Improved intermediate asymptotics for the heat equation[☆,☆☆]

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Abstract

This letter is devoted to results on intermediate asymptotics for the heat equation. We study the convergence towards a stationary solution in self-similar variables. By assuming the equality of some moments of the initial data and of the stationary solution, we get improved convergence rates using entropy / entropy-production methods. We establish the equivalence of the exponential decay of the entropies with new, improved functional inequalities in restricted classes of functions. This letter is the counterpart in a linear framework of a recent work on fast diffusion equations, see [8]. Results extend to the case of a Fokker-Planck equation with a general confining potential.

Key words: Heat equation, Fokker-Planck equation, Ornstein-Uhlenbeck equation, intermediate asymptotics, self-similar variables, stationary solutions, large time behavior, rate of convergence, entropy, Poincaré inequality, logarithmic Sobolev inequality, interpolation inequalities
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Consider the *heat equation* in the euclidean space,

$$\frac{\partial u}{\partial t} = \Delta u \quad t > 0\,,\ x \in \mathbb{R}^d \tag{1}$$

with an initial condition $u_0 \in L^1(\mathbb{R}^d)$. By writing $u = u_+ - u_-$ where u_+ and u_- are respectively the positive and negative parts of u and solving (1) with initial data $(u_0)_+$ and $(u_0)_-$, we may reduce the problem to the case of a nonnegative function, corresponding to a nonnegative initial condition u_0, without restriction. The heat equation being linear, we can assume without loss of generality that u_0 is a probability measure so that in the sequel of this note $\int_{\mathbb{R}^d} u_0\,dx = 1 = \int_{\mathbb{R}^d} u(t,x)\,dx$ for any $t \geq 0$. Getting decay rates and even an asymptotic expansion for large values of t is completely standard, see for instance [13]. However, a few details and some notations will be useful for later purpose.

First of all, as a straightforward consequence of the expression of the Green function, $G(t,x,y) := (4\pi t)^{-d/2}\,e^{-\frac{|x-y|^2}{4t}}$, any solution u of (1) can be written as $u(t,x) = \int_{\mathbb{R}^d} u_0(y)\,G(t,x,y)\,dy$ and therefore uniformly decays like $O(t^{-d/2})$ since, as $t \to \infty$, $u(t,x) \sim G(t,x,0)$. It is also classical to estimate the decay of $u(t,\cdot) - G(t,\cdot,0)$ in various $L^p(\mathbb{R}^d)$ norms. Such estimates are called *intermediate asymptotics* estimates. The point is to determine the first term of an asymptotic expansion of the solution as $t \to \infty$. For instance, as we shall see below, it can be proved that $\|u(t,\cdot) - G(t,\cdot,0)\|_{L^1(\mathbb{R}^d)} = O(t^{-1/2})$ as $t \to \infty$.

The *entropy method* can be used among various other approaches to obtain such an estimate. It relies on the logarithmic Sobolev inequality and goes as follows. First consider the time-dependent rescaling

$$u(t,x) = R^{-d}\,v\,(\log R, x/R) \quad \text{with} \quad R = R(t) := \sqrt{1+2\,t}\,, \quad t > 0\,,\ x \in \mathbb{R}^d\,. \tag{2}$$

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If u is a solution of (1), then v solves the *Fokker-Planck equation*

