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Robustness of equilibrium price dispersion in finite market games

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Abstract

We propose an approach to restricting the set of equilibria in a market game and use it to assess the robustness of the price dispersion results obtained by Koutsougeras [2003, J. Econ. Theory 108, 169–175] in the multiple trading posts setup. More precisely, we perturb the initial game by the introduction of transaction costs and our main results are the following. (i) No equilibrium with price dispersion of the game with costless transactions can be approached by equilibria with positive transaction costs as costs get arbitrarily small. (ii) When this type of perturbation is considered the set of equilibrium outcomes is not affected by the number of trading posts. In addition, the analysis hints at conditions required for non-zero transaction costs to serve as a source of price dispersion in this class of exchange economies.

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1. Introduction

General equilibrium theory provides an analysis of how markets work under perfect, price taking competition. Underlying perfect competition is the idea that agents take advantage of profitable exchange opportunities, leading in particular homogeneous goods to be traded at the same market price (the so-called 'law of one price'). Given the centrality of this "law" in economic theory, a large body of theoretical research has emerged to identify the conditions required for price uniformity, or symmetricaly, explaining persistent price dispersion. The literature has shown in particular that price uniformity may fail in the presence of information, or search costs (e.g., [35, 45, 8]). More recently Koutsougeras [23] showed that absent such frictions the mere existence of multiple trading-posts may give rise to equilibrium price dispersion. This stems from imperfect competition when traders internalize the price impact of their trade, preventing them to take advantage from arbitrage opportunities across markets.

One may ask whether this result implies that the law of one price cannot (reasonably) be justified—and thus imposed *a priori*—in organized but *imperfectly competitive* markets. To address this question, we offer a discussion of price uniformity (or lack thereoff) in economies where imperfect competition is the sole departure from the walrasian benchmark. Precisely, we consider the recent extension of the strategic market games approach initiated by ([40, 41]) to multiple markets ([6, 5, 23, 24]) and propose a natural robustness requirement for this class of trading games. We use this requirement to qualify the recent results about equilibrium price dispersion ([5, 23]) and eventually provide some support for the law of one price. The gist of our analysis is that the (possible) failure of the law of one price as well as related properties of the multiple posts setup are not immune to the introduction of arbitrarily small transaction costs.

Before explaining our approach further, it may be useful to give some intuition as to why a grain of transaction costs can restore price uniformity. In the original game with costless trade, agents can place 'wash-sales' trades, that is bids and offers that cancel each other *on a same trading post*. Although such trades matter for strategic equilibria because they affect the relative thickness of trading posts, any agent is indifferent as to the amount of his own wash-sales [see 32,31]. In other words, fixing others' strategies, an agent's allocation only depends on his net trades. With strictly positive transaction costs, agents also care about their gross trades, and never choose to be active on both sides of a given post. In the limit when transaction costs vanish, this leads to the selection—among the best response strategies—of the unique strategy minimizing gross trade, that is that without wash-sales. With multiple trading posts, wash-sales allow for equilibrium price dispersion as an agent's attempt to lower his bid on the expensive post may adversely impact his sales on this same post.[1] Hence, only equilibria satisfying

[1] This is related to Bloch and Ferrer [5, prop. 9] who show in a specific model that the law of one price holds when wash-sales are precluded. We discuss in more details our contribution compared to theirs in section 1.1.

the law of one price survive the introduction of small transaction costs.

Our robustness approach rests on the idea that the assumption of costless trade is a simplification. In particular, a property that would not hold—even approximately—once arbitrarily small transaction costs are explicitly modeled may be viewed as artificial. Following this line of reasoning, we perturb the initial game by the introduction of (a simple form of) transaction costs and state in our main result that no equilibrium with price dispersion of the game with costless transactions can be approached by equilibria with positive transaction costs as costs get arbitrarily small—in short, we say that such an equilibrium is not "robust". We also show that the set of robust outcomes—viz, those associated with robust equilibria—does not depend on the number of trading posts.

For tractability, we first derive our results under the assumption that transaction costs are linear in the amount of goods sent to the market. This specification follows [36], who introduce transaction costs in the bid-offer market game of [15]. While [36] focus on the issue of the existence of an equilibrium with active trade when transaction costs are not too prohibitive, we use transaction costs to define perturbed games and to reduce the set of equilibria. The assumption of proportional transaction costs is convenient and usual in the modeling of exchange economies,[2] and allows us to make our point in the most transparent way. Further, for the type of market games under study, it may be viewed as a property of some costs associated with monetary exchange, such as the opportunity cost of money holdings or the interest paid on borrowed money. One noteworthy implication of the linear specification is the law of one price holds even with non-zero transaction costs.

We consider more general specifications in section 4, and show that our main message extends to relatively general transaction cost functions, including some fixed cost specification. Formally, we identify conditions that are sufficient and tight to eliminate wash-sales trade and restore price uniformity in the limit. Essentially, what is required is that agents perceive a cost from simultaneously buying the same good in the same market place at the same price, which seems a reasonable characterization of many trade activities. Note however that, even though price uniformity obtains in the limit, our general specification leaves room for transaction costs themselves to induce price dispersion by affecting arbitrage strategies. Here, our analysis of the perturbed games suggests conditions that are necessary for transaction costs to serve as a separate source of dispersion. Namely, trading costs should not depend only on the total quantity traded per good, and should affect both sellers and buyers.

Most of our results are presented for the inside money case, in which agents are not constrained by initial money holdings. To emphasize the critical role of wash-sales we show in section 3 that our basic results also hold for the commodity money variant, which shows that binding liquidity constraints do not induce price dispersion

[2] See e.g. [21, 2] in general equilibrium theory, [11, 46] in financial economics, or Samuelson's "iceberg cost" specification in international trade theory [39, 26].

in *monetary* market games.[3] Further, to better clarify the connection between wash-sales and price dispersion we provide in section 4 an example showing that wash-sales on one side of the market (e.g. by net buyers only) are sufficient for price dispersion, and show that the elimination of wash-sales on only one trading post is sufficient to induce uniform prices. In that section, we also consider additional perturbations (such as ε-NE and complexity costs) and discuss how price dispersion in the limit relates to the relative importance of the different pertubations.

The rest of the paper is organized as follows. In the remainder of this section we discuss the related literature. In section 2 we introduce the framework and some definitions. Our main results are presented in section 3, under a linear transaction costs specification. The analysis is extended to more general transaction costs functions in section 4, while section 5 further discusses the generality of our robustness approach. Section 6 concludes. Some proofs are relegated to an appendix.

1.1. Relation to the literature

From a broad perspective, this paper contributes to the non-cooperative approach to market economies. One approach, which nicely captures aspects of relatively decentralized economies in which matching frictions are of primary importance, is to depict trade in markets between agents who meet randomly and agree on pairwise specific prices. This modelling strategy has been applied—among other things—to discuss the strategic foundations of the perfect competitive outcome in general exchange economies with a continuum of traders (see the monographs by [29, 17], and also [28, 27]). In contrast, we follow the strategic market game tradition where trade is mediated by independent and organized clearing markets.[4] As in [5, 23], agents can trade a given good on different trading posts, potentialy at different prices. In a related setting, [6] analyze the incentives of trading groups to form separate markets, under the restriction that no agent participates to more than one market.

With unrestricted participation, [25, 23] show that price dispersion can be sustained in equilibrium when the number of agents is finite. This is analyzed further in [24], who shows that price dispersion comes with real effects, and in [5], who show that generically prices are different on different trading posts. [24] further shows that the law of one price does not fail drastically as the number of agents goes to infinity, provided that no agent is constrained by his endowment. This set of results suggests that the canonical market game where agents are assumed to trade on a *single* market at a *single* price is not

[3]Such constraints can still generate (robust) price dispersion in the market game with multilateral trade of [1], as their example 1 (p. 139) does not involve wash-sales.

[4]We take the view that the diversity of non walrasian frameworks is partly a reflection of the diversity of real markets—some markets (e.g. stock markets or markets for agricultural goods) being more centralized and organized than others (e.g. markets for restaurant or some labor markets). The different approaches are thus complementary in increasing our understanding of the way markets work. More related to our point, these different frameworks may also lead to different notions of 'market power'.

theoretically justified, and should be replaced by the (more general) multiple trading post variant. In contrast we offer a justification for the one price assumption even with finite agents, and our irrelevance result suggests that there is no loss of generality in working with the single trading post market game.

In this literature, our work is mostly related to Bloch and Ferrer [5]. One contribution of our paper is to show that the failure of the law of one price identified in [25, 23] has its source not in market power *per se*, but in the fact that agents buy and sell a commodity on the same market and at the same price. Relatedly, [5] consider limited setup with two goods for which they can fully work with first order conditions, and derive the law of one price for the case of corner endowments and the Buy-or-Sell specification—that is, cases in which agents *cannot* play wash-sales. However, they impose strong assumptions on utility functions,[5] and implicitely restrict attention to equilibria with non binding endowments constraints—a restriction which in light of the examples in [1] and [24, section 4] is not innocuous for price dispersion. Compared to theirs, our contribution is twofold. Firstly, we neatly identify the connection between wash-sales and the failure of the law of one price and show that wash-sales are the critical source of price dispersion in any monetary (with commodity or inside money) market games. This holds true even if (some) agents have linear preferences, or in the case of binding endowments constraints. The former case is noteworthy, as the literature has shown that linear market games economies exhibit peculiar properties (e.g. [12]). Regarding the latter, we also derive our main results for the commodity money market game, in which agents' trading strategies are constrained by their money holding. Our results therefore show that binding liquidity constraints (in money or commodities) do not induce price dispersion in monetary market games. Our section 4 also contains additional results on the intimate connection between wash-sales and price dispersion. Second, we provide a rationale for the Buy-or-Sell specification assumed in [5]. This is more than a technical microfoundation as, from an abstract standpoint, one may argue that the usual, Buy-and-Sell model is more general in that it places fiewer restrictions on agents behavior.[6] To be sure, we are aware of no other attempt to rationalize the Buy-or-Sell version, with the exception of [30] who provides conditions for consumers to be active on one side of the market in a large market game with demand uncertainty.

We argue that 'good' equilibria in strategic market games should be robust to arbitrarily small transaction costs. By eliminating the indeterminacy of best replies associated with wash-sales, this argument cuts down one source of the multiplicity of Nash equilibria in market games.[7] Our approach differs from previous works on equilibrium

[5] [5] assume that for any agent h, the utility function u_h is strictly increasing and strictly concave, and satisfies the boundary conditions $\lim_{x^i \to 0} \frac{\partial u_h}{\partial x^i} = +\infty$, where x^i is h's consumption of good i.

[6] Shapley and Shubik [41] note that although situtations without wash-sales seem more 'realistic' it is not obvious how the restriction can be implemented with anonymous agents, since an agent could allways 'split' himself and disguise his trades. This type of behavior would not arise in our setting since with small transaction costs agents have strictly no incentives to buy and sell on a same post.

[7] See the formal analyses in [14, 31]. In the context of a market game with a continuum of players, [30] shows that the multiplicity of best responses disappears with demand uncertainty. The underlying

notions in strategic market games (e.g., [15, 12, 9, 47]), that focus exclusively on the problem of 'no trade' at some posts. Following [15], one popular approach has been to consider equilibria obtained as the limit of perturbed games in which some outside agency places vanishingly small bids and offers on each trading post. More recently, [16] use some strong notion of evolutionary stability to rule out equilibria in which some markets are inactive. Instead, our criterion applies to equilibria with active trade.

It is also instructive to relate this paper to the recent literature on complexity and dynamic bargaining games ([38, 19, 10]). In particular, Sabourian [38] and Gale and Sabourian [19] use complexity costs to select competivive outcomes in a class of decentralized markets with a finite number of agents. While the underlying reasons for multiplicity are clearly different, we also use small costs to select strategies, and equilibria, with attractive features. Besides, one may also argue that strategies with wash-sales might be pruned by complexity considerations. To be precise, consider two strategies that differ only in that the latter implies more market orders (say, a buy and a sell) than the former (say, only a buy order). Then, the latter might be viewed as more 'complex', and eliminated by small complexity costs. This (weak) ordering of strategies would suffice to obtain our results (see the discussion in section 4).

Lastly, our work is related to the literature explaining price dispersion in homogeneous markets. Following Stigler [44], many models have been introduced in which price dispersion arises because of limited information or costly consumer search. With *ex ante* identical agents, different prices can emerge in equilibrium in costly, noisy search environments where consumers may observe more than one price, as first shown by [8], and in directed search models where workers search on the job ([13]) or simultaneously apply to multiple firms ([20]). Price dispersion can also arise when information about prices is mediated by a central 'clearing house', if not all consumers ([45]), or firms ([3]), access the clearing house. These contributions emphasize the role of limited information or matching frictions.[8] In constrast, our research question is whether imperfect competition (in the sense of departure from price taking) *alone* can result in price dispersion. In a classical complete information, general-equilibrium model of imperfect competition, we show that this can only arise under extreme assumptions on transaction costs. This suggests that price dispersion is much more likely explained by limitation on information or rationality.

2. General setting and definitions

To fix ideas, we present our analysis for the multiple trading posts extension of the inside money market game of [32], and cover the commodity money case as an extension.

reason differs from ours, as 'wash-sales' cannot be defined in his setup with uncertainty.

[8] More recently, contributions have also emphasized the potential of bounded rationality [4, 43] or belief heterogeneity [34] in explaining dispersion.

We consider an exchange economy with a finite set $\mathcal{H}$ of agents, indexed by $h = 1, ..., H$, and $L+1$ goods, indexed by $i = 1, ..., L+1$, where good $L+1$ represents money. Each agent $h \in \mathcal{H}$ has endowment $e_h \in I\!R^L_+$, and preferences described by a utility function $u_h : I\!R^L_+ \longrightarrow I\!R_+$ defined over consumption bundles. We require u_h to be continuous, strictly increasing in the consumption of goods $1, \ldots, L$.[9] Good $L+1$ is an inside money with no direct utility. Agents have no initial money endowment, but can issue inside money at no cost. To avoid over issuance, it is postulated that an agent that goes bankrupt—that is whose (monetary) gains from sales do not cover his bids—has all his bids and offers confiscated (see [31] for a discussion).

