delay for a player with breakthrough time τ who would not be threatened by preemption.[11] By contrast, if it were common knowledge that both players had a breakthrough at time τ, each player at time τ would be indifferent between making a move or abstaining. Subgame-perfect-equilibrium maturation delays would then be zero, leading to full dissipation of players' rents.[12]

We next restrict the cross effects of maturation delays and breakthrough times on payoffs.

Assumption 2 *For each τ,*

$$(\ln L)_{11}(m, \tau) < (\ln L)_{12}(m, \tau), \quad M(\tau) \geq m > 0.$$

Equivalently, the mapping $(t, \tau) \mapsto L(t - \tau, \tau)$ is strictly log-supermodular over the relevant range: the relative returns to increasing the moving time t are increasing in the breakthrough time τ. This form of complementarity captures the idea that progress in the development technology is not too drastic. In particular, each player's stand-alone moving time, $\tau + M(\tau)$, is strictly increasing in τ, despite the fact that a player with a late breakthrough may have access to a better development technology.

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[8] This specification of payoffs encompasses Hopenhayn and Squintani's (2014), in which the payoff only depends on the maturation delay m, while Bobtcheff and Mariotti (2012) study a limit case where the payoff only depends on the moving time $\tau + m$.

[9] Throughout the paper, we use subscripts to denote partial derivatives and dots to denote time derivatives.

[10] This assumption, which is mathematically convenient to establish Lemmas 1–2 below, may seem strong as it rules out that ideas may immediately have value. Note, however, that any payoff function $\hat{L}(\cdot, 0)$ with $\hat{L}(0, \tau) > 0$ and that otherwise satisfies all our assumptions can be arbitrarily closely approximated by a function $L(\cdot, \tau)$ that satisfies all our assumptions, including the requirement that $L(0, \tau) = 0$. All that is takes is that is that the slope of $L(\cdot, \tau)$ in the neighborhood of zero be large enough.

[11] We borrow this terminology from Katz and Shapiro (1987).

[12] See Fudenberg and Tirole (1985) or Simon and Stinchcombe (1989) for a formalization of this result.

2.1.2 Information

The players' breakthrough times $\tilde{\tau}^a$ and $\tilde{\tau}^b$ are independently drawn from a continuously differentiable distribution G with positive density $\dot{G}$ over $[0, \infty)$.[13] Given our interpretation of time zero, the independence of breakthrough times should be understood conditionally on the event that takes place at this time and makes breakthroughs possible. Whereas the breakthrough distribution is common knowledge, the breakthrough time of each player is her private information, or type. Thus, as in Hendricks (1992), Bobtcheff and Mariotti (2012), or Hopenhayn and Squintani (2014), competition is only potential: a player never knows she has an active opponent until it is too late and she has been preempted. Unlike in Hopenhayn and Squintani (2014), the rate at which players have breakthroughs can vary over time.

2.1.3 Strategies and Equilibria

A strategy for player i is a function $\sigma^i : [0, \infty) \to [0, \infty)$ that specifies, for each possible value τ^i of her breakthrough time, the time $\sigma^i(\tau^i) \geq \tau^i$ at which she plans to make a move. Player i's payoff if her type is τ^i, player j's strategy is σ^j, and player i plans to make a move at time $t^i \geq \tau^i$ is

$$V^i(t^i, \tau^i, \sigma^j) \equiv \{\mathbf{P}[\sigma^j(\tilde{\tau}^j) > t^i] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = t^i]\} L(t^i - \tau^i, \tau^i). \tag{1}$$

A pair (σ^a, σ^b) is an equilibrium if for all i, τ^i, and $t^i \geq \tau^i$,

$$V^i(\sigma^i(\tau^i), \tau^i, \sigma^j) \geq V^i(t^i, \tau^i, \sigma^j). \tag{2}$$

Because players' breakthrough and moving times can be arbitrarily large, the only zero-probability events in equilibrium could be some player unexpectedly making a move. As this effectively terminates the game, any equilibrium is Bayesian perfect.

2.1.4 Comparison with First-Price Auctions

As is clear from the definition (1) of payoffs, our priority race bears a strong formal analogy with a first-price procurement auction with risk-averse bidders (Arozamena and Cantillon (2004)): in this context, τ^i is bidder i's monetary cost of providing a good, t^i is the price offered by bidder i, and L is the bidders' common utility function for wealth. However, our model has three novel features. First, and most importantly for the scope of our results, the payoff function L is type- and hence time-dependent, reflecting that the development technology may be nonstationarity in our setting. Second, unlike a standard utility function for wealth, each payoff function $L(\cdot, \tau)$ reaches a maximum at the stand-alone maturation delay $M(\tau)$. Third, whereas costs in procurement-auction models are typically distributed over a bounded interval, or there is a reserve price set by the buyer, breakthrough times

[13] This is in line with patent-race models, which typically assume an infinite horizon. By contrast, it would be somewhat artificial to assume that players' productive lifespan is both finite and commonly known.

in our model can take arbitrarily large values. As we shall see in Section 3, these last two features imply that, although the differential equation that characterizes equilibrium is the same in our model as in a first-price procurement auction, the boundary condition it is subject to is of a very different nature and calls for a specific analysis.

2.2 Examples

The following examples suggest the applicability of our general framework. We will use them in Section 4 to illustrate the economic implications of our model.

2.2.1 Example 1: Publication Delays and Patent Allowances

In academics, a benefit of more maturation time is faster publication. Specifically, let us interpret m as the delay between a breakthrough and the first submission for publication. Submission guarantees priority, but a publication delay $D(m, \tau)$ adds to the maturation delay m, reflecting the number and length of refereeing rounds.[14] We assume that there are positive returns to maturation, $D_1 < 0$: the more elaborate the submission, the shorter the publication delay. We also assume that there are decreasing returns to maturation, $D_{11} > 0$, and that very immature works get stuck in the publication process, $\lim_{m \downarrow 0} \{D(m, \tau)\} = \infty$ for all τ. Researchers compete for a unit publication prize and discount future rewards at rate r, so that $L(m, \tau) = \exp(-r[m + D(m, \tau)])$. Then $M(\tau)$ minimizes total time $m + D(m, \tau)$ from breakthrough to publication.[15] An alternative interpretation is that players are inventors who choose how long to wait before applying for a patent. Then, the more elaborate the application, the higher the probability $\pi \equiv \exp(-rD)$ that it succeeds.[16]

2.2.2 Example 2: Journal Rankings and Researchers' Incentives

In academics, another benefit of more maturation time is to allow one to publish her work in more prestigious outlets. Specifically, let journals be continuously ranked over $[0, \infty)$. There is some noise in the publication process: a paper matured for m time units is published in journal $J(m, \tilde{\varepsilon})$, for some exogenous random variable $\tilde{\varepsilon}$. We assume that there are positive returns to maturation, $J_1 > 0$: for any realization of $\tilde{\varepsilon}$, the more elaborate the paper, the higher the rank of the journal in which it is published. We also assume that very immature papers are published in the worst outlet, $J(0, \varepsilon) = 0$ for all ε. A researcher's instantaneous payoff from preempting her opponent depends on the final outlet j through a reward function $S(j, \tau)$ such that $S(0, \tau) = 0$ for all τ, which stands for monetary incentives,

[14] Alternatively, $D(m, \tau)$ can be interpreted as the time it takes the profession to recognize the importance of a paper's contribution.

[15] Assumption 1 follows from $D_1 < 0$ and $D_{11} > 0$. Assumption 2 moreover requires that $D_{11} > D_{12}$ over the relevant range.

[16] Empirically, this allowance rate is significantly different from 1. Carley, Hegde, and Marco (2015) document that, between 1996 and 2005, 55.8% of US patent applications ended up in a patent without continuation procedures, and that only 11.4% did so without further examination.

career prospects, or future funding for research. Researchers discount future rewards at rate r, so that $L(m, \tau) = \exp(-rm)\mathbf{E}[S(J(m, \tilde{\varepsilon}), \tau)]$.

2.2.3 Example 3: Innovation Quality

Our setting is more generally suited for the study of innovation quality. We just sketch here the relevant model, which is described in more detail in Section 4.2. Let $Q(m, \tau)$ be the quality of an innovation achieved by a researcher with breakthrough time τ who spends m time units maturing her breakthrough; quality may depend on the breakthrough time because of progress in the development technology. A researcher's instantaneous payoff from preempting her opponent depends on the achieved level of quality q through an inverse demand function $P(q)$. Researchers discount future rewards at rate r, so that $L(m, \tau) = \exp(-rm)P(Q(m, \tau))$.

3 Equilibrium Characterization

In this section, we establish that the general priority race described in Section 2.1 has a unique equilibrium. This equilibrium is characterized as the unique solution to an ordinary differential equation (ODE) subject to an appropriate boundary condition. We will rely on this characterization result to spell out the economic implications of our model in Section 4.

3.1 The Fundamental ODE

We first establish three intuitive yet useful properties of equilibria.

Lemma 1 *In any equilibrium,*

(i) $0 < \sigma^i(\tau^i) - \tau^i \leq M(\tau^i)$ *for all* i *and* τ^i*.*

(ii) σ^i *is strictly increasing for all* i*.*

(iii) $\sigma^a(0) = \sigma^b(0)$*.*

Property (i) is twofold. First, a player always prefers to wait before making a move in equilibrium because, no matter when she does so, her opponent's breakthrough may occur later on and thus represent no threat. Second, there is no point for her in waiting longer than in the absence of competition because slightly reducing her maturation delay would then strictly increase her payoff from preempting while not increasing preemption risk. Property (ii) in its weak form follows from a revealed-preference argument, using the complementarity between breakthrough and moving times encapsulated in Assumption 2. That any equilibrium must be separating follows from the fact that, if there were an atom at t^i in the equilibrium distribution of player i's moving time, then player j would never want to make a move at t^i or slightly later than t^i, for fear of being preempted. This would imply that player i faces no preemption risk over some time interval starting at t^i and thus that some types of player i who make a move at t^i would be strictly better off waiting slightly longer to do so. Finally, property (iii) reflects that, if one had $\sigma^i(0) < \sigma^j(0)$, player i, were she to have a breakthrough at time zero, would be strictly better off waiting slightly longer to make a move while still avoiding preemption risk.

For each i, let $\phi^i \equiv (\sigma^i)^{-1}$ be the inverse of σ^i, which is well defined and continuous over $\sigma^i([0,\infty))$ by Lemma 1(ii). Also let $\sigma(0) \equiv \sigma^a(0) = \sigma^b(0)$, which is well defined by Lemma 1(iii). Then the following regularity result holds.

Lemma 2 *In any equilibrium,*

(i) σ^i *is continuous, so that* $\sigma^i([0,\infty)) = [\sigma(0),\infty)$.

(ii) ϕ^i *is differentiable over* $[\sigma(0),\infty)$.

We use Lemma 2(ii) to derive a system of ODEs for the inverses ϕ^a and ϕ^b of σ^a and σ^b. Because σ^a and σ^b are strictly increasing according to Lemma 1(ii), and the breakthrough distribution has no atom by assumption, the probability of a tie is zero. Given (1), the problem faced by type τ^i of player i can then be written as

$$\max_{t^i \in [\tau^i,\infty)} \{\mathbf{P}[\sigma^j(\tilde{\tau}^j) > t^i] L(t^i - \tau^i, \tau^i)\} = \max_{t^i \in [\tau^i,\infty)} \{[1 - G(\phi^j(t^i))] L(t^i - \tau^i, \tau^i)\}. \quad (3)$$

We know from Lemma 1(i) that a zero maturation delay is inconsistent with equilibrium, so that the solution to problem (3) must be interior. The first-order condition is

$$[1 - G(\phi^j(t^i))] L_1(t^i - \tau^i, \tau^i) = \dot{G}(\phi^j(t^i)) \dot{\phi}^j(t^i) L(t^i - \tau^i, \tau^i).$$

Thus player i with breakthrough time τ^i balances the expected marginal benefit from an infinitesimal additional delay $\mathrm{d}t^i$, which is equal to the probability that player j plans to make a move after time $t^i + \mathrm{d}t^i$, $1 - G(\phi^j(t^i + \mathrm{d}t^i))$, multiplied by the marginal payoff gain, $L_1(t^i - \tau^i, \tau^i)\,\mathrm{d}t^i$, with the corresponding expected marginal cost, which is equal to the probability that player j makes a move during $[t^i, t^i + \mathrm{d}t^i]$, $\dot{G}(\phi^j(t^i))\dot{\phi}^j(t^i)\,\mathrm{d}t^i$, multiplied by the foregone payoff, $L(t^i - \tau^i, \tau^i)$. In equilibrium, this first-order condition must hold for $\tau^i = \phi^i(t^i)$, leading to the following system of ODEs:

$$\dot{\phi}^j(t) = \frac{1 - G}{\dot{G}}(\phi^j(t)) \frac{L_1}{L}(t - \phi^i(t), \phi^i(t)), \quad t \geq \sigma(0), \quad i = a, b. \quad (4)$$

By construction, $\phi^a(\sigma(0)) = \phi^b(\sigma(0)) = 0$. Together with (4), this common initial condition rules out asymmetric equilibria.

Lemma 3 *Every equilibrium is symmetric.*

Consider accordingly a symmetric equilibrium with common strategy σ and let $\phi \equiv \sigma^{-1}$.

The system (4) then reduces to the ODE

$$\dot{\phi}(t) = f(t, \phi(t)), \quad t \geq \sigma(0), \tag{5}$$

where, by definition,

$$f(t, \tau) \equiv \frac{1 - G}{\dot{G}}(\tau) \frac{L_1}{L}(t - \tau, \tau), \quad t > \tau. \tag{6}$$

Three observations are worth making at this stage of the analysis. First, (5) holds for all t larger than $\sigma(0)$, which is endogenous. In turn, by Lemma 1(i), $\sigma(0)$ must be such that ϕ stays in the domain $\mathcal{D} \equiv \{(t, \tau) : 0 \leq \tau < t \leq \tau + M(\tau)\}$. This global boundary condition is a novel feature of our model.[17] Second, by Assumption 1, $\lim_{t \downarrow \tau} \{f(t, \tau)\} = \infty$ and $\lim_{t \uparrow \tau + M(\tau)} \{f(t, \tau)\} = 0$ for all $\tau \geq 0$, so that the vector field induced by (6) is outward-pointing at the upper and lower boundaries $\tau = t$ and $\tau + M(\tau) = t$ of $\mathcal{D}$. This reinforces our first observation that the choice of $\sigma(0)$ is key to ensure that ϕ stays in $\mathcal{D}$. Third, a maturation delay $t - \phi(t)$ equal to the stand-alone maturation delay $M(\phi(t))$ cannot arise in equilibrium because, according to our second observation, this would cause ϕ to leave $\mathcal{D}$ through its lower boundary at time t. Thus ϕ must stay in the interior of $\mathcal{D}$, reflecting that preemption risk is never eliminated. These properties are illustrated in Figure 1.

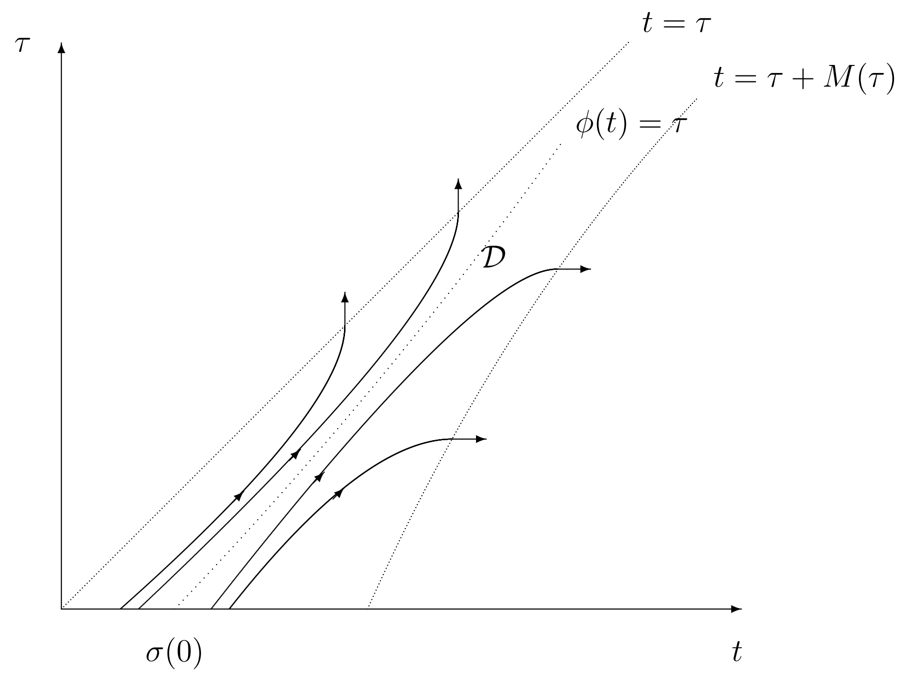

Figure 1 The domain $\mathcal{D}$ and the vector field induced by (6).

It remains to check the second-order conditions for optimality.

[17]If breakthroughs must occur before some common knowledge time T, types close to T face extreme preemption risk and their equilibrium maturation delays must converge to zero. See Lambrecht and Perraudin (2003), Arozamena and Cantillon (2004), and Anderson, Friedman, and Oprea (2010) for related boundary conditions in real-option and auction contexts.

Lemma 4 *Let ϕ be a solution to (5) that stays in $\mathcal{D}$ and let $\sigma \equiv \phi^{-1}$. Then (σ, σ) is an equilibrium.*

3.2 Existence and Uniqueness of Equilibrium

We are now ready to state the first central result of this section.

Theorem 1 *An equilibrium exists.*

The logic of Theorem 1 is as follows. By Lemmas 1–4, an equilibrium exists if and only if, for some $\sigma_0 \in [0, M(0)]$, the solution to (5) with initial condition $(\sigma_0, 0)$ stays in $\mathcal{D}$. Now, if this were not so, all the solutions to (5), indexed by their initial conditions, would leave $\mathcal{D}$. Because the vector field induced by (6) is continuous, we would thereby obtain a continuous mapping from $[0, M(0)]$ onto the space consisting of the lower and upper boundaries of $\mathcal{D}$. This, however, is impossible, as the former space is connected, whereas the latter has two connected components.[18] Hence an equilibrium must exist.

This argument is nonconstructive and does not ensure that a unique equilibrium exists. A priori, multiple initial conditions may be consistent with equilibrium because there is no obvious terminal condition to pin down the behavior of players with late breakthrough times. Moreover, intuitively, if player i decided to uniformly increase her moving time, the value of waiting for player j should also increase. Given these strategic complementarities, it would be natural to expect multiple equilibria to arise in our model. It may thus come as a surprise that, under relatively mild conditions, a unique equilibrium exists. To formulate our uniqueness result, we impose two further restrictions on the payoff functions $\{L(\cdot, \tau) : \tau \geq 0\}$. The first restriction is that stand-alone maturation delays do not diverge to infinity.

Assumption 3 $\liminf_{\tau \to \infty} \{M(\tau)\} < \infty$.

The second restriction is that the functions $\{L(\cdot, \tau) : \tau \geq 0\}$ uniformly satisfy a stronger concavity property than log-concavity. Formally, recall from Avriel (1972) or Caplin and Nalebuff (1991) that a nonnegative real-valued function h defined over an interval $\mathcal{I} \subset \mathbb{R}$ is ρ-concave for some $\rho > 0$ if $h(\lambda x + (1-\lambda) y) \geq [\lambda h(x)^\rho + (1-\lambda) h(y)^\rho]^{1/\rho}$ for all $(\lambda, x, y) \in [0,1] \times \mathcal{I} \times \mathcal{I}$. By Hölder's inequality, a higher value of ρ corresponds to a stronger notion of concavity: for instance, if $\rho = 1$, h is concave, whereas in the limiting case $\rho = 0$, h is log-concave. Our second restriction can then be stated as follows.

Assumption 4 *There exists $\rho > 0$ such that, for each τ, $L(\cdot, \tau)$ is ρ-concave over $[0, M(\tau)]$.*

Under these additional assumptions, the following result holds.

[18] The proof of Theorem 1 is reminiscent of the retraction principle of Ważewski (1947), see also Hartman (1964, Chapter X, Theorem 2.1). For related existence theorems for antifunnels, see Hubbard and West (1991, Theorem 4.7.3). An important difference, however, is that the function f in (6) does not satisfy a global Lipschitz condition over $\mathcal{D}$, so that a specific argument is needed.

Theorem 2 *The equilibrium is unique.*

Thus our model exhibits saddle-path stability: there exists a unique trajectory of (5) that stays in $\mathcal{D}$, a requirement that plays an analogous role in our model as the transversality condition in the Ramsey growth model. The intuition for this result is that two solutions to (5) with different initial conditions are not only ordered but also tend to drift apart from each other. Indeed, because the functions $\{L(\cdot, \tau) : \tau \geq 0\}$ are uniformly log-concave, if there are two equilibria with strategies $\sigma_1 > \sigma_2$, then the marginal gain L_1/L from slightly delaying one's move relative to the loss if she is preempted is higher under σ_1 than under σ_2. As a result, $\dot{\sigma}_1 > \dot{\sigma}_2$: the less a player feels threatened by preemption, the more she is willing to wait at the margin. Assumption 4 yields a lower bound on the rate at which σ_1 drifts apart from σ_2, namely,

$$\sigma_1(\tau) - \sigma_2(\tau) \geq [\sigma_1(0) - \sigma_2(0)] \exp\left(\rho \int_0^\tau \frac{\dot{G}}{1 - G}(\theta) \, \mathrm{d}\theta\right)$$

for all τ. As $\rho > 0$, the right-hand side of this inequality diverges to infinity because the breakthrough rate $\dot{G}/(1 - G)$ is not integrable over $[0, \infty)$, although it may tend to zero at infinity.[19] Thus even arbitrarily small differences in initial maturation delays eventually translate into arbitrarily large differences in maturation delays. This, under Assumption 3, is inconsistent with the fact that players' equilibrium maturation delays cannot exceed their stand-alone maturation delays. Hence the equilibrium must be unique.[20]

Key to this reasoning is the minimal rationality requirement that players should never wait in equilibrium more than in the absence of competition. Imposing this requirement for increasing values of the breakthrough time narrows down the set of possible equilibrium trajectories. Asymptotically, this funneling effect pins down a unique equilibrium. The role of dominance considerations in this mechanism might be thought of in analogy with the global-game literature.[21]

4 Economic Implications

In this section, we study how the strength of competition and the characteristics of research projects affect the length of maturation delays and the quality of innovations.

4.1 Maturation Delays

4.1.1 Dynamic Analysis

A useful benchmark for our analysis is the stationary single-race model of Hopenhayn and

[19] Observe that ρ can be small as one wishes as long as it remains positive. This line of argument builds on uniqueness results for antifunnels (Hubbard and West (1991)).

[20] We argue in Appendix C that the equilibrium is likely to be unique in a much wider set of circumstances than those delineated by Assumptions 3–4.

[21] See, for instance, Carlsson and van Damme (1993) or Morris and Shin (2003).

Squintani (2014), in which the players' payoffs do not depend on their breakthrough times and the latter are exponentially distributed with parameter $\lambda > 0$. The ODE (5) then writes as $\dot{\phi}(t) = (1/\lambda)(\dot{L}/L)(t - \phi(t))$, the only solution of which that stays in $\mathcal{D}$ is $\phi_\lambda(t) \equiv t - M_\lambda$ for a constant maturation delay $M_\lambda \equiv (\dot{L}/L)^{-1}(\lambda)$. Intuitively, if player j selects the same maturation delay regardless of her breakthrough time, she makes a move at rate λ. By waiting herself an amount of time M_λ before making a move, player j balances the marginal payoff gain from an infinitesimal additional delay $\mathrm{d}t$, $\dot{L}(M_\lambda)\,\mathrm{d}t$, with the corresponding expected marginal cost, which is equal to the probability that player j makes a move in the next instant conditional on not having doing so before, $\lambda\,\mathrm{d}t$, multiplied by the foregone payoff, $L(M_\lambda)$. This unique equilibrium exhibits no time dependence.

Yet, in practice, research hardly takes place in such a stationary environment (Stephan (1996)). Indeed, human-capital accumulation and technological progress constantly modify the research environment, making researchers more innovative, that is, more likely to have breakthroughs, and also more efficient, that is, more able at developing their breakthroughs. Such knowledge spillovers are pervasive in science and innovation: advances in a given field spread to sometimes distant fields on which they shed a new light, enabling researchers or inventors to make progress there as well.

To assess the impact of such factors on the evolution of equilibrium maturation delays $\mu(\tau) \equiv \sigma(\tau) - \tau$, we invert (5) to get

$$\dot{\mu}(\tau) = \frac{\dot{G}}{1-G}(\tau)\,\frac{L}{L_1}(\mu(\tau), \tau) - 1. \tag{7}$$

The constant 1 in (7) reflects that if the moving time did not change as a function of the breakthrough time, a unit increase in the latter would mechanically decrease the maturation delay by one. We see from (7) the two forces driving the evolution of the maturation delay. The first one is the breakthrough rate $\dot{G}/(1-G)$: if it increases over time, each player anticipates her opponent to become more innovative, implying higher preemption risk and thus lower incentives for her to wait before making a move. The second one is the growth potential L_1/L:[22] if it increases over time, that is, if $(L_1/L)_2 > 0$, then players with late breakthroughs are more willing to wait and incur preemption risk than players with early breakthroughs. The following result holds.

Proposition 1 *If the mapping*

$$\tau \mapsto \frac{\dot{G}}{1-G}(\tau)\,\frac{L}{L_1}(m, \tau) - 1 \tag{8}$$

has a positive (negative) derivative over $\mathcal{T}_m \equiv \{\tau : m < M(\tau)\}$ *for all* $m > 0$, *then* $\mu(\tau)$ *is strictly decreasing (increasing) in* τ.

The logic of Proposition 1 relies on the following equilibrium argument. One may at first glance expect from (7) that a positive derivative for (8) induces moving times to grow

[22] Note that this is the reciprocal of Aumann and Kurz's (1977) fear-of-ruin index for utilities of wealth.

at an increasing rate, and thus maturation delays to increase. This, however, overlooks the equilibrium requirement that players should never wait more in equilibrium than in the absence of competition. Indeed, what a positive derivative for (8) actually implies is that, if maturation delays were to start increasing, then they would do so at an increasing speed, eventually exceeding stand-alone maturation delays. As this is inconsistent with equilibrium, maturation delays must decrease.[23]

According to Proposition 1, an increasing growth potential mitigates the negative impact on maturation delays of an increasing breakthrough rate. By contrast, this negative impact is reinforced if the growth potential decreases over time.[24] Thus, if players become increasingly innovative, equilibrium maturation delays increase only if so does the growth potential of breakthroughs. It is instructive to revisit Example 1 in the light of this insight.

Example 1 If the publication-delay function satisfies $D_{12} > 0$, additional maturation time decreases more publication delays for early than for late breakthroughs. A reason why this may happen is that, through improving refereeing and editorial standards, the publication process becomes a better substitute to the amount of time researchers spend developing their breakthroughs. If, besides, $D_2 < 0$, then total time from breakthrough to publication at the stand-alone maturation delay, $M(\tau) + D(M(\tau), \tau)$, is decreasing in τ. However, $D_{12} > 0$ implies that $(L_1/L)_2 < 0$; therefore, according to Proposition 1, equilibrium maturation delays decrease if researchers become increasingly innovative: a more efficient publication process reinforces their fear of preemption. In equilibrium, the impact on total time from breakthrough to publication, $\mu(\tau) + D(\mu(\tau), \tau)$, is a priori ambiguous. We show in Appendix B.1 that, if the breakthrough rate increases enough, the equilibrium maturation delay may decrease so much that, as a result of the lowering elaboration of submissions, total time from breakthrough to publication tends to increase. In the patent interpretation, $(L_1/L)_2 < 0$ if $(\ln \pi)_{12} < 0$, that is, if, in terms of allowance rate, the relative returns to increasing the maturation delay are decreasing over time. This may happen because the patent office becomes better at screening applications based on their intrinsic merit. Then, if inventors become increasingly innovative, they wait less in equilibrium before applying for a patent. The elaboration of applications may decrease so much as to make patent applications less likely to succeed over time. Note that this effect takes place not in spite of, but because of the research community (the pool of inventors) and the publication process (the patent office) becoming more efficient over time. These results indicate a possible alternative interpretation of the slowdown in the publication process documented by Ellison (2002) for top economics journals over the period 1970–2000, and of the finding by Card and DellaVigna (2013) that

[23]The usual Lyapunov method would consist in working with the state (σ, τ) and in checking the sign over $\{(m, \tau) : 0 < m < M(\tau)\}$ of the scalar product of the gradient of the maturation delay relative to the state, $(1, -1)$, and of the law of motion of the state, $(\dot{G}L/[(1-G)L_1], 1)$. However, for each τ, $(L/L_1)(m, \tau)$ varies from zero to infinity as m varies from zero to $M(\tau)$, making this method inconclusive. Our approach, by contrast, focuses, not on the sign of this scalar product, but on how it varies with τ.

[24]Both scenarios are consistent with Assumption 2, which only states that $(L_1/L)_1 < (L_1/L)_2$ and thus imposes no a priori restriction on the sign of $(L_1/L)_2$ as $(L_1/L)_1 < 0$ by Assumption 4.

in recent years the number of submissions to these journals has dramatically increased while their acceptance rates has decreased. These results also suggest that enhancing the efficiency of the editorial process may counterproductively increase publication delays.

4.1.2 Comparative Statics

Our comparative-statics results rely on two partial orders over the sets of breakthrough distributions and of payoff functions that are directly motivated by (7).