$$\frac{\partial v}{\partial t} = \Delta v + \nabla \cdot (x\, v) \tag{3}$$

with same initial condition $v(t = 0, \cdot) = u_0$. Let $v_\infty(x) := (2\pi)^{-d/2}\, e^{-|x|^2/2}$ be the unique stationary solution of (3) with mass 1, and define $d\mu := v_\infty\, dx$ as the Gaussian measure. We denote by $L^p(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d, d\mu)$ the Lebesgue spaces corresponding respectively to Lebesgue's measure and to the Gaussian measure. Understanding the intermediate asymptotics for u amounts to study the convergence of v to v_∞, as $t \to \infty$. Define the *entropy* by $\mathcal{E}_1[w] := \int_{\mathbb{R}^d} w \log w\, d\mu$. Let v be a solution of (3) and define $w(t, \cdot) := v(t, \cdot)/v_\infty$, $w_0 := w(t = 0, \cdot)$. Then $\frac{d}{dt}\, \mathcal{E}_1[w(t, \cdot)] = -\mathcal{I}_1[w(t, \cdot)]$ where $\mathcal{I}_1$ is the *Fisher information* defined by $\mathcal{I}_1[w] := \int_{\mathbb{R}^d} w\, |\nabla \log w|^2\, d\mu$. Gross' *logarithmic Sobolev inequality* exactly amounts to $\mathcal{E}_1[v/v_\infty] \le \frac{1}{2}\, \mathcal{I}_1[v/v_\infty]$ and so, it follows that

$$\mathcal{E}_1[w(t, \cdot)] \le \mathcal{E}_1[w_0]\, e^{-2\,t} \quad \forall\, t \ge 0\,.$$

By the *Csiszár-Kullback inequality*, see for instance [17], we get $\|v(t, \cdot) - v_\infty\|^2_{L^1(\mathbb{R}^d)} \le \frac{1}{4}\, \mathcal{E}_1[w(t, \cdot)]$ and deduce that

$$\|v(t, \cdot) - v_\infty\|_{L^1(\mathbb{R}^d)} \le \frac{1}{2}\, \sqrt{\mathcal{E}_1[w_0]}\, e^{-t} \quad \forall\, t \ge 0\,.$$

Undoing the change of variables (2) and observing that $u_\infty(t, x) := R(t)^{-d}\, v_\infty\,(x/R(t)) = G(t + 1/2, \cdot, 0)$, we finally get

$$\|u(t, \cdot) - u_\infty(t, \cdot)\|_{L^1(\mathbb{R}^d)} \le \frac{1}{2}\, \sqrt{\frac{\mathcal{E}_1[w_0]}{1 + 2\,t}} \quad \forall\, t \ge 0\,,$$

which establishes the claimed estimate, namely: $\|u(t, \cdot) - G(t, x, 0)\|_{L^1(\mathbb{R}^d)} \le O\left(t^{-1/2}\right)$ as $t \to \infty$. Such an estimate is quite classical. The above method is known as the *Bakry-Emery method* or *entropy / entropy-production method* and also provides a proof of the logarithmic Sobolev inequality. See [16, 3] for some references on this topic, in the context of partial differential equations.

By combining $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$ estimates using Hölder's inequality, we get that

$$\|u(t, \cdot) - G(t, \cdot, 0)\|_{L^p(\mathbb{R}^d)} \le O\Big(t^{-\frac{1}{2p}\,(1+(p-1)\,d)}\Big) \quad \text{as} \quad t \to \infty\,.$$

In a $L^2(\mathbb{R}^d)$ framework, a much more detailed description can be achieved using a spectral decomposition. If v is a solution of (3), then $w = v/v_\infty$ is a solution of the *Ornstein-Uhlenbeck equation*

$$\frac{\partial w}{\partial t} = \Delta w - x \cdot \nabla w \tag{4}$$

with initial data $w_0 = u_0/v_\infty$. Notice that $\int_{\mathbb{R}^d} w_0\, d\mu = 1$ and, as a consequence, $\int_{\mathbb{R}^d} w(t, \cdot)\, d\mu = 1$ for all $t \ge 0$. Define by $(H_k)_{k\in\mathbb{N}^d}$ the sequence of Hermite type polynomials (see for instance [19]) acting on $x = (x_1, x_2 \ldots x_d) \in \mathbb{R}^d$, such that $H_k(x) := \prod_{j=1}^d h_{k_j}(x_j)$ where $h_n(y) := (-1)^n\, (n!)^{-1/2}\, e^{y^2/2}\, \frac{d^n}{dy^n}(e^{-y^2/2})$, $y \in \mathbb{R}$ and $k = (k_1, ..., k_d) \in \mathbb{N}^d$. These functions provide an orthonormal family of eigenfunctions in $L^2(\mathbb{R}^d, d\mu)$ which spans the eigenspaces of the Ornstein-Uhlenbeck operator, that is $-(\Delta H_k - x \cdot \nabla H_k) = |k|\, H_k$, where $|k| := \sum_{j=1}^d k_j$. Up to a scaling, $(h_n)_{n\in\mathbb{N}}$ is the usual family of Hermite polynomials on $\mathbb{R}$.