Trade is organized as follows. For any commodity $i = 1, ..., L$, there are $K^i \geq 1$ trading posts where good i is exchanged for money (good $L+1$). Trading posts are indexed by (i, s), where $s = 1, ..., K^i$. We let $\mathcal{K}$ denote the set of trading posts in the economy, and $K = \sum_{i=1}^{L} K^i$ the aggregate number of trading posts.

An agent's strategy, σ_h, specifies for each trading post a non negative offer of commodity, $q_h^{i,s}$, and a non negative bid in term of money, $b_h^{i,s}$. Let S_h denote the strategy set of agent h, and $S = S_1 \times \ldots \times S_H$ the set of strategy profiles, with generic element $\sigma = (\sigma_h)_{h \in \mathcal{H}}$. To single out the strategy of a given agent h, we will sometimes write the strategy profile as (σ_h, σ_{-h}).

Given a strategy profile $\sigma \in S$, define

$$B^{i,s} = \sum_{\mathcal{H}} b_h^{i,s} \qquad \text{and} \qquad Q^{i,s} = \sum_{\mathcal{H}} q_h^{i,s},$$

and, for a given $h \in \mathcal{H}$,

$$B_h^{i,s} = \sum_{\mathcal{H}\backslash\{h\}} b_{h'}^{i,s} \qquad \text{and} \qquad Q_h^{i,s} = \sum_{\mathcal{H}\backslash\{h\}} q_{h'}^{i,s}.$$

On trading post (i, s), prices are formed according to the standard Shapley-Shubik rule:

$$p^{i,s} = \begin{cases} B^{i,s}/Q^{i,s} & \text{if } Q^{i,s} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Subsequently we use the convention $1/p^{i,s} = 0$ whenever $p^{i,s} = 0$.

We follow Rogawski and Shubik [36] in modeling transaction costs as consuming part of the commodities offered in transaction. Further, to state our results in the most transparent way, we first restrict ourselves to the following linear iceberg cost specification. When a quantity q of commodity $i \in \{1, ..., L\}$ is offered on post (i, s), an additional quantity $\eta \cdot q$ ($\eta \geq 0$) is needed in order to place this offer (a fraction of the good is 'lost in transaction'). We consider more general specifications in section 4. However, we view this simple specification as a natural starting point for two reasons. (i) Our formulation is in the spirit of the Shapley-Shubik approach in that it preserves

[9] The assumption that u_h is strictly increasing in *all* goods $1, \ldots, L$ is made to simplify the exposition. We can relax it to: "u_h is non decreasing in all its arguments (i.e. final consumption of commodities $1, \ldots, L+1$) and strictly increasing in at least one argument".

the independance of trading posts. (ii) It captures in the simplest way the idea that many costs incurred in transacting increase with the volume of trade. This is true for some direct costs associated with the distribution of goods, such as transportation, storage or packaging costs, but also—and more importantly—for costs associated with monetary exchange such as the opportunity cost of money holdings, or interest rate paid on borrowed money.

Given this specification, agent h chooses his strategy σ_h in the set[10]

$$S_h = \left\{ \left(b_h^{i,s}, q_h^{i,s} \right) \in I\!R_+^{2K} \quad | \quad \sum_{s=1}^{K_i} \left(q_h^{i,s} + \eta q_h^{i,s} \right) \leq e_h^i \right\},$$

and, given others' strategies, does not go bankrupt whenever

$$D_h \left(\sigma_h, \sigma_{-h} \right) := \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} - \sum_{i=1}^{L} \sum_{s=1}^{K_i} q_h^{i,s} p^{i,s} \leq 0. \tag{2}$$

Final allocations are then determined as follows:

$$x_h^i \left(\sigma \right) = \begin{cases} e_h^i - \sum_{s=1}^{K_i} \left(1 + \eta \right) q_h^{i,s} + \sum_{s=1}^{K_i} b_h^{i,s} / p^{i,s} & \text{if (2) holds,} \\ e_h^i - \sum_{s=1}^{K_i} \left(1 + \eta \right) q_h^{i,s} & \text{otherwise.} \end{cases} \tag{3}$$

It easily follows from (3) that (2) holds with equality at the optimum for h.

The case $\eta = 0$ corresponds to the game analysed in [23]. We will refer to this as the initial game, Γ, and to the game with $\eta > 0$ as Γ_η.

The rest of this section is devoted to the formal statement of the law of one price, and to our robustness requirement.

Definition 1. *A trading post (i, s) is active if $p^{i,s} > 0$, or equivalently $B^{i,s} > 0$ and $Q^{i,s} > 0$.*

Definition 2. *A strategy profile $\sigma \in S$ satisfies the law of one price (LOP) if it induces, for any good $i = 1, \ldots, L$, prices that are uniform across active trading-posts:*

$$\left(p^{i,s} p^{i,r} > 0 \Longrightarrow p^{i,s} = p^{i,r} \right) \qquad \forall\ i, r, s.$$

We say that a Nash equilibrium of the initial game is robust if it can be approached by equilibria of the perturbed games with strictly positive transaction costs as transaction costs vanish. Formally,

Definition 3. *A Nash equilibrium σ of the game Γ is "robust" if there exists a sequence $\{^n\eta, ^n\sigma\}_{n=1}^{\infty}$ where $^n\eta \in I\!R_+$ and $^n\sigma \, (\in S)$ is a NE of the perturbed games $\Gamma_{^n\eta}$ such that $\lim_{n\to\infty} {}^n\eta = 0$ and $\lim_{n\to\infty} {}^n\sigma = \sigma$.*

[10]To ease the exposition, we do not write S_h^η for the strategy set although it does depend on η. Note that strategy sets for any $\eta > 0$ are included into strategy sets for $\eta = 0$, so that any equilibria belongs to this larger set, $S_1^0 \times \cdots \times S_H^0$. No confusion should result.

Finally, we also introduce the set of strategies for which h is not simulatenously active on both sides of a given trading post:

$$\bar{S}_h := S_h \cap \left\{ \left(b_h^{i,s}, q_h^{i,s} \right) \in I\!R_+^{2K} \quad | \quad b_h^{i,s}.q_h^{i,s} = 0 \quad \forall (i,s) \in \mathcal{K} \right\}.$$

The corresponding set of strategy profiles is $\bar{S} := \bar{S}_1 \times \ldots \times \bar{S}_H$.

3. The linear case

This section presents our main results using the above linear iceberg cost specification. We analyse the game with strictly positive transaction costs in section 3.1, and present the robustness results in section 3.2. Section 3.3 shows that these results extend to the game with outside money.

3.1. The perturbed games

We start with one important intermediate result stating that when $\eta > 0$ it is never a best reply to buy and sell on the same trading post.

Lemma 1. *Let $\eta > 0$. Any individual best reply in Γ_η satisfies $\sigma_h \in \bar{S}_h$.*

Proof. Assume the contrary, viz that there exist a post (i,s) such that $b_h^{i,s} > 0$ and $q_h^{i,s} > 0$ for a candidate best reply σ_h. We construct a profitable deviation $\hat{\sigma}_h$ by subtracting a small amount of wash-sales (conveniently defined) on that post (i,s). Formally, for $\delta > 0$ consider the deviation with $\hat{q}_h^{i,s} = q_h^{i,s} - \delta$ and $\hat{b}_h^{i,s} = b_h^{i,s} - \delta p^{i,s}$. As $b_h^{i,s} \cdot q_h^{i,s} > 0$ and $\sigma_h \in S_h$, we can choose $\delta > 0$ (small enough) such that $\hat{\sigma}_h \in S_h$. Now, substituting $\hat{\sigma}_h$ for σ_h affects neither prices nor the (net) commodity bundle obtained through trading.[11] To see that $\hat{p}^{i,s} = p^{i,s}$ note that the initial price $p^{i,s}$ is defined by

$$p^{i,s} \left(Q_h^{i,s} + q_h^{i,s} \right) = B_h^{i,s} + b_h^{i,s}. \tag{4}$$

Substracting $\delta p^{i,s}$ from both sides and rearranging yields

$$p^{i,s} \left(Q_h^{i,s} + q_h^{i,s} - \delta \right) = B_h^{i,s} + b_h^{i,s} - \delta p^{i,s}, \tag{5}$$

and, eventually $p^{i,s} = \frac{B_h^{i,s} + \hat{b}_h^{i,s}}{Q_h^{i,s} + \hat{q}_h^{i,s}} = \hat{p}^{i,s}$. Further, the agent net trade is unaffected as

$$\frac{\hat{b}_h^{i,s}}{\hat{p}^{i,s}} - \hat{q}_h^{i,s} = \frac{b_h^{i,s} - \delta p^{i,s}}{p^{i,s}} - \left(q_h^{i,s} - \delta \right) = \frac{b_h^{i,s}}{p^{i,s}} - q_h^{i,s}. \tag{6}$$

Now, straightforward manipulations show that $D_h(\hat{\sigma}_h, \sigma_{-h}) = D_h(\sigma_h, \sigma_{-h})$, implying that $\hat{\sigma}_h$ satisfies (2) and is admissible. Furthermore, by playing $\hat{\sigma}_h$, agent h gets the same allocation $x_h^j(\hat{\sigma}_h, \sigma_{-h}) = x_h^j(\sigma_h, \sigma_{-h})$ for all goods $j \neq i$, and $x_h^i(\hat{\sigma}_h, \sigma_{-h}) = x_h^j(\sigma_h, \sigma_{-h}) + \eta\delta > x_h^j(\sigma_h, \sigma_{-h})$. The contradiction follows. □

[11] In that sense, $(b_h^{i,s} - \hat{b}_h^{i,s}, q_h^{i,s} - \hat{q}_h^{i,s})$ are 'wash-sales' trades.

When $\eta = 0$, the proof of lemma 1 states the standard result that agent h can obtain his preferred consumption bundle through a continuum of best reply strategies parameterized by the amount of wash-sales (see [32, 31]). By contrast, when $\eta > 0$, agents seek to minimize their gross trades and (endogenously) choose to be active on only one side of a given trading post.

The next result shows that the law of one price cannot be violated when transaction costs are positive.

Proposition 1. *Let $\eta > 0$. Any equilibrium of the market game Γ_η satisfies the law of one price.*

Proof. The proof proceeds in two steps. **Step 1.** We first show that in a candidate equilibrium of Γ_η with $p^{i,s} > p^{i,r} > 0$, it must hold that

$$\frac{B_h^{i,s}}{B_h^{i,s} + b_h^{i,s}} > \frac{B_h^{i,r}}{B_h^{i,r} + b_h^{i,r}} \qquad \forall h \in \mathcal{B}(i,s), \tag{7}$$

where $\mathcal{B}(i,s) := \left\{ h \in \mathcal{H} \quad | \quad b_h^{i,s} > 0 \right\}$ denotes the set of (equilibrium) bidders on post (i,s). The proof is by contradiction. Consider $h \in \mathcal{B}(i,s)$ such that

$$\frac{B_h^{i,s}}{B_h^{i,s} + b_h^{i,s}} \leq \frac{B_h^{i,r}}{B_h^{i,r} + b_h^{i,r}}. \tag{8}$$

First note that $q_h^{i,s} = 0$ by lemma 1. We consider a deviation shifting a small amount of money from (i,s) to (i,r). For $\eta > 0$, consider the deviation $\hat{\sigma}_h$ defined by the substitution of $\hat{b}_h^{i,s} = b_h^{i,s} - \eta$ and $\hat{b}_h^{i,r} = b_h^{i,r} + \eta$ for $b_h^{i,s}$ and $b_h^{i,r}$ in σ_h. This is well defined for $\eta > 0$ small enough, because $b_h^{i,s} > 0$. We first check that $\hat{\sigma}_h$ satisfies the no bankruptcy condition $D_h(\hat{\sigma}_h, \sigma_{-h}) \leq 0$. By the definition of equilibrium, we have that $D_h(\sigma_h, \sigma_{-h}) = 0$. Using $\hat{q}_h^{i,s} = q_h^{i,s} = 0$ and $\hat{q}_h^{i,r} = q_h^{i,r}$, straightforward manipulation then yields

$$D_h(\hat{\sigma}_h, \sigma_{-h}) = D_h(\sigma_h, \sigma_{-h}) + q_h^{i,r}\left(p^{i,r} - \hat{p}^{i,r}\right). \tag{9}$$

Now,

$$p^{i,r} - \hat{p}^{i,r} = \frac{B_h^{i,r} + b_h^{i,r}}{Q^{i,r}} - \frac{B_h^{i,r} + \hat{b}_h^{i,r}}{Q^{i,r}} = \frac{b_h^{i,r} - \hat{b}_h^{i,r}}{Q^{i,r}} = -\frac{\eta}{Q^{i,r}} < 0 \tag{10}$$

so that $D_h(\hat{\sigma}_h, \sigma_{-h}) < 0$. It remains to show that this admissible deviation (for η small enough) is indeed preferred by h. First note that final consumption for good $j \neq i$ remains unchanged. Define $\hat{x}_h^i(\eta) := x_h^j(\hat{\sigma}_h(\eta), \sigma_{-h})$. Clearly $\hat{x}_h^i(0) = x_h^i(\sigma_h, \sigma_{-h})$, the (putative) equilibrium consumption. Furthermore,

$$\begin{aligned} \tfrac{\mathrm{d}\hat{x}_h^i(\eta)}{d\eta}\Big|_{\eta=0^+} &= -\tfrac{\partial x_h^i(.)}{\partial b_h^{i,s}} + \tfrac{\partial x_h^i(.)}{\partial b_h^{i,r}} = -\tfrac{Q^{i,s}}{b_h^{i,s}+B_h^{i,s}} \tfrac{B_h^{i,s}}{B_h^{i,s}+b_h^{i,s}} + \tfrac{Q^{i,r}}{b_h^{i,r}+B_h^{i,r}} \tfrac{B_h^{i,r}}{B_h^{i,r}+b_h^{i,r}} \\ &= -\tfrac{B_h^{i,s}}{b_h^{i,s}+B_h^{i,s}} \tfrac{1}{p^{i,s}} + \tfrac{B_h^{i,r}}{b_h^{i,r}+B_h^{i,r}} \tfrac{1}{p^{i,r}}. \end{aligned}$$

It easily follows from $p^{i,s} > p^{i,r}$ and (8) that this is strictly positive, so that for η small enough $u_h(\hat{\sigma}_h, \sigma_{-h}) > u_h(\sigma_h, \sigma_{-h})$, contradicting the assumption that σ_h is a best reply. **Step 2.** Now, assume that an equilibrium of Γ_η violates the law of one price, that is there exist i, s and r such that $p^{i,s} > p^{i,r} > 0$. First note that, by (7), $b_h^{i,s} > 0$ implies $b_h^{i,r} > 0$, so that $\mathcal{B}(i,s) \subseteq \mathcal{B}(i,r)$. Now,

$$\begin{aligned} H-1 &= \sum_{\mathcal{H}} \tfrac{B_h^{i,s}}{B^{i,s}} = \sum_{\mathcal{B}(i,s)} \tfrac{B_h^{i,s}}{B^{i,s}} + \sum_{\mathcal{H}\backslash\mathcal{B}(i,s)} 1 \\ &> \sum_{\mathcal{B}(i,s)} \tfrac{B_h^{i,r}}{B^{i,r}} + \sum_{\mathcal{H}\backslash\mathcal{B}(i,s)} 1 \geq \sum_{\mathcal{B}(i,r)} \tfrac{B_h^{i,r}}{B^{i,r}} + \sum_{\mathcal{H}\backslash\mathcal{B}(i,r)} 1 = H-1. \end{aligned}$$

where the first inequality comes from (7). A contradiction. □

3.2. Robustness results

We are now ready to state our (non-)robustness results. Our first main result shows that the law of one price holds for any robust equilibrium.