First, let $\underline{G}$ and $\overline{G}$ be two breakthrough distributions with densities $\dot{\overline{G}}$ and $\dot{\underline{G}}$. By analogy with the first-price-auction literature (Lebrun (1998), Maskin and Riley (2000), Arozamena and Cantillon (2004)), let $\overline{G}$ dominate $\underline{G}$ in the *breakthrough-rate order* if

$$\frac{\dot{\underline{G}}}{1-\underline{G}} > \frac{\dot{\overline{G}}}{1-\overline{G}} \tag{9}$$

over $(0,\infty)$, so that, in particular, $\overline{G}$ first-order stochastically dominates $\underline{G}$: breakthroughs tend to occur later under $\overline{G}$ than under $\underline{G}$.

Next, let $\underline{L}$ and $\overline{L}$ be two payoff functions satisfying Assumptions 1–4, with stand-alone maturation-delay functions $\underline{M}$ and $\overline{M}$. By analogy with the fear-of-ruin literature (Aumann and Kurz (1977), Foncel and Treich (2005)), let $\overline{L}$ dominate $\underline{L}$ in the *growth-rate order* if

$$\frac{\overline{L}_1}{\overline{L}} > \frac{\underline{L}_1}{\underline{L}} \tag{10}$$

over $\{(m,\tau) : 0 < m < \underline{M}(\tau)\}$, so that, in particular, $\underline{M} \leq \overline{M}$:[25] payoffs grow with maturation delays at a higher rate under $\overline{L}$ than under $\underline{L}$.

Then the following comparative-statics result holds.

Proposition 2 *Let $\underline{\mu}$ ($\overline{\mu}$) be the equilibrium maturation delay under the distribution $\underline{G}$ ($\overline{G}$) or the payoff function $\underline{L}$ ($\overline{L}$). Then, if $\overline{G}$ dominates $\underline{G}$ in the breakthrough-rate order, or if $\overline{L}$ dominates $\underline{L}$ in the growth-rate order, then $\overline{\mu}(\tau) > \underline{\mu}(\tau)$ for all τ.*

Proposition 2 reflects the tension in our model between the risk of preemption and the fear of preemption. When players are more innovative, preemption risk is higher because breakthroughs occur more often, which speeds up the maturation process. By contrast, when players compete on research projects with a higher growth potential, the fear of preemption is lower because the marginal gain from slightly delaying one's move relative to the loss if she is preempted is higher, which slows down the maturation process. We use Examples 1 and 2 to illustrate these effects.

Example 1 A first implication of Proposition 2 is that academic submissions or patent applications are less elaborate when researchers or inventors are more innovative, leading to longer publication delays or lower allowance rates. To illustrate the second half of Proposition

[25] Condition (10) does not imply that $\underline{L}$ uniformly generates higher payoffs than $\overline{L}$. Indeed, as $\underline{L}(0,\tau) = \overline{L}(0,\tau) = 0$ for all τ, one can have $\overline{L}(m,\tau) < \underline{L}(m,\tau)$ for m close to zero, as in Example 1 below.

2, fix the publication-delay function, but allow the discount rate to vary. When $\underline{r} < \overline{r}$, the corresponding payoff functions $\underline{L}$ and $\overline{L}$ are related by $\underline{L} = \overline{L}^{\underline{r}/\overline{r}}$, so that $\overline{L}$ dominates $\underline{L}$ in the growth-rate order. Hence, in this example, when competition takes place between relatively impatient researchers, such as assistant professors on a tenure track, maturation delays are longer and initial submissions are more elaborate than when it takes place between relatively patient researchers, who for instance already have tenure. This reflects that more impatient researchers are more concerned by having their papers stuck in the publication process than less impatient ones. To ensure faster publication, they are thus more willing to incur risk by letting their breakthroughs mature more than the latter before submitting their work. A similar conclusion can be drawn in the patent interpretation, where an inventor may be more or less impatient due to more or less severe cash constraints.

Example 2 Proposition 2 allows one to study the impact of the shape of the reward function and of the noise distribution on equilibrium maturation delays. To fix ideas, suppose that S does not depend on the breakthrough time and is quadratic in journal ranking, $S(j) \equiv aj + bj^2/2$, and that $J(m, \varepsilon) = m\varepsilon$ for all m and ε.[26] Then the growth potential

$$\frac{\dot{L}}{L}(m) = -r + \frac{1 + (b/a)\{\mathbf{E}[\tilde{\varepsilon}^2]/\mathbf{E}[\tilde{\varepsilon}]\}m}{m + (b/a)\{\mathbf{E}[\tilde{\varepsilon}^2]/\mathbf{E}[\tilde{\varepsilon}]\}m^2/2}$$

is increasing in b/a and $\mathbf{E}[\tilde{\varepsilon}^2]/\mathbf{E}[\tilde{\varepsilon}]$. Concretely, a higher b/a ratio corresponds to a more convex reward function S, putting relatively more weight on leading journals like "top five" journals in Economics or "series A" journals in Physics. Such a reward function makes researchers more willing to incur preemption risk, as the marginal gain from the perspective of publishing in a higher-ranked journal, which on average requires more maturation, is higher relative to the loss if one is preempted. This, according to Proposition 2, leads to longer equilibrium maturation delays. A higher $\mathbf{E}[\tilde{\varepsilon}^2]/\mathbf{E}[\tilde{\varepsilon}]$ ratio has a similar impact. Intuitively, this is because the convexity of the researchers' reward function creates an option value of waiting that is higher, the higher this ratio is.[27]

More Than Two Players The impact of increased competition is similar to that of an increase in the breakthrough-rate order. Indeed, suppose there are $N > 2$ symmetric players whose breakthrough times are independently drawn from the distribution G. In a symmetric equilibrium with common strategy $\sigma = \phi^{-1}$, the problem faced by type τ^i of player i is

$$\max_{t^i \in [\tau^i, \infty)} \left\{ \prod_{j \neq i} \mathbf{P}[\sigma(\tilde{\tau}^j) > t^i] L(t^i - \tau^i, \tau^i) \right\} = \max_{t^i \in [\tau^i, \infty)} \{[1 - G(\phi(t^i))]^{N-1} L(t^i - \tau^i, \tau^i)\},$$

[26] It is straightforward to check that Assumptions 1–4 hold for this example.

[27] An alternative interpretation of the above function J is that it describes the random output from research. A short maturation delay then corresponds to a low-return, low-risk research strategy, while a long maturation delay corresponds to high-return, high-risk research strategy. This echoes the tradeoff emphasized by March (1991) between exploitation of already-existing ideas along conventional lines and exploration of new ideas. A more convex function encourages the second type of research strategy, for instance by rewarding long-term rather than short-term successes. Azoulay, Graff Zivin, and Manso (2011) provide evidence on the impact of different incentive schemes in academic life sciences.

so that ϕ satisfies

$$\dot{\phi}(t) = \frac{1}{N-1} f(t, \phi(t)), \quad t \geq \sigma(0),$$

where f is given by (6). Increasing the number of players thus amounts to multiply the breakthrough rate, which, by Proposition 2, leads to shorter maturation delays.[28]

The contrast between the prospection phase and the development phase highlighted in Proposition 2 suggests that, if the quality of research outputs is a function of the amount of time researchers spend developing their breakthroughs, they should be induced to engage in projects with higher growth potential rather than in projects where breakthroughs are more frequent. If the development technology varies over time, however, an explicit modeling of quality is required, a task to which we now turn.

4.2 Innovation Quality

4.2.1 Technology, Maturation, and Quality

By the quality of an innovation, we mean its capacity to create value for those who use it or consume it. In the context of our model, this can be thought of as the result of two factors. One is the time τ at which a player has a breakthrough, which determines the technology she will have at her disposal to develop it; the other is the amount of time m she devotes to this task. We denote by $Q(m, \tau)$ the resulting quality level.[29]

Assumption 5 *The function Q is continuous over $[0, \infty) \times [0, \infty)$ and thrice continuously differentiable over $(0, \infty) \times (0, \infty)$. For all m and τ,*

$$Q(0, \tau) = 0 \text{ and } Q_1(m, \tau) > 0.$$

Moreover, for all $m > 0$ and $\tau \geq 0$,

$$Q_2(m, \tau) > 0 \text{ and } \lim_{\tau \to \infty} \{Q(m, \tau)\} < \infty.$$

Thus, in terms of quality, the returns to maturation and to technology are positive. A later breakthrough allows a player to save on the maturation time required to achieve any given quality level, reflecting that she can build on an increasing body of knowledge to develop her breakthrough.

We next restrict the cross effects of maturation delays and breakthrough times on quality.

[28]Formally, each player behaves as if she were facing a single opponent with breakthrough time drawn from the distribution $G_{(1/N-1)} \equiv 1-(1-G)^{N-1}$ of the first-order statistic for a sample of $N-1$ independent breakthrough times drawn from G.

[29]In practice, quality is not directly observable, but must rather be inferred from multiple characteristics of innovations. For instance, in the case of patents, Lanjouw and Schankerman (2004) construct a composite index of patent quality using as indicators the number of claims, forward citations to the patent, backward citations in the patent application, and family size.

Assumption 6 *For all m and τ,*

$$Q_{11}(m,\tau) \le Q_{12}(m,\tau).$$

Equivalently, the mapping $(t,\tau) \mapsto Q(t-\tau,\tau)$ is supermodular: in terms of quality, the returns to increasing the moving time t are increasing in the breakthrough time τ. It is useful to interpret this form of complementarity in light of the maturation-technology wedge $(Q_1 - Q_2)(m,\tau)$ between the returns to maturation and the returns to technology. This quantity represents the (possibly negative) difference in the qualities of the innovations achieved by a type-τ player and a type-$\tau + \mathrm{d}\tau$ player when they both make a move at time $t = m + \tau$, so that the former benefits from a longer maturation delay and the latter from a better development technology.[30] Assumption 6 states that this quality differential decreases in m, reflecting that the type-$\tau + \mathrm{d}\tau$ player can exploit her technological advantage over a longer period of time.

4.2.2 Dynamic Analysis

To assess how quality $\chi(\tau) \equiv Q(\mu(\tau),\tau)$ evolves in equilibrium, we change variables and directly work in terms of quality. The maturation delay $T(q,\tau) - \tau$ required to deliver an innovation of quality q given a breakthrough time τ is implicitly defined by

$$Q(T(q,\tau) - \tau, \tau) = q. \tag{11}$$

To develop the analogy with the evolution of maturation delays, it is helpful to take advantage of (11) to redefine the players' payoff as a function of q and τ, which leads to

$$H(q,\tau) \equiv L(T(q,\tau) - \tau, \tau). \tag{12}$$

The following assumption parallels Assumption 4.

Assumption 7 *There exists $\rho > 0$ such that, for each τ, $H(\cdot,\tau)$ is ρ-concave over* $[0, Q(M(\tau),\tau)]$.

Using the identity $\mu(\tau) = T(\chi(\tau),\tau) - \tau$ along with (7) and (12) yields an ODE for the evolution of equilibrium quality,

$$\dot{\chi}(\tau) = \frac{\dot{G}}{1-G}(\tau)\,\frac{H}{H_1}(\chi(\tau),\tau) - \frac{T_2}{T_1}(\chi(\tau),\tau). \tag{13}$$

By analogy with (7), the first term on the right-hand side of (13) reflects preemption risk, measured by the breakthrough rate $\dot{G}/(1-G)$, and growth potential, measured by H_1/H in quality units. In line with the logic of Proposition 1, when the ratio of these two terms

[30] Indeed, one has $Q(m,\tau) - Q(m-\mathrm{d}\tau, \tau+\mathrm{d}\tau) = (Q_1 - Q_2)(m,\tau)\,\mathrm{d}\tau + o(\mathrm{d}\tau)$. The quantity $(Q_1 - Q_2)(m,\tau)$ is economically meaningful because the factors m and τ are measured in the same time units.

increases with the breakthrough time, this tends, other things equal, to make quality fall over time.

A possibly countervailing factor, however, comes from the second term on the right-hand side of (13), which, unlike the corresponding constant 1 in (7), can be affected by the passage of time. To interpret this term, notice that, just as 1 trivially measures by how much a player must decrease her maturation delay when her breakthrough time increases by one unit, so as to keep the same moving time, T_2/T_1 measures by how much the quality of her innovation decreases when her breakthrough time increases by one unit, subject to the same constraint. That is, for a given pair (q_0, τ_0) and a fixed moving time $T(q_0, \tau_0)$, $(T_2/T_1)(q_0, \tau_0)\, \mathrm{d}\tau$ represents how many quality units a player would have produced on top of q_0, had she had her breakthrough at $\tau_0 - \mathrm{d}\tau$ instead of τ_0.

For a given quality level q_0, variations of the marginal rate of substitution $(T_2/T_1)(q_0, \tau_0)$ with respect to the breakthrough time τ_0 reflect how changes in technology affect how fast innovations are developed.[31] For instance, if $(T_2/T_1)_2 > 0$, technological progress makes additional maturation delay become increasingly productive, which tends, other things equal, to make quality increase over time.

To determine which assumptions on Q guarantee that $(T_2/T_1)_2 > 0$, note from (11) that T_2/T_1 is linked to the maturation-technology wedge by

$$\frac{T_2}{T_1}\,(q,\tau) = (Q_1 - Q_2)(T(q,\tau) - \tau, \tau). \tag{14}$$

Under Assumption 5, the maturation delay $T(q, \tau) - \tau$ required to achieve an innovation of quality q decreases in the breakthrough time τ; this, under Assumption 6, tends to increase $(T_2/T_1)(q, \tau)$ as the maturation-technology wedge $(Q_1 - Q_2)(m, \tau)$ is then decreasing in the maturation delay m. A sufficient condition for $(T_2/T_1)_2 > 0$ is thus that $Q_{12} > Q_{22}$. This holds whenever the difference in the qualities of the innovations achieved by a type-τ player and a type-$\tau + \mathrm{d}\tau$ player when they both make a move at time $t = m + \tau$ increases with τ, reflecting that the technological advantage of the type-$\tau + \mathrm{d}\tau$ player decreases over time.

The following result parallels Proposition 1.

Proposition 3 *If the mapping*

$$\tau \mapsto \frac{\dot{G}}{1-G}\,(\tau)\,\frac{H}{H_1}\,(q,\tau) - \frac{T_2}{T_1}\,(q,\tau) \tag{15}$$

has a positive (negative) derivative over $\mathcal{T}_q \equiv \{\tau : q < Q(M(\tau), \tau)\}$ *for all* $q > 0$, *then* $\chi(\tau)$ *is strictly decreasing (increasing) in* τ.

4.2.3 Comparative Statics

As a starting point, it is helpful to observe that, if researchers become more innovative,

[31] To use a mechanical analogy, $(T_2/T_1)(q_0, \tau_0)$ represents the speed at which small additional delays are converted into quality along the isochrone curve $T(q, \tau) = T(q_0, \tau_0)$, so that variations of $(T_2/T_1)(q_0, \tau_0)$ with respect to τ_0 can be interpreted as an acceleration.

the quality of innovations unambiguously deteriorates. Indeed, we know from Proposition 2 that, if $\overline{G}$ dominates $\underline{G}$ in the breakthrough-rate order, then equilibrium maturation delays are uniformly longer under $\overline{G}$ than under $\underline{G}$; under Assumption 5, innovation-quality levels are then uniformly higher under $\overline{G}$ than under $\underline{G}$. The key here is that the development technology is held fixed while the intensity of competition is varied.

But what about a change in the development technology itself? To address this question, we specify Example 3 using a parametric model of quality, inspired by the innovation-and-growth literature, in which researchers may operate away from the technological frontier.[32]

Example 3 The technological frontier is represented by a function $\xi : [0, \infty) \to [0, \infty)$; for each s, $\xi(s)$ is the speed at which the quality of an innovation increases at time s when the frontier technology is used to develop it. The distance to the frontier is measured by a parameter $d \in [0, 1]$; $1 - d$ is the fraction of increases in ξ that are assimilated by a researcher after having a breakthrough.[33] Then quality is given by

$$Q(m, \tau; d) \equiv \xi(\tau)m + (1 - d) \int_{\tau}^{\tau+m} [\xi(s) - \xi(\tau)] \,\mathrm{d}s \tag{16}$$

and payoffs are given by

$$L(m, \tau; d) \equiv \exp(-rm)P(Q(m, \tau; d)). \tag{17}$$

Assumptions 5–6 are satisfied if ξ is bounded, with $\xi > 0$ and $\dot{\xi} > 0$.[34] If $d = 0$, a researcher, no matter when she had a breakthrough, can use at any point in time the frontier technology to develop it. By contrast, if $d = 1$, a researcher with a time-τ breakthrough is locked in with the time-τ technology. For intermediate values of d, a researcher benefits to some extent from ongoing technological progress while developing her breakthrough, but cannot catch up with the most recent advances. The maturation-technology wedge $\xi(\tau) - d\dot{\xi}(\tau)m$ is decreasing in the distance d to the frontier: the less a researcher can assimilate new technological advances after having a breakthrough, the larger the technological advantage enjoyed by a type-$\tau + \mathrm{d}\tau$ player over a type-τ player.

To study the impact on innovation quality of a change in the distance to the frontier, it is first tempting to exploit our results on equilibrium maturation delays, using the ODE (7). From (17), we have

$$\frac{L_1}{L}\,(m, \tau; d) = -r + \frac{Q_1}{Q}\,(m, \tau; d)\,\epsilon_P(Q(m, \tau; d)), \tag{18}$$

where ϵ_P is the elasticity of consumers' inverse demand function for quality.

[32]See Aghion and Howitt (2005) for a survey of this literature.

[33]To use Cohen and Levinthal's (1990) terminology, $1 - d$ is the "absorptive capacity" of a researcher.

[34]Detailed calculations for this example are gathered in Appendix B.2. We provide there conditions on the functions ξ and P that ensure that Assumptions 1–7 are satisfied. These conditions state that technological progress is not too drastic and that the inverse demand for innovation quality becomes sufficiently insensitive to quality increases, the higher quality is.

It is easy to check that the quality $Q(m, \tau; d)$ and the sensitivity $(Q_1/Q)(m, \tau; d)$ of quality to maturation are decreasing in the distance d to the frontier. Therefore, if the elasticity $\epsilon_P(q)$ is increasing in quality q, we get that $L(\cdot, \cdot; d)$ decreases with d in the growth-rate order. Then, according to Proposition 2, a lower distance to the frontier leads to an equilibrium with longer maturation delays and, therefore, higher quality levels. This is for instance the case if P is sufficiently convex, that is, if the benefit of a quality increment increases fast enough with quality. Then, when inventors become closer to the technological frontier, they have an incentive to develop their breakthroughs even more.

However, a more natural assumption is that the elasticity $\epsilon_P(q)$ is decreasing in quality q, which requires P to be sufficiently concave. Then it need no longer be the case that $L(\cdot, \cdot; d)$ decreases with d in the growth-rate order, and Proposition 2 no longer applies: depending on the inverse demand for quality, being closer to the frontier may lead to longer or shorter maturation delays in equilibrium.[35] The intuition is that, as the marginal returns are decreasing, inventors may prefer to let their breakthroughs mature less, despite the development technology being more efficient.

We shall instead directly reason in terms of quality, using the ODE (13). The distance to the frontier affects the evolution of quality both through the marginal rate of substitution of breakthrough time for quality, T_2/T_1, and through the growth potential of breakthrough measured in quality units, H_1/H. As for T_2/T_1, (14) yields, in the current specification,

$$\frac{T_2}{T_1}(q, \tau; d) = \xi(\tau) - d\dot{\xi}(\tau)[T(q, \tau; d) - \tau],$$

which is decreasing in d because the maturation delay required to achieve an innovation of quality q, $T(q, \tau; d) - \tau$, is increasing in d, a force that tends to reduce quality. The behavior of H_1/H is less clear-cut. Indeed, from (12) and (17), we have

$$\frac{H_1}{H}(q, \tau; d) = -rT_1(q, \tau; d) + \frac{P'}{P}(q),$$

which varies with d inversely to the increase in maturation delay required to produce an additional unit of quality, $T_1(q, \tau; d) - 1$. When the technological frontier ξ increases at a decreasing rate, the latter is strictly increasing in d, which reduces the growth potential of breakthroughs and further contributes to reduce quality. Hence the following result.

Proposition 4 *If $\ddot{\xi} < 0$ and $\overline{d} > \underline{d}$, then $\chi(\tau; \underline{d}) > \chi(\tau; \overline{d})$ for all τ.*

Intuitively, when the distance to the frontier decreases, technology becomes more labor enhancing. This mitigates the impact of preemption risk and reduces the sensitivity of quality to competition. This result complements the endogenous-growth literature, according to which product-market competition should be more growth enhancing for firms that are closer to the technological frontier.[36] In our model, higher breakthrough rates always have a

[35]That the latter scenario is not only a theoretical possibility is illustrated in Appendix B.2.2.

[36]See Aghion, Harris, and Vickers (1997), Aghion, Harris, Howitt, and Vickers (2001), and Aghion, Bloom, Blundell, Griffith, and Howitt (2005).

Schumpeterian effect: they intensify the race for priority, leading inventors to prematurely apply for patents. Proposition 4 shows that the resulting negative effect on innovation quality is less pronounced, the lower the distance to the technological frontier.

5 Asymmetric Priority Races

In this section, we study the impact of asymmetries between players. To keep the analysis tractable, we simplify the model by assuming that players' payoffs do not directly depend on their breakthrough times; that is, we abstract from changes in the technology affecting the development of breakthroughs. This implies that the quality of a research output is only a function of the time spent maturing it.

5.1 The Hare and the Tortoise

Suppose first that players have constant but different breakthrough rates $\lambda^a > \lambda^b$: a, the *hare*, is a relatively more innovative researcher than b, the *tortoise*. The game otherwise remains the same as in Section 2, the only difference being that $\tilde{\tau}^a$ and $\tilde{\tau}^b$ now have different distributions. The payoff $L(m, \tau) \equiv L(m)$ is independent of τ and we let M be the point at which it reaches its maximum. In an equilibrium with continuous strategies (σ^a, σ^b), $\phi^a \equiv (\sigma^a)^{-1}$ and $\phi^b \equiv (\sigma^b)^{-1}$ solve the following system of ODEs:[37]

$$\dot{\phi}^j(t) = \frac{1}{\lambda^j} \frac{\dot{L}}{L}(t - \phi^i(t)), \quad t \geq \sigma(0), \quad i = a, b. \tag{19}$$

As in the symmetric case, the initial condition $\sigma(0) = \sigma^a(0) = \sigma^b(0)$ of that system must be chosen in such a way that neither ϕ^a nor ϕ^b leave $\mathcal{D}$. Letting $\mu^i(\tau) \equiv \sigma^i(\tau) - \tau$ be player i's equilibrium maturation delay, we can now state the central result of this section.

Theorem 3 *There exists a unique continuous equilibrium. In this equilibrium, $\mu^a(\tau)$ ($\mu^b(\tau)$) is strictly increasing (decreasing) in τ. In particular, $\mu^a(\tilde{\tau}^a) > \mu^b(\tilde{\tau}^b)$ unless $\tilde{\tau}^a = \tilde{\tau}^b = 0$.*

A key insight of Theorem 3 is that the hare always lets her breakthroughs mature more than the tortoise, no matter when their respective breakthroughs occur: echoing a theme in March (1991), more innovative researchers endogenously behave more ambitiously in the exploration of new ideas and thus succeed or fail more spectacularly than less innovative ones. This leads to the prediction that, within a group of competing researchers with different innovative abilities, speed of discovery and quality of research output should be positively correlated.[38] In our model, this effect only arises because of competition between researchers.

[37] As explained in Appendix A, Lemma 1 carries over to this asymmetric context. Our proof of Lemma 2(i), however, does not extend to the case of asymmetric players, so that we have to postulate continuity.

[38] Note, however, that, according to Proposition 2, this correlation would be reversed if it were computed across noncompeting groups of homogenous researchers: indeed, groups with more innovative researchers are more competitive, leading to shorter maturation delays and lower-quality research outputs.

Indeed, if they were not threatened by preemption, all researchers would adopt the same maturation delay M, leading to identical high-quality research outputs. A further implication of Theorem 3 is that, if a breakthrough occurs early on, the resulting maturation delay is less sensitive to the identity of the player who has it than if it occurs later on. Thus later breakthroughs result in higher heterogeneity in the quality of research outputs: over time, the hare experiences a flight to quality, contrary to the tortoise.

We now study how the equilibrium reacts to changes in breakthrough rates. Consider an increase in the hare's breakthrough rate, holding the tortoise's constant. The direct effect of such an increase can be seen upon writing (19) for $i = b$: if the hare's behavior as summarized by $\dot{\phi}^a(t)$ is held fixed, an increase in λ^a triggers a downward shift of the tortoise's maturation delay $t - \phi^b(t)$. Of course, things are complicated by the fact that the hare's behavior also varies when her own breakthrough rate increases. Yet, the following result shows that facing a more challenging opponent unambiguously makes the tortoise more cautious, thereby deteriorating the quality of her research output.

Proposition 5 *Let $\underline{\mu}^i$ ($\overline{\mu}^i$) be player i's equilibrium maturation delay when the hare's breakthrough rate is $\underline{\lambda}^a$ ($\overline{\lambda}^a$) and the tortoise's breakthrough rate is λ^b. Then, if $\overline{\lambda}^a > \underline{\lambda}^a$, $\overline{\mu}^b(\tau) < \underline{\mu}^b(\tau)$ for all τ.*

As for the hare, an increase in her breakthrough rate is a mixed blessing. More frequent breakthroughs increase her competitive edge and, other things equal, allow her to take more time to let them mature. However, by Proposition 5, the tortoise reacts to an increase in the hare's breakthrough rate by letting her own breakthroughs mature even less, which makes her a tougher opponent. Which of these effects dominates is a priori unclear and depends on when the hare has her breakthrough. If this occurs early on, which is more likely as she has become more innovative, the second, strategic effect dominates because one must have $\sigma^a(0) = \sigma^b(0)$ in equilibrium. Players are thus caught in a vicious circle: the fact that the tortoise behaves more cautiously in equilibrium compels the hare to do the same when she has an early breakthrough.[39]

Asymmetric breakthrough rates may reflect innate ability differences between researchers, or specialization. In that respect, the results of this section shed some light on the findings of Borjas and Doran (2012) on the post-1992 influx of highly-skilled Soviet mathematicians on the scientific production of US mathematicians. These authors not only document a large drop in the publication rate of US mathematicians whose research agenda overlapped most with Soviet mathematicians—a crowding-out effect that could simply reflect increased competition for scarce journal space—but they also show that the quality of their papers, as measured by the number of citations they generated or their likelihood of becoming "home runs," significantly fell as well. This second effect is consistent with our finding that, when

[39] Unlike in Proposition 5, however, one cannot translate this local insight into a global comparative-statics result. We show in Appendix B.3 that, depending on the parameters of the model, for late breakthrough times, an increase in the hare's breakthrough rate may increase or decrease her equilibrium maturation delay.

she faces a hare, a tortoise lets her breakthroughs mature less than if she were facing an opponent of equal strength, and the less so, the more innovative the hare is. Thus one may interpret the drop in quality documented by Borjas and Doran (2012) as resulting from US mathematicians adopting a more cautious research strategy for fear of being preempted by their more talented Soviet opponents.

When competitors are research labs or firms instead of individuals, an advantage in innovative ability may typically result from better funding, higher interdisciplinarity, stronger leadership, and other organizational features.[40] The amount of R&D, for instance, has been found to rise in a roughly proportional way with firm size.[41] One may thus postulate that, in a mechanical way, the breakthrough rate varies monotonically with firm size. Interestingly, however, it is also typically found in the data that the number of patents and innovations per dollar of R&D tends to decline with firm size or the amount of R&D. Although Proposition 5 does not directly speak to this issue, it suggests a new mechanism, whereby a firm having higher R&D expenditures tends to make its competitors more cautious; this, in turn, reduces its own R&D productivity, when the latter is measured, not by the ratio of patents to R&D, but, rather, by the ratio of their quality to R&D.

5.2 The Ant and the Grasshopper

Other respects in which players may differ are the type of research projects they undertake, how able they are at developing them, or the incentives they face. Suppose accordingly that players have a constant common breakthrough rate λ but different payoff functions L^a and L^b such that L^a dominates L^b in the growth-rate order: a, the *ant*, works on a project with higher growth potential than b, the *grasshopper*. As pointed out in Section 4.1.2, in the case of academic competition, the higher standards of a may reflect that, unlike b, she does not have tenure yet, or that she belongs to an institution with a steeper incentive scheme. The payoffs $L^a(m,\tau) \equiv L^a(m)$ and $L^b(m,\tau) \equiv L^b(m)$ are independent of τ and are assumed to reach their maximum at the same point M.[42] In an equilibrium with continuous strategies (σ^a, σ^b), $\phi^a \equiv (\sigma^a)^{-1}$ and $\phi^b \equiv (\sigma^b)^{-1}$ solve the following system of ODEs:

$$\dot{\phi}^j(t) = \frac{1}{\lambda} \frac{\dot{L}^i(t - \phi^i(t))}{L^i(t - \phi^i(t))}, \quad t \geq \sigma(0), \quad i = a, b. \tag{20}$$

By analogy with Theorem 3, there exists a unique continuous equilibrium, in which the ant always lets her breakthroughs mature more than the grasshopper, no matter when their respective breakthroughs occur. Unlike in the case of the hare and the tortoise, however, the ant tends to behave more cautiously, and the grasshopper less cautiously, than if they were each facing an opponent with similar payoff function. Indeed, being confronted to an

[40] See, in the case of medical science, Hollingsworth and Hollingsworth (2000).

[41] See Cohen and Klepper (1996) and Cohen (2010) for discussions of the relevant empirical evidence.

[42] This holds, for instance, if $L^b = h \circ L^a$ for some differentiable and strictly increasing function $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$ and $h(l)/l$ is strictly decreasing in l (Foncel and Treich (2005, Proposition 1)).

ant inclines a grasshopper to let her breakthroughs mature more, and thus has a positive impact on the quality of her research output because facing a less hasty competitor with higher standards than her owns reduces the competitive pressure she is exposed to.