If w_0 satisfies the orthogonality condition

$$\int_{\mathbb{R}^d} w_0\, H_k\, d\mu = 0 \quad \forall\, k \in \mathbb{N}^d \text{ such that } 0 < |k| < n\,, \tag{5}$$

then an improved rate of convergence follows, in the sense that

$$\|w(t, \cdot) - 1\|_{L^2(\mathbb{R}^d, d\mu)} \le e^{-n\,t}\, \|w_0 - 1\|_{L^2(\mathbb{R}^d, d\mu)} \quad \forall\, t \ge 0\,.$$

If (5) initially holds, we indeed have $\int_{\mathbb{R}^d} w(t, \cdot)\, H_k\, d\mu = 0$ for any $t \ge 0$ and any $k \in \mathbb{N}^d$such that $0 < |k| < n$. Then, since $\frac{d}{dt}\|w(t, \cdot) - 1\|^2_{L^2(\mathbb{R}^d, d\mu)} = -2 \int_{\mathbb{R}^d} |\nabla w(t, \cdot)|^2\, d\mu$, the conclusion holds using the following result.

Proposition 1 (Improved Poincaré inequality). *Assume that $w \in L^2(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w\,d\mu = 1$ and the condition $\int_{\mathbb{R}^d} w\,H_k\,d\mu = 0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. Then the following inequality holds, with optimal constant:*

$$\|w-1\|^2_{L^2(\mathbb{R}^d,\,d\mu)} \le \frac{1}{n}\,\|\nabla w\|^2_{L^2(\mathbb{R}^d,\,d\mu)}\,.$$

The proof is no more than a straightforward rewriting of the Rayleigh quotient $\|\nabla w\|^2_{L^2(\mathbb{R}^d,\,d\mu)}/\|w-1\|^2_{L^2(\mathbb{R}^d,\,d\mu)}$ under the appropriate orthogonality condition. Notice that polynomials H_k are of degree $|k|$ so that the Condition (5) can be rephrased in terms of moment conditions. See [13, 14] for further results in this direction.

It is natural to search for improved estimates of convergence also in $L^p(\mathbb{R}^d)$ with $p \in [1,2)$ by looking for improved functional inequalities whenever condition (5) is fulfilled. We may for instance quote [2] in which improvements on the constant, but not on the rates, have been achieved for $p = 1$.

For any $p \in (1,2]$, consider the *generalized entropy*

$$\mathcal{E}_p[w] := \int_{\mathbb{R}^d} \frac{w^p-1}{p-1}\,d\mu\,.$$

This definition is consistent with the definition of $\mathcal{E}_1$ because, under the condition $\int_{\mathbb{R}^d} w\,d\mu = 1$, $\mathcal{E}_p[w] = \int_{\mathbb{R}^d} \frac{w^p-w}{p-1}\,d\mu \to \mathcal{E}_1[w]$ as $p \to 1$. The functional $\mathcal{E}_p$ controls the convergence in $L^p(\mathbb{R}^d,\,d\mu)$ using a generalized Csiszár-Kullback inequality. In [9, 4], it has been proved that $\|w-1\|^2_{L^p(\mathbb{R}^d,\,d\mu)} \le \frac{1}{p}\,2^{2/p}\,\max\{\|w\|^{2-p}_{L^p(\mathbb{R}^d,\,d\mu)}, 1\}\,\mathcal{E}_p[w]$, for any $p \in [1,2]$. Since $\|w\|_{L^1(\mathbb{R}^d,\,d\mu)} = 1$, we have $1 \le \|w\|^p_{L^p(\mathbb{R}^d,\,d\mu)} = 1 + (p-1)\,\mathcal{E}_p[w]$, and so

$$\|w-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w]\right) \quad \text{with} \quad \mathcal{A}_p(s) := \frac{2^{1/p}}{\sqrt{p}}\left[1+(p-1)\,s\right]^{1-p/2}\sqrt{s}\,. \tag{6}$$