Theorem 2. *Price dispersion is not a robust property. More precisely if an equilibrium of* Γ *features price dispersion, then it is not robust.*

Proof. The proof is by contradiction. Assume there exists a robust equilibrium ${}^*\sigma$ of Γ with dispersed prices. By failure of the LOP, there exist i, s and r such that

$$ {}^*p^{i,s} = \frac{{}^*B^{i,s}}{{}^*Q^{i,s}} >^* p^{i,r} = \frac{{}^*B^{i,r}}{{}^*Q^{i,r}} > 0. \tag{11}$$

By robustness, there exists a sequence $\{{}^n\sigma\}_{n=1}^{\infty}$ of equilibria of perturbed games with vanishing costs such that ${}^n\sigma \to^* \sigma$, implying in particular

$$\lim_{n\to\infty} \frac{{}^nB^{i,s}}{{}^nQ^{i,s}} = \frac{{}^*B^{i,s}}{{}^*Q^{i,s}} \quad \text{and} \quad \lim_{n\to\infty} \frac{{}^nB^{i,r}}{{}^nQ^{i,r}} = \frac{{}^*B^{i,r}}{{}^*Q^{i,r}}. \tag{12}$$

Now, proposition 1 implies that $\frac{{}^nB^{i,s}}{{}^nQ^{i,s}} = \frac{{}^nB^{i,r}}{{}^nQ^{i,r}} \quad \forall n$. By unicity of the limit, we have that $\lim_{n\to\infty} \frac{{}^nB^{i,s}}{{}^nQ^{i,s}} = \lim_{n\to\infty} \frac{{}^nB^{i,r}}{{}^nQ^{i,r}}$, which is in contradiction with (11). □

The example in [23] shows that price uniformity is not a (necessary) property of equilibria for the economy in the limit, Γ. In contrast, theorem 2 states that price uniformity does hold for the limit economy obtained as transaction costs vanish.[12]

We now turn to our second main result, stating that the set of robust equilibrium allocations—viz, allocations associated with robust equilibria—does not depend on the number of trading posts. This investigation is motivated by the analysis in [24] indicating that the set of equilibrium outcomes expands as the number of posts is increased.

Theorem 3. *Provided that* $K^i \geq 1$, *the number of trading posts is irrelevant for robust allocations.*

[12]See Gale [18] for the discussion of 'the economy in the limit' and 'the limit economy'.

Proof. See the appendix. □

Remark 1. *Proposition 5 and 6 in [24] assert the equivalence* for interior equilibria[13] *between equilibrium allocations of the one trading post setup and uniform prices allocations of the multiple trading post setup. Given theorem 2 above, this suggests theorem 3. Note however that this is not sufficient to show the result. First, robust equilibria are* not interior *(lemma 1). Secondly, we need to check that the robustness property is not affected by the number of trading posts.*

Finally, note that not all "uniform price" equilibria are robust. Intuitively, only those equilibria where agents do not place wash sales trades may be immune to our robustness requirement. The general implication of our robustness test on the structure of the set of equilibria of the single trading post setup is beyond the scope of this paper, though. Here, we simply state the following:

Proposition 4. *Let σ^* be a robust equilibrium. Then $\sigma^* \in \bar{S}$. Furthermore, there is no agent that buys and sells a same good (on different posts) at σ^*.*

Proof. We first show that $\sigma^* \in \bar{S}$. First note that only relative bids matter in equilibrium, so that we can consider w.l.o.g. equilibria with normalized bids, in the set

$$S^0 := \left\{ \left(b_h^{i,s}, q_h^{i,s}\right) \in I\!R_+^{2KH} \quad | \quad \sum_{s=1}^{K_i'} q_h^{i,s} \leq e_h^i \quad \forall i \; \forall h, \quad \sum_{h,i,s} b_h^{i,s} \leq 1 \right\}.$$

The set of normalized strategy profiles without wash sales, $\bar{S}^0 := S^0 \cap \bar{S}$, is a closed subset of $\bar{S}$. Now consider a converging sequence ${}^n\sigma \to \sigma$. Lemma 1 implies that ${}^n\sigma \in \bar{S}^0 \quad \forall n$. As $\bar{S}^0$ is closed, we have $\sigma \in \bar{S}^0$. We now show the second part. Assume that there exists $h \in \mathcal{H}$ and two active posts i, s for good i such that $b_h^{i,r} > 0$ and $q_h^{i,s} > 0$. By the robustness of σ^*, we have $b_h^{i,r} = q_h^{i,s} = 0$ and $p^{i,r} = p^{i,s}$. Now, let $x_h^+ \equiv \frac{b_h^{i,r}}{p^{i,r}} - q_h^{i,s}$ and assume that, say, $x_h^+ \geq 0$. (the opposite case is similar). The quantity x_h^+ is the net contribution of h's trades on these two trading posts to his equilibrium consumption. This net trade is covered by a quantity of money $m_h^+ \equiv p^{i,r} x_h^+ > 0$ obtained from trades in all remaining markets. Consider the strategy $\hat{\sigma}_h$ obtained from σ_h^* by setting $\hat{q}_h^{i,s} = 0$ and $\hat{b}_h^{i,r} = m_h^+$. This strategy is feasible, and leaves h's budget position unaltered since it only affects prices on posts (i, r) and (i, s). Now, $\hat{b}_h^{i,r} < b_h^{i,r}$ implies that $\hat{p}^{i,r} < p^{i,r}$, and thus $\hat{b}_h^{i,r}/\hat{p}^{i,r} > x_h^+$. Hence, $\hat{\sigma}_h$ implements extra consumption compared to the equilibrium strategy σ_h^*. A contradiction. □

3.3. The market game with commodity money

In this section, we show that our results extend to the multiple post extension of the market game with commodity money of Dubey and Shubik [15]. There are two

[13] An equilibrium is interior if all traders submit strictly positive bids and offers on all posts.

reasons to this analysis. First, equilibrium price dispersion can arise in this setup too ([25]). Secondly, given that the distinctive feature of this framework lies in the presence of (money) liquidity constraints, one might expect price dispersion to obtain under weaker conditions. We show that this is not the case.

Let Γ' denote the game with commodity money and no transaction costs, and Γ'_η denote the game with a given $\eta > 0$. We only describe the differences with the inside money case. Good $L+1$ is an outside commodity money that may enter utility. Any agent $h \in \mathcal{H}$ has an initial money endowment, $e_h^{L+1} \geq 0$, that limits his bidding capacity. Agent h's strategy space is thus:

$$S_h = \left\{ \left(b_h^{i,s}, q_h^{i,s} \right) \in I\!R_+^{2K} \quad | \quad \sum_{s=1}^{K_i} (1+\eta)\, q_h^{i,s} \leq e_h^i, \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} \leq e_h^{L+1} \right\}.$$

Final allocations are determined for any commodity $i \in \{1, .., L\}$ by

$$x_h^i (\sigma) = e_h^i - (1+\eta) \sum_{s=1}^{K_i} q_h^{i,s} + \sum_{s=1}^{K_i} b_h^{i,s} / p^{i,s}, \tag{13}$$

and, for the $L+1^{\text{th}}$ commodity (money), by:

$$x_h^{L+1} (\sigma) = e_h^{L+1} - \sum_{i=1}^{L} \sum_{s=1}^{K_i} b_h^{i,s} + \sum_{i=1}^{L} \sum_{s=1}^{K_i} q_h^{i,s} p^{i,s}. \tag{14}$$

To show that our results extend in a straightforward way to Γ', it is useful to rewrite final money holdings (14) as

$$x_h^{L+1} (\sigma_h, \sigma_{-h}) = e_h^{L+1} - D_h (\sigma_h, \sigma_{-h}), \tag{15}$$

with $D_h (\sigma_h, \sigma_{-h})$ introduced in (2). The main building blocks of the previous analysis can then be easily adapted:

Lemma 2. *Let $\eta > 0$. Any individual best reply in Γ'_η satisfies $\sigma_h \in \bar{S}_h$.*

Proof. Consider the deviation in the proof of lemma 1. It satisfies the liquidity constraint because it reduces the agent's aggregate bids. Further, the consumption of money is unaffected, $\hat{x}_h^{L+1} = x_h^{L+1} (\sigma_h, \sigma_{-h})$ because $D_h (\hat{\sigma}_h, \sigma_{-h}) = D_h (\sigma_h, \sigma_{-h})$ by the definition of wash-sales. The proof follows. □

Proposition 5. *Let $\eta > 0$. Any equilibrium of Γ'_η satisfies the law of one price.*

Proof. We simply need to check step 1 in the proof of proposition 1, that is to show that condition (7) holds $\forall h \in \mathcal{B}(i,s)$ for a candidate equilibrium in which $p^{i,s} > p^{i,r} > 0$. The deviation $\hat{\sigma}_h$ in the proof of proposition 1 amounts to shifting a small quantity of money from one post to another. Thus, $\hat{\sigma}_h$ satisfies the liquidity constraint because the amount of bids is unchanged. Further, $x_h^{L+1} (\hat{\sigma}_h, \sigma_{-h}) - x_h^{L+1} (\sigma_h, \sigma_{-h}) = D_h (\sigma_h, \sigma_{-h}) - D_h (\hat{\sigma}_h, \sigma_{-h}) \geq 0$ by (9) and (10). Step 1 and the proof follows. □

Other proofs are (almost) unaffected. In particular, we have:

Theorem 6. *If an equilibrium of Γ' features price dispersion, then it is not robust.*

Theorem 7. *The set of robust allocations for Γ' is independent of the number of trading posts (provided that $K^i \geq 1$).*

The fact that we obtain similar results for the market game with money liquidity constraints of Dubey and Shubik [15] and for that with perfect costless inside money of Postlewaite and Schmeidler [32] suggests the following observations. First, the price dispersion results illustrated in [23] for the latter framework and in [25] for the former have the very same source (namely, wash sales). Secondly, the existence of money liquidity constraints *per se* does not induce price dispersion in this framework. The intuition for this hinges on the assumption that there is a unique means of transaction (money). Hence, although the trading structure allows for one good to be purchased or sold on different locations—and potentially at different prices—there is only one way to transact. This amounts to assuming an upper bound on the degree of price inconsistency. It is is worth mentioning that the price dispersion result of Amir et al. [1], which is of a different nature, should not be affected by our perturbations.

4. Generalized transaction costs

We now generalize the specification of transaction costs and identify tight sufficient conditions for our main result to hold. In particular, we show that our analysis can incorporate some form of non convexities, and discuss fixed costs specifications. Further, we study the extent to which strictly positive transaction costs may lead to equilibrium price dispersion in this setup.

Our initial specification exhibits several features. *(i)* Transaction costs are paid in commodities—some goods are simply 'lost in transaction'. *(ii)* Costs are only paid by sellers and, *(iii)* they are linear. Note that these assumptions embed two conceptually distinct issues: how transaction costs are paid (commodity, money or 'time'), and how they depend on the agent's gross position (bid and offer) on the market.

Assumption *(i)* is made to simplify the exposition. In particular, we could consider the alternative—and seemingly more natural—case of monetary costs, but this would require the introduction of additional details that would only obscure our point.[14] To discuss the importance of *(ii)* and *(iii)*, we assume in this section that transaction costs are paid in utility terms. This is harmless, given our focus on arbitrarily small costs. In line with the market game approach, we maintain the assumption that costs are independent between trading posts. Hence, we consider a general function $c : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ such that $c(b, q) \geq 0$ represents the (normalized) transaction cost borne

[14] E.g., for the inside money case, if part of the transaction costs is paid in money, one has to introduce a modelling device to redistribute (directly or through the market) the monetary cost so that agents do not go bankrupt in equilibrium. This is not needed for the commodity money case.

when posting (b, q) on a given trading post. Given the interpretation, it is natural to assume that no cost is incurred when the agent does not participate to the trading post $(c(0,0) = 0)$, and to restrict attention to non decreasing functions.[15] Also given the focus on negligible transaction costs, we assume that c takes finite values in the sense that $c(b,q) < \infty$ if $b < \infty$ and $q < \infty$. We denote by $\mathcal{C}$ the set of such transaction costs functions. To make explicit the dependence on c in the limiting argument, we refer in this section to 'c-robust' equilibria, and use the notation $\Gamma_{\eta c}$ for the perturbed games.