The following result parallels Proposition 5.

Proposition 6 *Let $\underline{\mu}^i$ ($\overline{\mu}^i$) be player i's equilibrium maturation delay when the ant's payoff function is $\underline{L}^a$ ($\overline{L}^a$) and the grasshopper's payoff function is L^b. Then, if $\overline{L}^a$ dominates $\underline{L}^a$ in the growth-rate order, $\overline{\mu}^a(\tau) > \underline{\mu}^a(\tau)$ for all τ.*

Hence an increase in the growth potential of the ant's breakthroughs leads her to let her breakthrough mature even more. This, in turn, induces the grasshopper to do the same, at least when she has an early breakthrough. In line with the examples discussed in Section 4.1.2, this positive contagion effect suggests that the high standards set by, or imposed upon, young researchers eager to get tenure may have a positive externality on the academic profession as a whole. Similarly, incentive schemes rewarding long-term rather than short-term successes, such as the ERC in Europe or the HHMI in the US (Azoulay, Graff Griffin, and Manso (2011)) mitigate the impact of preemption risk, even for researchers who do not directly benefit from them.

6 Concluding Remarks

We would like to close this paper with two final observations.

The first pertains to the implication of our analysis for empirical work on R&D. A key ingredient of our model is the researchers' breakthrough rate, which measures their raw innovative ability. This quantity and how it evolves over time are hard to measure empirically. Indeed, what we observe in practice is not the fecundity of R&D itself, but only the output of research, that is, for instance, the number of publications per scientist, or the number of patents per dollar of R&D expenditure. This creates an identification problem in interpreting changes in the patent to R&D ratio. An interesting implication of our analysis is that the measurement of real changes in research productivity is made even more complex by the race for priority. Indeed, the more innovative the research community becomes, the faster scientists or inventors disclose their results or patent their innovations. Hence an observed growth in the patent to R&D ratio may actually overstate the real growth in research productivity, in contrast with the conclusions of empirical work that has attempted to disentangle real from apparent changes in this variable (Lanjouw and Schankerman (2004)).

The second is that, whereas we have focused throughout our analysis on winner-take-all priority races, being preempted does not necessarily mean in practice that a researcher should lose all the fruit of her work. For instance, Stephan (1996) notes that "A number of institutional arrangements have evolved in science to help minimize risk or provide some insurance against risk." Notable examples of institutions that play this role are second-tier or field journals. If publications in these journals are taken into account in tenure or hiring decisions, they act as a safety net by securing some rents to preempted researchers. This in turn induces researchers to spend more time maturing their ideas, thereby increasing the overall quality of the research published in first-tier journals.[43] Yet an interesting tradeoff is that this insurance mechanism may conflict with the positive impact of steep incentive schemes on the maturation of ideas. The optimal design of rewards in a dynamic competitive research environment is an important topic for future investigations.

[43] A formal analysis is provided in Appendix B.4.

Appendix A: Proofs of the Main Results

Proof of Lemma 1. (i) Suppose first, by way of contradiction, that $\sigma^i(\tau^i) = \tau^i$ for some i and τ^i. Then, according to (1), type τ^i's equilibrium payoff is zero. Yet, according to (1) again, type τ^i could secure a payoff $\{\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \tau^i + m] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = \tau^i + m]\} L(m, \tau^i)$ by waiting an amount of time $m > 0$. As $L(m, \tau^i) > 0$ for all $m > 0$, this implies that $\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \tau^i + m] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = \tau^i + m] = 0$ and hence $\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \tau^i + m] = 0$ for any such m. As a result, one must have $\mathbf{P}[\sigma^j(\tilde{\tau}^j) \leq \tau^i] = 1$, which is impossible as

$$\mathbf{P}[\sigma^j(\tilde{\tau}^j) \leq \tau^i] = \int_0^{\tau^i} 1_{\{\sigma^j(\tau^j) \leq \tau^i\}} \, \mathrm{d}G(\tau^j) \leq G(\tau^i) < 1. \tag{21}$$

This contradiction establishes that $\sigma^i(\tau^i) - \tau^i > 0$ for all i and τ^i.

Suppose next, by way of contradiction, that $\sigma^i(\tau^i) - \tau^i > M(\tau^i)$ for some i and τ^i. Then, as L is strictly decreasing over $[M(\tau^i), \infty)$, one has, for each $\varepsilon \in (0, \sigma^i(\tau^i) - \tau^i - M(\tau^i)]$ such that $\mathbf{P}[\sigma^j(\tilde{\tau}^j) = \sigma^i(\tau^i) - \varepsilon] = 0$,

$$\begin{aligned}
V^i(\sigma^i(\tau^i), \tau^i, \sigma^j) &= \{\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \sigma^i(\tau^i)] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = \sigma^i(\tau^i)]\} L(\sigma^i(\tau^i) - \tau^i, \tau^i) \\
&\leq \mathbf{P}[\sigma^j(\tilde{\tau}^j) \geq \sigma^i(\tau^i)] L(\sigma^i(\tau^i) - \tau^i, \tau^i) \\
&< \mathbf{P}[\sigma^j(\tilde{\tau}^j) \geq \sigma^i(\tau^i) - \varepsilon] L(\sigma^i(\tau^i) - \varepsilon - \tau^i, \tau^i) \\
&= \{\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \sigma^i(\tau^i) - \varepsilon] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = \sigma^i(\tau^i) - \varepsilon]\} L(\sigma^i(\tau^i) - \varepsilon - \tau^i, \tau^i) \\
&= V^i(\sigma^i(\tau^i) - \varepsilon - \tau^i, \tau^i, \sigma^j),
\end{aligned}$$

which is ruled out by (2). This contradiction establishes that $\sigma^i(\tau^i) - \tau^i \leq M(\tau^i)$ for all i and τ^i. The result follows.

(ii) We first prove that σ^i is nondecreasing for all i, that is, that $\sigma^i(\hat{\tau}^i) \geq \sigma^i(\tau^i)$ for all τ^i and $\hat{\tau}^i > \tau^i$. By Lemma 1(i), the result is obvious if $\hat{\tau}^i \geq \sigma^i(\tau^i)$ or $\sigma^i(\hat{\tau}^i) \geq \tau^i + M(\tau^i)$. Thus suppose that $\sigma^i(\tau^i) > \hat{\tau}^i$ and $\tau^i + M(\tau^i) > \sigma^i(\hat{\tau}^i)$. It follows from the first of these inequalities that $\sigma^i(\tau^i)$ is a feasible moving time for type $\hat{\tau}^i$, just like $\sigma^i(\hat{\tau}^i)$ is a feasible moving time for type τ^i as $\sigma^i(\hat{\tau}^i) \geq \hat{\tau}^i > \tau^i$. Hence, by (2),

$$V^i(\sigma^i(\tau^i), \tau^i, \sigma^j) \geq V^i(\sigma^i(\hat{\tau}^i), \tau^i, \sigma^j), \tag{22}$$

$$V^i(\sigma^i(\hat{\tau}^i), \hat{\tau}^i, \sigma^j) \geq V^i(\sigma^i(\tau^i), \hat{\tau}^i, \sigma^j). \tag{23}$$

Because $\sigma(\tilde{\tau}^j) \geq \tilde{\tau}^j$ and G has unbounded support, both $\mathbf{P}[\sigma^j(\tilde{\tau}^j) > t^i] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = t^i]$ and $\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \hat{t}^i] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = \hat{t}^i]$ are positive. Moreover, by assumption, $\sigma^i(\tau^i) > \hat{\tau}^i$, and, by Lemma 1(i), $\sigma^i(\hat{\tau}^i) > \hat{\tau}^i$, so that both $L(\sigma^i(\tau^i) - \hat{\tau}^i, \hat{\tau}^i)$ and $L(\sigma^i(\hat{\tau}^i) - \hat{\tau}^i, \hat{\tau}^i)$ are positive. Multiplying (22) by (23) and rearranging using (1) then yields

$$\frac{L(\sigma^i(\tau^i) - \tau^i, \tau^i)}{L(\sigma^i(\tau^i) - \hat{\tau}^i, \hat{\tau}^i)} \geq \frac{L(\sigma^i(\hat{\tau}^i) - \tau^i, \tau^i)}{L(\sigma^i(\hat{\tau}^i) - \hat{\tau}^i, \hat{\tau}^i)}. \tag{24}$$

By assumption and by Lemma 1(i), $\hat{\tau}^i < \sigma^i(\hat{\tau}^i) < \tau^i + M(\tau^i)$ and $\hat{\tau}^i < \sigma^i(\tau^i) \leq \tau^i + M(\tau^i)$. Thus, according to (24), to establish that $\sigma^i(\hat{\tau}^i) \geq \sigma^i(\tau^i)$, we only need to show that the mapping $t \mapsto L(t-\tau^i,\tau^i)/L(t-\hat{\tau}^i,\hat{\tau}^i)$ is strictly decreasing over $(\hat{\tau}^i,\tau^i+M(\tau^i)]$. This in turn holds if its derivative is strictly negative at all t in this interval, that is, if

$$(\ln L)_1(t-\tau^i,\tau^i)-(\ln L)_1(t-\hat{\tau}^i,\hat{\tau}^i)=\int_{\tau^i}^{\hat{\tau}^i}[(\ln L)_{11}-(\ln L)_{12}](t-\tau,\tau)\,\mathrm{d}\tau<0$$

for any such t, which follows from Assumption 2, owing to the fact that $t-\tau\leq M(\tau)$ for all $\tau\geq\tau^i$ as $t\leq\tau^i+M(\tau^i)$. This establishes that σ^i is nondecreasing for all i.

Suppose next, by way of contradiction, that $\hat{\tau}^i>\tau^i$ and yet $\sigma^i(\hat{\tau}^i)=\sigma^i(\tau^i)$. Then σ^i is constant over $[\tau^i,\hat{\tau}^i]$ and the distribution of player i's equilibrium moving time has an atom at $\sigma^i(\tau^i)$. The following claim then holds.

Claim 1 *There exists $\varepsilon_0>0$ such that*

$$\sigma^j(\tau^j)\notin(\sigma^i(\tau^i),\sigma^i(\tau^i)+\varepsilon_0),\quad \tau^j\in[0,\sigma^i(\tau^i)). \tag{25}$$

(Observe that the interval $[0,\sigma^i(\tau^i))$ is nonempty as $\sigma^i(\tau^i)\geq\sigma^i(0)>0$ by Lemma 1(i) along with the fact that σ^i is nondecreasing.) Suppose Claim 1 established. Then the only types of player j who can make a move during $(\sigma^i(\tau^i),\sigma^i(\tau^i)+\varepsilon_0)$ are those such that $\tau^j\geq\sigma^i(\tau^i)$. But it follows from Lemma 1(i) that $\sigma^j(\sigma^i(\tau^i))=\sigma^i(\tau^i)+\varepsilon_1$ for some $\varepsilon_1>0$. Hence, as σ^j is nondecreasing, player j never makes a move during $(\sigma^i(\tau^i),\sigma^i(\tau^i)+\varepsilon_0\wedge\varepsilon_1)$. Because $\sigma^i(\hat{\tau}^i)=\sigma^i(\tau^i)\leq\tau^i+M(\tau^i)<\hat{\tau}^i+M(\hat{\tau}^i)$ by Lemma 1(i), one has, letting $\hat{t}^i\equiv[\sigma^i(\tau^i)+(\varepsilon_0\wedge\varepsilon_1)/2]\wedge[\hat{\tau}^i+M(\hat{\tau}^i)]$ and using the fact that $L(\cdot,\hat{\tau}^i)$ is strictly increasing over $[0,M(\hat{\tau}^i)]$,

$$\begin{aligned}
V^i(\sigma^i(\hat{\tau}^i),\hat{\tau}^i,\sigma^j)&=V^i(\sigma^i(\tau^i),\hat{\tau}^i,\sigma^j)\\
&=\{\mathbf{P}[\sigma^j(\tilde{\tau}^j)>\sigma^i(\tau^i)]+\alpha\mathbf{P}[\sigma^j(\tilde{\tau}^j)=\sigma^i(\tau^i)]\}L(\sigma^i(\tau^i)-\hat{\tau}^i,\hat{\tau}^i)\\
&\leq\mathbf{P}[\sigma^j(\tilde{\tau}^j)\geq\sigma^i(\tau^i)]L(\sigma^i(\tau^i)-\hat{\tau}^i,\hat{\tau}^i)\\
&<\mathbf{P}[\sigma^j(\tilde{\tau}^j)\geq\sigma^i(\tau^i)]L(\hat{t}^i-\hat{\tau}^i,\hat{\tau}^i)\\
&=\{\mathbf{P}[\sigma^j(\tilde{\tau}^j)>\hat{t}^i]+\alpha\mathbf{P}[\sigma^j(\tilde{\tau}^j)=\hat{t}^i]\}L(\hat{t}^i-\hat{\tau}^i,\hat{\tau}^i)\\
&=V^i(\hat{t}^i,\hat{\tau}^i,\sigma^j),
\end{aligned}$$

which is ruled out by (2). This contradiction establishes that σ^i is strictly increasing for all i. The result follows.

To complete the proof of Lemma 1(ii), it remains to prove Claim 1. If $\sigma^j(\sigma^i(\tau^i)-)\leq\sigma^i(\tau^i)$, (25) directly follows from the fact that σ^j is nondecreasing. Thus suppose that $\sigma^j(\sigma^i(\tau^i)-)=\sigma^i(\tau^i)+\varepsilon_1^-$ for some $\varepsilon_1^->0$. Then, as σ^j is nondecreasing, there exists $\delta_1>0$ such that $\sigma^j(\tau^j)>\sigma^i(\tau^i)+\varepsilon_1^-/2$ for all $\tau^j\geq\sigma^i(\tau^i)-\delta_1$. Consider now types $\tau^j<\sigma^i(\tau^i)-\delta_1$. By Lemma 1(i), among these, we only need to be concerned by those such that $\tau^j\geq[\sigma^i(\tau^i)-M(\tau^j)]\vee 0$. Thus let $A\equiv\{\tau^j:[\sigma^i(\tau^i)-M(\tau^j)]\vee 0\leq\tau^j<\sigma^i(\tau^i)-\delta_1\}$ and assume that A is nonempty. Observe that, because $\tau^j+M(\tau^j)$ is strictly increasing in τ^j, A is an interval. We now show that there exists some $\varepsilon_0^->0$ such that any type in A is strictly better off making a move before time $\sigma^i(\tau^i)$ than making a move during $(\sigma^i(\tau^i),\sigma^i(\tau^i)+\varepsilon_0^-)$, from which Claim 1 follows for $\varepsilon_0\equiv\varepsilon_0^-\wedge(\varepsilon_1^-/2)$. For any type $\tau^j\in A$, making a move at time $\sigma^i(\tau^i)-\varepsilon$ yields a payoff $\mathbf{P}[\sigma^i(\tilde{\tau}^i)\geq\sigma^i(\tau^i)-\varepsilon]L(\sigma^i(\tau^i)-\varepsilon-\tau^j,\tau^j)$

for all $\varepsilon \in (0, \sigma^i(\tau^i) - \tau^j)$ such that $\mathbf{P}[\sigma^i(\tilde{\tau}^i) = \sigma^i(\tau^i) - \varepsilon] = 0$, whereas making a move at time $\sigma^i(\tau^i) + \varepsilon'$ yields at most a payoff $\mathbf{P}[\sigma^i(\tilde{\tau}^i) \geq \sigma^i(\tau^i) + \varepsilon']L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)$ for all $\varepsilon' > 0$. Suppose, by way of contradiction, that

$$\begin{aligned}&\forall \varepsilon_0^- \in (0, \infty) \;\; \exists \varepsilon' \in [0, \varepsilon_0^-] \;\; \exists \tau^j \in A \;\; \forall \varepsilon \in (0, \sigma^i(\tau^i) - \tau^j)\\ &\mathbf{P}[\sigma^i(\tilde{\tau}^i) \geq \sigma^i(\tau^i) + \varepsilon']L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j) \geq \mathbf{P}[\sigma^i(\tilde{\tau}^i) \geq \sigma^i(\tau^i) - \varepsilon]L(\sigma^i(\tau^i) - \varepsilon - \tau^j, \tau^j).\end{aligned}$$

Then, a fortiori,

$$\begin{aligned}&\forall \varepsilon_0^- \in (0, \infty) \;\; \exists \varepsilon' \in [0, \varepsilon_0^-] \;\; \exists \tau^j \in A \;\; \forall \varepsilon \in (0, \sigma^i(\tau^i) - \tau^j)\\ &\{\mathbf{P}[\sigma^i(\tilde{\tau}^i) \geq \sigma^i(\tau^i) - \varepsilon] - \mathbf{P}[\sigma^i(\tilde{\tau}^i) = \sigma^i(\tau^i)]\}L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)\\ &\quad \geq \mathbf{P}[\sigma^i(\tilde{\tau}^i) \geq \sigma^i(\tau^i) - \varepsilon]L(\sigma^i(\tau^i) - \varepsilon - \tau^j, \tau^j),\end{aligned}$$

so that, letting ε go to zero,

$$\begin{aligned}&\forall \varepsilon_0^- \in (0, \infty) \;\; \exists \varepsilon' \in [0, \varepsilon_0^-] \;\; \exists \tau^j \in A\\ &\mathbf{P}[\sigma^i(\tilde{\tau}^i) \geq \sigma^i(\tau^i)]\left[1 - \frac{L(\sigma^i(\tau^i) - \tau^j, \tau^j)}{L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)}\right] \geq \mathbf{P}[\sigma^i(\tilde{\tau}^i) = \sigma^i(\tau^i)].\end{aligned}$$

Because the distribution of player i's equilibrium moving time has an atom at $\sigma^i(\tau^i)$, this implies that

$$\inf_{\varepsilon_0^- \in (0,\infty)} \sup_{\varepsilon' \in [0, \varepsilon_0^-]} \sup_{\tau^j \in A} \left\{1 - \frac{L(\sigma^i(\tau^i) - \tau^j, \tau^j)}{L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)}\right\} > 0. \tag{26}$$

We now show that for ε_0^- and, thus, ε', close enough to zero, the supremum over $\tau^j \in A$ in (26) is achieved at the supremum $\sigma^i(\tau^i) - \delta_1$ of A. To do so, we only need to check that, for ε' close enough to zero, the mapping $\tau^j \mapsto L(\sigma^i(\tau^i) - \tau^j, \tau^j)/L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)$ is strictly decreasing over A. This in turn holds if its derivative is negative at all τ^j in this interval, that is, if

$$\begin{aligned}&\frac{L_1(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j) - L_2(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)}{L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)} - \frac{L_1(\sigma^i(\tau^i) - \tau^j, \tau^j) - L_2(\sigma^i(\tau^i) - \tau^j, \tau^j)}{L(\sigma^i(\tau^i) - \tau^j, \tau^j)}\\ &\quad = \int_{\sigma^i(\tau^i)}^{\sigma^i(\tau^i)+\varepsilon'} \left[\frac{(L_{11} - L_{12})L - (L_1 - L_2)L_1}{L^2}\right](t - \tau^j, \tau^j)\,\mathrm{d}t\\ &\quad = \int_{\sigma^i(\tau^i)}^{\sigma^i(\tau^i)+\varepsilon'} [(\ln L)_{11} - (\ln L)_{12}](t - \tau^j, \tau^j)\,\mathrm{d}t\end{aligned}$$

is negative for any such τ^j. (Observe that ε' must be uniform in τ^j.) Because $\sigma^i(\tau^i) - \tau^j > \delta_1 > 0$ for all $\tau^j \in A$ and because L is positive and thrice continuously differentiable over $[\sigma^i(\tau^i) - \tau^j, \infty) \times A$ by Assumption 1, there exists a bound $C > 0$ such that, for each $\varepsilon' \in [0, \varepsilon_0^-]$ and for each $(t, \tau^j) \in [\sigma^i(\tau^i), \sigma^i(\tau^i) + \varepsilon'] \times A$,

$$[(\ln L)_{11} - (\ln L)_{12}](t - \tau^j, \tau^j) \leq [(\ln L)_{11} - (\ln L)_{12}](\sigma^i(\tau^i) - \tau^j, \tau^j) + C[t - \sigma^i(\tau^i)].$$

Thus, for each $(\varepsilon', \tau^j) \in [0, \varepsilon_0^-] \times A$,

$$\int_{\sigma^i(\tau^i)}^{\sigma^i(\tau^i)+\varepsilon'} [(\ln L)_{11} - (\ln L)_{12}](t - \tau^j, \tau^j)\,\mathrm{d}t$$
$$\leq \left(\max_{\hat{\tau}^j \in \mathrm{cl}\,A} \left\{ [(\ln L)_{11} - (\ln L)_{12}](\sigma^i(\tau^i) - \hat{\tau}^j, \hat{\tau}^j) \right\} + \frac{C}{2}\,\varepsilon' \right) \varepsilon'. \quad (27)$$

Now, because $\sigma^i(\tau^i) - \hat{\tau}^j \geq \delta_1 > 0$ for all $\hat{\tau}^j \in \mathrm{cl}A$, the function to be maximized in (27) is continuous over the compact interval $\mathrm{cl}A$ as L is thrice continuously differentiable over the relevant range. Thus it attains its maximum. Moreover, by Assumption 2, this maximum must be negative, as $\sigma^i(\tau^i) - \hat{\tau}^j \leq M(\hat{\tau}^j)$ for all $\hat{\tau}^j \in \mathrm{cl}A$. Therefore, letting ε_0^- and, thus, ε', be close enough to zero, we get that the left-hand side of (27) is negative for all $\tau^j \in A$, as desired. It follows that, for such ε_0^- and ε',

$$\sup_{\tau^j \in A} \left\{ 1 - \frac{L(\sigma^i(\tau^i) - \tau^j, \tau^j)}{L(\sigma^i(\tau^i) + \varepsilon' - \tau^j, \tau^j)} \right\} = 1 - \frac{L(\delta_1, \sigma^i(\tau^i) - \delta_1)}{L(\delta_1 + \varepsilon', \sigma^i(\tau^i) - \delta_1)}.$$

In turn, because $L(\cdot, \sigma^i(\tau^i) - \delta_1)$ is strictly increasing over $[0, M(\sigma^i(\tau^i) - \delta_1)]$,

$$\sup_{\varepsilon' \in [0, \varepsilon_0^-]} \left\{ 1 - \frac{L(\delta_1, \sigma^i(\tau^i) - \delta_1)}{L(\delta_1 + \varepsilon', \sigma^i(\tau^i) - \delta_1)} \right\} = 1 - \frac{L(\delta_1, \sigma^i(\tau^i) - \delta_1)}{L(\delta_1 + \varepsilon_0^-, \sigma^i(\tau^i) - \delta_1)}$$

for all small enough $\varepsilon_0^- \in [0, M(\sigma^i(\tau^i) - \delta_1) - \delta_1]$. (Note that $M(\sigma^i(\tau^i) - \delta_1) + \sigma^i(\tau^i) - \delta_1 > M(\tau^j) + \tau^j \geq \sigma^i(\tau^i)$ for all $\tau^j \in A$, so that this interval is not reduced to $\{0\}$.) As $\delta_1 > 0$, one can let ε_0^- go to zero, and we get that the left-hand side of (26) is zero. This contradiction establishes Claim 1. The result follows.

(iii) Suppose, by way of contradiction, that $\sigma^i(0) < \sigma^j(0)$ for some i. By Lemma 1(i), $\sigma^j(0) \leq M(0)$ and hence $\sigma^i(0) < M(0)$. By Lemma 1(ii), $\sigma^j(\tau) > \sigma^j(0)$ for all $\tau > 0$, and hence $\sigma^j(\tau) > \hat{t}^i$ for all $\tau \geq 0$ and $\hat{t}^i \in (\sigma^i(0), \sigma^j(0))$. It follows that for any such $\hat{t}^i$

$$V^i(\sigma^i(0), 0, \sigma^j) = L(\sigma^i(0), 0) < L(\hat{t}^i, 0) = V^i(\hat{t}^i, 0, \sigma^j),$$

which is ruled out by (2). This contradiction establishes that $\sigma^a(0) = \sigma^b(0)$. The result follows.

Observe for future reference that the conclusions of Lemma 1 more generally hold when $\tilde{\tau}^a$ and $\tilde{\tau}^b$ are independently drawn from continuously differentiable distributions G^a and G^b with positive densities $\dot{G}^a$ and $\dot{G}^b$ over $[0, \infty)$. The only change in the proofs is in (21), where 'G^j' should be substituted to 'G' throughout. ■

Proof of Lemma 2. (i) The proof goes through a series of steps.

Step 1 We first prove that, in any equilibrium, if the players' equilibrium strategies have discontinuity points, then the corresponding gaps in the distributions of their moving times $\sigma^a(\tilde{\tau}^a)$ and $\sigma^b(\tilde{\tau}^b)$ cannot overlap. Suppose, by way of contradiction, that they do. Then, because these distributions have no atoms by Lemma 1(ii) along with the assumption that the breakthrough distribution has no atoms, there would exist some player i and some discontinuity point τ^i of σ^i such that, for some $\varepsilon > 0$, with probability 1 player j does not make a move during $[\sigma^i(\tau^i-), \sigma^i(\tau^i-) + \varepsilon]$. One must have $\sigma^i(\tau^i-) < \tau^i + M(\tau^i)$, for, otherwise, one would have $\sigma^i(\tau^i+) > \tau^i + M(\tau^i)$ as σ^i is discontinuous at τ^i, and hence some type $\hat{\tau}^i$ of player i close to but above τ^i would have a maturation delay strictly longer than $M(\hat{\tau}^i)$, which is impossible by Lemma 1(i). (Here we exploit the fact that $M(\tau)$ is continuous and even twice continuously differentiable in τ, as $\dot{M}(\tau) = -L_{12}(M(\tau), \tau)/L_{11}(M(\tau), \tau)$ by the implicit function theorem.) As a result, one cannot have $\tau^i = 0$, for, otherwise, player i with type 0 would be strictly better off making a move at time $[\sigma^i(0) + \varepsilon/2] \wedge M(0)$, as she would thereby increase her payoff from moving first, while still avoiding any preemption risk. Thus one can choose $\hat{\tau}^i < \tau^i$ close enough to τ^i such that $\sigma^i(\tau^i-) < \hat{\tau}^i + M(\hat{\tau}^i)$ and

$$L([\sigma^i(\tau^i-) + \varepsilon - \hat{\tau}^i] \wedge M(\hat{\tau}^i), \hat{\tau}^i) > \frac{\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \sigma^i(\hat{\tau}^i)]}{\mathbf{P}[\sigma^j(\tilde{\tau}^j) > \sigma^i(\tau^i-)]} L(\sigma^i(\hat{\tau}^i) - \hat{\tau}^i, \hat{\tau}^i).$$

One then has, letting $\hat{t}^i \equiv [\sigma^i(\tau^i-) + \varepsilon] \wedge [\hat{\tau}^i + M(\hat{\tau}^i)]$ and using the facts that $L(\cdot, \hat{\tau}^i)$ is strictly increasing over $[0, M(\hat{\tau}^i)]$ and that the distribution of $\sigma^j(\tilde{\tau}^j)$ has no atom and does not charge the interval $[\sigma^i(\tau^i-), \sigma^i(\tau^i-) + \varepsilon]$,

$$\begin{aligned} V^i(\sigma^i(\hat{\tau}^i), \hat{\tau}^i, \sigma^j) &= \mathbf{P}[\sigma^j(\tilde{\tau}^j) > \sigma^i(\hat{\tau}^i)] L(\sigma^i(\hat{\tau}^i) - \hat{\tau}^i, \hat{\tau}^i) \\ &< \mathbf{P}[\sigma^j(\tilde{\tau}^j) > \sigma^i(\tau^i-)] L(\hat{t}^i - \hat{\tau}^i, \hat{\tau}^i) \\ &= \mathbf{P}[\sigma^j(\tilde{\tau}^j) > \hat{t}^i] L(\hat{t}^i - \hat{\tau}^i, \hat{\tau}^i) \\ &= V^i(\hat{t}^i, \hat{\tau}^i, \sigma^j), \end{aligned}$$

which is ruled out by (2). This contradiction establishes that the gaps, if any exists, in the distributions of $\sigma^i(\tilde{\tau}^j)$ and $\sigma^j(\tilde{\tau}^j)$ cannot overlap. This notably rules out discontinuous symmetric equilibria.