Next, assume that $\int_{\mathbb{R}^d} w\,H_k\,d\mu = 0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$ and consider the *generalized Poincaré inequalities,* with $p \in [1,2]$, namely

$$\mathcal{E}_p[w] \le \mathcal{B}_{n,p}\int_{\mathbb{R}^d} \left|\nabla w^{p/2}\right|^2 d\mu \quad \forall\, w \in H^1(\mathbb{R}^d,\,d\mu)\,. \tag{7}$$

Such inequalities have been established for $n = 1$ by W. Beckner in [5] with optimal constant $\mathcal{B}_{1,p} = 2/p$ for the Gaussian measure. By the same method, it has been shown in [1] that for a larger class of measures $d\mu$, if (7) holds for $p = 1$ and $p = 2$, for some positive constants $\mathcal{B}_{n,1}$ and $\mathcal{B}_{n,2}$ respectively, then it also holds for any $p \in (1,2)$ with

$$\mathcal{B}_{n,p} = \tfrac{1}{p-1}\left[1-((2-p)/p)^{\mathcal{B}_{n,1}/(2\,\mathcal{B}_{n,2})}\right]\mathcal{B}_{n,2}\,. \tag{8}$$

By the logarithmic Sobolev inequality and the improved Poincaré inequality, see Proposition 1, we know that $\mathcal{B}_{n,1} \le 2$ and $\mathcal{B}_{n,2} = 1/n$. Hence it follows that $\mathcal{B}_{n,p} \le \frac{1}{p-1}\left[1-((2-p)/p)^n\right]\frac{1}{n}$. On the other hand, as in [3], if w is a solution of (4), then

$$\frac{d}{dt}\,\mathcal{E}_p[w(t,\cdot)] = -\frac{4}{p}\int_{\mathbb{R}^d} \left|\nabla w^{p/2}\right|^2 d\mu\,. \tag{9}$$

If (5) is satisfied, we conclude using (7) and (6) that any solution of (4) with initial data w_0 satisfies

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\,e^{-2\,\lambda(n,p)\,t} \quad \text{and} \quad \|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\,e^{-\lambda(n,p)\,t} \quad \forall\, t \ge 0\,,$$

with $\lambda(n,p) := \frac{2}{p}\,n\,(p-1)\left[1-((2-p)/p)^n\right]^{-1}$. The last estimate holds because, for any $t \ge 0$,

$$\|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w(t,\cdot)]\right) \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\,e^{-2\,\lambda(n,p)\,t}\right) \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)e^{-\lambda(n,p)\,t}$$

Notice that $\lambda(1,p) = 1$ and $\lambda(n,2) = n$. Nothing is gained as $p \to 1$, since $\lim_{p\to1}\lambda(n,p) = 1$ is independent of n.

On the other hand, by Hölder's inequality, we have for free that $\|w-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \|w-1\|_{L^2(\mathbb{R}^d,\,d\mu)}$. Hence, if w is a solution of (4) with initial data w_0, we know that $\|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le e^{-n\,t}\,\|w_0-1\|_{L^2(\mathbb{R}^d,\,d\mu)}$ as $t \to \infty$, for any

$p \in [1,2]$, if (5) is satisfied. By interpolation, we recover the rates of [13, 14]. However, this is not satisfactory since neither $\|w_0 - 1\|_{L^p(\mathbb{R}^d,\,d\mu)}$ nor $\mathcal{E}_p[w_0]$ are involved in the right hand side of the above estimate.