For general costs functions, the key property required to single out strategies without wash sales is:

Assumption (A1). *$\forall (b,q) > (0,0)$, $\forall (b',q') \neq (0,0)$ with $(b',q') < (b,q)$ it holds that $c(b,q) > c(0,q')$ and $c(b,q) > c(b',0)$.*

Lemma 3. *Fix $c \in \mathcal{C}$. If assumption (A1) holds, then $\forall \eta > 0$ any individual best reply in $\Gamma_{\eta c}$ satisfies $b_h^{i,s} \cdot q_h^{i,s} = 0$.*

Proof. 3 Assume the contrary, that is $\exists \sigma \in \mathrm{NE}(\Gamma_{\eta c})$ such that $\exists h, i, k$ with $b_h^{i,k}.q_h^{i,k} > 0$. We distinguish two cases, depending on whether $b_h^{i,k} - p^{i,k} q_h^{i,k} \geq 0$ or $b_h^{i,k} - p^{i,k} q_h^{i,k} < 0$. We consider only the former case (the reasoning for the latter case is analogous). Given that h is a net buyer on post (i,k), he can obtain the same allocation by altering his strategy on this post such that $\left(b_h'^{i,k}, q_h'^{i,k}\right) = \left(b_h^{i,k} - p^{i,k} q_h^{i,k}, 0\right)$. The transaction cost on post (i,k) associated with this strategy is $c\left(b_h^{i,k} - p^{i,k} q_h^{i,k}, 0\right)$. By optimality σ_h' cannot be preferred to σ^h, wich in turn implies that $c\left(b_h^{i,k} - p^{i,k} q_h^{i,k}, 0\right) = c\left(b_h^{i,k}, q_h^{i,k}\right)$. Now, by assumption 4 for $b = b_h^{i,k}$, $q = q_h^{i,k}$ and $b' = b_h^{i,k} - p^{i,k} q_h^{i,k}$ it must hold that $c\left(b_h^{i,k}, q_h^{i,k}\right) > c\left(b_h^{i,k} - p^{i,k} q_h^{i,k}, 0\right)$. A contradiction. □

Assumption (A1) is in our view a weak condition, and is satisfied by many usual specifications (we discuss some of them below). For the moment, let us simply mention that it is satisfied if c is strictly increasing in at least one argument, or if agents incur a fixed cost on each side of a trading post.

4.1. Robust equilibria

We now study the robust properties of Γ under (A1). We first show that our main result easily extends to transaction costs function satisfying this property.

Proposition 8. *Consider a transaction cost function $c \in \mathcal{C}$ that satisfies (A1). Then any c-robust equilibrium of Γ satisfies the law of one price.*

Proof. Let ${}^*\sigma$ be a c-robust NE of Γ. Then there exists a sequence $\{{}^n\eta, {}^n\sigma\}_{n=1}^{\infty}$ such that ${}^n\eta \to 0$, ${}^n\sigma \to {}^*\sigma$, ${}^n\eta > 0 \quad \forall n \in \mathbb{N}$ and ${}^n\sigma$ is a NE of the perturbed game $\Gamma_{{}^n\eta c}$. By

[15]Formally, whenever $(q', b') > (q, b)$ (where '>' is understood component-wise), we have $c(q', b) \geq c(q, b)$ and $c(q', b) \geq c(q, b)$.

lemma 3 we have that ${}^n b_h^{i,s} \cdot {}^n q_h^{i,s} = 0 \quad \forall n \in \mathbb{N}, \forall i, s, h$. This implies that ${}^* b_h^{i,s} \cdot {}^* q_h^{i,s} = 0$ $\forall i, s, h$, because ${}^n b_h^{i,s} \cdot {}^n q_h^{i,s} \to {}^* b_h^{i,s} \cdot {}^* q_h^{i,s}$. Now, the argument in the proof of proposition 1 can be used to show that ${}^*\sigma$ must satisfy the LOP. □

The next result states a converse to proposition 8, and shows that assumption (A1) characterizes exactly the set of (non decreasing) transaction costs specifications that restore price uniformity in any monetary market game.

Proposition 9. *Assume that $c \in \mathcal{C}$ violates (A1). Then one can construct an economy modelled as a market game admitting a c-robust equilibrium with price dispersion.*

Proof. We give a sketch of the proof. Violation of assumption (A1) implies either that $\exists q > 0$, $b > b' > 0$ with $c(b, q) = c(b', 0)$ or that $\exists b > 0$, $q > q' > 0$ with $c(b, q) = c(0, q')$. Both cases are symmetric; we consider the former. Let $\gamma = c(b, q)$ and $F = [b', b] \times [0, q]$. By the monotonicity of c, $c(b'', q'') = \gamma \ \forall (b'', q'') \in F$. The gist of the proof consists in exploiting this fact to construct (a market game and) a dispersed price equilibrium that cannot be 'destroyed' by removing an agent's wash-sales. To simplify the proof, we assume that $c(b'', q'') = 0$ for any trade that is strictly lower than a trade in F, that is $c(b'', q'') = 0 \ \forall (b'', q'') \in G = [0, b') \times [0, q]$.[16] The structure (agents, markets and type of trades) of the example is the same as that of example 1 below. Let $F' = \left[b' + \frac{b-b'}{4}, b - \frac{b-b'}{4}\right] \times \left[0, \frac{q}{2}\right]$. We proceed in three steps. *Step (i).* We first claim that we can construct strategies satisfying the FOCs (18) such that $\left(b_h^k, q_h^k\right) \in F'$ and $\left(b_h^k - p^k q_h^k, 0\right) \in F'$ for any trades conducted by agents α and β, and $\left(b_h^k, q_h^k\right) \in G$ for any trades conducted by agents 0 and 1. Indeed, for any $\delta > 0$ we can find strategy profiles satisfying the relevant equations and implying price dispersion with wash-sales "as small as required" in the sense that $0 < \frac{q_h^k}{Q^k} < \delta$ for $h = \alpha, \beta$.[17] This class of examples is such that as that $\delta \to 0$, (a) $\frac{p^s}{p^r} \to 1$ (price dispersion vanishes), (b) $\frac{q_1^r}{Q^r}$ and $\frac{b_1^s}{B^s} \to 0$ (agent 1 vanishes) and, (c) $\frac{b_\alpha^r}{B^r}$, $\frac{b_\alpha^s}{B^s}$, $\frac{b_\beta^r}{B^r}$ and $\frac{b_\beta^s}{B^s} \to \frac{1}{2}$. Now, inspection of Eq. (18) shows that for any $\left(\lambda^r, \lambda^s, \lambda^b, \lambda^q\right) \in \mathbb{R}_{++}^4$, the strategy profile defined by $\left(\lambda^r \lambda^b b_h^r, \lambda^r \lambda^q q_h^r, \lambda^s \lambda^b b_h^s, \lambda^s \lambda^q q_h^s\right)$ also satisfies the FOCs. Using these 4 degrees of freedom, and choosing $\delta > 0$ small enough, one can find σ^* satisfying the FOCs such that any trade by agents 0 and 1 are in G, any trade by agents α and β are in F', and the equivalent trades without wash-sales, $(b_\alpha^r - p^r q_\alpha^r, 0)$ and $\left(b_\beta^s - p^s q_\beta^s, 0\right)$ are also in F'. (To see this, note that one can choose λ^s and λ^r s.t. $B^s = B^r = \frac{b'+b}{2}$, and then pick λ^q such that Q^s and $Q^r \leq \frac{q}{2}$. This normalization delivers the required condition for actual trades for δ small enough; that the equivalent no wash sales trades for α and β are also in F' follows from the fact that wash-sales become as small as required). For any h, we denote by $\bar{\sigma}_h^* \in \bar{S}_h$ the strategy obtained by removing wash-sales from

[16] The rationale for this simplification is that this amounts to allowing for the maximum economies on transaction costs that can be realised with trades below F, which should enlarge the set of potential profitable deviations when transaction costs are positive.

[17] We have constructed a Mathematica spreadsheet that gives us such a strategy profile with price dispersion for any small δ.

σ_h^*. *Step (ii).* Given σ^*, one can easily find utility functions and endowments such that all FOCs (18) are satisfied, and no endowment constraint binds. Further, the utility functions can be taken to be strictly increasing and concave. *Step (iii).* To conclude, we claim that σ^* is indeed c-robust. Assume the contrary. This implies in particular that there exists $1 > \bar{\eta} > 0$ such that for any $0 < \eta < \bar{\eta}$, σ^* is not a NE of $\Gamma_{\eta c}$ (Otherwise, one could use $\{^n\sigma\} \equiv \{\sigma^*\}$ to approach σ^*). Consider the sequence $\{\eta_n\} \equiv \{(\bar{\eta})^n\}$. Then for any $n > 1$, there exists $h \in \mathcal{H}$ (h may depend on n) with a deviation $^n\hat{\sigma}_h$ s. t.

$$\eta_n \left\{TC\left(\sigma_h^*\right) - TC\left(^n\hat{\sigma}_h\right)\right\} > u_h\left(\sigma_h^*, \sigma_{-h}^*\right) - u_h\left(^n\hat{\sigma}_h, \sigma_{-h}^*\right) \geq 0, \tag{16}$$

where $TC\left(\sigma_h\right)$ stands for the total transaction cost of σ_h. W.l.o.g. we can take the deviations to be without wash-sales, that is $^n\hat{\sigma}_h \in \bar{S}_h$. Using the finiteness of $\mathcal{H}$ and the compacteness of $\bar{S}_h$, one can find an invariant agent h' and extract a subsequence $\{^{n'}\hat{\sigma}_{h'}\}_{n' \in I\!N}$ that converges to a limit $\hat{\sigma}_h^* \in \bar{S}_h$. It follows from (16) that $u_{h'}\left(\hat{\sigma}_h^*, \sigma_{-h}^*\right) = u_{h'}\left(\sigma_h^*, \sigma_{-h}^*\right)$, viz $\hat{\sigma}_h^* \in \bar{S}_h$ is a best reply to σ_{-h}^*; the unicity result derived in lemma 5 (appendix B) then implies that $\hat{\sigma}_h^* = \bar{\sigma}_h^*$. The properties of σ_h^* and $\bar{\sigma}_h^*$ therefore imply that if $h' \in \{\alpha, \beta\}$, $TC\left(^{n'}\hat{\sigma}_{h'}\right) = \gamma = TC\left(\sigma_{h'}^*\right)$ for n' large. But then, the strict inequality in (16) cannot be satisfied along the sequence $\{^{n'}\hat{\sigma}_{h'}\}_{n' \in I\!N}$, a contradiction. A similar contradiction results if $h' \in \{0, 1\}$ under our simplifying assumption on the behavior of c in G.[18] □

As an illustration, it is useful to apply those results to the special but important class of fixed cost specification.[19] Formally, consider $c \in \mathcal{C}$ such that

$$c\left(b, q\right) = \gamma_{bq} \cdot \mathbf{1}_{\{(q,b)>0\}} + \gamma_b \cdot \mathbf{1}_{\{b>0\}} + \gamma_q \cdot \mathbf{1}_{\{q>0\}}, \tag{17}$$

with $\mathbf{1}_X$ the indicator function of X, and $\left(\gamma_{bq}, \gamma_b, \gamma_q\right) \in \mathbb{R}_+^3$. In that case, assumption (A1) amounts to $\gamma_b > 0$ and $\gamma_q > 0$. Proposition 8 then implies that, under a fixed cost specification our result hold as one would expect when agents face a cost to initiate trade on each side of the market:

Corollary 10. *Let c be of the fixed cost form (17) with $\gamma_b > 0$ and $\gamma_q > 0$. Then any c-robust equilibrium of Γ satisfies the LOP.*

On the other hand, a 'participation cost' specification, in which agents simply pay a cost to participate to the market ($\gamma_b = \gamma_q = 0$), does not eliminate price dispersion. Indeed, the 'participation cost' case corresponds to the most extreme violation of (A1),

[18] This simplification allows us to use σ^* as a candidate equilibrium when $\eta > 0$ is sufficiently small. More generally, σ^* would be approached by a sequence with $^n\sigma^* \neq \sigma^*$. For a fixed c, and given utility functions derived in step (ii), the relevant sequence can be numerically computed (as a function of η) if the behavior of c in G is sufficiently regular (say, satisfying the property that the set of points in $[0, b']$ where $c\left(., 0\right)$ is not derivable, and the set of points in $[0, q]$ where $c\left(0, .\right)$ is not derivable, are finite).

[19] We thank the associate editor and a referee for suggesting this discussion.

and is the only specification that virtualy does not eliminate (in the limit) any equilibrium of any market game.[20] Finally, one may ask what happens for the intermediate case in which agents face a fixed cost on only one side of the market ($\gamma_b > 0 = \gamma_q$ or $\gamma_q > 0 = \gamma_b$). Such a specification will eliminate some dispersed prices equilibria of some market games. For instance, this would eliminate the dispersed prices example in [5] and [23]. However, as example 1 in section 5.1 makes clear, it will not eliminate price dispersion for all market games. One conclusion to draw from this discussion is that it is sufficient for our results that there is a cost associated with each 'side' of the market. This seems (to us) a reasonable characterization of many trade activities. For instance, in financial markets, there is typically a positive cost—e.g., time cost or communication cost—associated with each order. Even with search costs, a fixed cost may be incurred to carry on two transactions.

To conclude, we discuss the generality of theorem 3. While the proof provided in appendix A hinges on the linearity assumption—essentially to construct the required sequence of NE for any trading structure once such a sequence is given for one trading structure—we conjecture that the irrelevance of the number of posts remains valid under general specifications satisfying (A1). This is based on the observation that any (c-)robust equilibrium for a given number of posts is a NE without wash-sales and as such can be associated with a NE without wash-sales of the associated game with a different number of posts.[21] However, there is no straightforward way to prove this conjecture (especially without imposing strong restrictions on the fundamentals of the economy). Here, we extend theorem 3 to a weakened form of the linear specification, and to the 'two sided' fixed costs specification:

Theorem 11. *Assume one of the following cases*

1. *Total costs have the 'commodity-wise linear' form* $\Sigma_{i\in\mathcal{I}}C^i\left(\Sigma_k q^{i,k}, \Sigma_k b^{i,k}\right)$, *where* $\forall i$, $C^i\left(.,.\right)$ *is non decreasing, and strictly increasing in (at least) one argument.*
2. *All* u_h *are concave, and transaction costs have the fixed cost specification (17) with* $\gamma_b.\gamma_q > 0$.