Step 2 From Step 1 along with the fact that σ^j can only have jump discontinuities, if player i's equilibrium strategy has a discontinuity point at τ^i, then $\tau^i > 0$ and the set $\phi^j((\sigma^i(\tau^i-), \sigma^i(\tau^i+)))$ is well defined. We first prove that in such a case $\sigma^j(\tau^j) = \tau^j + M(\tau^j)$ for all $\tau^j \in \phi^j((\sigma^i(\tau^i-), \sigma^i(\tau^i+)))$. Indeed, suppose, by way of contradiction, that $\sigma^j(\tau^j) < \tau^j + M(\tau^j)$ for such a type τ^j. One then has, letting $\hat{t}^j = \sigma^i(\tau^i+) \wedge [\tau^j + M(\tau^j)]$ and using the facts that $L(\cdot, \tau^j)$ is strictly increasing over $[0, M(\tau^j)]$ and that the distribution of $\sigma^i(\tilde{\tau}^i)$ has no atom and does not charge the interval $[\sigma^i(\tau^i-), \sigma^i(\tau^i+)]$,

$$\begin{aligned} V^j(\sigma^j(\tau^j), \tau^j, \sigma^i) &= \mathbf{P}[\sigma^i(\tilde{\tau}^i) > \sigma^j(\tau^j)] L(\sigma^j(\tau^j) - \tau^j, \tau^j) \\ &< \mathbf{P}[\sigma^i(\tilde{\tau}^i) > \hat{t}^j] L(\hat{t}^j - \tau^j, \tau^j) \\ &= V^j(\hat{t}^j, \tau^j, \sigma^i), \end{aligned}$$

which is ruled out by (2). This contradiction establishes the claim. Now, consider type $\overline{\tau}^j \equiv \sup \phi^j((\sigma^i(\tau^i-), \sigma^i(\tau^i+)))$. Because $\sigma^j(\tau^j) = \tau^j + M(\tau^j)$ for all $\tau^j < \overline{\tau}^j$ close enough to $\overline{\tau}^j$, it follows from Lemma 1(i)–(ii) that $\sigma^j(\overline{\tau}^j) = \overline{\tau}^j + M(\overline{\tau}^j)$, so that $\overline{\tau}^j = \phi^j(\overline{\tau}^j + M(\overline{\tau}^j)) = \phi^j(\sigma^i(\tau^i+))$. Finally, consider type $\underline{\tau}^j \equiv \inf \phi^j((\sigma^i(\tau^i-), \sigma^i(\tau^i+)))$. Because $\sigma^j(\tau^j) = \tau^j + M(\tau^j)$ for all $\tau^j > \underline{\tau}^j$ close enough to $\underline{\tau}^j$, σ^j is discontinuous at $\underline{\tau}^j$ if $\sigma^j(\underline{\tau}^j) <$

$\underline{\tau}^j + M(\underline{\tau}^j) = \sigma^i(\tau^i-)$. Suppose, by way of contradiction, that this is the case. Then, applying the above reasoning at $\underline{\tau}^j$ and interchanging the roles of i and j, we have that $\overline{\tau}^i \equiv \sup \phi^i((\sigma^j(\underline{\tau}^j-), \sigma^j(\underline{\tau}^j+))) = \phi^i(\sigma^j(\underline{\tau}^j+))$ and $\sigma^i(\overline{\tau}^i) = \overline{\tau}^i + M(\overline{\tau}^i)$. But $\sigma^j(\underline{\tau}^j+) = \sigma^i(\tau^i-)$ so $\overline{\tau}^i = \phi^i(\sigma^i(\tau^i-))$ and hence $\sigma^i(\overline{\tau}^i) = \sigma^i(\tau^i-)$ and $\overline{\tau}^i = \tau^i$. Thus, as $\sigma^i(\overline{\tau}^i) = \overline{\tau}^i + M(\overline{\tau}^i)$, we get that $\sigma^i(\tau^i-) = \tau^i + M(\tau^i)$, which is impossible as noted in Step 1. This contradiction establishes that $\sigma^j(\underline{\tau}^j) = \underline{\tau}^j + M(\underline{\tau}^j)$, so that $\underline{\tau}^j = \phi^j(\underline{\tau}^j + M(\underline{\tau}^j)) = \phi^j(\sigma^i(\tau^i-))$. Overall, we have shown that $\sigma^j(\tau^j) = \tau^j + M(\tau^j)$ for all $\tau^j \in \phi^j([\sigma^i(\tau^i-), \sigma^i(\tau^i+)])$. Observe for future reference that $\phi^j(\sigma^i(\tau^i-)) = \underline{\tau}^j < \tau^i$. This follows from the fact that $\underline{\tau}^j < \overline{\tau}^j$ as $\sigma^j(\underline{\tau}^j) = \sigma^i(\tau^i-) < \sigma^i(\tau^i+) = \sigma^j(\overline{\tau}^j)$, and that, moreover, $\overline{\tau}^j \leq \tau^i$, for, otherwise, $\sigma^j(\overline{\tau}^j) = \overline{\tau}^j + M(\overline{\tau}^j) > \tau^i + M(\tau^i) \geq \sigma^i(\tau^i+)$, which is impossible as $\sigma^j(\overline{\tau}^j) = \sigma^i(\tau^i+)$.

Step 3 Suppose as in Step 2 that player i's equilibrium strategy has a discontinuity point at $\tau^i > 0$. Then consider type $\tilde{\tau} = \sup\{\tau \in [0, \tau^i) : \sigma^i(\tau) \geq \sigma^j(\tau)\}$, which is well defined as $\sigma^i(0) = \sigma^j(0)$, and strictly less than τ^i as $\sigma^j(\tau) > \sigma^i(\tau^i-) > \sigma^i(\tau)$ for all $\tau \in (\phi^j(\sigma^i(\tau^i-)), \tau^i)$ by Step 2. Observe that σ^j must be continuous over $(\tilde{\tau}, \tau^i]$, for, otherwise, it would follow from Step 2 that $\sigma^i(\tau) = \tau + M(\tau) \geq \sigma^j(\tau)$ for some $\tau \in (\tilde{\tau}, \tau^i)$, in contradiction with the definition of $\tilde{\tau}$. We now show that $\sigma^i(\tilde{\tau}+) = \sigma^j(\tilde{\tau}+)$. Clearly one must have $\sigma^i(\tilde{\tau}+) \leq \sigma^j(\tilde{\tau}+)$, for, otherwise, one would have $\sigma^i(\tau) > \sigma^j(\tau)$ for some $\tau \in (\tilde{\tau}, \tau^i)$, in contradiction with the definition of $\tilde{\tau}$. So suppose, by way of contradiction, that $\sigma^i(\tilde{\tau}+) < \sigma^j(\tilde{\tau}+)$. By Lemma 1(iii) and Step 1, one must have $\tilde{\tau} > 0$. Then, according to the definition of $\tilde{\tau}$, there exists a strictly increasing sequence $\{\tau_n\}$ converging to $\tilde{\tau}$ such that $\sigma^i(\tau_n) \geq \sigma^j(\tau_n)$ for all n. Thus $\sigma^j(\tilde{\tau}+) > \sigma^i(\tilde{\tau}+) > \sigma^i(\tau_n) \geq \sigma^j(\tau_n)$, which shows that σ^j is discontinuous at $\tilde{\tau}$. But it then follows from Step 2, interchanging the roles of i and j, that $\sigma^i(\tilde{\tau}+) \geq \sigma^j(\tilde{\tau}+)$. Together with the fact that $\sigma^i(\tilde{\tau}+) \leq \sigma^j(\tilde{\tau}+)$, this contradiction shows that $\sigma^i(\tilde{\tau}+) = \sigma^j(\tilde{\tau}+) \equiv \tilde{\sigma}$, as claimed.

Step 4 Define τ^i, $\tilde{\tau}$, and $\tilde{\sigma}$ as in Step 3. Consider the functions ϕ^i and ϕ^j. As for ϕ^j, it is continuous and strictly increasing over $(\tilde{\sigma}, \sigma^i(\tau^i-))$ because σ^j is strictly increasing and continuous over $(\tilde{\tau}, \tau^i)$ by Lemma 1(ii) and Step 3. As for ϕ^i, it may not be defined over the entire interval $(\tilde{\sigma}, \sigma^i(\tau^i-))$ because σ^i may have discontinuity points in $(\tilde{\tau}, \tau^i)$. Yet, one can straightforwardly extend ϕ^i to all of $(\tilde{\sigma}, \sigma^i(\tau^i-))$ by requiring it to be constant over any interval $[\sigma^i_-, \sigma^i_+]$ corresponding to a discontinuity point of σ^i. Call $\overline{\phi}^i$ the function generated in this way, which is continuous and nondecreasing. We first establish that $\overline{\phi}^i$ and ϕ^j are Lipschitz over $(\tilde{\sigma}, \sigma^i(\tau^i-))$.

We start with ϕ^j and study to that effect the incentives of player i. Two cases must be distinguished. First, if $t \in (\tilde{\sigma}, \sigma^i(\tau^i-)) \cap \sigma^i([0, \infty))$, then $\phi^i(t)$ is well defined. Because ϕ^j is continuous over $(\tilde{\sigma}, \sigma^i(\tau^i-))$, so is the maximization problem faced by any type of player i that belongs to $(\tilde{\tau}, \tau^i)$. Hence, by Berge's maximum theorem, the maximizer correspondence is upper hemicontinuous in player i's type. In the present case, this notably implies that we can without loss of generality assume that there exists a strictly increasing sequence $\{t_n\}$ converging to t in $(\tilde{\sigma}, \sigma^i(\tau^i-)) \cap \sigma^i([0, \infty))$. (This amounts to make the convention that $\sigma^i(\tau) = \sigma^i(\tau-)$ at any discontinuity point $\tau \in (\tilde{\tau}, \tau^i)$ of σ^i.) By Lemma 1(i), $t > \phi^i(t)$. Thus for n large enough, type $\phi^i(t)$ could deviate and make a move at time t_n as type $\phi^i(t_n)$ does. It follows from (2) along with the fact that the distribution of $\sigma^j(\tilde{\tau}^j)$ has no atom by Lemma

1(ii) that

$$\begin{aligned}[1-G(\phi^j(t))]L(t-\phi^i(t),\phi^i(t)) &= \mathbf{P}[\sigma^j(\tilde{\tau}^j)>t]L(t-\phi^i(t),\phi^i(t)) \\ &= V^i(t,\phi^i(t),\sigma^j) \\ &\geq V^i(t_n,\phi^i(t),\sigma^j) \\ &= \mathbf{P}[\sigma^j(\tilde{\tau}^j)>t_n]L(t_n-\phi^i(t),\phi^i(t)) \\ &= [1-G(\phi^j(t_n))]L(t_n-\phi^i(t),\phi^i(t))\end{aligned}$$

for n large enough. Rearranging and using the fact that ϕ^j is strictly increasing, we get

$$0<G(\phi^j(t))-G(\phi^j(t_n))\leq[1-G(\phi^j(t))]\,\frac{L(t-\phi^i(t),\phi^i(t))-L(t_n-\phi^i(t),\phi^i(t))}{L(t_n-\phi^i(t),\phi^i(t))}$$

for n large enough. Dividing through by $t-t_n$ and letting t_n increase to $t>\phi^i(t)$, we conclude that

$$0\leq D_-[G\circ\phi^j](t)\leq[1-G(\phi^j(t))]\,\frac{L_1}{L}\,(t-\phi^i(t),\phi^i(t)),\quad t\in(\tilde{\sigma},\sigma^i(\tau^i-))\cap\sigma^i([0,\infty)),\tag{28}$$

where $D_-[G\circ\phi^j](t)$ is the lower left Dini derivative of $G\circ\phi^j$ at t. As $t-\phi^i(t)$ is bounded away from zero over $(\tilde{\sigma},\sigma^i(\tau^i-))\cap\sigma^i([0,\infty))$, it follows that $D_-[G\circ\phi^j]$ is bounded over this set. Now, turn to the case where $t\in(\tilde{\sigma},\sigma^i(\tau^i-))\setminus\sigma^i([0,\infty))$, assuming that this set is nonempty. Note that it is then composed of a countable number of intervals. Over each of these intervals, one has $\phi^j(t)=t-M(\phi^j(t))$ by Step 2. Because $\tau+M(\tau)$ is twice continuously differentiable in τ, with positive derivative $[(L_{11}-L_{12})/L_{11}](M(\tau),\tau)$ according to the implicit function theorem and Assumptions 1–2, we get that ϕ^j is continuously differentiable over $(\tilde{\sigma},\sigma^i(\tau^i-))\setminus\sigma^i([0,\infty))$, with a bounded derivative. Because G is also continuously differentiable, there exists a constant K such that

$$0\leq D_-[G\circ\phi^j](t)\leq K,\quad t\in(\tilde{\sigma},\sigma^i(\tau^i-))\setminus\sigma^i([0,\infty)).\tag{29}$$

Combining the bounds (28) and (29), we get that $G\circ\phi^j$ is Lipschitz over $(\tilde{\sigma},\sigma^i(\tau^i-))$ (see for instance Giorgi and Komlósi (1992, Lemma 1.15)). Moreover, because $\dot{G}$ is continuous and positive over $[0,\infty)$, G^{-1} is locally Lipschitz over $[0,1)$. Hence ϕ^j is Lipschitz over $(\tilde{\sigma},\sigma^i(\tau^i-))$.

We consider next ϕ^i and study to that effect the incentives of player j. Again, two cases must be distinguished. First suppose that $t\in(\tilde{\sigma},\sigma^i(\tau^i-))\cap\sigma^i([0,\infty))$, and consider an approximating sequence $\{t_n\}$ as above. By Lemma 1(i), $t>\phi^j(t)$. Thus for n large enough, type $\phi^j(t)$ could deviate and make a move at time t_n as type $\phi^j(t_n)$ does. Proceeding in a similar way as above, and using the fact that $\phi^i=\overline{\phi}^i$ over $(\tilde{\sigma},\sigma^i(\tau^i-))\cap\sigma^i([0,\infty))$, we get

$$0<G(\overline{\phi}^i(t))-G(\overline{\phi}^i(t_n))\leq[1-G(\overline{\phi}^i(t))]\,\frac{L(t-\phi^j(t),\phi^j(t))-L(t_n-\phi^j(t),\phi^j(t))}{L(t_n-\phi^j(t),\phi^j(t))}$$

for n large enough. Dividing through by $t-t_n$ and letting t_n increase to $t>\phi^i(t)=\overline{\phi}^i(t)$, we conclude that

$$0\leq D_-[G\circ\overline{\phi}^i](t)\leq[1-G(\overline{\phi}^i(t))]\,\frac{L_1}{L}\,(t-\phi^j(t),\phi^j(t)),\quad t\in(\tilde{\sigma},\sigma^i(\tau^i-))\cap\sigma^i([0,\infty)),\tag{30}$$

As $t - \phi^j(t)$ is bounded away from zero over $(\tilde{\sigma}, \sigma^i(\tau^i-)) \cap \sigma^i([0,\infty))$, it follows that $D_-[G \circ \overline{\phi}^i]$ is bounded over this set. Now, if $t \in (\tilde{\sigma}, \sigma^i(\tau^i-)) \setminus \sigma^i([0,\infty))$, then $\overline{\phi}^i$ is constant over $(t-\varepsilon, t]$ for some $\varepsilon > 0$ according to our convention. Thus

$$D_-[G \circ \overline{\phi}^i](t) = 0, \quad t \in (\tilde{\sigma}, \sigma^i(\tau^i-)) \setminus \sigma^i([0,\infty)). \tag{31}$$

Combining the bounds (30) and (31), and reasoning as in the case of ϕ^j, we obtain that $\overline{\phi}^i$ is Lipschitz over $(\tilde{\sigma}, \sigma^i(\tau^i-))$.

Step 5 Define the functions $\overline{\phi}^i$ and ϕ^j as in Step 4. Because they are Lipschitz over $(\tilde{\sigma}, \sigma^i(\tau^i-))$, they are absolutely continuous and thus almost everywhere differentiable over this interval. Their derivatives, where they exist, can be evaluated as follows. Consider first some $t \in (\tilde{\sigma}, \sigma^i(\tau^i-)) \setminus \sigma^i([0,\infty))$. If $\overline{\phi}^i$ and ϕ^j are differentiable at t, then $\dot{\overline{\phi}}^i(t) = 0$ and $\dot{\phi}^j(t) = 1/[1 + \dot{M}(\phi^j(t))]$. Consider next some $t \in (\tilde{\sigma}, \sigma^i(\tau^i-)) \cap \sigma^i([0,\infty))$. If $\overline{\phi}^i$ and ϕ^j are differentiable at t, then $\dot{\overline{\phi}}^i(t)$ (respectively $\dot{\phi}^j(t)$) is obtained by differentiating the mapping $\hat{t} \mapsto [1 - G(\overline{\phi}^i(\hat{t}))]L(\hat{t} - \phi^j(t), \phi^j(t))$ (respectively $\hat{t} \mapsto [1 - G(\phi^j(\hat{t}))]L(\hat{t} - \overline{\phi}^i(t), \overline{\phi}^i(t))$) and requiring that the resulting derivative, whenever it exists, be equal to zero at $\hat{t} = t$, as implied by optimality. (Observe that in this case $\overline{\phi}^i(t) = \phi^i(t)$.) For any such t, this yields

$$\dot{\overline{\phi}}^i(t) = \frac{1-G}{\dot{G}}\,(\overline{\phi}^i(t))\,\frac{L_1}{L}\,(t - \phi^j(t), \phi^j(t)), \tag{32}$$

$$\dot{\phi}^j(t) = \frac{1-G}{\dot{G}}\,(\phi^j(t))\,\frac{L_1}{L}\,(t - \overline{\phi}^i(t), \overline{\phi}^i(t)). \tag{33}$$

Define now the quantity

$$R(t) \equiv \ln\!\left(\frac{1 - G(\phi^j(t))}{1 - G(\overline{\phi}^i(t))}\right), \quad t \in (\tilde{\sigma}, \sigma^i(\tau^i-)). \tag{34}$$

Using that $\overline{\phi}^i$ and ϕ^j are absolutely continuous over $(\tilde{\sigma}, \sigma^i(\tau^i-))$, that $\dot{G}$ is continuous, that $G \circ \overline{\phi}^i$ and $G \circ \phi^j$ are bounded away from 1 over $(\tilde{\sigma}, \sigma^i(\tau^i-))$, and thus that the logarithm function is Lipschitz over the corresponding range of $(1 - G \circ \phi^j)/(1 - G \circ \overline{\phi}^i)$, we get that R is absolutely continuous over $(\tilde{\sigma}, \sigma^i(\tau^i-))$ and hence is equal to the integral of its derivative, which is well defined almost everywhere. Now, for each $t \in (\tilde{\sigma}, \sigma^i(\tau^i-)) \setminus \sigma^i([0,\infty))$ such that $\overline{\phi}^i$ and ϕ^j are differentiable at t, we have $\dot{\overline{\phi}}^i(t) = 0$ and $\dot{\phi}^j(t) = 1/[1 + \dot{M}(\phi^j(t))]$, and, therefore,

$$\dot{R}(t) = -\left[\frac{\dot{G}}{(1-G)(1+\dot{M})}\right](\phi^j(t)),$$

which is negative as $\dot{G}$ is bounded away from zero and $1 + \dot{M}$ is positive. Similarly, for each $t \in (\tilde{\sigma}, \sigma^i(\tau^i-)) \cap \sigma^i([0,\infty))$ such that $\overline{\phi}^i$ and ϕ^j are differentiable at t, we have, using (32)–(33),

$$\dot{R}(t) = \frac{\dot{G}}{1-G}\,(\overline{\phi}^i(t))\,\dot{\overline{\phi}}^i(t) - \frac{\dot{G}}{1-G}\,(\phi^j(t))\,\dot{\phi}^j(t)$$

$$= \frac{L_1}{L}(t - \phi^j(t), \phi^j(t)) - \frac{L_1}{L}(t - \overline{\phi}^i(t), \overline{\phi}^i(t))$$

$$= \int_{\phi^j(t)}^{\overline{\phi}^i(t)} \left[\frac{(L_{11} - L_{12})L - (L_1 - L_2)L_1}{L^2}\right](t - \tau, \tau) \, \mathrm{d}\tau$$

$$= \int_{\phi^j(t)}^{\overline{\phi}^i(t)} [(\ln L)_{11} - (\ln L)_{12}](t - \tau, \tau) \, \mathrm{d}\tau,$$

which is negative according to Assumption 2, owing to the fact that $\phi^j < \overline{\phi}^i$ over $(\tilde{\sigma}, \sigma^i(\tau^i-))$ according to the definition of $\tilde{\sigma}$ in Step 2, and to the fact that $t - \tau \leq M(\tau)$ for all $\tau \geq \phi^j(t)$ as $t \leq \phi^j(t) + M(\phi^j(t))$. We thus obtain that R is strictly decreasing over $(\tilde{\sigma}, \sigma^i(\tau^i-))$. Now, as observed in Step 2, $\phi^j(\sigma^i(\tau^i-)) < \tau^i = \overline{\phi}^i(\sigma^i(\tau^i-))$. Thus, by (34), $R(\sigma^i(\tau^i-)) > 0$ and, therefore, as R is strictly decreasing over $(\tilde{\sigma}, \sigma^i(\tau^i-))$, $R(\tilde{\sigma}+) > 0$. This, however, is ruled out by the fact that, as shown in Step 3, $\sigma^i(\tilde{\tau}+) = \sigma^j(\tilde{\tau}+) = \tilde{\sigma}$, so that $\overline{\phi}^i(\tilde{\sigma}+) = \phi^j(\tilde{\sigma}+) = \tilde{\tau}$ and thus, by (34) again, $R(\tilde{\sigma}+) = 0$. This contradiction establishes that σ^i is continuous for all i. The result follows.

(ii) It follows from Lemma 2(i) that ϕ^a and ϕ^b are defined and absolutely continuous over $[\sigma(0), \infty)$, and that they satisfy (4) almost everywhere in $(\sigma(0), \infty)$. As a result,

$$\phi^i(t) = \phi^i(t_0) + \int_{t_0}^{t} \frac{1 - G}{\dot{G}}(\phi^i(s)) \frac{L_1}{L}(s - \phi^j(s), \phi^j(s)) \, \mathrm{d}s, \quad t > t_0 > \sigma(0), \quad i = a, b. \tag{35}$$

But because, for each i, $s - \phi^j(s)$ is bounded away from zero over any compact subinterval of $(\sigma(0), \infty)$, the integrand in (35) is continuous in s over any such interval. Thus, by the fundamental theorem of calculus, one may differentiate (35) everywhere with respect to t to get that (4) holds for all $t > \sigma(0)$. To conclude the proof, observe that for each i, ϕ^i is continuous at $\sigma(0)$ by Lemma 1(ii), and that so is the integrand in (35) and thus $\dot{\phi}^i$ as $\sigma(0) > 0$ by Lemma 1(i). This implies that $\dot{\phi}^i$ can be continuously extended at $\sigma(0)$. The result follows. ■

Proof of Lemma 3. Suppose, by way of contradiction, that $\phi^i(t) > \phi^j(t)$ for some $t > \sigma(0)$. As $\phi^i(\sigma(0)) = \phi^j(\sigma(0))$, $t_0 \equiv \sup\{s \in [\sigma(0), t) : \phi^i(s) = \phi^j(s)\}$ is well defined and strictly less than t. Moreover, $\phi^i(t_0) = \phi^j(t_0)$ and $\phi^i > \phi^j$ over (t_0, t). Integrating (4) yields

$$\ln\left(\frac{1 - G(\phi^j(t))}{1 - G(\phi^i(t))}\right) = \int_{t_0}^{t} \left[\frac{L_1}{L}(s - \phi^j(s), \phi^j(s)) - \frac{L_1}{L}(s - \phi^i(s), \phi^i(s))\right] \mathrm{d}s$$

$$= \int_{t_0}^{t} \int_{\phi^j(s)}^{\phi^i(s)} \left[\frac{(L_{11} - L_{12})L - (L_1 - L_2)L_1}{L^2}\right](s - \tau, \tau) \, \mathrm{d}\tau \, \mathrm{d}s$$

$$= \int_{t_0}^{t} \int_{\phi^j(s)}^{\phi^i(s)} [(\ln L)_{11} - (\ln L)_{12}](s - \tau, \tau) \, \mathrm{d}\tau \, \mathrm{d}s. \tag{36}$$

Because $\phi^i(t) > \phi^j(t)$, the left-hand side of (36) is positive. However, the right-hand side of (36) is negative according to Assumption 2, owing to the fact that $\phi^j < \phi^i$ over (t_0, t) and to the fact that, for each $s \in (t_0, t)$, $s - \tau \leq M(\tau)$ for all $\tau \geq \phi^j(s)$ as $s \leq \phi^j(s) + M(\phi^j(s))$. This contradiction establishes that $\phi^i(t) = \phi^j(t)$ for all $t > \sigma(0)$. The result follows. ■

Proof of Lemma 4. We must check that, for any type $\tau = \phi(t)$ of player i, making a move at time $\sigma(\tau) = t$ is a best response if player j plays the strategy σ. Observe first that making a move at $\hat{t} < \sigma(0)$ cannot be a best response, either because $\hat{t} < \phi(t)$, so that this deviation is not feasible, or, when $\hat{t} \geq \phi(t)$, because

$$V^i(\sigma(0), \phi(t), \sigma) = L(\sigma(0) - \phi(t), \phi(t)) > L(\hat{t} - \phi(t), \phi(t)) = V^i(\hat{t}, \phi(t), \sigma)$$

as σ is strictly increasing, $L(\cdot, \phi(t))$ is strictly increasing over $[0, M(\phi(t))]$, and $\hat{t} - \phi(t) < \sigma(0) - \phi(t) \leq M(0) - \phi(t) \leq M(\phi(t))$. According to this observation, we can focus on deviations by type $\phi(t)$ such that she mimics the behavior of an other type $\phi(\hat{t})$ by making a move at time $\hat{t}$. We can further restrict $\hat{t}$ to be such that $\phi(t) \leq \hat{t} \leq \phi(t) + M(\phi(t))$, for other deviations are either not feasible or can be shown not to be best responses along the lines of Lemma 1(i). Now, writing (5) at $\hat{t}$ yields

$$\begin{aligned}
\frac{L_1}{L}(\hat{t} - \phi(t), \phi(t)) &= \frac{L_1}{L}(\hat{t} - \phi(\hat{t}), \phi(\hat{t})) + \int_{\phi(\hat{t})}^{\phi^i(t)} \left[\frac{(L_{12} - L_{11})L + (L_1 - L_2)L_1}{L^2}\right](\hat{t} - \tau, \tau)\,\mathrm{d}\tau \\
&= \frac{\dot{G}}{1 - G}(\phi(\hat{t}))\,\dot{\phi}(\hat{t}) + \int_{\phi(\hat{t})}^{\phi(t)} [(\ln L)_{12} - (\ln L)_{11}](\hat{t} - \tau, \tau)\,\mathrm{d}\tau \\
&\gtreqless \frac{\dot{G}}{1 - G}(\phi(\hat{t}))\,\dot{\phi}(\hat{t})
\end{aligned}$$

if $\phi(t) \gtreqless \phi(\hat{t})$, that is, $t \gtreqless \hat{t}$, according to Assumption 2, owing to the fact that $\hat{t} - \tau \leq M(\tau)$ for all τ between $\phi(\hat{t})$ and $\phi(t)$ as $\hat{t} \leq M(\phi(\hat{t})) - \phi(\hat{t})$ by construction, and $\hat{t} \leq M(\phi(t)) - \phi(t)$ by assumption. That is, for player i with type $\phi(t)$, the expected incremental payoff of slightly delaying her move is positive at all $\hat{t} < t$ and negative at all $\hat{t} > t$. As a result, the second-order condition for problem

$$\max_{\hat{t} \in [\phi(t), \infty)} \{[1 - G(\phi(\hat{t}))] L(\hat{t} - \phi(t), \phi(t))\}$$

or, equivalently, as σ is strictly increasing,

$$\max_{\hat{t} \in [\phi(t), \infty)} \{V^i(\hat{t}, \phi(t), \sigma)\},$$

is satisfied when the first-order condition (5) is satisfied, for $\hat{t} = t$. The result follows. ■

Proof of Theorem 1. We only need to show that there exists at least one value of $\sigma_0 \in (0, M(0))$ such that the solution to (5) with initial condition $(\sigma_0, 0)$ stays in $\mathcal{D}$. It is helpful for the purpose of this proof to consider the ODE for $\sigma = \phi^{-1}$. Specifically, for each $\sigma_0 \in (0, M(0))$, consider the following initial value problem:

$$\dot{\sigma}(\tau) = \frac{1}{f(\sigma(\tau), \tau)}, \quad \tau \geq 0, \tag{37}$$

$$\sigma(0) = \sigma_0. \tag{38}$$

It is easy to check from the definition (6) of f that, over the interior $\text{int}\mathcal{D}'$ of the domain $\mathcal{D}' \equiv \{(\tau, t) : 0 \leq \tau < t \leq \tau + M(\tau)\}$, the mapping $(\tau, t) \mapsto 1/f(t, \tau)$ is continuous and locally Lipschitz in σ. Hence, by the Cauchy–Lipschitz theorem, for each $\sigma_0 \in (0, M(0))$, problem (37)–(38) has a unique maximal solution $\sigma(\cdot, 0, \sigma_0)$ in $\text{int}\,\mathcal{D}'$ (see for instance Perko (2001, Section 2.2, Theorem, and Section 2.4, Theorem 1)). Define also the degenerate solutions $\sigma(\cdot, 0, 0) \equiv \{(0,0)\}$ and $\sigma(\cdot, 0, M(0)) \equiv \{(0, M(0))\}$ for $\sigma_0 = 0$ and $\sigma_0 = M(0)$, respectively. For each $\sigma_0 \in [0, M(0)]$, let

$$\tau(\sigma_0) \equiv \sup\{\tau \geq 0 : (\tau', \sigma(\tau', 0, \sigma_0)) \in \text{int}\,\mathcal{D}' \text{ for all } \tau' \in (0, \tau)\},$$

with $\sup \emptyset = 0$ by convention, so that $\tau(0) = \tau(M(0)) = 0$. The proof is complete if we show that $\tau(\sigma_0) = \infty$ for some $\sigma_0 \in (0, M(0))$. To this end, define

$$L_0 \equiv \{\sigma_0 \in [0, M(0)] : \tau(\sigma_0) < \infty \text{ and } \sigma(\tau(\sigma_0), 0, \sigma_0) = \tau(\sigma_0)\},$$
$$U_0 \equiv \{\sigma_0 \in [0, M(0)] : \tau(\sigma_0) < \infty \text{ and } \sigma(\tau(\sigma_0), 0, \sigma_0) = \tau(\sigma_0) + M(\tau(\sigma_0))\}.$$

Clearly $L_0 \neq \emptyset$ as $0 \in L_0$, $U_0 \neq \emptyset$ as $M(0) \in U_0$, and $L_0 \cap U_0 = \emptyset$. From the non-crossing property of the solutions to (37)–(38) over $\text{int}\,\mathcal{D}'$, L_0 and U_0 are intervals. If we knew that both L_0 and U_0 were relatively open in $[0, M(0)]$, then, because $[0, M(0)]$ is connected and thus cannot be the union of two disjoint open sets, we could argue that there must exist some $\sigma_0 \in [0, M(0)]$ such that $\sigma_0 \notin L_0 \cup U_0$. Given the definitions of $\tau(\sigma_0)$, L_0, and U_0, it would follow that $\tau(\sigma_0) = \infty$. More precisely, as L_0 and U_0 are intervals, one would have $L_0 = [0, \underline{\sigma}_0)$ and $U_0 = (\overline{\sigma}_0, M(0)]$, where $0 < \underline{\sigma}_0 \leq \overline{\sigma}_0 < M(0)$, so that $\tau(\sigma_0) = \infty$ if and only if $\sigma_0 \in \Sigma_0 \equiv [\underline{\sigma}_0, \overline{\sigma}_0]$. The proof that this is indeed the case relies on the following claim, the proof of which can be found below.