Consider first the case $p = 1$. An alternative approach is suggested by the method of [7, 6], which applies to the fast diffusion equation $\frac{\partial u}{\partial t} = \Delta u^m$ for $m < 1$. By assuming some uniform bound on the initial data, which is preserved along the evolution, it is possible to relate the asymptotic rate for intermediate asymptotics with the spectrum of the linearized operator. We can indeed observe that $\|w_0 - 1\|^2_{L^2(\mathbb{R}^d,\,d\mu)} \le \|w_0 - 1\|_{L^1(\mathbb{R}^d,\,d\mu)}\, \|w_0 - 1\|_{L^\infty(\mathbb{R}^d,\,d\mu)} \le \frac{1}{2}\sqrt{\mathcal{E}_1[w_0]}\, \|w_0 - 1\|_{L^\infty(\mathbb{R}^d,\,d\mu)}$ using Hölder's inequality and the Csiszár-Kullback inequality. This proves that

$$\|w(t,\cdot) - 1\|^2_{L^1(\mathbb{R}^d,\,d\mu)} \le \tfrac{1}{2}\, \|w_0 - 1\|_{L^\infty(\mathbb{R}^d,\,d\mu)}\, \sqrt{\mathcal{E}_1[w_0]}\, e^{-n\,t} \quad \text{as } t \to \infty$$

if (5) is satisfied initially. Still, this provides neither an estimate of $\int_{\mathbb{R}^d} w(t,\cdot)\, \log w(t,\cdot)\, d\mu$ nor a functional inequality which improves upon the logarithmic Sobolev inequality. To prove such an inequality, we keep following the strategy of [6]. A simple but key idea is to observe that the functions defined for any $p \in [1,2]$ by $h_p(0) = 1$, $h_p(1) = p/2$ and, for any $s \in (0,1) \cup (1,\infty)$ by $h_p(s) := [s^p - 1 - p\,(s-1)]/[(p-1)\,|s-1|^2]$ if $p > 1$, $h_1(s) := [s\,\log s - (s-1)]/|s-1|^2$, are continuous, nonnegative, decreasing on $\mathbb{R}^+$ and achieve their maximum at 0. Define on $L^\infty(\mathbb{R}^d)$ the functional

$$\mathcal{H}_p[w] := \|w\|^{2-p}_{L^\infty(\mathbb{R}^d)} \sup_{x \in \mathbb{R}^d} h_p(w(x)) = \|w\|^{2-p}_{L^\infty(\mathbb{R}^d)}\, h_p\left(\inf_{x\in\mathbb{R}^d} w(x)\right).$$

Theorem 2 (Improved logarithmic Sobolev inequality). *Assume that $w \in L^\infty_+(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w\, d\mu = 1$ and satisfies the condition $\int_{\mathbb{R}^d} w\, H_k\, d\mu = 0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. Then the following inequality holds, with optimal constant:*

$$\int_{\mathbb{R}^d} w\, \log w\, d\mu \le \frac{\mathcal{H}_1[w]}{n} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{w}\, d\mu\,.$$

Proof. We may indeed observe that by the Poincaré inequality and using the definition of $\mathcal{H}_1$, we get

$$\int_{\mathbb{R}^d} \frac{|\nabla w|^2}{w}\, d\mu \ge \frac{1}{\|w\|_{L^\infty(\mathbb{R}^d)}} \int_{\mathbb{R}^d} |\nabla w|^2\, d\mu \ge \frac{n}{\|w\|_{L^\infty(\mathbb{R}^d)}} \int_{\mathbb{R}^d} |w-1|^2\, d\mu \ge \frac{n}{\mathcal{H}_1[w]} \int_{\mathbb{R}^d} w \log w\, d\mu\,.$$

The optimality of the constant can be checked by a lengthy but elementary computation using the functions $w_\varepsilon^k := H_k(x)\,\chi(x\,\varepsilon^{1/(2n)}) + C_\varepsilon^k$ for some smooth truncation function χ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $B(0,1)$ and $\chi \equiv 0$ in $\mathbb{R}^d \setminus B(0,2)$. Here for $k \in \mathbb{N}^d$ is such that $|k| = n$ and the constant C_ε^k is chosen so that $\int_{\mathbb{R}^d} w_\varepsilon^k\, d\mu = 1$. □

As a consequence of the Maximum Principle applied to the heat equation (1) and the fact that to $u_0 = v_\infty$ corresponds a self-similar solution of (1), namely $u(t,x) = G(t+\frac{1}{2}, x, 0)$, we have the estimate

$$\mathcal{H}_1[w(t,\cdot)] \le \mathcal{H}_1[w_0] \quad \forall\, t \ge 0\,.$$

By applying Theorem 2, we obtain a new result of decay for $\mathcal{E}_1[w(t,\cdot)]$ with a constant which is exactly $\mathcal{E}_1[w_0]$, to the price of a rate which is less than $2\,n$.