Then the number of trading posts is irrelevant for robust allocations.

Proof. Case 1 follows from the proof of theorem 3 since that proof only uses deviations for which the total bids and offers for a given commodity (and hence costs) are constant and the fact that robust equilibria satisfy the LOP. Case 2 is a corrolary of the general claim, established in appendix B, that under this specification any $\sigma \in NE\left(\Gamma\right)$ without wash-sales is c-robust. □

[20] To be precise, one can easily show that if $c \in \mathcal{C}$ does not eliminate any NE of any economy, then $c\left(b,q\right) = c\left(0,0\right)\cdot\mathbf{1}_{\{(q,b)>0\}}$. Conversely, for any $c \equiv \gamma\cdot\mathbf{1}_{\{(q,b)>0\}}$, if all u_h are concave it is a straightforward exercise to show that if a NE σ is such $u_h\left(\sigma\right) > u_h\left(e_h\right)$,$\forall h$, then σ is c-robust.

[21] There is no clear reason why the robustness property should be affected by the number of posts. Indeed, using simple examples with convex and concave costs, we have not been able to provide a counterexample to this conjecture.

The specifications covered by theorem 11 are reasonable, though more restrictive than those satisfying (A1). In particular, case 1 covers some type of commodity-specific transport costs, or an interest rate that varies with the total amount of money borrowed. On the other hand, as noted above, case 2 would be relevant for trade activities in which a fixed cost is borne for each transaction.

4.2. On transaction costs and price dispersion (the perturbed games)

Propositions 8 and 9 show that (A1) is sufficient and tight for price uniformity to prevail as transaction costs vanish. For the linear case, proposition 1 stated a stronger (and potentially surprising) result: the law of one price holds even for non negligible costs. This specific result does not extend to general transaction costs functions. In brief, for a fixed $c \in \mathcal{C}$ satisfying (A1), agents may have no interest in arbitraging price differences because of the transaction costs component of doing so. Hence, our analysis is not in conflict with the conventional wisdom that transaction costs are a major source of price dispersion.

This section provides a short discussion of this type of price dispersion. Precisely, we present standard specifications that are inconsistent with price dispersion even when costs are strictly positive. The results suggest that to serve as a source of price dispersion, transaction costs must depend in a complicated way on the agent's position on *each* side of the market. Note that this possible failure of the law of one price will be driven merely by the presence of transaction costs and not by wash-sales. This implies in particular that the extent of (potential) price dispersion is directly linked to the magnitude of transaction costs, in the following sense:

Corollary 12. *Fix $c \in \mathcal{C}$ satisfying (A1). Let $\pi > 0$. Then $\exists \bar{\eta} > 0$ such that $\forall \eta < \bar{\eta}$, in any NE of $\Gamma_{\eta c}$ price dispersion is bounded above by π, that is*

$$\max_{\{s,r|p^{i,r}>0,p^{i,s}>0\}} \left(\frac{p^{i,s}}{p^{i,r}} - 1 \right) < \pi \qquad \forall i \in \{1, \ldots, L\}.$$

Proof. Follows from proposition 8. □

With this in mind, consider first the case in which the transaction cost function c depends only on one argument (either b or q). Condition (A1) then simply amounts to assuming that c is strictly increasing. For this case, the law of one price holds not only in the limit, but also for non negligible costs:

Proposition 13. *Assume that TCs are given by $c(q)$ or $c(b)$ where $c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a strictly increasing function. Then the LOP holds for any equilibrium of $\Gamma_{\eta c}$ ($\eta > 0$).*

Proof. Similar to that of proposition 1 □

Note that no restriction are needed on the shape of $c(.)$ aside from strict monotonicity. In particular, this is compatible with the usual assumption of convex costs, but also with the presence of non-convexities (return to scale) in the transaction technology.[22] Furthermore, transaction costs are not required to vary smoothly when the agent offer (or, bid) vary. Hence (upward) jumps, or kinks in the cost function are allowed.

Consider now the case in which both buyers and sellers incur a transaction cost. Proposition 1 can then be extended for the commodity-wise linear form and for the two-sided fixed cost specification:

Proposition 14. *Assume that transaction costs have either the commodity-wise linear form of theorem 11, or the fixed cost specification (17) with* $\gamma_b.\gamma_q > 0$*. Then the LOP holds for any equilibrium of* $\Gamma_{\eta c}$ *(*$\eta > 0$*).*

Proof. Direct adaptation of the proof of proposition 1. □

Intuitively, those cases do not allow for dispersed prices because they have the property that marginal transaction costs among the trading posts where one trades a given commodity are necessarily equal. In contrast, two-sided transaction costs (satisfying (A1)) could lead to price dispersion in the presence of non linearities. In that respect, our emphasis on the robustness of the law of one price does not conflict with standard theories stressing the role of transaction costs in explaining price dispersion. Indeed, propositions 13 and 14 above yield the additional insight that, in the setup under study, price dispersion requires both buyers *and* sellers to incur some (non linear) costs in their trading activity.

5. Discussion

5.1. More on wash sales and price dispersion

The previous analysis underscores the important role of wash-sales. In this section we provide two additional results to further our understanding of the connection between (the structure of) wash-sales and (the possibility of) price dispersion. The first result shows that wash-sales on all trading posts are necessary for equilibrium price dispersion; the second result shows that wash-sales on one side (buying or selling side) are sufficient.

Proposition 15. *Assume that there exists* one active *trading post for good* i *(say,* (i, n)*) in which no agent simultaneously buy and sell. Then* $p^{i,k} = p^{i,n}$ for any active post *for good* i*.*

Proof. We distinguish two cases, depending on whether there exists $\exists s$ such that $p^{i,s} > p^{i,n} > 0$, or $\exists r$ such that $p^{i,n} > p^{i,r} > 0$. Consider first the latter case. Then, step 1 and 2 of theorem 1 apply (with (i, n) in place of (i, s)) as they only make use of the fact

[22] For instance a standard cost function covered by proposition 9 is $c(0) = 0$ and $c(q) = \gamma + C(q)$ for $q > 0$, where $\gamma > 0$ is a fixed cost and $C(q)$ is increasing and convex.

that there are no wash-sales on the post with the highest price. The result obtains for this case. Consider now the case in which $\exists s$ such that $p^{i,s} > p^{i,n} > 0$. By a reasoning similar but symmetric to that in step 1, one can show that

$$\frac{Q_h^{i,s}}{Q_h^{i,s} + q_h^{i,s}} < \frac{Q_h^{i,n}}{Q_h^{i,n} + q_h^{i,n}} \qquad \forall h \in \mathcal{Q}(i,n).$$

Also, one can easily verify that the argument in step 2, with offers in place of bids and (i,n) in place of (i,s), can be used to obtain a contradiction. The result follows. □

The above result shows that equilibrium price dispersion requires not only that (some) agents place wash sales on the trading posts with extreme prices, but also on any active post. In that precise sense, there must be 'a lot' of wash sales to support non uniform prices. Regarding the structure of transaction costs, one implication is that if positive transaction costs (satisfying (A1)) are incurred on any single post, the law of one price holds in the limit.

We now provide an example with dispersion in which wash-sales are placed on one side of the market (here, net buyers). Regarding our robustness approach, this shows that a specification in which one side but not the other incur a fixed cost to participate to the market will not eliminate price heterogeneity. The example may be of independent interest, as examples found elsewhere in the literature [e.g., 5, 23] involve wash-sales on both sides of each trading posts, and leave open the question of the necessity of this specific pattern.

Example 1. *The example involves two goods $\{1,2\}$, four agents $\mathcal{H} = \{0,1,\alpha,\beta\}$, and two trading posts/markets $\{r,s\}$. We consider a 'barter' market game, so that on each post each good can be exchanged against the other. An agent thus solves*

$$\begin{cases} \max_{\left(b_h^r, q_h^r, b_h^s, q_h^s\right) \in S_h} u\left(x_h^1, x_h^2\right) \\ s.t. \quad \begin{aligned} x_h^1 &= e_h^1 + \frac{q_h^r}{Q^r} B^r - b_h^r + \frac{q_h^s}{Q^s} B^s - b_h^s \\ x_h^2 &= e_h^2 + \frac{b_h^r}{B^r} Q^r - q_h^r + \frac{b_h^s}{B^s} Q^s - q_h^s \end{aligned} \end{cases}$$

where b_h^k (resp., q_h^k) is the quantity of good 1 (resp., good 2) put on post k. We further specify $u\left(x_h^1, x_h^2\right) = \ln x_h^1 + \ln x_h^2$ $\forall h \in \mathcal{H}$, and fix endowments

$$e_0 = \left(2, \frac{144}{35}\right), \quad e_1 = \left(\frac{267}{145}, \frac{16}{15}\right), \quad e_\alpha = \left(\frac{9237}{841}, \frac{26161}{5887}\right), \quad e_\beta = \left(\frac{29227}{8410}, \frac{20918}{12615}\right).$$

For this example, FOCs are sufficient and necessary for equilibrium (see for instance [5]). They are given by

$$\frac{\partial u_h}{\partial x_h^1} \Big/ \frac{\partial u_h}{\partial x_h^2} = \frac{B_h^k}{Q_h^k}\left(\frac{Q^k}{B^k}\right)^2 = \frac{B_h^k}{B^k}\frac{Q^k}{Q_h^k}\frac{1}{p^k} \qquad \forall k = r,s \quad \forall h \in \mathcal{H}. \tag{18}$$

We claim that the following strategy profile forms an equilibrium with $\frac{p^s}{p^r} = \frac{15}{14}$:

$$\sigma_0 = \left(0, \frac{5}{7}, 0, \frac{2}{5}\right), \qquad \sigma_1 = \left(\frac{7}{29}, 0, 0, \frac{1}{15}\right),$$
$$\sigma_\alpha = \left(\frac{5285}{841}, \frac{30}{7}, \frac{588}{841}, 0\right), \quad \sigma_\beta = \left(\frac{399}{841}, 0, \frac{1935}{841}, \frac{23}{15}\right).$$

This yields aggregate bids and offers $B^r = 7$, $Q^r = 5$, $B^s = 3$, $Q^s = 2$, prices $p^r = \frac{7}{5} < p^s = \frac{3}{2}$, and final allocations $\left(x_0^1, x_0^2\right) = \left(\frac{18}{5}, 3\right)$, $\left(x_1^1, x_1^2\right) = \left(\frac{17}{10}, \frac{34}{29}\right)$, $\left(x_\alpha^1, x_\alpha^2\right) = \left(10, \frac{4300}{841}\right)$ and $\left(x_\beta^1, x_\beta^2\right) = \left(3, \frac{1680}{841}\right)$. To see that this forms a NE, one can check that all first order conditions (18) are satisfied, and that no agent is constrained by his endowment in equilibrium (i.e. $b_h^r + b_h^s < e_h^1$ and $q_h^r + q_h^s < e_h^1$ for any $h \in \mathcal{H}$). In this equilibrium there is on each post one agent playing wash-sales: agent α on post r and agent β on post s. Both agents are net buyers (of good 2), as $b_\alpha^r - p^r q_\alpha^r = \frac{239}{841} > 0$ and $b_\beta^s - p^s q_\beta^s = \frac{7}{8410} > 0$.

5.2. More perturbations

Our robustness approach builds on one type of perturbation—transaction costs. One may ask how our conclusions would be affected when considering different or additional perturbations.[23] We provide some insights into this issue here. One should keep in mind though that the main point of the paper is not to discuss all types of perturbations, but to stress that price dispersion is not robust to one particular, economically motivated one.

One alternative approach is to weaken the (Nash) equilibrium concept, along the lines of Radner's [33] ε-Nash equilibrium, and see what happens when both perturbations vanish. In the context of our framework, the ε-NE can be motivated as one way to introduce some grain of bounded rationality. Formaly, define

Definition 4. *An ε-Nash equilibrium σ of the game Γ_η is a strategy profile such that no agent $h \in \mathcal{H}$ has a deviation $\hat{\sigma}_h \in S_h$ with $u_h(\hat{\sigma}_h, \sigma_{-h}) > u_h(\sigma_h, \sigma_{-h}) + \varepsilon$.*

Clearly, a robust equilibrium (in our sense) can still be attained as a limit of ε-NE. This is true irrespective of the relative importance of transaction costs and maximizing errors. The more relevant question is whether equilibria with wash-sales (and, potentialy, dispersed prices) can be supported in the limit.

For general transaction costs satisfying (A1), the type of equilibria sustained depends on the relative importance of transaction costs and maximizing errors as *both* perturbations become small. For instance, if one allows maximizing errors to fall sufficiently rapidly as transaction costs fall, only c-robust (without wash-sales) equilibria will survive. In constrast, if departure from perfect optimization are relatively

[23]We thank the associate editor for raising this relevant issue, and from suggesting the discussion of ε-Nash equilibria.

more significant than transaction costs, equilibria with wash-sales will also be sustained in the limit. To make the discussion more precise, fix a Nash equilibrium σ^* of a game Γ, and a transaction cost function c satisfying (A1). Here, we further assume to simplify that c is continuous. For an arbitrary strategy σ_h, let $C(\sigma_h)$ denote the *total* normalized cost of σ_h, that is $C(\sigma_h) = \sum_{i,k} c(b_h^{i,k}, q_h^{i,k})$, and define $\bar{C}_h(\sigma_h^*) = C(\sigma_h^*) - C(\bar{\sigma}_h^*)$, with $\bar{\sigma}_h^*$ the strategy without wash-sales associated with σ_h^*.[24] One can interpret $\bar{C}_h(\sigma_h^*)$ as a measure of the cost of wash-sales in σ_h^*. Note that by definition $\bar{C}_h(\sigma_h^*) \geq 0, \forall h \in \mathcal{H}$, and that (since (A1) holds) the equilibrium σ^* is without wash-sales if and only if $\max_{h\in\mathcal{H}}\{\bar{C}_h(\sigma_h^*)\} = 0$. We expect equilibria with more costly wash-sales—higher value of $\max_{h\in\mathcal{H}}\{\bar{C}_h(\sigma_h^*)\}$—to be less easily supported. To account for the presence of both perturbations, we use sequences $\{\varepsilon_n, \eta_n\}_{n\in I\!N}$ such that $\varepsilon_n, \eta_n > 0$, $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \eta_n = 0$, and $\lim_{n\to\infty} \varepsilon_n/\eta_n \in I\!R_+ \cup \{\infty\}$ exists. The next result provides some restrictions on how transaction costs and maximizing errors must behave for σ^* to be approximated by ε_n-NE of the perturbed games $\Gamma_{\eta_n c}$.