Claim 2 *For each $(\tau_1, \sigma_1) \in \text{int}\,\mathcal{D}'$, the terminal value problem*

$$\dot{\sigma}(\tau) = \frac{1}{f(\sigma(\tau), \tau)}, \quad \tau \leq \tau_1, \tag{39}$$
$$\sigma(\tau_1) = \sigma_1 \tag{40}$$

has a unique solution $\sigma(\cdot, \tau_1, \sigma_1)$ in $\mathcal{D}'$ over $[0, \tau_1]$.

Suppose Claim 2 established. We show that U_0 is relatively open in $[0, M(0)]$. As U_0 is an interval that contains $M(0)$, we only need to show that if $\sigma_0 \in U_0$, then $\sigma_0' \in U_0$ for some $\sigma_0' < \sigma_0$. For each $\varepsilon > 0$, consider the solution $\sigma(\cdot, \tau(\sigma_0), \tau(\sigma_0) + M(\tau(\sigma_0)) - \varepsilon)$ to (39)–(40) with terminal condition $\sigma(\tau(\sigma_0)) = \tau(\sigma_0) + M(\tau(\sigma_0)) - \varepsilon$. By Claim 2, this solution can be maximally extended down to 0, and $\sigma_0'(\varepsilon) \equiv \sigma(0, \tau(\sigma_0), \tau(\sigma_0) + M(\tau(\sigma_0)) - \varepsilon) \in (0, \sigma_0)$ from the non-crossing property of the solutions to (37)–(38) over $\text{int}\,\mathcal{D}'$. Now, consider the solution $\sigma(\cdot, 0, \sigma_0'(\varepsilon))$ to (37)–(38) with initial condition $\sigma(0) = \sigma_0'(\varepsilon)$, so that $\sigma(\tau(\sigma_0), 0, \sigma_0'(\varepsilon)) = \tau(\sigma_0) + M(\tau(\sigma_0)) - \varepsilon$, and suppose that $\sigma_0'(\varepsilon) \notin U_0$ for all $\varepsilon > 0$, so that $\sigma(\cdot, 0, \sigma_0'(\varepsilon))$ never leaves $\mathcal{D}'$ through its upper boundary $t = \tau + M(\tau)$. Notice that, because $L_1(M(\tau(\sigma_0)), \tau(\sigma_0)) = 0$ and $M(\tau)$ is (twice) continuously differentiable in τ as observed in the proof of Lemma 2, there exist an open ball $\mathcal{B}$ with radius $\eta > 0$ centered at $(\tau(\sigma_0), \tau(\sigma_0) + M(\tau(\sigma_0)))$ that does not intersect the lower boundary $t = \tau$ of $\mathcal{D}'$, and a number

$$K > \max_{\tau \in \{\tau' \geq 0 : (\tau', \tau' + M(\tau')) \in \mathcal{B}\}} \{1 + \dot{M}(\tau)\} \equiv \delta_{\mathcal{B}} \tag{41}$$

such that $1/f(\sigma,\tau) \geq K$ for all $(\sigma,\tau) \in \mathcal{B} \cap \operatorname{int}\mathcal{D}'$. As the slope of the portion of the upper boundary of $\mathcal{D}'$ contained in $\mathcal{B}$ is at most $\delta_{\mathcal{B}}$, the segment of slope K connecting $(\tau(\sigma_0), \tau(\sigma_0) + M(\tau(\sigma_0)) - \varepsilon)$ to $(\tau(\sigma_0) + \varepsilon/(K - \delta_{\mathcal{B}}), \tau(\sigma_0) + M(\tau(\sigma_0)) + \delta_{\mathcal{B}}\varepsilon/(K - \delta_{\mathcal{B}}))$ is entirely contained in $\mathcal{B}$ for all $\varepsilon \in (0, (K - \delta_{\mathcal{B}})\eta \cos \arctan \delta_{\mathcal{B}})$. For any such ε, because $\sigma(\cdot, 0, \sigma_0'(\varepsilon))$ does not leave $\mathcal{D}'$ through its upper boundary, it must eventually leave $\mathcal{B}$. But, because of the above observation, and because $\sigma(\tau(\sigma_0), 0, \sigma_0'(\varepsilon)) = \tau(\sigma_0) + M(\tau(\sigma_0)) - \varepsilon$, it cannot do so before time $\tau(\sigma_0) + \varepsilon/(K - \delta_{\mathcal{B}})$. In particular,

$$\begin{aligned}
\sigma\left(\tau(\sigma_0) + \frac{\varepsilon}{K - \delta_{\mathcal{B}}}, 0, \sigma_0'(\varepsilon)\right) &\geq \tau(\sigma_0) + M(\tau(\sigma_0)) - \varepsilon + K\left[\tau(\sigma_0) + \frac{\varepsilon}{K - \delta_{\mathcal{B}}} - \tau(\sigma_0)\right] \\
&= \tau(\sigma_0) + M(\tau(\sigma_0)) + \frac{\delta_{\mathcal{B}}\varepsilon}{K - \delta_{\mathcal{B}}} \\
&\geq \tau(\sigma_0) + \frac{\varepsilon}{K - \delta_{\mathcal{B}}} + M\left(\tau(\sigma_0) + \frac{\varepsilon}{K - \delta_{\mathcal{B}}}\right), \qquad (42)
\end{aligned}$$

where the last inequality follows from the definition (41) of $\delta_{\mathcal{B}}$. But (42) implies that $\sigma(\cdot, 0, \sigma_0'(\varepsilon))$ must leave $\mathcal{D}'$ through its upper boundary at a time $\tau \leq \tau(\sigma_0) + \varepsilon/(K - \delta_{\mathcal{B}})$, a contradiction. It follows that $\sigma_0'(\varepsilon) \in U_0$ for all $\varepsilon > 0$ close enough to zero, which proves the claim as $\sigma_0'(\varepsilon) < \sigma_0$ for any such ε. The proof that L_0 is relatively open is similar, and is therefore omitted. Hence the result.

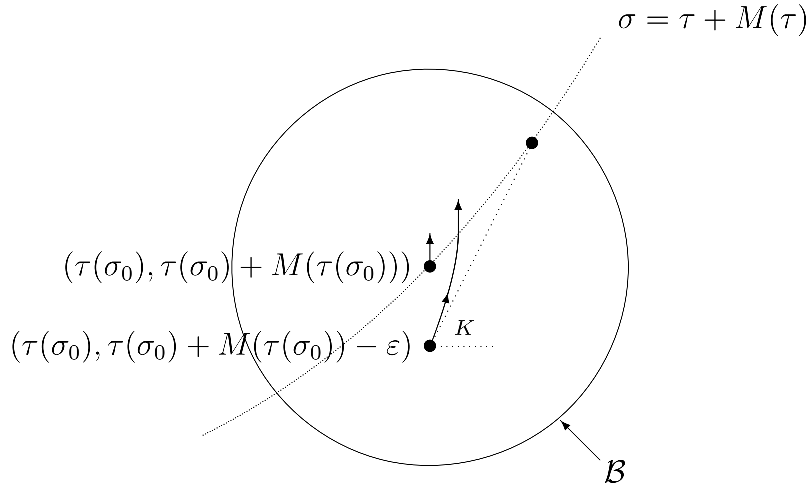

Figure 2 Illustration of the last step of the proof.

To complete the proof of Theorem 1, it remains to prove Claim 2. By the Cauchy–Lipschitz theorem, the terminal value problem (39)–(40) has a unique maximal solution $\sigma(\cdot, \tau_1, \sigma_1)$ in $\mathcal{D}'$. Because $(\tau_1, \sigma_1) \in \operatorname{int}\mathcal{D}'$,

$$\tau_0 \equiv \inf\{\tau \leq \tau_1 : (\sigma(\tau', \tau_1, \sigma_1), \tau') \in \operatorname{int}\mathcal{D}' \text{ for all } \tau' \in (\tau, \tau_1)\} < \tau_1.$$

We now show that $\tau_0 = 0$, which concludes the proof. Suppose, by way of contradiction, that $\tau_0 > 0$. Then, either $\sigma(\cdot, \tau_1, \sigma_1)$ leaves $\mathcal{D}'$ through its lower boundary, so that $\sigma(\tau_0, \tau_1, \sigma_1) =$

τ_0, or $\sigma(\cdot, \tau_1, \sigma_1)$ leaves $\mathcal{D}'$ through its upper boundary, so that $\sigma(\tau_0, \tau_1, \sigma_1) = \tau_0 + M(\tau_0)$. In the first case, there exist $\varepsilon_0 \in (0, \tau_1 - \tau_0)$ and $k < 1$ such that $(\partial\sigma/\partial\tau)(\tau_0 + \varepsilon, \tau_1, \sigma_1) < k$ for all $\varepsilon \in (0, \varepsilon_0)$. For any such ε, we have

$$\sigma(\tau_0 + \varepsilon, \tau_1, \sigma_1) < \sigma(\tau_0, \tau_1, \sigma_1) + k\varepsilon = \tau_0 + k\varepsilon < \tau_0 + \varepsilon,$$

so that $(\tau_0 + \varepsilon, \sigma(\tau_0 + \varepsilon, \tau_1, \sigma_1)) \notin \text{int}\mathcal{D}'$, which is impossible according to the definition of τ_0. In the second case, there exist $\varepsilon_0 \in (0, \tau_1 - \tau_0)$ and, by analogy with (41), a number

$$K > \max_{\tau \in [\tau_0, \tau_0 + \varepsilon_0]} \{1 + \dot{M}(\tau)\} \equiv \delta_0 \tag{43}$$

such that $(\partial\sigma/\partial\tau)(\tau_0 + \varepsilon, \tau_1, \sigma_1) > K$ for all $\varepsilon \in (0, \varepsilon_0)$. For any such ε, we have

$$\begin{aligned}\sigma(\tau_0 + \varepsilon, \tau_1, \sigma_1) &< \sigma(\tau_0 + \varepsilon_0, \tau_1, \sigma_1) - K(\varepsilon_0 - \varepsilon) \\ &< \tau_0 + \varepsilon_0 + M(\tau_0 + \varepsilon_0) - K(\varepsilon_0 - \varepsilon) \\ &\leq \tau_0 + M(\tau_0) - (K - \delta_0)\varepsilon_0 + K\varepsilon,\end{aligned}$$

where the second inequality follows from the definition of τ_0, and the third inequality follows from the definition (43) of δ_0. Letting ε go to zero then yields $\sigma(\tau_0, \tau_1, \sigma_1) \leq \tau_0 + M(\tau_0) - (K - \delta_0)\varepsilon_0 < \tau_0 + M(\tau_0)$, which is impossible according to the definition of τ_0. These contradictions establish Claim 2. The result follows. ∎

Proof of Theorem 2. As a preliminary, observe that, for each τ, as $L(\cdot, \tau)$ is ρ-concave over $[0, M(\tau)]$ by Assumption 4 and $\rho > 0$, $L(\cdot, \tau)^\rho$ is concave over this interval and so

$$1 - \frac{LL_{11}}{L_1^2}(m, \tau) \geq \rho, \quad \tau \geq 0, \quad M(\tau) > m \geq 0. \tag{44}$$

The logic of the proof is then similar to Hubbard and West (1991, Exercise 4.7#3). Consider two equilibria (σ_1, σ_1) and (σ_2, σ_2) with $\sigma_1(0) \geq \sigma_2(0)$ and define

$$g(\tau) \equiv \sigma_1(\tau) - \sigma_2(\tau), \quad \tau \geq 0. \tag{45}$$

By construction,

$$M(\tau) > \sigma_1(\tau) - \tau \geq \sigma_2(\tau) - \tau > 0 \tag{46}$$

for all $\tau \geq 0$, where the middle inequality follows from the non-crossing property of the solutions to (37)–(38) over $\text{int}\mathcal{D}'$ along with the assumption that $\sigma_1(0) \geq \sigma_2(0)$. From (45)–(46), we get that

$$0 \leq g(\tau) \leq M(\tau) \tag{47}$$

for all $\tau \geq 0$. Furthermore, for any such τ, we have

$$\begin{aligned}\dot{g}(\tau) &= \frac{\dot{G}}{1 - G}(\tau)\left[\frac{L}{L_1}(\sigma_1(\tau) - \tau, \tau) - \frac{L}{L_1}(\sigma_2(\tau) - \tau, \tau)\right] \\ &\geq \frac{\dot{G}}{1 - G}(\tau) \min_{m \in [\sigma_2(\tau) - \tau, \sigma_1(\tau) - \tau]} \left\{\left(\frac{L}{L_1}\right)_1(m, \tau)\right\}[\sigma_1(\tau) - \sigma_2(\tau)] \\ &= \frac{\dot{G}}{1 - G}(\tau) \min_{m \in [\sigma_2(\tau) - \tau, \sigma_1(\tau) - \tau]} \left\{1 - \frac{LL_{11}}{L_1^2}(m, \tau)\right\} g(\tau) \\ &\geq \frac{\dot{G}}{1 - G}(\tau)\, \rho g(\tau),\end{aligned} \tag{48}$$

where the first equality follows from (6) and (37), the second equality follows from (45), and the second inequality follows from (44). Integrating (48) yields

$$g(\tau) \geq g(0) \exp\left(\rho \int_0^\tau \frac{\dot{G}}{1-G}\,(\theta)\,\mathrm{d}\theta \right) = \frac{1}{[1-G(\tau)]^\rho}\, g(0) \tag{49}$$

and thus, by (47),

$$0 \leq g(0) \leq M(\tau)[1-G(\tau)]^\rho$$

for all τ, so that

$$0 \leq g(0) \leq \liminf_{\tau \to \infty} \{M(\tau)[1-G(\tau)]^\rho\} = 0,$$

where the equality follows from Assumption 3 along with the assumption that $\rho > 0$. This shows that $g(0) = 0$ and thus that $\sigma_1(0) = \sigma_2(0)$. From the uniqueness part of the Cauchy–Lipschitz theorem, we finally obtain that $\sigma_1 = \sigma_2$. Hence the result. ■

Proof of Proposition 1. To simply notation, let $B \equiv \dot{G}/(1-G)$. Suppose first that the mapping (8) has a positive derivative over $\mathcal{T}_m$ for all m. The proof that $\mu(\tau)$ is strictly decreasing in τ consists of two steps.

Step 1 We first show that one cannot have $\dot{\mu} \geq 0$ over an interval $[\tau_0, \infty)$. According to (7), we have

$$\ddot{\mu}(\tau) = E(\mu(\tau), \tau) + B(\tau)\dot{\mu}(\tau)F(\mu(\tau), \tau), \tag{50}$$

where

$$E(m, \tau) \equiv \dot{B}(\tau)\,\frac{L}{L_1}\,(m, \tau) + B(\tau)\left(\frac{L}{L_1}\right)_2 (m, \tau)$$

and

$$F(m, \tau) \equiv \left(\frac{L}{L_1}\right)_1 (m, \tau).$$

Imposing that the mapping (8) has a positive derivative over $\mathcal{T}_m$ for all m is equivalent to imposing that $E > 0$ over $\{(m, \tau) : 0 < m < M(\tau)\}$. Moreover, observe that $F > 0$ over this domain because, for each τ, $L(\cdot, \tau)$ is strictly log-concave over $[0, M(\tau)]$ by Assumption 4. Therefore, according to (50), $\dot{\mu}(\tau) \geq 0$ implies that $\ddot{\mu}(\tau) > 0$. Thus, if one had $\dot{\mu} \geq 0$ over an interval $[\tau_0, \infty)$, $\dot{\mu}(\tau)$ would have a well-defined limit $\dot{\mu}(\infty) \in (0, \infty]$ as τ goes to infinity. But then, as $\liminf_{\tau \to \infty} \{M(\tau)\} < \infty$ by Assumption 3, $\mu(\tau)$ would exceed $M(\tau)$ at some time τ, which is impossible. This contradiction establishes that there are arbitrarily large times τ_0 such that $\dot{\mu}(\tau_0) < 0$.

Step 2 Fix $\tau_0 > 0$ such that $\dot{\mu}(\tau_0) < 0$, and let us show that $\dot{\mu} < 0$ over $[0, \tau_0]$. Suppose the contrary holds, and let $\tau_1 \equiv \sup\{\tau < \tau_0 : \dot{\mu}(\tau) \geq 0\}$. Then $\dot{\mu}(\tau_1) = 0$ and $\ddot{\mu}(\tau_1) \leq 0$.

Letting $\dot{\mu}(\tau_1) = 0$ in (50) yields

$$\ddot{\mu}(\tau_1) = E(\mu(\tau_1), \tau_1) > 0,$$

a contradiction. Hence $\dot{\mu} < 0$ over $[0, \tau_0]$, as claimed. Because τ_0 can be arbitrarily large by Step 1, we have $\dot{\mu} < 0$ over $[0, \infty)$.

The proof that $\mu(\tau)$ is strictly increasing in τ when the mapping (8) has a negative derivative over $\mathcal{T}_m$ for all m is similar. The only modification to Step 1 consists in observing that if $\dot{\mu} > -1$ is strictly decreasing over some interval $[\tau_0, \infty)$, then $\dot{\mu}(\tau)$ has a well-defined finite limit $\dot{\mu}(\infty)$ as τ goes to infinity. Hence the result. ■

Proof of Proposition 2. Consider first a change in the breakthrough distribution. Suppose, by way of contradiction, that $\overline{\mu}(\tau_0) \leq \underline{\mu}(\tau_0)$ for some $\tau_0 \geq 0$. Then, because $(L/L_1)(\cdot, \tau)$ is strictly increasing over $[0, \underline{M}(\tau))$ by Assumption 4, it follows from (7) and (9) that

$$\dot{\underline{\mu}}(\tau_0) = \frac{\dot{\underline{G}}}{1 - \underline{G}}(\tau_0) \frac{L}{L_1}(\underline{\mu}(\tau_0), \tau_0) - 1 > \frac{\dot{\overline{G}}}{1 - \overline{G}}(\tau_0) \frac{L}{L_1}(\overline{\mu}(\tau_0), \tau_0) - 1 = \dot{\overline{\mu}}(\tau_0), \tag{51}$$

so that $\underline{\mu}(\tau) > \overline{\mu}(\tau)$ for all $\tau > \tau_0$ close enough to τ_0. We now show that $\underline{\mu} > \overline{\mu}$ over (τ_0, ∞). Suppose the contrary holds, and let $\tau_1 \equiv \inf\{\tau > \tau_0 : \underline{\mu}(\tau) \leq \overline{\mu}(\tau)\}$. Then $\underline{\mu}(\tau_1) = \overline{\mu}(\tau_1)$ and $\dot{\underline{\mu}}(\tau_1) \leq \dot{\overline{\mu}}(\tau_1)$. Proceeding as for (51), however, shows that $\underline{\mu}(\tau_1) = \overline{\mu}(\tau_1)$ implies that $\dot{\underline{\mu}}(\tau_1) > \dot{\overline{\mu}}(\tau_1)$, a contradiction. Hence the claim. Now, by analogy with (45), define

$$g(\tau) \equiv \underline{\mu}(\tau) - \overline{\mu}(\tau), \quad \tau \geq \tau_0. \tag{52}$$

We have by construction

$$\underline{M}(\tau) > \underline{\mu}(\tau) \geq \overline{\mu}(\tau) > 0 \tag{53}$$

for all $\tau \geq \tau_0$. From (52)–(53), we get that

$$0 \leq g(\tau) \leq \underline{M}(\tau) \tag{54}$$

for all $\tau \geq \tau_0$. Furthermore, for any such τ, we have

$$\begin{aligned} \dot{g}(\tau) &= \frac{\dot{\underline{G}}}{1 - \underline{G}}(\tau) \frac{L}{L_1}(\underline{\mu}(\tau), \tau) - \frac{\dot{\overline{G}}}{1 - \overline{G}}(\tau) \frac{L}{L_1}(\overline{\mu}(\tau), \tau) \\ &> \frac{\dot{\underline{G}}}{1 - \underline{G}}(\tau) \left[\frac{L}{L_1}(\underline{\mu}(\tau), \tau) - \frac{L}{L_1}(\overline{\mu}(\tau), \tau) \right] \\ &\geq \frac{\dot{\underline{G}}}{1 - \underline{G}}(\tau)\, \rho g(\tau) \end{aligned} \tag{55}$$

for all $\tau \geq \tau_0$, where the first inequality follows from (9), and the second inequality follows along the same lines as (48). Fixing some $\varepsilon > 0$ and integrating (55) yields

$$g(\tau) \geq g(\tau_0 + \varepsilon) \exp\left(\rho \int_{\tau_0 + \varepsilon}^{\tau} \frac{\dot{\underline{G}}}{1 - \underline{G}}(\theta)\, \mathrm{d}\theta \right) = \left[\frac{1 - \underline{G}(\tau_0 + \varepsilon)}{1 - \underline{G}(\tau)} \right]^{\rho} g(\tau_0 + \varepsilon),$$

and thus, by (54),

$$0 \leq g(\tau_0+\varepsilon) \leq \underline{M}(\tau)\left[\frac{1-\underline{G}(\tau)}{1-\underline{G}(\tau_0+\varepsilon)}\right]^{\rho}$$

for all $\tau \geq \tau_0$, so that

$$0 \leq g(\tau_0+\varepsilon) \leq \liminf_{\tau\to\infty}\left\{\underline{M}(\tau)\left[\frac{1-\underline{G}(\tau)}{1-\underline{G}(\tau_0+\varepsilon)}\right]^{\rho}\right\} = 0,$$

where the equality follows from Assumption 3 along with the assumption that $\rho > 0$. This, however, is impossible, as $g > 0$ over (τ_0, ∞). This contradiction establishes that $\overline{\mu}(\tau) > \underline{\mu}(\tau)$ for all τ.

Consider next a change in the payoff function. Suppose, by way of contradiction, that $\overline{\mu}(\tau_0) \leq \underline{\mu}(\tau_0)$ for some $\tau_0 \geq 0$. Then, because both $(\underline{L}/\underline{L}_1)(\cdot,\tau)$ and $(\overline{L}/\overline{L}_1)(\cdot,\tau)$ are strictly increasing over $[0, \underline{M}(\tau))$ by Assumption 4 along with the fact that $\underline{M}(\tau) \leq \overline{M}(\tau)$, it follows from (7) and (10) that

$$\dot{\underline{\mu}}(\tau_0) = \frac{\dot{G}}{1-G}(\tau_0)\,\frac{\underline{L}}{\underline{L}_1}(\underline{\mu}(\tau_0),\tau_0) - 1 > \frac{\dot{G}}{1-G}(\tau_0)\,\frac{\overline{L}}{\overline{L}_1}(\overline{\mu}(\tau_0),\tau_0) - 1 = \dot{\overline{\mu}}(\tau_0),$$

so that $\underline{\mu}(\tau) > \overline{\mu}(\tau)$ for all $\tau > \tau_0$ close enough to τ_0. As in the case of a change in the breakthrough distribution, we can deduce from this that $\overline{\mu} > \underline{\mu}$ over (τ_0, ∞). Defining g as in (52), the analogues of (53)–(54) hold. For each $\tau \geq \tau_0$, we have

$$\begin{aligned}\dot{g}(\tau) &= \frac{\dot{G}}{1-G}(\tau)\left[\frac{\underline{L}}{\underline{L}_1}(\underline{\mu}(\tau),\tau) - \frac{\overline{L}}{\overline{L}_1}(\overline{\mu}(\tau),\tau)\right]\\ &> \frac{\dot{G}}{1-G}(\tau)\left[\frac{\underline{L}}{\underline{L}_1}(\underline{\mu}(\tau),\tau) - \frac{\underline{L}}{\underline{L}_1}(\overline{\mu}(\tau),\tau)\right]\\ &\geq \frac{\dot{G}}{1-G}(\tau)\,\rho g(\tau)\end{aligned} \tag{56}$$

for all $\tau \geq \tau_0$, where the first inequality follows from (10) and the second inequality follows along the same lines as (48). The remainder of the proof is as in the case of a change in the breakthrough distribution. Hence the result. ■

Proof of Proposition 3. Let $B \equiv \dot{G}/(1-G)$ as in the proof of Proposition 1. Suppose first that the mapping (15) has a positive derivative over $\mathcal{T}_q$ for all q. The proof that $\chi(\tau)$ is strictly decreasing in τ consists of two steps.

Step 1 We first show that one cannot have $\dot{\chi} \geq 0$ over an interval $[\tau_0, \infty)$. According to (13), we have

$$\ddot{\chi}(\tau) = X(\chi(\tau),\tau) + \dot{\chi}(\tau)Y(\chi(\tau),\tau), \tag{57}$$

where

$$X(q,\tau) \equiv \dot{B}(\tau)\,\frac{H}{H_1}(q,\tau) + B(\tau)\left(\frac{H}{H_1}\right)_2(q,\tau) - \left(\frac{T_2}{T_1}\right)_2(q,\tau)$$

and

$$Y(q,\tau) \equiv B(\tau)\left(\frac{H}{H_1}\right)_1(q,\tau) - \left(\frac{T_2}{T_1}\right)_1(q,\tau).$$

Imposing that the mapping (15) has a positive derivative over $\mathcal{T}_q$ for all q is equivalent to imposing that $X > 0$ over $\{(q,\tau) : 0 < q < Q(M(\tau),\tau)\}$. Moreover, observe that $Y > 0$ over this domain because, as observed above, $(H/H_1)_1(q,\tau)$ is positive over this domain by Assumption 7, and $(T_2/T_1)_1(q,\tau) = (Q_{11} - Q_{12})(T(q,\tau) - \tau,\tau)T_1(q,\tau)$ is nonnegative by Assumptions 5–6. Therefore, according to (57), $\dot{\chi}(\tau) \geq 0$ implies that $\ddot{\chi}(\tau) > 0$. Thus, if one had $\dot{\chi} \geq 0$ over an interval $[\tau_0,\infty)$, $\dot{\chi}(\tau)$ would have a well-defined limit $\dot{\chi}(\infty) \in (0,\infty]$ as τ goes to infinity. But then, as $\liminf_{\tau\to\infty}\{M(\tau)\} < \infty$ by Assumption 3 and $\lim_{\tau\to\infty}\{Q(m,\tau)\} < \infty$ for all m by Assumption 5, $\mu(\tau)$ would exceed $M(\tau)$ at some time τ, which is impossible. This contradiction establishes that there are arbitrarily large times τ_0 such that $\dot{\chi}(\tau_0) < 0$.

Step 2 Fix $\tau_0 > 0$ such that $\dot{\chi}(\tau_0) < 0$, and let us show that $\dot{\chi} < 0$ over $[0,\tau_0]$. Suppose the contrary holds, and let $\tau_1 \equiv \sup\{\tau < \tau_0 : \dot{\chi}(\tau) \geq 0\}$. Then $\dot{\chi}(\tau_1) = 0$ and $\ddot{\chi}(\tau_1) \leq 0$. Letting $\dot{\chi}(\tau_1) = 1$ in (57) yields

$$\ddot{\chi}(\tau_1) = X(\chi(\tau_1),\tau_1) > 0,$$

a contradiction. Hence $\dot{\chi} < 0$ over $[0,\tau_0]$, as claimed. Because τ_0 can be arbitrarily large by Step 1, we have $\dot{\chi} < 0$ over $[0,\infty)$.

The proof that $\chi(\tau)$ is strictly increasing in τ when the mapping (15) has a negative derivative over $\mathcal{T}_q$ for all q is similar. The only modification to Step 1 consists in observing that if $\dot{\chi}$ is strictly decreasing over some interval $[\tau_0,\infty)$, then $\dot{\chi}(\tau)$ has a well-defined limit $\dot{\chi}(\infty) \in [-\infty,0)$ as τ goes to infinity. The claim follows. Hence the result. ■

Proof of Proposition 4. We use identity (14) throughout the proof. We will rely on the following claim.

Claim 3 *For all $q > 0$, $\tau \geq 0$, and $d \in [0,1]$,*

$$T_1(q,\tau;d) > 0, \tag{58}$$
$$T_3(q,\tau;d) > 0. \tag{59}$$

For all $m > 0$, $\tau \geq 0$, and $d \in [0,1]$,

$$(Q_1 - Q_2)_1(m,\tau;d) \leq 0, \tag{60}$$
$$(Q_1 - Q_2)_3(m,\tau;d) < 0. \tag{61}$$

Finally, if $\ddot{\xi} < 0$, then, for all $q > 0$, $\tau \geq 0$, and $d \in [0,1]$,

$$T_{13}(q,\tau;d) > 0. \tag{62}$$

Proof. Inequality (58) follows from Assumption 5 along with the implicit definition (11) of $T(q,\tau;d)$, that is, in the present context,

$$Q(T(q,\tau;d)-\tau,\tau;d)=q. \tag{63}$$

Inequality (59) follows from Assumption 5 and (63), observing from (16) that $Q_3(m,\tau;d)<0$ as $\dot{\xi}>0$. Inequality (60) follows from (16), which yields $(Q_1-Q_2)_1(m,\tau;d)=-d\dot{\xi}(\tau)\leq 0$ as $\dot{\xi}>0$. Inequality (61) follows from (16), which yields $(Q_1-Q_2)_3(m,\tau;d)=-\dot{\xi}(\tau)m<0$ as $\dot{\xi}>0$. Note that (58)–(61) do not require $\ddot{\xi}<0$.