Corollary 3 (Improved decay rate of the entropy). *Let w be a solution of* (4) *with a nonnegative bounded initial data $w_0 \in L^1(\mathbb{R}^d,\, d\mu)$ such that $\int_{\mathbb{R}^d} w_0\, d\mu = 1$ and* (5) *is satisfied. Then*

$$\mathcal{E}_1[w(t,\cdot)] \le \mathcal{E}_1[w_0]\, e^{-n\,t/\mathcal{H}_1[w_0]} \quad \forall\, t \ge 0\,.$$

This result is actually equivalent to Theorem 2, as follows by differentiating the above inequality at $t = 0$ (for which equality is trivially satisfied) and using the fact that $-\int_{\mathbb{R}^d} |\nabla w_0|^2/w_0\, d\mu = \frac{d}{dt}\, \mathcal{E}_1[w(t,\cdot)]_{|t=0} \le \mathcal{E}_1[w_0]\, \frac{d}{dt}\, e^{-n\,t/\mathcal{H}_1[w_0]}{}_{|t=0}$. What we have achieved is a global, improved exponential decay of the entropy $\mathcal{E}_1$ in a restricted class of functions. To simplify even further, for any $\varepsilon \in (0,1)$ and $n \in \mathbb{N}^*$, consider the set $\mathcal{X}^n_\varepsilon := \{w \in L^1(\mathbb{R}^d,\, d\mu)\, :\, 1-\varepsilon \le w \le 1+\varepsilon$ *a.e. and the condition* $\int_{\mathbb{R}^d} w\, H_k\, d\mu = 0$ *holds for any* $k \in \mathbb{N}^d$ *such that* $0 < |k| < n\,\}$, which is appropriate to handle the optimality case corresponding to $\varepsilon \to 0_+$. The best constant in Theorem 2 is indeed asymptotically equivalent to the sharp rate of convergence in Corollary 3, in the sense that $\lim_{\varepsilon\to 0_+} \inf_{w \in \mathcal{X}^n_\varepsilon} n/\mathcal{H}_1[w] = \lim_{\varepsilon \to 0_+} n/[(1+\varepsilon)\, h(1-\varepsilon)] = 2\,n$.

For simplicity, we have considered only the case $p = 1$, but the method also applies to any $p \in (1,2)$. We obtain an improved version of (7) under the restriction that $w \in L^1(\mathbb{R}^d, d\mu)$ is bounded nonnegative and the condition $\int_{\mathbb{R}^d} w\, H_k\, d\mu = 0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. With $\mathcal{B}_{n,1} = 4\,\mathcal{H}_1[w]/n$ and $\mathcal{B}_{n,2} = 1/n$, we get $\mathcal{B}_{n,p} \le \mathcal{K}[n,p,w] := (n\,(p-1))^{-1}\left[1 - ((2-p)/p)^{2\,\mathcal{H}_1[w]}\right]$ by (8). Using the entropy / entropy-production identity (9), the fact that $\mathcal{K}[n,p,w(t,\cdot)] \le \mathcal{K}[n,p,w_0]$ and the generalized Csiszár-Kullback inequality (6), we obtain

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\, e^{-\frac{4\,t}{p\,\mathcal{K}[n,p,w_0]}} \quad \text{and} \quad \|w-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\, e^{-\frac{2\,t}{p\,\mathcal{K}[n,p,w_0]}} \quad \forall\, t \ge 0\,. \tag{10}$$