Proposition 16. *Let σ^* be a NE of Γ, and c a continuous transaction cost function satisfying (A1). Then (i) σ^** cannot *be approached by any sequence of ε_n-NE of the perturbed games $\Gamma_{\eta_n c}$ if $\lim_{n\to\infty} \varepsilon_n/\eta_n < \max_{h\in\mathcal{H}}\{\bar{C}_h(\sigma_h^*)\}$. (ii) σ^* is approachable by a sequence of ε_n-NE of the perturbed games $\Gamma_{\eta_n c}$ if $\lim_{n\to\infty} \varepsilon_n/\eta_n > \max_{h\in\mathcal{H}}\{C(\sigma_h^*)\}$.*

Proof. See the appendix. □

One implication of proposition 16 is that only c-robust equilibria survive when $\lim_{n\to\infty} \frac{n_\varepsilon}{n_\eta} = 0$, while any Nash equilibrium of Γ survives when $\lim_{n\to\infty} \frac{n_\varepsilon}{n_\eta}$ is sufficiently large.[25]

Another way to allow for bounded rationality, and to perturb the initial game, is to introduce complexity costs directly on strategies (see, e.g., [10]). In our setup, one could measure the complexity of a (market) strategy by the number of (market) orders required to implement it. Formally, let $\gamma(\sigma_h) = \gamma.\#(\sigma_h)$, where $\gamma > 0$, and $\#(\sigma_h)$ denotes the number of strictly positive elements in σ_h. If such complexity costs are the only perturbation, it is easily seen that wash-sales will be eliminated, and that the law of one price will be restored in the limit. (To see this, note that this case is formally equivalent to the fixed cost specification in (17) with $\gamma_{bq} = 0$ and $\gamma_b = \gamma_q = \gamma$). The same conclusion obtains if complexity costs and transaction costs are jointly considered, irrespective of the relative importance of both type of perturbations:

Proposition 17. *Fix $c \in \mathcal{C}$ satisfying (A1). Let σ^* be a NE of Γ that can be approximated by a sequence of NE of perturbed games with transaction costs and with*

[24] Formally, $\left(\hat{b}_h^{i,k}, \hat{q}_h^{i,k}\right) = \left(b_h^{i,k} - p^{i,k} q_h^{i,k}, 0\right)$ whenever $b_h^{i,k} \geq p^{i,k} q_h^{i,k}$ (net buyer on post (i,k)), and $\left(\hat{b}_h^{i,k}, \hat{q}_h^{i,k}\right) = \left(0, q_h^{i,k} - b_h^{i,k}/p^{i,k}\right)$ otherwise (net seller).

[25] This is not surprising. Our approach provides a way to *reduce* the set of potential equilibrium outcomes, by breaking indifference in the agents' best replies. The ϵ-NE, on the contrary, can potentialy *enlarge* the set of equilibria. As more weight is put on the latter than the former in the limit argument, more equilibria can be sustained.

γ_n-complexity costs. Then σ^* satisfies the LOP.

This shows that our approach is robust to some alternative or additional perturpations. Now, if in turn both complexity costs and maximizing errors are considered, the outcome depends on the relative importance of both type of pertubations. Note that the results obtained for this case are sharper than, but qualitatively similar to, those of proposition 16.

Proposition 18. *Let u_h be concave. (i) If $\lim_{n \longrightarrow \infty} \frac{\varepsilon_n}{\gamma_n} < 1$, no equilibrium with wash-sales is approachable by a sequence of ε_n-NE of the perturbed games with γ_n-complexity costs, while any equilibrium without wash-sales is approachable. (ii) If $\lim_{n \longrightarrow \infty} \frac{\varepsilon_n}{\gamma_n} > 1$, any NE in which any agent places wash-sales on at most (one side of) one market is approachable. (iii) More generally, any NE in which any agent places wash-sales on at most m $(\leq 2K)$ sides of trading posts is approachable if $\lim_{n \longrightarrow \infty} \frac{\varepsilon_n}{\gamma_n} > m$. (In particular, any NE is approachable if $\lim_{n \longrightarrow \infty} \frac{\varepsilon_n}{\gamma_n} > 2K$).*

Proof. See the appendix. □

Regarding the distribution of prices, one conclusion to draw from the above discussion is that the law of one price is more likely to hold when transaction (or, complexity) costs are more significant than departure from full optimizing behavior. On the other hand price dispersion may arise if maximizing errors are large relative to other type of perturbations.[26] Note however that equilibria without wash-sales—a subset of the equilibria satifying the law of on price—are robust *irrespective* of the relative importance of all the above pertubations.

More generally, the original market game setup can be perturbed in many directions. As suggested by the above discussion, wash-sales (and price dispersion) will be preserved by some type of perturbations, and eliminated by others.[27] Now, if one requires a candidate equilibrium to be robust to any admissible perturbation (e.g., in the spirit of the strategic stability of [22]), one may conclude that price dispersion is not a very robust property.

6. Concluding remarks

The present paper introduces a natural approach to restricting the set of equilibria in a market game, by requiring equilibria to be robust to arbitrarily small transaction costs. In the context of (two versions of) the multiple trading posts variant of the canonical market game, we show that price uniformity holds in any equilibrium satisfying the requirement. In short, the failure of the law of one price—emphasized by [25, 23]—is not very robust.

[26] For contributions who focus on bounded rationality to explain price dispersion, see [4, 43].

[27] One interesting and open research question is how the introduction of "trembles" would affect equilibrium strategies and outcomes.

Our results may be used in assessing the usefulness of the multiple trading posts variant, as opposed to the canonical, single trading post market game. In this respect, we suggest the following interpretation. In some sense, the multiple trading posts variant might be seen as a generalization of the canonical market game. However this generalization might be misleading, in that its unique impact is to give rise to unreasonable equilibria (in the precise sense that all "new" equilibria are killed by the introduction of arbitrarily small transaction costs). This suggests that, without additional assumptions, there is no loss of generality in working with the single trading post version in which the law of one price is posited.

At this point, it is worth stressing that our results do *not* diminish the importance of the contribution of [23, 24]. Rather, one possible conclusion of the current paper is that more research is needed for the multiple post market game to offer a convincing and separate explanation for price dispersion. We believe in particular that introducing market manipulation considerations in this framework may shed light on some deviations from the law of one price that arise in integrated and well organized markets and cannot easily be explained by informational or search frictions, such as the well documented persistent price divergence between identical, 'twin-securities' [37, 42].

More generally, the approach also has implications for the canonical market game. In essence, only equilibria in which agents do not place wash sales trades should be robust. The analysis of the structure of the set of robust equilibria is the subject of future research. An interesting result is that the indeterminacy result of Peck et al. [31], obtained for *interior* Nash equilibria, does not extend to robust equilibria.[28]

A. Proof of proposition 3

We first introduce additional notations to make explicit the dependence on the number of trading posts. To this end, denote $\mathbf{K} = \left(K^1, \ldots, K^L\right) \in \mathbb{N}_+^L$. Further define, in strategy space,

$$\begin{aligned} \mathrm{NE}_\eta(\mathbf{K}) &= \text{set of equilibria of } \Gamma_\eta^{\mathbf{K}}, \\ \mathrm{NE}(\mathbf{K}) &= \text{set of equilibria of } \Gamma^{\mathbf{K}}, \\ \mathrm{RE}(\mathbf{K}) &= \text{subset of } \mathrm{NE}(\mathbf{K}) \text{ that are robust}, \end{aligned}$$

and, in allocation space,

$$\begin{aligned} \mathrm{NA}_\eta(\mathbf{K}) &= \left\{\mathbf{x} \in \mathbb{R}_+^{LH} \mid \exists \sigma \in \mathrm{NE}_\eta(\mathbf{K}) \quad \mathbf{x} = \mathbf{x}(\sigma)\right\}, \\ \mathrm{NA}(\mathbf{K}) &= \left\{\mathbf{x} \in \mathbb{R}_+^{LH} \mid \exists \sigma \in \mathrm{NE}(\mathbf{K}) \quad \mathbf{x} = \mathbf{x}(\sigma)\right\}, \\ \mathrm{RA}(\mathbf{K}) &= \left\{\mathbf{x} \in \mathbb{R}_+^{LH} \mid \exists \sigma \in \mathrm{RE}(\mathbf{K}) \quad \mathbf{x} = \mathbf{x}(\sigma)\right\}, \end{aligned}$$

where $\mathbf{x}(\sigma) := \left(x_h^i(\sigma)\right)$. Proposition 3 then rewrites:

Proposition 19. *Let* $\mathbf{K}', \mathbf{K} \in \mathbb{N}_+^L$. *Then* $\mathrm{RA}(\mathbf{K}') = \mathrm{RA}(\mathbf{K})$.

[28] We investigate this, and the issue of existence of a robust equilibrium in a companion paper [7].

We first show the following intermediate result.

Lemma 4. *Let* $\eta > 0$ *and* $\mathbf{K}', \mathbf{K} \in I\!N_{++}^{L}$. *Then* $\mathrm{NA}_{\eta}(\mathbf{K}') = \mathrm{NA}_{\eta}(\mathbf{K})$.

Proof. (By induction and permutations), it is sufficient to prove the result for the case $\mathbf{K} = (K^1 - 1, K^2, \ldots, K^L)$ and $\mathbf{K}' = (K^1, \ldots, K^L)$. First note that $\mathrm{NA}_{\eta}(\mathbf{K}') \supseteq \mathrm{NA}_{\eta}(\mathbf{K})$ because one can always add an inactive post. Hence, we need to show that $\mathrm{NA}_{\eta}(\mathbf{K}') \subseteq \mathrm{NA}_{\eta}(\mathbf{K})$. Fix an allocation $\mathbf{x} \in \mathrm{NA}_{\eta}(\mathbf{K}')$, and let $\sigma' \in \mathrm{NE}_{\eta}(\mathbf{K}')$ be one equilibrium such that $\mathbf{x} = \mathbf{x}(\sigma')$. W.l.o.g. we assume that all $(1, s)$ posts are active at σ (otherwise, a mere permutation in the label of posts is sufficient). Consider the strategy profile $\sigma \in S(\mathbf{K})$ constructed from σ' by transferring all trades posted on post $(1, K_1)$ to post $(1, 1)$:

$$\left(b_h^{1,1}, q_h^{1,1}\right) = \left(b_h^{\prime 1,1} + b_h^{\prime 1,K_1}, q_h^{\prime 1,1} + q_h^{\prime 1,K_1}\right), \tag{19}$$

$$\left(b_h^{i,s}, q_h^{i,s}\right) = \left(b_h^{\prime i,s}, q_h^{\prime i,s}\right) \quad \forall h \quad \forall (i, s) \neq (1, 1). \tag{20}$$

We claim that σ is an equilibrium and that it implements $\mathbf{x}$. We proceed in two steps. **Step 1.** We first check that σ leads to the final allocation $\mathbf{x}$. First note that by theorem 1, σ' satisfies the LOP, so that

$$p^{\prime 1,1} = \frac{B^{\prime 1,1}}{Q^{\prime 1,1}} = \frac{B^{\prime 1,K_1}}{Q^{\prime 1,K_1}} = p^{\prime 1,K_1}. \tag{21}$$

Prices induced by σ are thus given by

$$p^{i,s} = \frac{B^{\prime i,s}}{Q^{\prime i,s}} = p^{\prime i,s} \quad \forall (i, s) \neq (1, 1), \tag{22}$$

$$p^{1,1} = \frac{B^{\prime 1,1} + B^{\prime 1,K_1}}{Q^{\prime 1,1} + Q^{\prime 1,K_1}} = p^{\prime 1,1} = p^{\prime 1,K_1}, \tag{23}$$

where the last equality follows from (21). One can easily check using (22) and (23) that the strategy profile σ satisfies all the relevant constraints, and that $\mathbf{x}(\sigma) = \mathbf{x}(\sigma')$.
Step 2. We now show that $\sigma \in \mathrm{NE}_{\eta}(\mathbf{K})$. Assume the contrary. Then there exists one agent, say h, and one deviation $\hat{\sigma}_h \in S_h(\mathbf{K})$ such that $u_h(\hat{\sigma}_h, \sigma_{-h}) > u_h(\sigma_h, \sigma_{-h})$. We shall use $\hat{\sigma}_h$ to construct a profitable deviation $\hat{\sigma}'_h$ to the equilibrium σ'. Define $\hat{\sigma}'_h \in S_h(\mathbf{K}')$ by

$$\left(\hat{b}_h^{\prime 1,1}, \hat{q}_h^{\prime 1,1}\right) = \left(\tau_b \hat{b}_h^{1,1}, \tau_q \hat{q}_h^{1,1}\right), \tag{24}$$

$$\left(\hat{b}_h^{\prime 1,K_1}, \hat{q}_h^{\prime 1,K_1}\right) = \left((1 - \tau_b)\, \hat{b}_h^{1,1}, (1 - \tau_q)\, \hat{q}_h^{1,1}\right), \tag{25}$$

$$\left(\hat{b}_h^{\prime i,s}, \hat{q}_h^{\prime i,s}\right) = \left(\hat{b}_h^{i,s}, \hat{q}_h^{i,s}\right) \quad \forall h \quad \forall (i, s) \neq (1, 1), (1, K_1), \tag{26}$$

where the weights τ_b and τ_q are given by

$$\tau_b = \frac{B_h^{\prime 1,1}}{B_h^{\prime 1,1} + B_h^{\prime 1,K_1}}, \qquad \tau_q = \frac{Q_h^{\prime 1,1}}{Q_h^{\prime 1,1} + Q_h^{\prime 1,K_1}}. \tag{27}$$