Consider now inequality (62). According to (63), we have

$$T_1(q,\tau;d) = \frac{1}{Q_1}\,(T(q,\tau;d)-\tau,\tau;d), \tag{64}$$

$$T_3(q,\tau;d) = -\frac{Q_3}{Q_1}\,(T(q,\tau;d)-\tau,\tau;d). \tag{65}$$

It follows from (64)–(65) that $T_{13}(q,\tau;d)>0$ if and only if

$$(Q_{11}Q_3-Q_{13}Q_1)(T(q,\tau;d)-\tau,\tau;d)>0.$$

An explicit computation using (16) yields that this in turn holds whenever

$$\xi(\tau)[\xi(\tau+m)-\xi(\tau)]+(1-d)\left\{[\xi(\tau+m)-\xi(\tau)]^2-\dot{\xi}(\tau+m)\int_{\tau}^{\tau+m}[\xi(s)-\xi(\tau)]\,\mathrm{d}s\right\}>0$$

for all $m>0$ and $\tau\geq 0$. The first term of this sum is positive as $\dot{\xi}>0$. The second term is bounded below by $(1-d)[\xi(\tau+m)-\xi(\tau)][\xi(\tau+m)-\xi(\tau)-\dot{\xi}(\tau+m)m]$ as $\dot{\xi}>0$, which is nonnegative for all $d\in[0,1]$ as $\ddot{\xi}<0$. The claim follows. ∎

Suppose, by way of contradiction, that $\chi(\tau_0;\underline{d})\leq\chi(\tau_0;\overline{d})$ for some $\tau_0\geq 0$. Then

$$\begin{aligned}(Q_1-Q_2)(T(\chi(\tau_0;\overline{d}),\tau_0;\overline{d})-\tau_0,\tau_0;\overline{d}) &\leq (Q_1-Q_2)(T(\chi(\tau_0;\underline{d}),\tau_0;\overline{d})-\tau_0,\tau_0;\overline{d})\\ &\leq (Q_1-Q_2)(T(\chi(\tau_0;\underline{d}),\tau_0;\underline{d})-\tau_0,\tau_0;\overline{d})\\ &< (Q_1-Q_2)(T(\chi(\tau_0;\underline{d}),\tau_0;\underline{d})-\tau_0,\tau_0;\underline{d}),\end{aligned} \tag{66}$$

where the first inequality follows from (58) and (60), the second inequality follows from (59)–(60), and the third inequality follows from (61). Moreover, recall that from

$$H(q,\tau;d)\equiv\exp(-r[T(q,\tau;d)-\tau])P(q),$$

we have

$$\frac{H_1}{H}\,(q,\tau;d)=-rT_1(q,\tau;d)+\frac{P'}{P}\,(q).$$

Assumption 7 implies that $(H_1/H)_1<0$ over the relevant range. Moreover, (62) implies that $(H_1/H)_3<0$ when $\ddot{\xi}<0$. Combining these two observations with the assumption that $\chi(\tau_0;\underline{d})\leq\chi(\tau_0;\overline{d})$, we have

$$\frac{H_1}{H}\,(\chi(\tau_0,\underline{d}),\tau_0;\underline{d})\geq\frac{H_1}{H}\,(\chi(\tau_0,\overline{d}),\tau_0;\underline{d})>\frac{H_1}{H}\,(\chi(\tau_0,\overline{d}),\tau_0;\overline{d}). \tag{67}$$

It follows from (66)–(67) that

$$\begin{aligned}\dot{\chi}(\tau_0;\overline{d}) &= \frac{\dot{G}}{1-G}(\tau_0)\,\frac{H}{H_1}(\chi(\tau_0,\overline{d}),\tau_0;\overline{d}) - (Q_1-Q_2)(T(\chi(\tau_0;\overline{d}),\tau_0;\overline{d})-\tau_0,\tau_0;\overline{d}) \\ &> \frac{\dot{G}}{1-G}(\tau_0)\,\frac{H}{H_1}(\chi(\tau_0,\underline{d}),\tau_0;\underline{d}) - (Q_1-Q_2)(T(\chi(\tau_0;\underline{d}),\tau_0;\underline{d})-\tau_0,\tau_0;\underline{d}) \\ &= \dot{\chi}(\tau_0;\underline{d}).\end{aligned}$$

As in the proof of Proposition 2, we can deduce from this that $\chi(\cdot;\overline{d}) > \chi(\cdot;\underline{d})$ over (τ_0,∞). Now, by analogy with (45), define

$$g(\tau) \equiv \chi(\tau;\overline{d}) - \chi(\tau,\underline{d}), \quad \tau \geq 0. \tag{68}$$

For each τ, let $M(\tau;\overline{d})$ be the unique maximum of $L(\cdot,\tau;d)$. By construction,

$$Q(M(\tau;\overline{d}),\tau;d) > \chi(\tau,\overline{d}) \geq \chi(\tau,\underline{d}) > 0 \tag{69}$$

for all $\tau \geq \tau_0$. From (68)–(69), we get that

$$0 \leq g(\tau) \leq Q(M(\tau;\overline{d}),\tau;d) \tag{70}$$

for all $\tau \geq \tau_0$. Furthermore, for any such τ, we have

$$\begin{aligned}\dot{g}(\tau) &= \frac{\dot{G}}{1-G}(\tau)\left[\frac{H}{H_1}(\chi(\tau,\overline{d}),\tau;\overline{d}) - \frac{H}{H_1}(\chi(\tau,\underline{d}),\tau;\underline{d})\right] \\ &\quad - [(Q_1-Q_2)(T(\chi(\tau;\overline{d}),\tau;\overline{d})-\tau,\tau;\overline{d}) - (Q_1-Q_2)(T(\chi(\tau;\underline{d}),\tau;\underline{d})-\tau,\tau;\underline{d})] \\ &> \frac{\dot{G}}{1-G}(\tau)\left[\frac{H}{H_1}(\chi(\tau,\overline{d}),\tau;\overline{d}) - \frac{H}{H_1}(\chi(\tau,\underline{d}),\tau;\overline{d})\right] \\ &\geq \frac{\dot{G}}{1-G}(\tau)\,\rho g(\tau)\end{aligned} \tag{71}$$

where the first inequality follows from (66) along with the fact that $(H_1/H)_3 < 0$ and the second inequality follows along the same lines as (48). The remainder of the proof is as in the proof of Proposition 2, bearing in mind that, as $\liminf_{\tau\to\infty}\{M(\tau;\overline{d})\} < \infty$ by Assumption 3 and $\lim_{\tau\to\infty}\{Q(m,\tau;d)\} < \infty$ for all m by Assumption 5, $\liminf_{\tau\to\infty}\{Q(M(\tau;\overline{d}),\tau;d)\} < \infty$. Hence the result. ■

Proof of Theorem 3. It is helpful to rewrite (19) using as new variables $\nu^a(t) \equiv t - \phi^a(t)$ and $\nu^b(t) \equiv t - \phi^b(t)$, yielding

$$\dot{\nu}^j(t) = 1 - \frac{1}{\lambda^j}\frac{\dot{L}}{L}(\nu^i(t)), \quad t \geq \sigma(0), \quad i = a, b. \tag{72}$$

Compared to (19), two simplifications arise. First, (72) is an autonomous system, in which time does not show up as an independent variable. Second, a continuous equilibrium exists if and only if there exists a solution to (72) that is entirely contained in the bounded set $(0,M]\times(0,M]$. To simplify notation, we will take advantage of the fact that (72) is autonomous to write it for any time $t \geq 0$. (Formally, this amounts to a time translation of length $\sigma(0)$, $\nu_0^i(t) \equiv \nu^i(t+\sigma(0))$. Without risk of confusion, we hereafter identify 'ν^i' and 'ν_0^i'.) To guide the intuition, the phase portrait of (72) is illustrated in Figure 3.

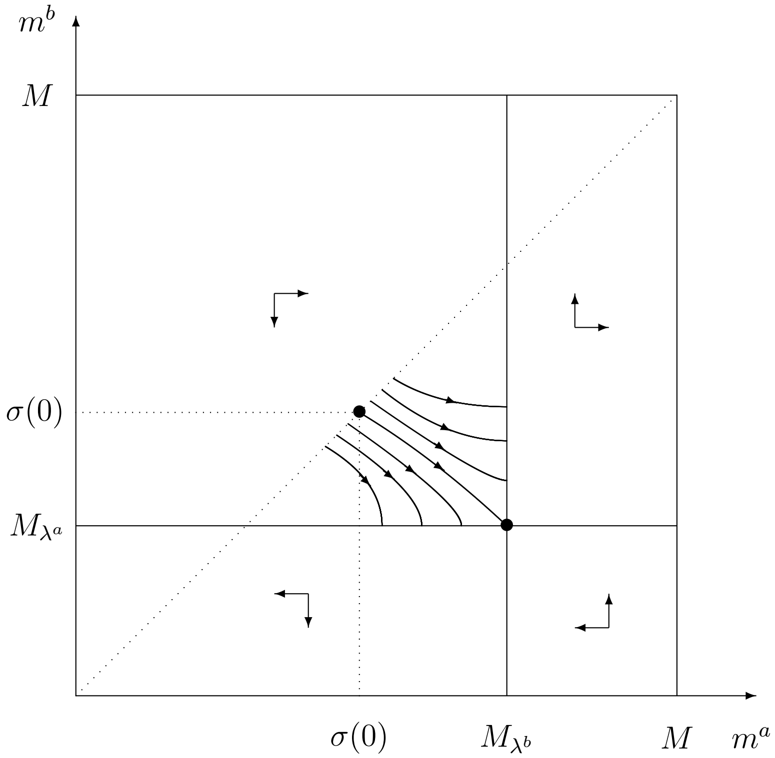

Figure 3 The phase portrait of (72) and the unique continuous equilibrium.

We first prove existence. To do so, we first need to establish two intermediary results. Regarding the second one, recall that for each $\lambda > 0$, $M_\lambda = (\dot{L}/L)^{-1}(\lambda)$, and that, by Assumption 4, $\dot{L}/L$ is strictly decreasing over $(0, M]$.

Claim 4 *In any continuous equilibrium,* $\mu^a(\tau) > \mu^b(\tau)$ *for all* $\tau > 0$.

Proof. We equivalently prove that $\nu^a(t) > \nu^b(t)$ for all $t > 0$. From (72) along with the fact that $\sigma^a(0) = \sigma^b(0) = \sigma(0)$, one has $\dot{\nu}^a(0) > \dot{\nu}^b(0)$ as $\lambda^a > \lambda^b$. Hence $\nu^a(t) > \nu^b(t)$ for all t close to but strictly greater than zero. We now show that $\nu^a > \nu^b$ over $(0, \infty)$. Suppose the contrary holds, and let $t_0 \equiv \inf\{t > 0 : \nu^a(t) \leq \nu^b(t)\}$. But then $\nu^a(t_0) = \nu^b(t_0)$ and $\dot{\nu}^a(t_0) \leq \dot{\nu}^b(t_0)$, in contradiction with (72). The claim follows. ■

Claim 5 *In any continuous equilibrium,*

$$\mu^a(\tau) > M_{\lambda^a} \ \ \textit{and} \ \ \mu^b(\tau) < M_{\lambda^b}, \quad \tau \geq 0. \tag{73}$$

Proof. We prove equivalently that $\nu^a(t) > M_{\lambda^a}$ and, symmetrically, that $\nu^b(t) < M_{\lambda^b}$ for all $t \geq 0$. Suppose, by way of contradiction, that $\nu^a(t) \leq M_{\lambda^a}$ for such t, and start with the case where $\nu^a(t) < M_{\lambda^a}$. Then, by definition of M_{λ^a}, $(\dot{L}/L)(\nu^a(t)) > \lambda^a$ and thus, by (72), $\dot{\nu}^b(t) < 1 - \lambda^a/\lambda^b < 0$. Hence $t^a \equiv \inf\{s > t : \nu^a(s) \geq M_{\lambda^a}\}$ must be finite, for, otherwise, (ν^a, ν^b) would eventually leave $(0, M] \times (0, M]$, in contradiction to the assumption that (σ^a, σ^b) is an equilibrium. Hence $\nu^a(t^a) = M_{\lambda^a}$ and $\dot{\nu}^a(t^a) \geq 0$. By (72), this implies that $(\dot{L}/L)(\nu^b(t^a)) \leq \lambda^a$, so that $\nu^b(t^a) \geq M_{\lambda^a} = \nu^a(t^a)$, which, as $t^a > 0$, is impossible by Claim 4. This contradiction establishes that $\nu^a(t) \geq M_{\lambda^a}$ for all $t \geq 0$. To complete the proof, we must rule out the case $\nu^a(t) = M_{\lambda^a}$. Suppose first, by way of contradiction, that this equality holds for some $t > 0$. Then, by (72) along with Claim 4, one has $\dot{\nu}^a(t) = 1 - (\dot{L}/L)(\nu^b(t))/\lambda^a < 1 - (\dot{L}/L)(\nu^a(t))/\lambda^a = 0$. Hence $\nu^a(s) < M_{\lambda^a}$ for all s close to but strictly greater than t, which is impossible according to the first part of the proof. Suppose finally, by way of contradiction, that $\nu^a(0) = M_{\lambda^a}$, that is, $\sigma(0) = M_{\lambda^a}$. Then, by (72), $\dot{\nu}^a(0) = 0$, so that $\dot{\nu}^b(0) < 0$ as $\lambda^a > \lambda^b$. Differentiating (72) then yields

$$\ddot{\nu}^a(0) = -\frac{1}{\lambda^a} \,\overset{\cdot\cdot}{\overbrace{(\ln L)}}\,(\sigma(0))\, \dot{\nu}^b(\sigma(0)) < 0.$$

Because $\nu^a(0) = M_{\lambda^a}$ and $\dot{\nu}^a(0) = 0$, we get that $\nu^a(t) < M_{\lambda^a}$ for all t close to but strictly greater than zero, which is again impossible according to the first part of the proof. These contradictions establish that $\nu^a(t) > M_{\lambda^a}$ for all $t \geq 0$. The proof that $\nu^b(t) < M_{\lambda^b}$ for all $t \geq 0$ is similar, and is therefore omitted. The claim follows. ■

In light of our summary of the constant-breakthrough case at the beginning of Section 4.1.1, the interpretation of Claim 5 is that the hare tends to behave less cautiously, and the tortoise more cautiously, than if they were each facing an opponent of equal strength. A direct implication of (73) is that $M_{\lambda^a} < \sigma(0) < M_{\lambda^b}$.

By Claim 5, the solution to (72) starting at (M, M) does not correspond to an equilibrium. Hence, as $\nu^a(0) = \nu^b(0)$ in any equilibrium, we can restrict the study of (72) to the open square $\mathcal{M} \equiv (0, M) \times (0, M)$. Given a point $\boldsymbol{m} \equiv (m^a, m^b)$ in $\mathcal{M}$, we denote by $\boldsymbol{\nu}(\cdot, \boldsymbol{m}) : t \mapsto (\nu^a(t, \boldsymbol{m}), \nu^b(t, \boldsymbol{m}))$ the solution to (72) passing through $\boldsymbol{m}$ at $t = 0$. This solution is defined over a maximal interval $[0, t_{\max}(\boldsymbol{m}))$, where $t_{\max}(\boldsymbol{m}) \in (0, \infty]$. We need to establish that there exists some $\boldsymbol{m} \equiv (m, m)$ with $m \in (M_{\lambda^a}, M_{\lambda^b})$ such that $t_{\max}(\boldsymbol{m}) = \infty$. We will use the following notation: $\mathcal{I}$ is the segment of the diagonal in $\mathcal{M}$ joining $(M_{\lambda^a}, M_{\lambda^a})$ to $(M_{\lambda^b}, M_{\lambda^b})$; $\mathcal{J}^a$ is the segment in $\mathcal{M}$ joining $(M_{\lambda^a}, M_{\lambda^a})$ to $(M_{\lambda^b}, M_{\lambda^a})$; $\mathcal{J}^b$ is the segment in $\mathcal{M}$ joining $(M_{\lambda^b}, M_{\lambda^b})$ to $(M_{\lambda^b}, M_{\lambda^a})$; finally $\mathcal{J} \equiv \mathcal{J}^a \cup \mathcal{J}^b$. The proof that an equilibrium exists consists of two steps.

Step 1 As a preliminary remark, note that any solution $\boldsymbol{\nu}(\cdot, \boldsymbol{m})$ to (72) starting at some point $\boldsymbol{m} \in \mathcal{R}^a \equiv ((0, M_{\lambda^b}] \times (0, M_{\lambda^a}]) \setminus \{(M_{\lambda^b}, M_{\lambda^a})\}$ is such that $t_{\max}(\boldsymbol{m}) < \infty$. Indeed, for any such $\boldsymbol{m}$, we have, according to (72), $(\partial \nu^a/\partial t)(t, \boldsymbol{m}) \leq 0$, $(\partial \nu^b/\partial t)(t, \boldsymbol{m}) \leq 0$, and $(\partial \boldsymbol{\nu}/\partial t)(t, \boldsymbol{m}) \neq (0, 0)$ for all $t \in [0, t_{\max}(\boldsymbol{m}))$. This shows that $\boldsymbol{\nu}(t, \boldsymbol{m})$ converges as t goes to $t_{\max}(\boldsymbol{m})$ to a point in the closure $\mathrm{Cl}\, \mathcal{R}^a \setminus \mathcal{J}^a$ of $\mathcal{R}^a \setminus \mathcal{J}^a$. As there is no critical point for (72) in $\mathrm{Cl}\, \mathcal{R}^a \setminus \mathcal{J}^a$, $t_{\max}(\boldsymbol{m})$ must be finite. Similarly, any solution $\boldsymbol{\nu}(\cdot, \boldsymbol{m})$ to (72) starting at some point $\boldsymbol{m} \in \mathcal{R}^b \equiv ([M_{\lambda^b}, M) \times [M_{\lambda^a}, M)) \setminus \{(M_{\lambda^b}, M_{\lambda^a})\}$ is such that $t_{\max}(\boldsymbol{m}) < \infty$. Observe that, from the above proof, any solution to (72) starting in $\mathcal{J} \setminus \{(M_{\lambda^b}, M_{\lambda^a})\}$ meets this set only once, at time zero.

Step 2 For each $\boldsymbol{m} \in \mathcal{I}$, let

$$t_{\mathcal{J}}(\boldsymbol{m}) \equiv \sup\{t \geq 0 : \nu^a(s, \boldsymbol{m}) \leq M_{\lambda^b} \text{ and } \nu^b(s, \boldsymbol{m}) \geq M_{\lambda^a} \text{ for all } s \in [0, t]\} \in [0, \infty].$$

Thus $t_{\mathcal{J}}(\boldsymbol{m})$ is the first time at which the trajectory $\boldsymbol{\nu}(\cdot, \boldsymbol{m})$ starting at $\boldsymbol{m} \in \mathcal{I}$ reaches $\mathcal{J}$. The case $t_{\mathcal{J}}(\boldsymbol{m}) = \infty$ corresponds to a trajectory that stays in the triangle delimited by $\mathcal{I} \cup \mathcal{J}$, and thus to an equilibrium. Suppose, by way of contradiction, that $t_{\mathcal{J}}(\boldsymbol{m}) < \infty$ for all $\boldsymbol{m} \in \mathcal{I}$. We must either have

$$\nu^a(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m}) = M_{\lambda^b} \text{ and } \nu^b(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m}) > M_{\lambda^a}$$

or

$$\nu^a(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m}) < M_{\lambda^b} \text{ and } \nu^b(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m}) = M_{\lambda^a},$$

because $(M_{\lambda^b}, M_{\lambda^a})$ is a critical point for (72) and thus cannot be reached in a finite time $t_{\mathcal{J}}(\boldsymbol{m})$ from any point $\boldsymbol{m} \in \mathcal{I}$. From Step 1, this implies in turn that $t_{\mathcal{J}}(\boldsymbol{m})$ is the unique solution of the equation $\text{Dist}\,(\boldsymbol{\nu}(t, \boldsymbol{m}), \mathcal{J}) = 0$. In other words, if $\nu^i(t, \boldsymbol{m}) = M_{\lambda^j}$ for some i, then $t = t_{\mathcal{J}}(\boldsymbol{m})$. We now prove that the function $t_{\mathcal{J}}$ is continuous over $\mathcal{I}$. Fix $\boldsymbol{m} \in \mathcal{I}$ and assume for instance that $\nu^a(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m}) < M_{\lambda^b}$ and $\nu^b(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m}) = M_{\lambda^a}$. Then

$$\frac{\partial(\nu^b - M_{\lambda^a})}{\partial t}\,(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m}) = 1 - \frac{1}{\lambda^b}\,\frac{\dot{L}}{L}\,(\nu^a(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m})) \neq 0. \tag{74}$$

Because L is twice continuously differentiable, the flow $(t, \boldsymbol{m}) \mapsto \boldsymbol{\nu}(t, \boldsymbol{m})$ associated to (72) is a continuously differentiable mapping (Perko (2001, Section 2.5, Theorem 1, Remark)). Thus, from (74) along with the implicit function theorem, $t_{\mathcal{J}}$ is continuous, and so is the mapping $\Psi : \mathcal{I} \to \mathcal{J} : \boldsymbol{m} \mapsto \boldsymbol{\nu}(t_{\mathcal{J}}(\boldsymbol{m}), \boldsymbol{m})$. Therefore, as $\mathcal{I}$ is connected, $\Psi(\mathcal{I})$ must be connected in $\mathcal{J}$. Because $\Psi(M_{\lambda^i}, M_{\lambda^i}) = (M_{\lambda^i}, M_{\lambda^i})$ for each i, this implies, given the structure of $\mathcal{J}$, that $(M_{\lambda^b}, M_{\lambda^a}) \in \Psi(\mathcal{I})$. This, however, is impossible because, as observed above, $(M_{\lambda^b}, M_{\lambda^a})$ is a critical point for (72). This contradiction establishes that there exists $\boldsymbol{m} \in \mathcal{I}$ such that $t_{\mathcal{J}}(\boldsymbol{m}) = \infty$ and thus $t_{\max}(\boldsymbol{m}) = \infty$, and hence that a continuous equilibrium exists.

Observe that, by Step 1, any continuous equilibrium must be such that the associated trajectory $\boldsymbol{\nu}(\cdot, \boldsymbol{m})$ of (72) stays in the interior of the triangle delimited by $\mathcal{I} \cup \mathcal{J}$. Thus, according to (72), $(\partial\nu^a/\partial t)(t, \boldsymbol{m}) > 0$ and $(\partial\nu^b/\partial t)(t, \boldsymbol{m}) < 0$, so that $\mu^a(\tau)$ is strictly increasing in τ and $\mu^b(\tau)$ is strictly decreasing in τ. Two consequences follow. First, as $\mu^a(0) = \mu^b(0) = \sigma(0)$, we get that $\mu^a(\tilde{\tau}^a) > \mu^b(\tilde{\tau}^b)$ unless $\tilde{\tau}^a = \tilde{\tau}^b = 0$. Second, as $(\mu^a(\tau), \mu^b(\tau)) \in (0, M) \times (0, M)$ for all τ, $(\mu^a(\tau), \mu^b(\tau))$ has a limit as τ goes to infinity, which must be $(M_{\lambda^b}, M_{\lambda^a})$, the unique critical point of (72) in the triangle delimited by $\mathcal{I} \cup \mathcal{J}$.

It remains to prove uniqueness. Suppose, by way of contradiction, that there exist two continuous equilibria. According to the above remark, this implies that there exist two distinct points $\boldsymbol{m}_1 \equiv (m_1, m_1)$ and $\boldsymbol{m}_2 \equiv (m_2, m_2)$ in $\mathcal{I}$ such that both $\boldsymbol{\nu}(t, \boldsymbol{m}_1)$ and

$\boldsymbol{\nu}(t, \boldsymbol{m}_2)$ converge to $(M_{\lambda^b}, M_{\lambda^a})$ as t goes to infinity. With no loss of generality, assume that $m_1 > m_2$. Let us first observe that

$$\nu^i(t, \boldsymbol{m}_1) > \nu^i(t, \boldsymbol{m}_2), \quad t \geq 0, \quad i = a, b. \tag{75}$$

Indeed, if this were not the case, there would for instance exist some $t > 0$ such that $\nu^a(t, \boldsymbol{m}_1) = \nu^a(t, \boldsymbol{m}_2)$ and $\nu^i(s, \boldsymbol{m}_1) > \nu^i(s, \boldsymbol{m}_2)$ for all $s \in [0, t)$ and $i = a, b$. But then, because $\dot{L}/L$ is strictly decreasing over $(0, M]$,

$$\begin{aligned}
\nu^a(t, \boldsymbol{m}_1) &= m_1 + \int_0^t \frac{\partial \nu^a}{\partial t}(s, \boldsymbol{m}_1)\,\mathrm{d}s \\
&= m_1 + \int_0^t \left[1 - \frac{1}{\lambda^a}\frac{\dot{L}}{L}(\nu^b(s, \boldsymbol{m}_1))\right]\mathrm{d}s \\
&> m_2 + \int_0^t \left[1 - \frac{1}{\lambda^a}\frac{\dot{L}}{L}(\nu^b(s, \boldsymbol{m}_2))\right]\mathrm{d}s \\
&= m_2 + \int_0^t \frac{\partial \nu^a}{\partial t}(s, \boldsymbol{m}_2)\,\mathrm{d}s \\
&= \nu^a(t, \boldsymbol{m}_2),
\end{aligned}$$

which is ruled out by assumption. This contradiction establishes (75). We now show that $\boldsymbol{\nu}(t, \boldsymbol{m}_1)$ and $\boldsymbol{\nu}(t, \boldsymbol{m}_1)$ tend to drift apart from each other, which concludes the proof. Define

$$g^{a,b}(t) \equiv \frac{1}{2}\left\|\boldsymbol{\nu}(t, \boldsymbol{m}_1) - \boldsymbol{\nu}(t, \boldsymbol{m}_2)\right\|^2, \quad t \geq 0.$$

Then, for each $t \geq 0$, we have, by (72),

$$\begin{aligned}
\dot{g}^{a,b}(t) &= \left\langle \boldsymbol{\nu}(t, \boldsymbol{m}_1) - \boldsymbol{\nu}(t, \boldsymbol{m}_2), \frac{\partial \boldsymbol{\nu}}{\partial t}(t, \boldsymbol{m}_1) - \frac{\partial \boldsymbol{\nu}}{\partial t}(t, \boldsymbol{m}_2)\right\rangle \\
&= \sum_{i=a,b} \frac{1}{\lambda^i}\left[\nu^i(t, \boldsymbol{m}_1) - \nu^i(t, \boldsymbol{m}_2)\right]\left[\frac{\dot{L}}{L}(\nu^j(s, \boldsymbol{m}_2)) - \frac{\dot{L}}{L}(\nu^j(s, \boldsymbol{m}_1))\right],
\end{aligned}$$

which is strictly positive according to (75) and the monotonicity of $\dot{L}/L$. This proves that $g^{a,b}$ is strictly increasing and in particular that $g^{a,b}(t) > g^{a,b}(0) = m_1 - m_2 > 0$ for all $t \geq 0$. This, however, is impossible because both $\boldsymbol{\nu}(t, \boldsymbol{m}_1)$ and $\boldsymbol{\nu}(t, \boldsymbol{m}_2)$ converge to $(M_{\lambda^b}, M_{\lambda^a})$ as t goes to infinity and thus $g^{a,b}(t)$ converges to zero as t goes to infinity. This contradiction establishes that there exists a unique continuous equilibrium. Hence the result. ■

Proof of Proposition 5. Observe first from the proof of Theorem 3 that

$$\lim_{\tau \to \infty}\{\overline{\mu}^b(\tau)\} = M_{\overline{\lambda}^a} < M_{\underline{\lambda}^a} = \lim_{\tau \to \infty}\{\underline{\mu}^b(\tau)\} \tag{76}$$

as $\overline{\lambda}^a > \underline{\lambda}^a$, so that the result holds for τ large enough. Suppose, by way of contradiction, that $\underline{\mu}^b(\tau_0) = \overline{\mu}^b(\tau_0)$ for some time τ_0 or, equivalently, that

$$\underline{\phi}^b(t_0) = \overline{\phi}^b(t_0) \tag{77}$$

for some time t_0, where, by (19),

$$\dot{\underline{\phi}}^a(t) = \frac{1}{\underline{\lambda}^a}\frac{\dot{L}}{L}(t-\underline{\phi}^b(t)), \quad \dot{\underline{\phi}}^b(t) = \frac{1}{\lambda^b}\frac{\dot{L}}{L}(t-\underline{\phi}^a(t)), \quad t \geq \underline{\sigma}(0), \tag{78}$$

and

$$\dot{\overline{\phi}}^a(t) = \frac{1}{\overline{\lambda}^a}\frac{\dot{L}}{L}(t-\overline{\phi}^b(t)), \quad \dot{\overline{\phi}}^b(t) = \frac{1}{\lambda^b}\frac{\dot{L}}{L}(t-\overline{\phi}^a(t)), \quad t \geq \overline{\sigma}(0). \tag{79}$$

The proof consists of two steps.