Alternatively, an elementary computation as in the proof of Theorem 2 gives a similar result:

$$\frac{4}{p^2}\int_{\mathbb{R}^d}\left|\nabla w^{p/2}\right|^2 d\mu = \int_{\mathbb{R}^d} w^{p-2}\,|\nabla w|^2\, d\mu \ge \frac{1}{\|w\|_{L^\infty(\mathbb{R}^d)}^{2-p}}\int_{\mathbb{R}^d}|\nabla w|^2\, d\mu \ge \frac{n}{\|w\|_{L^\infty(\mathbb{R}^d)}^{2-p}}\int_{\mathbb{R}^d}|w-1|^2\, d\mu \ge \frac{n}{\mathcal{H}_p[w]}\,\mathcal{E}_p[w]$$

if $\int_{\mathbb{R}^d} w\, d\mu = 1$ and the condition $\int_{\mathbb{R}^d} w\, H_k\, d\mu = 0$ holds for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. This proves that

$$\mathcal{E}_p[w] \le \frac{4}{p^2}\,\frac{\mathcal{H}_p[w]}{n}\int_{\mathbb{R}^d}\left|\nabla w^{p/2}\right|^2 d\mu\,.$$

Using (9) and (6), this proves that any solution of (4) with initial data in $w_0 \in L^1 \cap L^\infty(\mathbb{R}^d, d\mu)$ satisfies

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\, e^{-n\,p\,t/\,\mathcal{H}_p[w_0]} \quad \text{and} \quad \|w-1\|_{L^p(\mathbb{R}^d,\,d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\; e^{-n\,p\,t/(2\,\mathcal{H}_p[w_0])} \quad \forall\, t \ge 0\,. \tag{11}$$

Comparing the rates of (10) and (11) is a natural question. In the limit $\varepsilon \to 0$, $\inf_{w\in\mathcal{X}^n_\varepsilon}\mathcal{H}_p[w] \sim \sup_{w\in\mathcal{X}^n_\varepsilon}\mathcal{H}_p[w] \to p/2$ and it follows that $\lim_{\varepsilon\to 0}\frac{4}{p\,\mathcal{K}[n,p,w_0]} = \frac{4}{p}\,n\,(p-1)/[1-((2-p)/p)^p] < 2\,n = \lim_{\varepsilon\to 0}\frac{n\,p\,t}{\mathcal{H}_p[w_0]}$. Hence, at least in the regime $\varepsilon \to 0$, (11) is a better estimate in terms of rates than (10). Undoing the change of variables (2), we have achieved a detailed result on improved u_0.

Corollary 4 (Improved intermediate asymptotics for the heat equation). *Let $p \in [1,2]$ and assume that u_0 is a probability measure such that $w_0 = u_0/v_\infty$ is bounded and satisfies the condition $\int_{\mathbb{R}^d} u_0\, H_k\, dx = 0$ for any $k \in \mathbb{N}^d$ such that $0 < |k| < n$. If u is the solution of* (1) *with initial condition u_0, then*

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_{L^p(\mathbb{R}^d)} \le (2\pi)^{-\frac{d}{2}\,(1-\frac{1}{p})}\,\mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\;(1+2\,t)^{-\frac{n\,p}{4\,\mathcal{H}_p[w_0]}-\frac{d}{2}\,(1-\frac{1}{p})} \quad \forall\, t \ge 0\,.$$

The proof relies on the remark that $\|u(t,\cdot) - u_\infty(t,\cdot)\|_{L^p(\mathbb{R}^d)} \le \|u_\infty(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{1}{p}}\,\|w(t,\cdot)-1\|_{L^p(\mathbb{R}^d,\,d\mu)}$ where $u_\infty(t,\cdot) := G(t+1/2,\cdot,0)$. The conclusion holds using $\|u_\infty(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = (2\pi\,R^2)^{-d/2}$ with $R = \sqrt{1+2\,t}$.