To show that $\hat{\sigma}'_h$ is feasible and yields the same allocation as $\hat{\sigma}_h$, we first compare prices. In view of (26), we have $\hat{p}'^{i,s} = \hat{p}^{i,s} \quad \forall (i,s) \neq (1,1), (1,K_1)$. Using successively Eq. (24)-(25), Eq. (27) and the definition of σ, the price on post $(1,1)$ can be computed as

$$\begin{aligned} \hat{p}'^{1,1} &= \tfrac{\hat{b}'^{1,1}_h + B'^{1,1}_h}{\hat{q}'^{1,1}_h + Q'^{1,1}_h} = \tfrac{\tau_b \hat{b}^{1,1}_h + B'^{1,1}_h}{\tau_q \hat{q}^{1,1}_h + Q'^{1,1}_h} = \tfrac{\tau_b}{\tau_q} \tfrac{\hat{b}^{1,1}_h + B'^{1,1}_h + B'^{1,K_1}_h}{\hat{q}^{1,1}_h + Q'^{1,1}_h + Q'^{1,K_1}_h} \\ &= \tfrac{\tau_b}{\tau_q} \tfrac{\hat{b}^{1,1}_h + B^{1,1}_h}{\hat{q}^{1,1}_h + Q^{1,1}_h} = \tfrac{\tau_b}{\tau_q} \hat{p}^{1,1}. \end{aligned} \tag{28}$$

Similarly, on post $(1,K_1)$:

$$\hat{p}'^{1,K_1} = \frac{1-\tau_b}{1-\tau_q} \hat{p}^{1,1}. \tag{29}$$

Now, to see that $\hat{\sigma}'_h$ is feasible, compute

$$D_h\left(\hat{\sigma}'_h, \sigma'_{-h}\right) - D_h\left(\hat{\sigma}_h, \sigma_{-h}\right) = \hat{q}^{1,1}_h \hat{p}^{1,1} - \hat{q}'^{1,1}_h \hat{p}'^{1,1} - \hat{q}'^{1,K_1}_h \hat{p}'^{1,K_1} = 0,$$

where the last equality follows from (24)-(25) and (28)-(29). To see that $\hat{\sigma}'_h$ yields the same allocation as $\hat{\sigma}_h$, we simply need to compute

$$x^1_h\left(\hat{\sigma}'_h, \sigma'_{-h}\right) - x^1_h\left(\hat{\sigma}_h, \sigma_{-h}\right) = \frac{\hat{b}'^{1,1}_h}{\hat{p}'^{1,1}} + \frac{\hat{b}'^{1,K_1}_h}{\hat{p}'^{1,K_1}} - \frac{\hat{b}^{1,1}_h}{\hat{p}^{1,1}} = 0$$

by (24)-(25) and (28)-(29). Hence, we have that $u_h\left(\hat{\sigma}'_h, \sigma'_{-h}\right) > u_h\left(\sigma'_h, \sigma'_{-h}\right)$, contradicting the fact that $\sigma' \in \mathrm{NE}_\eta(\mathbf{K}')$. Thus, $\sigma \in \mathrm{NE}_\eta(\mathbf{K})$. □ □

We now prove the result. Let $\mathbf{x} \in \mathrm{RA}(\mathbf{K})$. There exists $\sigma \in \mathrm{RE}(\mathbf{K})$ with $\mathbf{x}(\sigma) = \mathbf{x}$ and a sequence $\{^n\eta, ^n\sigma\}$ converging to $(0,\sigma)$ with $^n\sigma \in \mathrm{NE}_{^n\eta}(\mathbf{K})$. Lemma 4 implies that for any n, there exist $^n\sigma' \in \mathrm{NE}_{^n\eta}(\mathbf{K}')$ such that $\mathbf{x}(^n\sigma') = \mathbf{x}(^n\sigma)$. Now, any $^n\sigma'$ might be viewed as an element of the compact set $\bar{S}(\mathbf{K}')$. Compactness implies that the sequence $\{^n\sigma'\}_{n=1}^\infty$ contains a subsequence $\{^z\sigma'\}_{z=1}^\infty$ which converges to an element σ' of $\bar{S}$. By continuity, (i) $u_h\left(., \sigma'_{-h}\right)$ is maximized for σ'_h for any h, so that $\sigma' \in \mathrm{NE}(\mathbf{K}')$, and, (ii) $\mathbf{x}(\sigma') = \lim_{z\to\infty} \mathbf{x}(^z\sigma') = \lim_{z\to\infty} \mathbf{x}(^z\sigma) = \mathbf{x}$. Further, by construction $\sigma' \in \mathrm{RE}(\mathbf{K}')$, whence $\mathbf{x} \in \mathrm{RA}(\mathbf{K}')$. This terminates the proof. □

B. Proof of theorem 11 (case 2)

We first show that under concave preferences, any trading post on which one agent participates is 'essential'.

Lemma 5. *Let u_h be concave. Fix σ_{-h}. Then there exists a unique best reply $\bar{\sigma}^*_h \in \bar{S}_h$. Further, there exists a constant $\rho > 0$ such that $\forall \bar{\sigma}_h \in \bar{S}_h$, if there exists a trading post (i,k) with the property that $\bar{b}^{*i,k}_h > 0$ while $\bar{b}^{i,k}_h = 0$, or $\bar{q}^{*i,k}_h > 0$ while $\bar{q}^{i,k}_h = 0$, then $u_h\left(\bar{\sigma}_h, \sigma_{-h}\right) < u_h\left(\bar{\sigma}^*_h, \sigma_{-h}\right) - \rho$.*

Proof. We first show that $\bar{\sigma}^*_h$ is unique. Assume that there exists two best replies without wash sales, $\bar{\sigma}_h \neq \bar{\sigma}'_h$. Let $\hat{\sigma}_h = \frac{1}{2}\bar{\sigma}_h + \frac{1}{2}\bar{\sigma}'_h$. Note that $\hat{\sigma}_h$ is feasible by the feasability of $\bar{\sigma}_h$ and $\bar{\sigma}'_h$. For any post (i,k), denote by $\chi^{i,k}(\sigma_h)$ the contribution of the trade in that post to h's allocation, and by $\Delta^{i,k}(\sigma_h)$ the contribution to h's money holdings. Note that $x^i_h(\sigma_h, \sigma_{-h}) = e^i_h + \sum_k \chi^{i,k}(\sigma_h)$, and that $D_h(\sigma_h, \sigma_{-h}) = \sum_{(i,k)} \Delta^{i,k}(\sigma_h)$. Further, $D_h(\bar{\sigma}_h, \sigma_{-h}) = D_h(\bar{\sigma}'_h, \sigma_{-h}) = 0$ since $\bar{\sigma}_h \neq \bar{\sigma}'_h$ are best replies. We claim that $D_h(\hat{\sigma}_h, \sigma_{-h}) \geq 0$ and $x^i_h(\hat{\sigma}_h, \sigma_{-h}) \geq \frac{1}{2}x^i_h(\bar{\sigma}_h, \sigma_{-h}) + \frac{1}{2}x^i_h(\bar{\sigma}'_h, \sigma_{-h}) \quad \forall i$, with at least one inequality strict. To show this, we distinguish several cases depending on h's position on each active trading post.

Case 1: $\bar{b}^{i,k}_h = \bar{b}'^{i,k}_h = 0$. Then

$$\chi^{i,k}(\sigma_h) = \frac{1}{2}\bar{q}^{i,k}_h + \frac{1}{2}\bar{q}'^{i,k}_h = \frac{1}{2}\chi^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\chi^{i,k}(\bar{\sigma}'_h), \tag{30}$$

$$\Delta^{\tau}(\hat{\sigma}_h) = \frac{\frac{1}{2}\bar{q}^{i,k}_h + \frac{1}{2}\bar{q}'^{i,k}_h}{Q^{i,k}_h + \left(\frac{1}{2}\bar{q}^{i,k}_h + \frac{1}{2}\bar{q}'^{i,k}_h\right)} B^{i,k}_h \geq \frac{1}{2}\Delta^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\Delta^{i,k}(\bar{\sigma}'_h). \tag{31}$$

Case 2. $\bar{b}^{i,k}_h > 0$ and $\bar{b}'^{i,k}_h > 0$. Then

$$\chi^{i,k}_h(\hat{\sigma}_h) = \frac{\frac{1}{2}\bar{b}^{i,k}_h + \frac{1}{2}\bar{b}'^{i,k}_h}{B^{i,k}_h + \left(\frac{1}{2}\bar{b}^{i,k}_h + \frac{1}{2}\bar{b}'^{i,k}_h\right)} Q^{i,k}_h \geq \frac{1}{2}\chi^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\chi^{i,k}(\bar{\sigma}'_h), \tag{32}$$

with strict inequality whenever $\bar{b}^{i,k}_h \neq \bar{b}'^{i,k}_h$, and

$$\Delta^{i,k}(\hat{\sigma}_h) = -\frac{1}{2}\bar{b}^{i,k}_h - \frac{1}{2}\bar{b}'^{i,k}_h = \frac{1}{2}\Delta^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\Delta^{i,k}(\bar{\sigma}'_h). \tag{33}$$

Case 3. $\bar{b}^{i,k}_h > 0$ and $\bar{b}'^{i,k}_h = 0$ (and symmetricaly for $\bar{b}^{i,k}_h = 0$ and $\bar{b}'^{i,k}_h > 0$). Then,

$$\chi^{i,k}(\sigma_h) = \frac{\frac{1}{2}\bar{b}^{i,k}_h}{B^{i,k}_h + \frac{1}{2}\bar{b}^{i,k}_h} Q^{i,k}_h + \frac{1}{2}\bar{q}'^{i,k}_h > \frac{1}{2}\chi^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\chi^{i,k}(\bar{\sigma}'_h), \tag{34}$$

$$\Delta^{\tau}(\hat{\sigma}_h) = -\frac{1}{2}\bar{b}^{i,k}_h + \frac{\frac{1}{2}\bar{q}'^{i,k}_h}{Q^{i,k}_h + \frac{1}{2}\bar{q}'^{i,k}_h} B^{i,k}_h \geq \frac{1}{2}\Delta^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\Delta^{i,k}(\bar{\sigma}'_h). \tag{35}$$

(Inequalities (31)-(32) follow from the strict concavity, and (34)-(35) from the strict monotonicity, of $f : x \mapsto \frac{x}{A+x}B$ over $\mathbb{R}_+$ $(A, B > 0)$). Summing (31), (33) and (35) over all trading posts, one gets $D_h(\hat{\sigma}_h, \sigma_{-h}) \geq \frac{1}{2}D_h(\bar{\sigma}'_h, \sigma_{-h}) + \frac{1}{2}D_h(\bar{\sigma}'_h, \sigma_{-h}) = 0$. Summing (30), (32) and (34) over posts for a given good, one gets $\sum_k \chi^{i,k}(\hat{\sigma}_h) \geq \frac{1}{2}\sum_i \chi^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\sum_i \chi^{i,k}(\bar{\sigma}'_h)$ and eventualy $x^i_h(\hat{\sigma}_h, \sigma_{-h}) \geq \frac{1}{2}x^i_h(\bar{\sigma}_h, \sigma_{-h}) + \frac{1}{2}x^i_h(\bar{\sigma}'_h, \sigma_{-h})$. Now, one can easily see that there must exist al least one $(i,k) \in \mathcal{K}$ s.t. either $\Delta^{i,k}(\hat{\sigma}_h) > \frac{1}{2}\Delta^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\Delta^{i,k}(\bar{\sigma}'_h)$ or $\chi^{i,k}(\hat{\sigma}_h) > \frac{1}{2}\chi^{i,k}(\bar{\sigma}_h) + \frac{1}{2}\chi^{i,k}(\bar{\sigma}'_h)$, for otherwise the only cases are case 1 with $\bar{q}^{i,k}_h = \bar{q}'^{i,k}_h$ and case 2 with $\bar{b}^{i,k}_h = \bar{b}'^{i,k}_h$, implying $\bar{\sigma}_h = \bar{\sigma}'_h$. Hence, as claimed, $D_h(\hat{\sigma}_h, \sigma_{-h}) \geq 0$ and $x^i_h(\hat{\sigma}_h, \sigma_{-h}) \geq \frac{1}{2}x^i_h(\bar{\sigma}_h, \sigma_{-h}) + \frac{1}{2}x^i_h(\bar{\sigma}'_h, \sigma_{-h}) \quad \forall i$, with at least one inequality strict. But this implies that there exist a feasible deviation,

$\hat{\sigma}'_h$ resulting in an allocation $x^i_h(\hat{\sigma}'_h, \sigma_{-h}) \geq \frac{1}{2}x^i_h(\bar{\sigma}_h, \sigma_{-h}) + \frac{1}{2}x^i_h(\bar{\sigma}'_h, \sigma_{-h})$ with at least one strict inequality. Using the strict monotonicity and the concavity of u_h, we get

$$u_h(\hat{\sigma}'_h, \sigma_{-h}) > u_h\left(\frac{1}{2}x^i_h(\bar{\sigma}_h, \sigma_{-h}) + \frac{1}{2}x^i_h(\bar{\sigma}'_h, \sigma_{-h})\right) \geq u_h(\bar{\sigma}_h, \sigma_{-h}).$$

This contradicts the assumed optimality of $\bar{\sigma}_h$.