Step 1 Suppose, by way of contradiction, that

$$\dot{\underline{\phi}}^b(t_0) \geq \dot{\overline{\phi}}^b(t_0). \tag{80}$$

Now, observe that, as $\overline{\lambda}^a > \underline{\lambda}^a$,

$$\dot{\underline{\phi}}^a(t_0) > \dot{\overline{\phi}}^a(t_0) \tag{81}$$

by (77)–(79), and that, as $\dot{L}/L$ is strictly decreasing over $(0, M]$,

$$\underline{\phi}^a(t_0) \geq \overline{\phi}^a(t_0) \tag{82}$$

by (78)–(80). Combining (81) with (82) yields that, for some $\varepsilon^a > 0$,

$$\underline{\phi}^a(t) > \overline{\phi}^a(t) \tag{83}$$

for all $t \in (t_0, t_0 + \varepsilon^a)$. Together with (78)–(79), this, as $\dot{L}/L$ is strictly decreasing over $(0, M]$, implies that

$$\dot{\underline{\phi}}^b(t) > \dot{\overline{\phi}}^b(t) \tag{84}$$

for all $t \in (t_0, t_0 + \varepsilon^a)$. Similarly, combining (77) with (84) yields that, for some $\varepsilon^b > 0$,

$$\underline{\phi}^b(t) > \overline{\phi}^b(t) \tag{85}$$

for all $t \in (t_0, t_0 + \varepsilon^b)$. Together with (78)–(79), this, as $\overline{\lambda}^a > \underline{\lambda}^a$ and $\dot{L}/L$ is strictly decreasing over $(0, M]$, implies that

$$\dot{\underline{\phi}}^a(t) > \dot{\overline{\phi}}^a(t) \tag{86}$$

for all $t \in (t_0, t_0 + \varepsilon^b)$. More generally, as long as (83) and (85) hold, so do (84) and (86). As a result, (83) and (85) hold for all $t > t_0$. This, however, is impossible because, according to (76), $t - \underline{\phi}^b(t) > t - \overline{\phi}^b(t)$ for t large enough. This contradiction establishes that, if (77) holds, then

$$\dot{\underline{\phi}}^b(t_0) < \dot{\overline{\phi}}^b(t_0). \tag{87}$$

Step 2 It is easily checked that, if (77) holds, then $\underline{\phi}^b(t_0) = \overline{\phi}^b(t_0) > 0$. Indeed, if one had $\underline{\phi}^b(t_0) = \overline{\phi}^b(t_0) = 0$, then it would follow from Lemma 1(iii) that $\underline{\phi}^a(t_0) = \overline{\phi}^a(t_0) = 0$ and thus, by (78)–(79), that $\dot{\underline{\phi}}^b(t_0) = \dot{\overline{\phi}}^b(t_0)$, which is ruled out by (87). Thus, in particular, $t_0 > \underline{\sigma}(0) \vee \overline{\sigma}(0)$. Note also that $\underline{\phi}^b$ and $\overline{\phi}^b$ cannot cross over $[\underline{\sigma}(0) \vee \overline{\sigma}(0), t_0)$, for, otherwise, given (87), there would exist some time t_1 in this interval such that $\underline{\phi}^b(t_1) = \overline{\phi}^b(t_1)$ and $\dot{\underline{\phi}}^b(t_1) \geq \dot{\overline{\phi}}^b(t_1)$, which is ruled out by Step 1. As a result, $\underline{\sigma}(0) < \overline{\sigma}(0)$ and

$$\overline{\phi}^b(t) < \underline{\phi}^b(t) \tag{88}$$

for all $t \in (\underline{\sigma}(0), t_0)$, where $\overline{\phi}^b \equiv 0$ over $(\underline{\sigma}(0), \overline{\sigma}(0))$. Now, observe that, as $\dot{L}/L$ is strictly decreasing over $(0, M]$,

$$\underline{\phi}^a(t_0) < \overline{\phi}^a(t_0) \tag{89}$$

by (78)–(79) and (87). Because $\underline{\sigma}(0) < \overline{\sigma}(0)$, however,

$$\overline{\phi}^a(t) < \underline{\phi}^a(t) \tag{90}$$

for all $t \in (\underline{\sigma}(0), \overline{\sigma}(0))$, where $\overline{\phi}^a \equiv 0$ over $(\underline{\sigma}(0), \overline{\sigma}(0))$. It follows from (89)–(90) that there must exist some time t_1 in $(\underline{\sigma}(0), t_0)$ such that $\underline{\phi}^a(t_1) = \overline{\phi}^a(t_1)$ and $\dot{\underline{\phi}}^a(t_1) \leq \dot{\overline{\phi}}^a(t_1)$. Together with (78)–(78), this, as $\overline{\lambda}^a > \underline{\lambda}^a$ and $\dot{L}/L$ is strictly decreasing over $(0, M]$, implies that $\underline{\phi}^b(t_1) < \overline{\phi}^b(t_1)$, which, according to (88), is impossible. This contradiction establishes that there is no time t_0 such that (77) holds. Hence the result. ■

Proof of Proposition 6. We first argue that $\overline{\mu}^a(\tau) > \underline{\mu}^a(\tau)$ for all $\tau \geq 0$. Observe first that, by analogy with (76),

$$\lim_{\tau \to \infty} \{\overline{\mu}^a(\tau)\} = \overline{M}^a_\lambda \equiv \left(\frac{\dot{\overline{L}}^a}{\overline{L}^a}\right)^{-1}(\lambda) > \left(\frac{\dot{\underline{L}}^a}{\underline{L}^a}\right)^{-1}(\lambda) \equiv \underline{M}^a_\lambda = \lim_{\tau \to \infty} \{\underline{\mu}^a(\tau)\}$$

as $\overline{L}^a$ dominates $\underline{L}^a$ in the growth-rate order, so that the result holds for τ large enough. Suppose, by way of contradiction, that $\underline{\mu}^a(\tau_0) = \overline{\mu}^a(\tau_0)$ for some time τ_0 or, equivalently, that

$$\underline{\phi}^a(t_0) = \overline{\phi}^a(t_0)$$

for some time t_0, where, by (20),

$$\dot{\underline{\phi}}^a(t) = \frac{1}{\lambda}\frac{\dot{L}^b}{L^b}(t - \underline{\phi}^b(t)), \quad \dot{\underline{\phi}}^b(t) = \frac{1}{\lambda}\frac{\dot{\underline{L}}^a}{\underline{L}^a}(t - \underline{\phi}^a(t)), \quad t \geq \underline{\sigma}(0), \tag{91}$$

and

$$\dot{\overline{\phi}}^a(t) = \frac{1}{\lambda}\frac{\dot{L}^b}{L^b}(t - \overline{\phi}^b(t)), \quad \dot{\overline{\phi}}^b(t) = \frac{1}{\lambda}\frac{\dot{\overline{L}}^a}{\overline{L}^a}(t - \overline{\phi}^a(t)), \quad t \geq \overline{\sigma}(0). \tag{92}$$

The proof then proceeds along the same lines as in Steps 1–2 of the proof of Proposition 5, exchanging the roles of a and b and of $\underline{\cdot}$ and $\overline{\cdot}$. Observe in particular that $\overline{\sigma}(0) > \underline{\sigma}(0)$. Hence the result. ■

Appendix B: Additional Proofs and Calculations

B.1 Dynamic Analysis of Example 1

Focusing for concreteness on the publication-delay interpretation of this example, we first give a sufficient condition for total time from breakthrough to publication evaluated at the stand-alone maturation delay to decrease in the breakthrough time.

Claim 6 *If $D_{12} > 0$ and $D(M(\tau), \tau)$ is nonincreasing in τ, then $M(\tau) + D(M(\tau), \tau)$ is strictly decreasing in τ.*

Proof. Observe that $M(\tau)$ satisfies $1 + D_1(M(\tau), \tau) = 0$, so that, as $D_{11}(M(\tau), \tau) > 0$, $\dot{M}(\tau) = -(D_{12}/D_{11})(M(\tau), \tau)$ and $D_{12}(M(\tau), \tau)$ have opposite signs. Hence $\dot{M} < 0$ if $D_{12} > 0$. The claim follows. ■

An example of a publication-delay function that satisfies the assumptions of Claim 6 is $D(m, \tau) = M(\tau)^2/m$, for some function M such that $0 > \dot{M} > -1$.

To illustrate the claim that total time from breakthrough to publication may increase in equilibrium despite the publication process becoming more efficient, consider the following specification for the publication delay:

$$D(m, \tau) = I(D(m, \infty), \tau). \tag{93}$$

One can interpret $D(\cdot, \infty)$ as an asymptotic publication-delay function, and $I(\cdot, \tau)$ as a deformation of the identity function over $[0, \infty)$. We shall assume that $\dot{D}(\cdot, \infty) < 0$, that $\ddot{D}(\cdot, \infty) > 0$, and that $\lim_{m \downarrow 0} \{D(m, \infty)\} = \infty$. As for I, we shall assume that (i) $I(0, \tau) = 0$ for all τ, (ii) $I_1 > 0$, which, together with $\dot{D}(\cdot, \infty) < 0$ ensures that $D_1 < 0$, (iii) $I_{11} > 0$, which, together with (ii) and $\ddot{D}(\cdot, \infty) > 0$ ensures that $D_{11} > 0$, (iv) $\lim_{\delta \to \infty} \{I(\delta, \tau)\} = \infty$ for all τ, which, together with $\lim_{m \downarrow 0} \{D(m, \infty)\} = \infty$ ensures that $\lim_{m \downarrow 0} \{D(m, \tau)\} = \infty$ for all τ, (v) $I_{12} < 0$, which, together with (ii) ensures that $D_{12} > 0$, and (vi) $I_1(\cdot, \tau)$ converges uniformly to 1 as τ goes to infinity. As discussed in the main text, Assumption (v) implies that the publication process becomes increasingly efficient. Yet, Assumption (vi) puts an upper bound to these efficiency gains; indeed, it implies that, as τ goes to infinity, $I(\cdot, \tau)$ converges to the identity function over $[0, \infty)$ uniformly over compact sets, and that, as a result, $D(\cdot, \tau)$ converges to $D(\cdot, \infty)$ uniformly over compact sets. The following claim then holds.

Claim 7 *Suppose that D is given by (93), and that Assumptions (i)–(vi) hold. Then, if $B \equiv \dot{G}/(1 - G)$ increases to infinity, $D(\mu(\tau), \tau)$ goes to infinity as τ goes to infinity.*

Proof. In the context of the present example, and given (93), (7) rewrites as

$$\dot{\mu}(\tau) = \frac{B(\tau)}{-r[1 + I_1(D(\mu(\tau), \infty), \tau) D_1(\mu(\tau), \infty)]} - 1. \tag{94}$$

Because B is increasing and $D_{12} > 0$ by Assumption (v), $\mu(\tau)$ is strictly decreasing in τ and thus converges to a nonnegative limit m as τ goes to infinity. If m were positive, then, by Assumption (vi), the denominator of the first term on the right-hand side of (94) would converge to a finite, nonnegative limit $-r[1+D_1(m,\infty)]$. Thus, as $B(\tau)$ goes to infinity as τ goes to infinity, so would $\dot{\mu}(\tau)$. But then, as $\liminf_{\tau\to\infty}\{M(\tau)\}<\infty$ by Assumption 3, $\mu(\tau)$ would exceed $M(\tau)$ at some time τ, which is impossible. Thus $m=0$ and $\mu(\tau)$ converges to zero as τ goes to infinity. To conclude, observe that, by Assumption (vi), there exists τ_0 and $\varepsilon<1$ such that, for each $\tau\geq\tau_0$,

$$D(\mu(\tau),\tau)=I(D(\mu(\tau),\infty),\tau)\geq\min_{\delta\in[0,D(\mu(\tau),\infty)]}\{I_1(\delta,\tau)\}\,D(\mu(\tau),\infty)\geq(1-\varepsilon)D(\mu(\tau),\infty),$$

which goes to infinity as τ goes to infinity. The claim follows. ■

B.2 Detailed Calculations for Example 3

B.2.1 Checking Assumptions

We first give restrictions on the technological frontier ξ and the inverse demand function P that ensure that the payoff function

$$L(m,\tau;d)\equiv\exp(-rm)P(Q(m,\tau;d)),$$

with $Q(m,\tau;d)$ defined by (16), satisfies Assumptions 1–4, and that the corresponding payoff function expressed in terms of quality,

$$H(q,\tau;d)\equiv\exp(-r[T(q,\tau)-\tau])P(q),$$

satisfies Assumption 7. The following assumptions are hereafter maintained without explicit reference. Regarding the technological frontier, we shall assume, as in the main text, that ξ is bounded, with $\xi>0$ and $\dot{\xi}>0$, and, as in Proposition 4, that $\ddot{\xi}<0$. As for the inverse demand function, we shall assume that P vanishes at the origin, that $P'>0$, and that P is ρ_P-concave for some $\rho_P>0$. Further restrictions on ξ and P will involve upper bounds on $\dot{\xi}$, reflecting that the technological frontier does not move too fast over time, as well as a concomitant lower bound on ρ_P, reflecting that consumers of research outputs feature a willingness to pay for quality increases that decreases fast enough with quality.

Assumption 1 That $L(\cdot,\tau;d)$ only vanishes at the origin follows from the fact that so do $Q(\cdot,\tau;d)$ and P. To find sufficient conditions under which $L_1(\cdot,\tau;d)$ has a unique zero $M(\tau)$, at which $L(\cdot,\tau;d)$ reaches a maximum and is strongly concave, it is useful to work with the log-derivative

$$\frac{L_1}{L}(m,\tau;d)=-r+Q_1(m,\tau;d)\,\frac{P'}{P}\,(Q(m,\tau;d)). \tag{95}$$

The following claim then holds.

Claim 8 *The payoff function satisfies Assumption 1 if $\dot{\xi}(0)<\xi(0)\rho_P r$.*

Proof. By (16), $Q_1(0,\tau;d)=\xi(\tau)>0$ and hence $\lim_{m\downarrow 0}\{(L_1/L)(m,\tau;d)\}=\infty$. We now show that $\lim_{m\to\infty}\{(L_1/L)(m,\tau;d)\}=-r$, which implies that $L(\cdot,\tau;d)$ has at least a zero. First, by (16) again, $\lim_{m\to\infty}\{Q_1(m,\tau;d)\} = d\xi(\tau) + (1-d)\lim_{s\to\infty}\{\xi(s)\} < \infty$ and $\lim_{m\to\infty}\{Q(m,\tau;d)\} = \infty$. Moreover, as P^{ρ_P} is an increasing concave function for $\rho_P > 0$, $\lim_{q\to\infty}\{(P'/P)(q)\} = (1/\rho_P)\lim_{q\to\infty}\{[(P^{\rho_P})'/P^{\rho_P}](q)\} = 0$. Substituting in (95) then yields $\lim_{m\to\infty}\{(L_1/L)(m,\tau;d)\} = -r$, as claimed. It remains to show that $L_1(m,\tau;d) = 0$ implies that $L_{11}(m,\tau;d) < 0$ or, equivalently, that $(L_1/L)_1(m,\tau;d) < 0$. According to (95), we have

$$\left(\frac{L_1}{L}\right)_1(m,\tau;d) = Q_{11}(m,\tau;d)\,\frac{P'}{P}\,(Q(m,\tau;d)) + Q_1(m,\tau;d)^2\left(\frac{P'}{P}\right)'(Q(m,\tau;d)). \quad (96)$$

For each $q > 0$,

$$\left(\frac{P'}{P}\right)'(q) = \left[\left(\frac{P''P}{P'^2} - 1\right)\left(\frac{P'}{P}\right)^2\right](q) < -\rho_P\left(\frac{P'}{P}\right)^2(q). \quad (97)$$

Substituting (97) into (96) and using (16) yields

$$\begin{aligned}\left(\frac{L_1}{L}\right)_1(m,\tau;d) &= Q_{11}(m,\tau;d)\,\frac{P'}{P}\,(Q(m,\tau;d)) + Q_1(m,\tau;d)^2\left(\frac{P'}{P}\right)'(Q(m,\tau;d))\\ &< \frac{P'}{P}\,(Q(m,\tau;d))\left[\dot{\xi}(0) - \xi(0)\rho_P\, Q_1(m,\tau;d)\,\frac{P'}{P}\,(Q(m,\tau;d))\right]\\ &= \frac{P'}{P}\,(Q(m,\tau;d))[\dot{\xi}(0) - \xi(0)\rho_P r]\\ &< 0,\end{aligned}$$

where the first inequality follows from (16), taking advantage of $\dot{\xi} > 0$ and $\ddot{\xi} < 0$, and the equality reflects that, by assumption, $(L_1/L)(m,\tau;d)$ as given by (95) is equal to zero. The claim follows. ■

Assumption 3 It is useful to check Assumption 3 before turning to Assumptions 2 and 4.

Claim 9 *The payoff function satisfies Assumption 3.*

Proof. Note first that

$$Q(M(\tau),\tau) = \left(\frac{P'}{P}\right)^{-1}\left(\frac{r}{Q_1(M(\tau),\tau)}\right) \quad (98)$$

is uniformly bounded in τ as $0 < \xi(0) < Q_1(m,\tau) < \lim_{s\to\infty}\{\xi(s)\} < \infty$ for all m and τ. Next, observe that, by (16),

$$Q(M(\tau),\tau) = \int_0^{M(\tau)} Q_1(m,\tau)\,\mathrm{d}m > \xi(0)M(\tau) \quad (99)$$

for all τ, which implies that Assumption 3 is satisfied as $Q(M(\tau),\tau)$ is uniformly bounded in τ. The claim follows. ■

Assumptions 4 and 7 Turning to Assumptions 4 and 7, we show that, given Assumption

3, the same condition that ensures that the payoff function satisfies Assumption 1 further ensures that it satisfies Assumption 4 and that the payoff function expressed in terms of quality satisfies Assumption 7.

Claim 10 *The payoff function satisfies Assumption 4 if* $\dot{\xi}(0) < \xi(0)\rho_P r$.

Proof. We must show that (44) holds for some $\rho > 0$. We have

$$1 - \frac{LL_{11}}{L_1^2} = \frac{Q_1^2(P'^2 - PP'') - Q_{11}PP'}{(-rP + Q_1P')^2}. \tag{100}$$

As shown in the proof of Claim 9, $Q(M(\tau), \tau)$ is uniformly bounded in τ. Hence, using (16), we get that the denominator of the right-hand side of (100) is uniformly bounded above over $\{(m, \tau) : 0 \leq m < M(\tau)\}$,

$$(-rP + Q_1P')^2 \leq \lim_{s\to\infty} \{\xi^2(s)\} \max_{q\in\left[0,\sup_{\tau'\in[0,\infty)}\{Q(M(\tau),\tau)\}\right]} \{P'^2(q)\}. \tag{101}$$

Thus we only need to check that the numerator of the right-hand side of (100) is bounded away from zero over this domain. Indeed, we have

$$\begin{aligned} Q_1^2(P'^2 - PP'') - Q_{11}PP' &= Q_1^2P'^2\left(1 - \frac{PP''}{P'^2}\right) - Q_{11}PP' \\ &\geq Q_1^2P'^2\rho_P - Q_{11}PP' \\ &\geq \frac{Q_1P'^2}{r}\left(Q_1\rho_P r - Q_{11}\right) \\ &> \frac{\xi(0)}{r} \max_{q\in\left[0,\sup_{\tau'\in[0,\infty)}\{Q(M(\tau),\tau)\}\right]} \{P'^2(q)\}\left[\xi(0)\rho_P r - \dot{\xi}(0)\right] \\ &> 0, \end{aligned}$$

where the first inequality follows from the ρ_P-concavity of P, the second inequality follows from the fact that $L(\cdot, \tau)$ is strictly increasing over $[0, M(\tau))$ for each τ, and the third inequality follows from (16), taking advantage of $\dot{\xi} > 0$ and $\ddot{\xi} < 0$, and from the fact that $Q(M(\tau), \tau)$ is uniformly bounded in τ. The claim follows. ■

Claim 11 *The payoff function expressed in terms of quality satisfies Assumption 7 if* $\dot{\xi}(0) < \xi(0)\rho_P r$.

Proof. By analogy with (44), we must show that

$$1 - \frac{HH_{11}}{H_1^2}(q, \tau) \geq \rho, \quad \tau \geq 0, \quad Q(M(\tau), \tau) > q \geq 0$$

for some $\rho > 0$. We have

$$1 - \frac{HH_{11}}{H_1^2} = \frac{P'^2 - PP'' + rT_{11}P^2}{(-rT_1P + P')^2}. \tag{102}$$

Proceeding as for (101), we get that the denominator of the right-hand side of (102) is uniformly bounded above over $\{(q,\tau) : 0 \le q < Q(M(\tau),\tau)\}$. Thus we only need to check that the numerator of the right-hand side of (102) is bounded away from zero over this domain. Indeed, following the same steps as in the proof of Claim 10, we have

$$\begin{aligned}
P'^2 - PP'' - rT_{11}P^2 &= P'^2\left(1 - \frac{PP''}{P'^2}\right) + rT_{11}P^2 \\
&\ge P'^2\rho_P + rT_{11}P^2 \\
&= P'^2\rho_P - r\,\frac{Q_{11}}{Q_1^3}\,P^2 \\
&> \frac{P'^2}{rQ_1}\,(Q_1\rho_P r - Q_{11}) \\
&\ge \frac{1}{r\lim_{s\to\infty}\{\xi(s)\}} \max_{q\in\left[0,\sup_{\tau'\in[0,\infty)}\{Q(M(\tau),\tau)\}\right]} \{P'^2(q)\}\,[\xi(0)\rho_P r - \dot{\xi}(0)] \\
&> 0.
\end{aligned}$$

The claim follows. ∎

Assumption 2 Finally, we consider Assumption 2.

Claim 12 *The payoff function satisfies Assumption 4 if* $\dot{\xi}(0) < [\xi(0)]^2/(P'/P)^{-1}(r/\xi(0))$.

Proof. We have, by (95),

$$(\ln L)_{11} - (\ln L)_{12} = (Q_{11} - Q_{12})\,\frac{P'}{P} + Q_1(Q_1 - Q_2)\left(\frac{P'}{P}\right)'. \tag{103}$$

We need to show that this quantity is negative over $\{(m,\tau) : 0 < m \le M(\tau)\}$. The first term on the right-hand side of (103) is nonpositive by Assumption 6. Because P is ρ_P-concave, the second term is negative if the maturation-technology wedge $(Q_1 - Q_2)(m,\tau)$ is positive over $\{(m,\tau) : 0 < m \le M(\tau)\}$. We have, over this domain,

$$\begin{aligned}
(Q_1 - Q_2)(m,\tau) &= \xi(\tau) - d\dot{\xi}(\tau)m \\
&\ge \xi(0) - \dot{\xi}(0)M(\tau) \\
&> \xi(0) - \frac{\dot{\xi}(0)}{\xi(0)}\,Q(M(\tau,\tau) \\
&= \xi(0) - \frac{\dot{\xi}(0)}{\xi(0)}\left(\frac{P'}{P}\right)^{-1}\left(\frac{r}{Q_1(M(\tau),\tau)}\right) \\
&\ge \xi(0) - \frac{\dot{\xi}(0)}{\xi(0)}\left(\frac{P'}{P}\right)^{-1}\left(\frac{r}{\xi(0)}\right) \\
&> \ 0,
\end{aligned}$$

where the second inequality follows from (99), the second equality follows from (98), the third inequality follows from (16), and the fourth inequality follows from the assumption that $\dot{\xi}(0) < [\xi(0)]^2/(P'/P)^{-1}(r/\xi(0))$. The claim follows. ∎

B.2.2 On Maturation and Quality

This appendix provides additional material for Section 4.2.3. We start with the following claim, which is used in the discussion of (18).

Claim 13 *For all $m > 0$, $\tau \geq 0$, and $d \in [0,1]$,*

$$\begin{aligned} Q_3(m,\tau;d) &< 0, \\ \left(\frac{Q_1}{Q}\right)_3 (m,\tau;d) &< 0. \end{aligned}$$

Proof. To simplify notation, rewrite (16) as $Q(m,\tau;d) = Q^f(m,\tau) - d[Q^f(m,\tau) - Q^l(m,\tau)]$, where Q^f and Q^l respectively correspond to the frontier case ($d = 0$) and to the lock-in case ($d = 1$). We then have

$$Q_3(m,\tau;d) = (Q^l - Q^f)(m,\tau;d) = \xi(\tau)m - \int_\tau^{\tau+m} \xi(s)\,\mathrm{d}s$$

and

$$\left(\frac{Q_1}{Q}\right)_3 (m,\tau;d) = \frac{Q_1^l Q^f - Q_1^f Q^l}{Q^2}(m,\tau;d) = \frac{\xi(\tau)}{Q(m,\tau,d)^2}\left[\int_\tau^{\tau+m} \xi(s)\,\mathrm{d}s - \xi(\tau+m)m\right],$$

which are both negative if $m > 0$ as $\dot{\xi} > 0$. The claim follows. ■

To illustrate the claim that being closer to the frontier can lead to shorter maturation delays in equilibrium, consider the following inverse demand function for quality:

$$P(q) \equiv \exp\left(\frac{q^{1-\kappa}}{1-\kappa}\right), \tag{104}$$

for some constant $\kappa \geq 2$, so that $(P'/P)(q) = 1/q^\kappa$. This specification of P is convenient for computational purposes, yet it requires a little bit of care, as P is not ρ_P-concave over $[0,\infty)$ for any $\rho_P > 0$, though it is log-concave over $[0,\infty)$ and $\rho_P(\eta)$-concave for some $\rho_P(\eta) > 0$ over any interval $[\eta,\infty)$ such that $\eta > 0$.

Whereas we cannot, therefore, directly apply Claims 8 and 10–11, the details are easy to fix. The following claim parallels Claim 8.

Claim 14 *With P given by (104), the payoff function satisfies Assumption 1 if $\dot{\xi}(0) < 2\xi(0)^{2-1/\kappa} r^{1/\kappa}$.*

Proof. Now (95) writes as:

$$\frac{L_1}{L}(m,\tau;d) = -r + \frac{Q_1(m,\tau;d)}{Q(m,\tau;d)^\kappa}. \tag{105}$$

From (105), it is easily checked along the lines of the proof of Claim 8 that $L(\cdot,\tau;d)$ has at least a zero. It remains to show that $L_1(m,\tau;d) = 0$ implies that $L_{11}(m,\tau;d) < 0$ or, equivalently, that $(L_1/L)_1(m,\tau;d) < 0$. According to (105), we have

$$\begin{aligned}\left(\frac{L_1}{L}\right)_1(m,\tau;d) &= \frac{Q_{11}(m,\tau;d)}{Q(m,\tau;d)^\kappa} - \kappa\,\frac{Q_1(m,\tau;d)^2}{Q(m,\tau;d)^{\kappa+1}}\\ &= \frac{1}{Q(m,\tau;d)^\kappa}\left[Q_{11}(m,\tau;d) - \kappa Q_1(m,\tau;d)^{2-1/\kappa} r^{1/\kappa}\right]\\ &< \frac{1}{Q(m,\tau;d)^\kappa}\left[\dot{\xi}(0) - \kappa\xi(0)^{2-1/\kappa} r^{1/\kappa}\right]\\ &< 0,\end{aligned}$$

where the second equality reflects that, by assumption, $(L_1/L)(Q(m,\tau;d))$ as given by (105) is equal to zero, and the first inequality follows from (16), taking advantage of $\dot{\xi} > 0$ and $\ddot{\xi} < 0$. The claim follows. ■

As for the analogues of Claims 10–11, we can use the trick described in Appendix C.1. The idea is to make sure that, in equilibrium, σ never enters a strip $\{(\tau,t) : t-\tau \leq \varepsilon\}$, for some $\varepsilon > 0$. An easy way to ensure this is to bound above the breakthrough rate $\dot{G}/(1-G)$ by some positive constant λ. Then there exist positive numbers ε and ζ such that

$$\begin{aligned}\frac{1}{f(m,\tau;d)} &\leq \frac{\lambda}{-r + Q_1(m,\tau;d)(P'/P)(Q(m,\tau;d))}\\ &< \frac{\lambda}{-r + \xi(0)(P'/P)(\lim_{s\to\infty}\{\xi(s)\}m)}\\ &< 1-\zeta\end{aligned}$$

for all $m \leq \varepsilon$ and τ, which implies that $\mu(\tau) > \varepsilon$ for all τ in any equilibrium. Note that $\dot{\xi}(0)$ does not appear in the above bound. As $\mu(\tau)$ and, therefore, $\chi(\tau;d)$, are bounded away from zero in any equilibrium, we get that the functions $\{L(\cdot,\tau;d) : \tau \geq 0\}$ and $\{H(\cdot,\tau;d) : \tau \geq 0\}$ are uniformly ρ-concave along any equilibrium trajectory. Letting $\dot{\xi}(0)$ be small enough then allows one to proceed along the same steps as in the proofs of Claim 10–11.

The upshot of this discussion is that if P is given by (104) and if, over time, players do not become too innovative and the technological frontier does not move too fast, our equilibrium-uniqueness (Theorem 2) and comparative-statics (Propositions 2 and 4) results still hold. Hence, to show that, when the inverse demand for quality is given by (104), being closer to the frontier leads to shorter maturation delays and yet higher quality levels in equilibrium, we only need to verify the following claim.