Up to now, we have considered the simple case of the harmonic potential, $V(x) = \frac{1}{2}\,|x|^2$. As in [1], the previous results can be extended to more general potentials as follows. Consider $V \in W^{1,2}_{\rm loc}\cap W^{2,2}_{\rm loc}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} e^{-V(x)}dx = 1$, and define the probability measure $d\mu(x) := e^{-V(x)}dx$ in $\mathbb{R}^d$, which generalizes the Gaussian measure. Under the above conditions on V, the logarithmic Sobolev inequality holds (resp. (7) for $p = 1$) for some positive constant (resp. for $\mathcal{B}_{1,1} > 0$). The Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + \nabla V\cdot\nabla$ is essentially self-adjoint on $L^2(d\mu)$, has a non-degenerate eigenvalue $\lambda_0 = 0$ and a spectral gap $\lambda_1 > 0$. According to [18, Theorem 2.1], N has a pure point spectrum without accumulation points. Since $\lim_{k\to\infty}\lambda_k = \infty$, then by [15, Theorem XIII.64], the eigenfunctions of N form a complete basis of $L^2(\mathbb{R}^d, d\mu)$. We shall denote the eigenvalues by λ_k, $k \in \mathbb{N}$, and by E_k the corresponding eigenspaces.

Theorem 2 adapts without changes. Assume that $w \in L^\infty_+(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w\, d\mu = 1$. Then

$$\int_{\mathbb{R}^d} w\,\log w\, d\mu \le \frac{\mathcal{H}_1[w]}{\lambda_n}\int_{\mathbb{R}^d}\frac{|\nabla w|^2}{w}\, d\mu$$

under the orthogonality condition: $w \in \left(\bigcup_{k=1}^{n-1} E_k\right)^\perp$, that is $\int_{\mathbb{R}^d} w\, f_k\, d\mu = 0$ for any $f_k \in E_k$, $k = 1, 2, \ldots n-1$. Next, consider the solution w of the Ornstein-Uhlenbeck equation

$$\frac{\partial w}{\partial t} = -\mathsf{N}\, w = \Delta w - \nabla V\cdot\nabla w\,, \tag{12}$$

with initial condition $w_0 \in \left(\bigcup_{k=1}^{n-1} E_k\right)^\perp \cap L^\infty(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} w_0\, d\mu = 1$. With the same definition as above for $\mathcal{E}_p$, for any solution of (12) with initial data w_0, (11) is now replaced by

$$\mathcal{E}_p[w(t,\cdot)] \le \mathcal{E}_p[w_0]\, e^{-\lambda_n\, p\, t/\, \mathcal{H}_p[w_0]} \quad \text{and} \quad \|w-1\|_{L^p(\mathbb{R}^d,\, d\mu)} \le \mathcal{A}_p\left(\mathcal{E}_p[w_0]\right)\; e^{-\lambda_n\, p\, t/\,(2\, \mathcal{H}_p[w_0])} \quad \forall\; t \ge 0\,.$$

Let us conclude this letter by some comments and open questions. It is standard in entropy / entropy-production methods that determining sharp rates of convergence in an evolution equation is equivalent to finding sharp constants in functional inequalities, as we have seen in the case of the heat equation: the rate of convergence in $L^2(\mathbb{R}^d,\, d\mu)$ is given by the Poincaré inequality, while the rate of convergence in entropy, which controls the $L^1(\mathbb{R}^d,\, d\mu)$ norm, is related with the logarithmic Sobolev inequality. This is also true for nonlinear diffusion equations, see for instance [12]. In this case, a breakthrough came from the observation that uniform norms can also be used, see [10, 7, 6], to the price of a restricted functional framework. This allows to relate nonlinear quantities of entropy type with spectral properties of the linearized problem, in an appropriate functional space and, again, to relate sharp rates with best constants, see [8]. As long as nonlinear evolution problems are concerned, only a few invariant quantities are usually available: the mass and the position of the center of mass of the solution, for instance. In linear evolution problems, we can impose an arbitrary number of orthogonality conditions, which are preserved along the evolution. Improved rates of convergence are then expected, even when measured with nonlinear quantities like the entropy. Various attempts have been done, see for instance [2], but the question has been left open for many years. Such ideas have been partially explored by R.J. McCann, including in the linear case (see [11]), based on considerations on an appropriate Hessian matrix. Our approach provides a simpler and elementary answer under restrictions which are natural in view of [6]. It also raises a number of questions concerning the optimality of the new functional inequalities from a variational point of view, the convergence of minimizing sequences and the symmetry of the eventual minimizers.

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