We now prove the rest of the lemma. For any two subsets $(\kappa_b, \kappa_q) \in \mathcal{P}(\mathcal{K})^2$ of existings posts, we let $\bar{S}_h^{(\kappa_b,\kappa_q)}$ be the subset of strategies in $\bar{S}_h$ for which h is constrained *not* to buy on trading posts in κ_b and *not* to sell on posts in κ_q. By a reasoning analogous to the above, one can show that $\max_{\bar{S}_h^{(\kappa_b,\kappa_q)}} u_h(\sigma_h, \sigma_{-h})$ is attained for a unique strategy, that we denote $\bar{\sigma}^*_h(\kappa_b, \kappa_q)$. Now, as $\bar{S}_h \equiv \cup_{(\kappa_b,\kappa_q)\in\mathcal{P}(\mathcal{K})^2} \bar{S}_h^{(\kappa_b,\kappa_q)}$, agent h's maximisation program can be expressed as

$$\max_{(\kappa_b,\kappa_q)\in\mathcal{P}(\mathcal{K})^2} u_h(\bar{\sigma}^*_h(\kappa_b, \kappa_q), \sigma_{-h}).$$

The claim then easily follows from the finitess of $\mathcal{P}(\mathcal{K})^2$ and the unicity of $\bar{\sigma}^*_h$. □

One consequence of lemma 5 is that any $\bar{\sigma}^* \in NE(\Gamma) \cap \bar{S}$ is c-robust. To show this, it is sufficient to show that $\sigma^* \in NE(\Gamma_\eta)$ when η (> 0) is small enough. Assume the contrary. Then, for η as small as needed, there exists an agent h and a deviation $\hat{\sigma}_h$, which w.l.o.g. can be taken without wash-sales, such that

$$\eta\{TC(\bar{\sigma}^*_h) - TC(\hat{\sigma}_h)\} > u_h(\bar{\sigma}^*_h, \sigma^*_{-h}) - u_h(\hat{\sigma}_h, \sigma^*_{-h}), \tag{36}$$

where $TC(\sigma_h)$ denotes the transaction cost of σ_h. In particular, (36) implies that $TC(\bar{\sigma}^*_h) > TC(\hat{\sigma}_h)$, which given the cost specification implies the existence of a post (i, k) satisfying the assumption of lemma 5 (otherwise, it is easily seen that $TC(\bar{\sigma}^*_h) \leq TC(\hat{\sigma}_h)$). Hence, the r.h.s. of (36) is superior to the constant $\rho > 0$; however the l.h.s. can be made arbitrarily small for η small enough. As η can be taken as small as needed, a contradiction results. This establishes the claim that $\bar{\sigma}^*$ is c-robust.

Now, consider a c-robust equilibrium σ of a game Γ with an arbitrary number of posts. By robustness, $\sigma \in \bar{S}(\mathbf{K})$ and satifies the LOP. Hence, the strategy profile σ' defined by $\left(b'^i_h, q'^i_h\right) = \left(\Sigma_{k=1}^{K_i} b^i_h, \Sigma_{k=1}^{K_i} q^i_h\right)$ $\forall h, i$ is a NE of the game with one post per good, which yields the same allocation (see the proof of lemma 4 above). Now, since $\sigma \in NE(\Gamma) \cap \bar{S}$, by proposition 4 there is no h that buys and sells a given good at σ, implying that σ' is without wash-sales. Hence, σ' is also robust. The result follows.

C. Proof of proposition 16

Proof. Formally, ${}^n\sigma^*$ is an ε_n-NE of $\Gamma_{\eta_n c}$ if and only if for any $h \in \mathcal{H}$ and any $\hat{\sigma}_h \in S_h$

$$u(\hat{\sigma}_h, {}^n\sigma^*_{-h}) - \eta_n C(\hat{\sigma}_h) \leq u({}^n\sigma^*_h, {}^n\sigma^*_{-h}) - \eta_n C({}^n\sigma^*_h) + \varepsilon_n. \tag{37}$$

Claim (ii) simply follows from the fact that (under the condition stated) σ^* itself is a ε_n-NE with $\eta_n c$-transaction costs, for n sufficiently large. Using (37) for ${}^n\sigma_h^* = \sigma^*$ and rearranging, one need to check that $\forall h \in \mathcal{H}$,

$$u\left(\hat{\sigma}_h, \sigma_{-h}^*\right) - u\left(\sigma_h^*, \sigma_{-h}^*\right) \leq \eta_n C\left(\hat{\sigma}_h\right) + \eta_n \left(\frac{\varepsilon_n}{\eta_n} - C\left(\sigma_h^*\right)\right), \qquad \forall \hat{\sigma}_h \in S_h. \tag{38}$$

Now, the l.h.s is negative since σ^* is a NE, and the r.h.s. is strictly positive for n sufficiently large because $\lim_{n\to\infty} \varepsilon_n/\eta_n > \max_{h\in\mathcal{H}}\{C\left(\sigma_h^*\right)\} \geq C\left(\sigma_h^*\right)$. This proves (ii). To prove (i), we proceed by contradiction. Assume that there exists a sequence $\{{}^n\sigma^*\}_{n\in I\!N}$ of ε_n-NE of the perturbed games $\Gamma_{\eta_n c}$ converging to σ^*. This implies in particular that, along the sequence, no agent h can increase his payoff by more than ε_n by deviating to ${}^n\bar{\sigma}_h^*$, the strategy without wash-sales associated to ${}^n\sigma_h^*$. Using (37) for $\hat{\sigma}_h =^n \bar{\sigma}_h^*$ and rearranging, it holds that, $\forall h \in \mathcal{H}$,

$$\eta_n \left(\left(C\left({}^n\sigma_h^*\right) - C\left({}^n\bar{\sigma}_h^*\right)\right) - \frac{\varepsilon_n}{\eta_n}\right) \leq u\left({}^n\sigma_h^*, {}^n\sigma_{-h}^*\right) - u\left({}^n\bar{\sigma}_h, {}^n\sigma_{-h}^*\right) = 0, \tag{39}$$

where the last equality follows from the definition of ${}^n\bar{\sigma}_h^*$. Now, one can easily see from the formulae of ${}^n\bar{\sigma}_h^*$ that ${}^n\bar{\sigma}_h^* \to \bar{\sigma}_h^*$ as ${}^n\sigma_h^* \to \sigma_h^*$. Thus, by the continuity of c, $C\left({}^n\sigma_h^*\right) - C\left({}^n\bar{\sigma}_h^*\right) \to \bar{C}_h\left(\sigma_h^*\right)$. Cond. (39) then implies that $\bar{C}_h\left(\sigma_h^*\right) < \varepsilon_n/\eta_n$ for n sufficiently large. This holds $\forall h \in \mathcal{H}$ (a finite set), contradicting the assumption that $\lim_{n\to\infty} \varepsilon_n/\eta_n < \max_{h\in\mathcal{H}}\{\bar{C}\left(\sigma_h^*\right)\}$. The claim follows. □

D. Proof of proposition 18

We first show that for $m \in \{0, 1, \ldots, 2K\}$, any equilibrium σ^* of Γ such that $\max_{h\in\mathcal{H}}\left\{\#\left(\sigma_h^*\right) - \#\left(\bar{\sigma}_h^*\right)\right\} = m$, where $\bar{\sigma}_h^*$ is the strategy without wash-sales associated with σ_h^*, is approachable when $\lim_{n\longrightarrow\infty} \frac{\varepsilon_n}{\gamma_n} > m$. This will establish (iii), (ii), and the second part of (i). The proof is by contradiction. Assume that $\lim_{n\longrightarrow\infty} \frac{\varepsilon_n}{\gamma_n} > m$, and that σ^* cannot be approximated by any sequence $\{{}^n\sigma^*\}_{n\in I\!N}$ of ε_n-NE of the games with γ_n-complexity costs. In particular, there exists N such that $\forall n \geq N$, σ^* is not an ε_n-NE of Γ_{γ_n}. (Otherwise, one can pick a subsequence $\{\varepsilon_{n'}, \gamma_{n'}\}$ and use $\{{}^{n'}\sigma^*\} \equiv \{\sigma^*\}$ to approach σ^*). Hence $\forall n \geq N, \exists h\left(n\right) \in \mathcal{H}, {}^n\hat{\sigma}_{h(n)} \in S_{h(n)}$ such that

$$u({}^n\hat{\sigma}_{h(n)}, \sigma_{-h(n)}^*) - \gamma_n\#({}^n\hat{\sigma}_{h(n)}) > u(\sigma_{h(n)}^*, \sigma_{-h(n)}^*) - \gamma_n\#(\sigma_{h(n)}^*) + \varepsilon_n.$$

W.l.o.g. we can take the deviations $\hat{\sigma}_{h(n)}$ to be without wash-sales. Since $\mathcal{H}$ is finite, $\exists h' \in \mathcal{H}$ such that one can extract from $\{{}^n\hat{\sigma}_{h(n)}\}_{n\in I\!N}$ a subsequence $\{{}^{n'}\hat{\sigma}_{h(n')}\}_{n'\in I\!N}$ with $h(n') = h' \quad \forall n'$. Note that, by unicity of the limit, $\{\varepsilon_{n'}/\eta_{n'}\}_{n'\in I\!N}$ converges to $\lim_{n\longrightarrow\infty} \frac{\varepsilon_n}{\gamma_n}$. The (sub)sequence $\{{}^{n'}\hat{\sigma}_{h'}\}_{n'\in I\!N}$ satisfies

$$\gamma_{n'}\#(\sigma_{h'}^*) - \gamma_{n'}\#({}^{n'}\hat{\sigma}_{h'}) - \varepsilon_{n'} > u(\sigma_{h'}^*, \sigma_{-h'}^*) - u({}^{n'}\hat{\sigma}_{h'}, \sigma_{-h'}^*) \geq 0, \tag{40}$$

where the last inequality in the first line follows from the fact that $\sigma^* \in NE(\Gamma)$. Rearranging the l.h.s., cond. (40) implies that

$$\gamma_{n'}\Big(\#(\bar{\sigma}^*_{h'}) - \#({}^{n'}\hat{\sigma}_{h'})\Big) + \Big(\gamma_{n'}\big(\#(\sigma^*_{h'}) - \#(\bar{\sigma}^*_{h'})\big) - \varepsilon_{n'}\Big) \geq 0, \tag{41}$$

The second term in the l.h.s. is strictly negative for n' sufficiently large (say, $n' > N'$), because $\#(\sigma^*_{h'}) - \#(\bar{\sigma}^*_{h'}) \leq m < \lim_{n' \longrightarrow \infty} \frac{\varepsilon_{n'}}{\gamma_{n'}} = \lim_{n \longrightarrow \infty} \frac{\varepsilon_n}{\gamma_n}$. Cond. (41) thus implies that $\#(\bar{\sigma}^*_{h'}) > \#({}^{n'}\hat{\sigma}_{h'})$, or equivalently, $\#(\bar{\sigma}^*_{h'}) - 1 \geq \#({}^{n'}\hat{\sigma}_{h'})$ for $n' > N'$. Now, by the compactness of $S_{h'}$, $\{{}^{n'}\hat{\sigma}_{h(n')}\}_{n' \in I\!N}$ admits a converging subsequence, $\{{}^{n''}\hat{\sigma}_{h'}\}_{n'' \in I\!N}$, converging to an element $\hat{\sigma}_{h'} \in S_{h'}$. Being the limit of strategies without wash-sales, $\hat{\sigma}_{h'}$ is without wash-sales (see the proof of propostion 4). We next claim that $u(\hat{\sigma}_{h'}, \sigma^*_{-h'}) = u(\sigma^*_{h'}, \sigma^*_{-h'})$ and that $\#(\bar{\sigma}^*_{h'}) \geq \#(\hat{\sigma}_{h'}) + 1$. The former follows from letting $n' \to \infty$ in (40). The latter follows from the observation that if a given element of $\hat{\sigma}_{h'}$ is strictly positive, the corresponding element in ${}^{n''}\hat{\sigma}_{h'}$ is also strictly positive for n'' sufficiently large (otherwise, ${}^{n''}\hat{\sigma}_{h'}$ cannot converge to $\hat{\sigma}_{h'}$), i.e., $\#({}^{n''}\hat{\sigma}_{h'}) \geq \#(\hat{\sigma}_{h'})$. Using $\#(\bar{\sigma}^*_{h'}) - 1 \geq \#({}^{n''}\hat{\sigma}_{h'})$ (for n'' large), this implies $\#(\bar{\sigma}^*_{h'}) \geq \#(\hat{\sigma}_{h'}) + 1$. Thus, $\hat{\sigma}_{h'}$ is without wash-sales, yields the same payoff as $\sigma^*_{h'}$ but differs from $\bar{\sigma}^*_{h'}$ (as $\#(\bar{\sigma}^*_{h'}) \geq \#(\hat{\sigma}_{h'}) + 1$). This contradicts the unicity of the best reply without wash-sales established in lemma 5.

We now prove the first part of (i). Consider an equilibrium with wash-sales, σ^*. Then $\exists h', i, k$ such that agent h' buys and sells on post (i, k). Assume that σ^* is approachable by a sequence $\{{}^n\sigma^*_{h'}\}$. Then, $\forall \hat{\sigma}_{h'} \in S_{h'}$

$$u(\hat{\sigma}_{h'}, {}^n\sigma^*_{-h'}) - \gamma_n \#(\hat{\sigma}_{h'}) \leq u({}^n\sigma^*_{h'}, {}^n\sigma^*_{-h'}) - \gamma_n \#({}^n\sigma^*_{h'}) + \varepsilon_n. \tag{42}$$

Note that $b^{*k}_{h'} q^{*k}_{h'} > 0$ implies that ${}^n b^{*k}_{h'}\, {}^n q^k_{h'} > 0$ for n large, that is h' is also active on both sides of (i, k) at ${}^n\sigma^*_{h'}$. Now, define ${}^n\tilde{\sigma}^*_{h'}$ as the strategy obtained from ${}^n\sigma^*_{h'}$ by removing wash-sales on post k *only*. Then either $\#({}^n\tilde{\sigma}^*_{h'}) = \#({}^n\sigma^*_{h'}) - 1$, or $\#({}^n\tilde{\sigma}^*_{h'}) = \#({}^n\sigma^*_{h'}) - 2$. Applying (42) to ${}^n\tilde{\sigma}^*_{h'}$, one gets $\gamma_n \left(\# \left({}^n\sigma^*_{h'}\right) - \# \left({}^n\tilde{\sigma}^*_{h'}\right)\right) \leq \varepsilon_n$, implying that $1 \leq \varepsilon_n / \gamma_n$ for n sufficiently large. But this contradicts the assumption that $\lim_{n \longrightarrow \infty} \frac{\varepsilon_n}{\gamma_n} < 1$.

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