Claim 15 *With P given by (104), an increase in d leads to an increase in $L(\cdot,\cdot;d)$ in the growth-rate order.*

Proof. We use the same notation as in the proof of Claim 13. The derivative of the right-hand side of (105) with respect to d has the same sign as

$$[\kappa Q_1^l(Q^f - Q^l) - (Q^f - Q^l)_1 Q^l + (1-d)(\kappa-1)(Q^f-Q^l)_1(Q^f-Q^l)](m,\tau;d). \quad (106)$$

We want to show that, for $\kappa \geq 2$, the quantity (106) is positive if $m > 0$, uniformly in

$(m, \tau; d)$. Because $d \in [0,1]$, $\kappa \geq 2$, and because

$$[(Q^f - Q^l)_1(Q^f - Q^l)](m, \tau) = [\xi(\tau + m) - \xi(\tau)]\left[\int_\tau^{\tau+m} \xi(s)\,\mathrm{d}s - \xi(\tau)m\right]$$

is positive if $m > 0$ as $\dot{\xi} > 0$, we can focus on the first two terms in (106). As

$$[Q_1^l(Q^f - Q^l)](m, \tau) = \xi(\tau)\left[\int_\tau^{\tau+m} \xi(s)\,\mathrm{d}s - \xi(\tau)m\right]$$

is positive if $m > 0$ as $\dot{\xi} > 0$, and as $\kappa \geq 2$ by assumption, we only need to find conditions on ξ such that the quantity

$$\left[\frac{(Q^f - Q^l)_1 Q^l}{Q_1^l(Q^f - Q^l)}\right](m, \tau) = \frac{\xi(\tau + m) - \xi(\tau)}{(1/m)\int_\tau^{\tau+m} \xi(s)\,\mathrm{d}s - \xi(\tau)} \equiv \Xi(m, \tau)$$

is bounded above by 2, uniformly in (m, τ). Indeed, as $\ddot{\xi} < 0$, it follows from the Hermite–Hadamard inequality that,[44] for all $m > 0$ and $\tau \geq 0$,

$$\frac{\xi(\tau) + \xi(\tau + m)}{2} < \frac{1}{m}\int_\tau^{\tau+m} \xi(s)\,\mathrm{d}s.$$

Rearranging and using the fact that $(1/m)\int_\tau^{\tau+m} \xi(s)\,\mathrm{d}s > \xi(\tau)$ as $\dot{\xi} > 0$, we get that $\sup_{m\in(0,\infty)}\{\Xi(m, \tau)\} \leq 2$ for all $\tau \geq 0$, as requested. The claim follows. ■

B.3 Further Results on the Hare and the Tortoise

We investigate here the impact on the hare's equilibrium maturation delay of an increase in her own breakthrough rate. To do so, a few technical observations are in order. Recall first from the proof of Theorem 3 that the unique continuous equilibrium corresponds to the unique trajectory of (72) that converges to its critical point $(M_{\lambda^b}, M_{\lambda^a})$. The vector field corresponding to (72) is given by

$$\boldsymbol{f}(\boldsymbol{m}) = \begin{pmatrix} 1 - (1/\lambda^a)(\dot{L}/L)(m^b) \\ 1 - (1/\lambda^b)(\dot{L}/L)(m^a) \end{pmatrix}$$

at any point $\boldsymbol{m} \equiv (m^a, m^b)$ in $\mathcal{M}$. For each i, let

$$\rho^i \equiv -\ddot{\overbrace{(\ln L)}}(M_{\lambda^i}) = \left(\frac{\dot{L}^2 - L\ddot{L}}{L^2}\right)(M_{\lambda^i}) \tag{107}$$

and

$$\delta \equiv \sqrt{\frac{\rho^a \rho^b}{\lambda^a \lambda^b}}. \tag{108}$$

[44] Observe, alternatively, that the uniform distribution over $[\tau, \tau+m]$ second-order stochastically dominates the two-point distribution that puts equal weights on τ and $\tau + m$.

By Assumption 1, $\delta > 0$. The Jacobian of $\boldsymbol{f}$ at its critical point $(M_{\lambda^b}, M_{\lambda^a})$ is

$$D\boldsymbol{f}(M_{\lambda^b}, M_{\lambda^a}) = \begin{pmatrix} 0 & \rho^b/\lambda^a \\ \rho^a/\lambda^b & 0 \end{pmatrix}$$

and its eigenvalues are therefore δ and $-\delta$. As $\delta > 0$, this shows that the critical point $(M_{\lambda^b}, M_{\lambda^a})$ of $\boldsymbol{f}$ is hyperbolic (Perko (2001, Section 2.6, Definition 1)). Suppose from now on that L is thrice continuously differentiable, so that $\boldsymbol{f}$ is twice continuously differentiable in the neighborhood of $(M_{\lambda^b}, M_{\lambda^a})$. Then, according to Hartman (1960, Theorem (IV)), there exists a C^1-diffeomorphism H from a neighborhood U of $(M_{\lambda^b}, M_{\lambda^a})$ onto an open set containing the origin such that H linearizes the system $\dot{\boldsymbol{\nu}} = \boldsymbol{f}(\boldsymbol{\nu})$, locally transforming it into the linear system $\dot{\boldsymbol{\nu}} = D\boldsymbol{f}(M_{\lambda^b}, M_{\lambda^a})\boldsymbol{\nu}$. Thus, for each $\boldsymbol{m_0} \in U$, one can locally write

$$H(\boldsymbol{\nu}(t, \boldsymbol{m_0})) = e^{D\boldsymbol{f}(M_{\lambda^b}, M_{\lambda^a})t} H(\boldsymbol{m_0}).$$

Now, let $\mathcal{S}$ be the stable manifold of the nonlinear system $\dot{\boldsymbol{\nu}} = \boldsymbol{f}(\boldsymbol{\nu})$ (Perko (2001, Section 2.7, Theorem)), the upper branch of which corresponds to the equilibrium trajectory (ν^a, ν^b). Then, according to Theorem 3, there exists $t_0 \geq 0$ such that $(\nu^a(t), \nu^b(t)) \in \mathcal{S} \cap U$ for all $t \geq t_0$. As H maps $\mathcal{S}$ onto the stable subspace $\{C\boldsymbol{\xi}_{-\delta} : C \in \mathbb{R}\}$ of the linear system $\dot{\boldsymbol{\nu}} = D\boldsymbol{f}(M_{\lambda^b}, M_{\lambda^a})\boldsymbol{\nu}$ associated to the eigenvalue $-\delta$, we get that there exists a nonzero scalar constant C such that for any large enough t,

$$(\nu^a(t), \nu^b(t)) = H^{-1}(\exp(-\delta t)C\boldsymbol{\xi}_{-\delta}).$$

From Proposition 3 again, along with the fact that the derivative of H^{-1} at the origin is the identity, it follows in turn that

$$\|(\nu^a(t), \nu^b(t)) - (M_{\lambda^b}, M_{\lambda^a}) - \exp(-\delta t)C\boldsymbol{\xi}_{-\delta})\| = o(\exp(-\delta t)),$$

which implies

$$\lim_{t\to\infty} \{\exp(\delta t)[(\nu^a(t), \nu^b(t)) - (M_{\lambda^b}, M_{\lambda^a})]\} = C\boldsymbol{\xi}_{-\delta}, \tag{109}$$

upon multiplying by $\exp(\delta t)$. Hence the equilibrium maturation delays $\nu^a(t)$ and $\nu^b(t)$ converge exponentially fast to their limit values M_{λ^b} and M_{λ^a}.

We now how to use this result to study the impact on the hare's equilibrium maturation delay of an increase in her own breakthrough rate. Observe first from the proof of Theorem 3 that a change in λ^a does not affect the limit M_{λ^b} of $\nu^a(t)$ as t goes to infinity. It thus follows from (109) that it is sufficient to study how the eigenvalue δ defined in (108) varies with λ^a: a higher value of δ translates into a faster convergence of $\nu^a(t)$ to M_{λ^b} as t goes to infinity and thus, as $\nu^a(t) < M_{\lambda^b}$ for all t, into asymptotically longer maturation delays. From (107)–(108) and the definition of M_λ, we just need to study the variations of the mapping

$$\lambda \mapsto -\frac{S'(S^{-1}(\lambda))}{\lambda} = -\frac{1}{\lambda(S^{-1})'(\lambda)},$$

where $S \equiv \dot{L}/L$. The derivative of this mapping has the same sign as

$$(S^{-1})'(\lambda) + \lambda(S^{-1})''(\lambda).$$

Example 1 Suppose that L is given by

$$L(m) = \exp\left(-r\left[m + M\ln\left(\frac{M}{m}\right)\right]\right), \quad M \geq m \geq 0, \tag{110}$$

that is, $D(m,\tau) = M\ln(M/m)$ in Example 1. Then

$$S^{-1}(\lambda) = M_\lambda = \frac{r}{\lambda + r}\,M,$$

so that

$$\begin{aligned}\text{sgn}\,((S^{-1})'(\lambda) + \lambda(S^{-1})''(\lambda)) &= \text{sgn}\left(-\frac{1}{(\lambda+r)^2} + \frac{2\lambda}{(\lambda+r)^3}\right)\\ &= \text{sgn}\,(\lambda - r).\end{aligned}$$

Thus, if $\lambda^a < r$, a small increase in the hare's breakthrough rate λ^a shortens her equilibrium maturation delay for large values of her breakthrough time, whereas the opposite is true if $\lambda^a > r$.

Example 3 Suppose that L is given by

$$L(m) = \exp(-rm)[\exp(\xi m) - 1], \quad m \geq 0, \tag{111}$$

where $r > \xi > 0$, that is, $Q(m,\tau) = \xi m$ and $P(q) = \exp(q) - 1$ in Example 3. Then

$$S^{-1}(\lambda) = M_\lambda = \frac{1}{\xi}\ln\left(\frac{r}{r-\xi}\right),$$

so that

$$\begin{aligned}\text{sgn}\,((S^{-1})'(\lambda) + \lambda(S^{-1})''(\lambda)) &= \text{sgn}\left(\frac{1}{\lambda+r} - \frac{1}{\lambda+r-\xi} + \lambda\left[\frac{1}{(\lambda+r-\xi)^2} - \frac{1}{(\lambda+r)^2}\right]\right)\\ &= \text{sgn}\left(\lambda\left(\frac{1}{\lambda+r-\xi} + \frac{1}{\lambda+r}\right) - 1\right)\\ &= \text{sgn}\,(\lambda^2 - r(r-\xi)).\end{aligned}$$

Thus, if $\lambda^a < \sqrt{r(r-\xi)}$, a small increase in the hare's breakthrough rate λ^a shortens the hare's equilibrium maturation delay for large values of her breakthrough time, whereas the opposite is true if $\lambda^a > \sqrt{r(r-\xi)}$.

The intuition for these results is as follows. According to Theorem 3, the tortoise's equilibrium maturation delay is close to M_{λ^a} when she has a late breakthrough. Now, when the payoff function L is given by (110) or (111), it is straightforward to see that the maturation delay M_{λ^a} is convex in λ^a. Therefore, the limit of the tortoise's maturation delay is less sensitive to an increase in the hare's breakthrough rate when the hare's initial breakthrough rate is high than when it is low. When the hare has a late breakthrough herself, she is thus less threatened by preemption at the margin in the former case than in the latter case, and she is ready to let her breakthrough mature more: the direct effect of an increase in the hare's breakthrough rate asymptotically dominates the indirect effect that works through the modification of the tortoise's equilibrium behavior. This prediction is reversed if the hare's initial breakthrough rate is initially lower, for then an increase in it has a large impact on the tortoise's limit equilibrium behavior. This second scenario, however, is perhaps less realistic than the first, for it is likely that, in practice, λ^a is large relative to r (in Example 1) or to $\sqrt{r(r-\xi)}$ (in Example 3).

B.4 Follower Value

We have focused on the case where there is no value in being a follower. Yet, in practice, being preempted on a given topic does not necessarily mean for a researcher that she should lose all opportunities to publish related work: indeed, standing on the shoulders of her predecessors, she can in turn publish follow-up research, the quality of which often depends on the latter's achievements. In our competitive context, it is natural to postulate that the leader's and the follower's contributions are substitutes, and that the leader sets the tone for future research as soon as she discloses her results: that is, the current value of becoming a follower is a decreasing function $F^c(m)$ of the leader's maturation delay m.[45] Assume that F^c is differentiable and that $L(M) > \exp(-rM)F^c(M) \geq 0$, where r denotes the players' discount rate.[46] Player i's payoff if her type is τ^i, player j's strategy is σ^j, and player i plans to make a move at time $t^i \geq \tau^i$ is

$$\begin{aligned} W^i(t^i, \tau^i, \sigma^j) \equiv \{&\mathbf{P}[\sigma^j(\tilde{\tau}^j) > t^i] + \alpha \mathbf{P}[\sigma^j(\tilde{\tau}^j) = t^i]\} L(t^i - \tau^i) \\ &+ \mathbf{P}[\sigma^j(\tilde{\tau}^j) < t^i]\, \mathbf{E}\big[\exp(-r[\sigma^j(\tilde{\tau}^j) - \tau^i]) F^c(\sigma^j(\tilde{\tau}^j) - \tilde{\tau}^j) \,|\, \sigma^j(\tilde{\tau}^j) < t^i\big]. \end{aligned}$$

In a symmetric separating equilibrium with common continuous strategy σ, the problem faced by type τ^i of player i is

$$\max_{t^i \in [\tau^i, \infty)} \left\{ [1 - G(\phi(t^i))] L(t^i - \tau^i) + \int_{\phi(\tau^i)}^{\phi(t^i)} \exp(-r[\sigma(\tau^j) - \tau^i]) F^c(\sigma(\tau^j) - \tau^j)\, \mathrm{d}G(\tau^j) \right\},$$

where $\phi \equiv \sigma^{-1}$. The first-order condition is

$$[1 - G(\phi(t^i))] \dot{L}(t^i - \tau^i) = \dot{G}(\phi(t^i)) \dot{\phi}(t^i) \big[L(t^i - \tau^i) - \exp(-r(t^i - \tau^i)) F^c(t^i - \phi(t^i))\big].$$

Note that the expected marginal cost from an additional delay $\mathrm{d}t^i$ is lower than when the follower's value is zero. In a symmetric equilibrium, this first-order condition must hold for $\tau^i = \phi(t^i)$, leading to the ODE

$$\dot{\phi}(t) = \frac{1 - G}{\dot{G}}(\phi(t)) \, \frac{\dot{L}}{L - F}(t - \phi(t)), \quad t \geq \sigma(0), \tag{112}$$

where $F(m) \equiv \exp(-rm)F^c(m)$ for all $m \in [0, M]$. Let M_F be the unique root of $L = F$. We look for an equilibrium in which players spend at least M_F time units maturing their breakthroughs, so that ϕ stays in the domain $\mathcal{D}_F \equiv \{(t, \tau) : M_F \leq \tau + M_F < t < \tau + M\}$.

[45] For simplicity, we suppose throughout this appendix that payoffs are independent of breakthrough times, and we let M be the point at which L reaches its maximum.

[46] Recall that L is evaluated in breakthrough-time terms and is therefore already a present value.

Assume for simplicity that L is concave over $[0, M]$, so that the function $\dot{L}/(L-F)$ is strictly decreasing over $(M_F, M]$. One can then straightforwardly adapt the proofs of Theorems 1–2 to show that there exists a unique equilibrium of the postulated form.

An increase in F has a mechanical effect on the domain $\mathcal{D}_F$: it shrinks as F increases. The following comparative-statics result then holds.

Proposition 7 *Let $\underline{\mu}$ ($\overline{\mu}$) be the equilibrium maturation delay under the follower value $\underline{F}$ ($\overline{F}$). Then, if $\overline{F} > \underline{F}$ over $(0, M)$, we have $\overline{\mu}(\tau) > \underline{\mu}(\tau)$ for all τ.*

Proof. Suppose, by way of contradiction, that $\overline{\mu}(\tau_0) \leq \underline{\mu}(\tau_0)$ for some $\tau_0 \geq 0$. Then, it follows from (112) that

$$\dot{\underline{\mu}}(\tau_0) = \frac{\dot{G}}{1-G}(\tau_0)\,\frac{L-\underline{F}}{\dot{L}}(\underline{\mu}(\tau_0)) > \frac{\dot{G}}{1-G}(\tau_0)\,\frac{L-\overline{F}}{\dot{L}}(\underline{\mu}(\tau_0)) = \dot{\overline{\mu}}(\tau_0),$$

so that $\underline{\mu}(\tau) > \overline{\mu}(\tau)$ for all $\tau > \tau_0$ close enough to τ_0. As in Proposition 2, we can deduce from this that $\underline{\mu} > \overline{\mu}$ over (τ_0, ∞). Defining g as in (52), the analogues of (53)–(54) hold; actually, g is bounded above by $M - M_{\underline{F}}$. For each $\tau \geq \tau_0$, we have

$$\begin{aligned}
\dot{g}(\tau) &= \frac{\dot{G}}{1-G}(\tau)\left[\frac{L-\underline{F}}{\dot{L}}(\underline{\mu}(\tau)) - \frac{L-\overline{F}}{\dot{L}}(\overline{\mu}(\tau))\right] \\
&\geq \frac{\dot{G}}{1-G}(\tau)\left[\frac{L-\underline{F}}{\dot{L}}(\underline{\mu}(\tau)) - \frac{L-\underline{F}}{\dot{L}}(\overline{\mu}(\tau))\right] \\
&\geq \frac{\dot{G}(\tau)}{1-G(\tau)}\,g(\tau),
\end{aligned}$$

where the inequality follows from the fact that $\overline{F} > \underline{F}$, and the second inequality follows along the same lines as (48), using the fact that

$$\overbrace{\left(\frac{L-\underline{F}}{\dot{L}}\right)}^{\cdot}(m) = 1 - \frac{\dot{\underline{F}}}{\dot{L}}(m) - \frac{(L-\underline{F})\ddot{L}}{\dot{L}^2}(m) \geq 1$$

for all $m \in [M_{\underline{F}}, M)$. The remainder of the proof is as in the proof of Proposition 2. Hence the result. ■

Appendix C: On the Uniqueness of Equilibrium

In this appendix, we show that Assumptions 3–4 and 7 can be considerably relaxed while preserving our equilibrium-uniqueness and comparative-statics results. The general idea is that the lower bounds (48), (55)–(56), or (71) on the derivative of the gap between two candidate equilibria, maturation delays, or quality levels are not particularly tight, as they are obtained by bounding below the derivative of $(L/L_1)(\cdot, \tau)$ or $(H/H_1)(\cdot, \tau)$ uniformly over $[0, M(\tau)]$ or $[0, Q(M(\tau), \tau)]$, without using the information that the relevant terms of the difference are equilibrium objects. We show in this appendix that taking into account this information can help improve our results in a significant way. We focus on the equilibrium-uniqueness problem; comparative-statics results can be handled in a similar way.

C.1 A Cheap but Useful Trick

A key difficulty that may arise in applications is that the functions $\{L(\cdot, \tau) : \tau \geq 0\}$ may fail to be uniformly ρ-concave. An example is provided in Appendix B.2.2: in the context of Example 2, it is easy to exhibit an inverse demand function P such that, for each τ, the resulting payoff function $L(\cdot, \tau)$ is log-concave, that is, 0-concave, but not ρ-concave over $[0, M(\tau)]$ for any $\rho > 0$. Here, the difficulty is only what arises at $m = 0$. Supposing for concreteness that $M(\tau)$ is bounded away from zero, a simple but effective trick is to try and find positive numbers ε, ρ, and ζ such that

(i) For all $m \leq \varepsilon$ and τ, $1/f(m, \tau) < 1 - \zeta$.

(ii) For each τ, $L(\cdot, \tau)$ is ρ-concave over $[\varepsilon, M(\tau)]$.

Condition (i) implies that, in equilibrium, σ never enters the strip $\{(\tau, t) : t - \tau \leq \varepsilon\}$, for, otherwise, σ would leave $\mathcal{D}'$ through its lower boundary $t = \tau$. Condition (ii) in turn states that the functions $\{L(\cdot, \tau) : \tau \geq 0\}$ are uniformly ρ-concave over the remaining relevant domain, so that the proof of Theorem 2 goes through unchanged. The power of this simple argument is illustrated in Appendix B.2.2.

C.2 A More Refined ρ-Concavity Argument

In Appendix C.1, the lack of ρ-concavity of the functions $\{L(\cdot, \tau) : \tau \geq 0\}$ was confined to a strip of height ε above the diagonal $t = \tau$. This allowed for a straightforward generalization of Theorem 2. What if, by contrast, the lack of ρ-concavity is more pervasive? To address this question, and to show that equilibrium uniqueness may still obtain, we consider a simple parametric example. Specifically, let $D(m, \tau) = M(\tau) \ln(M(\tau)/m)$ in Example 1, so that

$$L(m, \tau) = \exp\left(-r\left[m + M(\tau) \ln\left(\frac{M(\tau)}{m}\right)\right]\right)$$

for some continuously differentiable function $M : [0, \infty) \to (0, \infty)$ such that $1 + \dot{M} > 0$. Assumptions 1–2 are satisfied. By contrast, if Assumption 3 is not satisfied, that is, if $\lim_{\tau \to \infty} \{M(\tau)\} = \infty$, then Assumption 4 is also not satisfied. Indeed, the ρ-concavity of

$L(\cdot, \tau)$ at each $m \in [0, M(\tau))$,

$$1 - \frac{LL_{11}}{L_1^2}(m, \tau) = \frac{M(\tau)}{r[M(\tau) - m]^2},$$

then goes to zero as τ goes to infinity. This implies that there is no $\rho > 0$ such that the functions $\{L(\cdot, \tau) : \tau \geq 0\}$ are uniformly ρ-concave, though they are uniformly log-concave. But, unlike in the situation dealt with in Appendix C.1, there exists no strip of finite height above the diagonal $t = \tau$ such that the functions $\{L(\cdot, \tau) : \tau \geq 0\}$ are uniformly ρ-concave, even for τ large enough. Thus no straightforward generalization of Theorem 2 seems forthcoming.

We now show how to use the key information that, in (48), the maturation delays in $[\sigma_1(\tau) - \tau, \sigma_2(\tau) - \tau]$ are equilibrium maturation delays, to derive a lower bound on the ρ-concavity of the payoff function, *evaluated along the equilibrium trajectory.* Let us assume for simplicity that breakthroughs are exponentially distributed, that is, $\dot{G}/(1 - G) = \lambda$ for some positive constant λ. Then the vector field associated to (37) is given by

$$\frac{1}{f(t, \tau)} = \frac{\lambda(t - \tau)}{r[M(\tau) - t + \tau]}. \tag{113}$$

Of particular interest are the isoclines of (113), that is, the curves $\tau \mapsto t_\alpha(\tau)$ defined by $1/f(t_\alpha(\tau), \tau) = \alpha$ for $\alpha \geq 0$. According to (113),

$$t_\alpha(\tau) = \tau + \frac{\alpha}{\alpha + \lambda/r} M(\tau) \tag{114}$$

for all τ. Observe that if (σ, σ) is an equilibrium, then, by (37), σ finds itself at any time τ on the $\dot{\sigma}(\tau)$-isocline, that is,

$$t_{\dot{\sigma}(\tau)}(\tau) = \sigma(\tau). \tag{115}$$

Moreover, by (114), one has, for each τ,

$$\dot{t}_{\dot{\sigma}(\tau)}(\tau) = 1 + \frac{\dot{\sigma}(\tau)}{\dot{\sigma}(\tau) + \lambda/r} \dot{M}(\tau). \tag{116}$$

From now on, let us assume that M is strictly convex, with $\lim_{\tau \to \infty} \{\dot{M}(\tau)\} = \infty$, so that Assumption 3 is clearly violated. Then the following property holds.

Claim 16 *If (σ, σ) is an equilibrium, then, for each τ,*

$$\dot{\sigma}(\tau) \geq 1 + \frac{\dot{\sigma}(\tau)}{\dot{\sigma}(\tau) + \lambda/r} \dot{M}(\tau). \tag{117}$$

Proof. Suppose by, way of contradiction, that

$$\dot{\sigma}(\tau_0) < 1 + \frac{\dot{\sigma}(\tau_0)}{\dot{\sigma}(\tau_0) + \lambda/r} \dot{M}(\tau_0) \tag{118}$$

for some τ_0. Then, applying (115)–(116) at $\tau = \tau_0$, it follows that $\sigma(\tau) < t_{\dot{\sigma}(\tau_0)}(\tau)$ for all $\tau > \tau_0$ close enough to τ_0. We show that in that case $\sigma < t_{\dot{\sigma}(\tau_0)}$ over (τ_0, ∞). Suppose the contrary holds, so that $\tau_1 \equiv \inf\{\tau > \tau_0 : \sigma(\tau) \geq t_{\dot{\sigma}(\tau_0)}(\tau)\} < \infty$. Then $\sigma(\tau_1) = t_{\dot{\sigma}(\tau_0)}(\tau_1)$. However, as $\sigma < t_{\dot{\sigma}(\tau_0)}$ over (τ_0, τ_1), we have $\dot{\sigma} < \dot{\sigma}(\tau_0)$ over (τ_0, τ_1) by definition of the isocline $t_{\dot{\sigma}(\tau_0)}$. Thus, using (115)–(116), (118), and the convexity of M, we get that

$$\sigma(\tau_1) < \sigma(\tau_0) + \dot{\sigma}(\tau_0)(\tau_1 - \tau_0) < \sigma(\tau_0) + \int_{\tau_0}^{\tau_1} \left[1 + \frac{\dot{\sigma}(\tau_0)}{\dot{\sigma}(\tau_0) + \lambda/r}\, \dot{M}(\tau)\right] \mathrm{d}\tau = t_{\dot{\sigma}(\tau_0)}(\tau_1),$$

a contradiction. Hence $\sigma < t_{\dot{\sigma}(\tau_0)}$ over (τ_0, ∞), as claimed, so that $\dot{\sigma} < \dot{\sigma}(\tau_0)$ over $(0, \infty)$. This, together with the fact that, by (114) along with the assumption that $\lim_{\tau\to\infty}\{\dot{M}(\tau)\} = \infty$, we have $\lim_{\tau\to\infty}\{\dot{t}_\alpha(\tau)\} = \infty$ for all $\alpha > 0$, implies that there exists $\alpha \in (0,1)$ such that $\sigma(\tau) < t_\alpha(\tau)$ for any large enough τ, so that $\dot{\sigma}(\tau) < \alpha < 1$ for any such τ. But then σ would eventually leave $\mathcal{D}'$ through its lower boundary $t = \tau$, which is impossible. This contradiction establishes the claim. ■

We are now ready to complete the uniqueness argument. According to (117), we have

$$\dot{\sigma}(\tau) \geq \alpha(\tau) \equiv \frac{\dot{M}(\tau) - \lambda/r + 1 + \sqrt{[\dot{M}(\tau) - \lambda/r + 1]^2 + 4\lambda/r}}{2} \tag{119}$$

for all τ. Thus $\sigma(\tau) = t_{\dot{\sigma}(\tau)}(\tau) \geq t_{\alpha(\tau)}(\tau)$, that is, according to (114) and (119),

$$\sigma(\tau) - \tau \geq \left[\frac{\dot{M}(\tau) - \lambda/r + 1 + \sqrt{[\dot{M}(\tau) - \lambda/r + 1]^2 + 4\lambda/r}}{\dot{M}(\tau) + \lambda/r + 1 + \sqrt{[\dot{M}(\tau) - \lambda/r + 1]^2 + 4\lambda/r}}\right] M(\tau).$$

This in turn provides a lower bound for the ρ-concavity of $L(\cdot, \tau)$ at $m = \sigma(\tau) - \tau$,

$$\begin{aligned} 1 - \frac{LL_{11}}{L_1^2}(\sigma(\tau) - \tau, \tau) &= \frac{M(\tau)}{r[M(\tau) - \sigma(\tau) + \tau]^2} \\ &\geq \frac{r\left[\dot{M}(\tau) + \lambda/r + 1 + \sqrt{[\dot{M}(\tau) - \lambda/r + 1]^2 + 4\lambda/r}\right]^2}{4\lambda^2 M(\tau)}. \end{aligned} \tag{120}$$

Fix some $\varepsilon > 0$. Because M is convex with $\lim_{\tau\to\infty}\{\dot{M}(\tau)\} = \infty$, there exists τ_0 such that $r\dot{M}(\tau) \geq 4\lambda(1+\varepsilon)$ for all $\tau \geq \tau_0$. It then follows from (120) that for any such τ,

$$1 - \frac{LL_{11}}{L_1^2}(\sigma(\tau) - \tau, \tau) \geq \frac{(1+\varepsilon)\dot{M}(\tau)}{\lambda M(\tau)}. \tag{121}$$

Observe that τ_0 and the bound (121) are independent of the equilibrium under consideration. Now, consider two equilibria (σ_1, σ_1) and (σ_2, σ_2) with $\sigma_1(0) \geq \sigma_2(0)$ and define the function g as in (45). Given that $\dot{G}/(1-G) = \lambda$, proceeding as in (48) yields

$$\dot{g}(\tau) \geq \lambda \min_{m \in [\sigma_2(\tau) - \tau, \sigma_1(\tau) - \tau]} \left\{1 - \frac{LL_{11}}{L_1^2}(m, \tau)\right\} g(\tau) \geq \frac{(1+\varepsilon)\dot{M}(\tau)}{M(\tau)}\, g(\tau) \tag{122}$$

for all $\tau \geq \tau_0$, where the second inequality follows from (121), using the fact that each

$m \in [\sigma_2(\tau) - \tau, \sigma_1(\tau) - \tau]$ is such that $m = \sigma(\tau) - \tau$ for some equilibrium (σ, σ). Integrating (122) yields

$$g(\tau) \geq \left[\frac{M(\tau)}{M(\tau_0)} \right]^{1+\varepsilon} g(\tau_0)$$

and thus, by (47),

$$0 \leq g(\tau_0) \leq M(\tau_0)^{1+\varepsilon} M(\tau)^{-\varepsilon}$$

for all $\tau \geq \tau_0$. As $\lim_{\tau \to \infty} \{M(\tau)\} = \infty$ and $\varepsilon > 0$, this shows that $g(\tau_0) = 0$ and thus that $\sigma_1(\tau_0) = \sigma_2(\tau_0)$. From the uniqueness part of the Cauchy–Lipschitz theorem, we finally obtain that $\sigma_1 = \sigma_2$. Hence the equilibrium is unique.

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