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Comparative Risk Aversion: A Formal Approach with Applications to Savings Behaviors*

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Abstract

We consider a formal approach to comparative risk aversion and applies it to intertemporal choice models. This allows us to ask whether standard classes of utility functions, such as those inspired by Kihlstrom and Mirman (1974), Selden (1978), Epstein and Zin (1989) and Quiggin (1982) are well-ordered in terms of risk aversion. Moreover, opting for this model-free approach allows us to establish new general results on the impact of risk aversion on savings behaviors. In particular, we show that risk aversion enhances precautionary savings, clarifying the link that exists between the notions of prudence and risk aversion.

Keywords: risk aversion, savings behaviors, precautionary savings.

JEL codes: D11, D81, D91.

1 Introduction

A common approach to study the role of risk aversion is to consider a particular class of preferences, presumably well-ordered in term of risk aversion, and then analyze the decisions that result from preferences within this class. In the context of intertemporal choice, a number of different classes of utility functions have been considered. The most popular choice

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consists of preferences à la Epstein and Zin (1989), while the framework in Kihlstrom and Mirman (1974) and Quiggin's (1982) anticipated utility theory provide alternative settings.

It occurs that predictions about the impact of risk aversion radically depends on the model that is chosen. For example, regarding the relation between risk aversion and precautionary savings in a simple two-period model, the preferences in Kihlstrom and Mirman (1974) and Quiggin (1982) lead to the conclusion that precautionary savings rise with risk aversion (Drèze and Modigliani (1972), Yaari (1987), and Bleichrodt and Eeckhoudt (2005)). On the contrary, this relation is ambiguous when Epstein and Zin's (1989) preferences are used (Kimball and Weil (2009)).[1]

The current paper makes three contributions. First, it discusses the extent to which the utility classes mentioned above are well-ordered in terms of risk aversion. In particular, we show that, when we consider marginal variations in risk, Epstein and Zin preferences are not well-ordered. Second, we suggest a model-free approach that makes it possible to discuss the role of risk aversion without focusing on any specific model of rationality. Third, we apply this setup to establish new general results on the role of risk aversion. In particular, we show that risk aversion enhances precautionary savings, clarifying the link that exists between risk aversion and prudence.

Our paper relies on an abstract procedure to define comparative risk aversion, which assumes no particular structure for the set of consequences. This definition is inspired by the seminal work of Yaari (1969). In a very natural way, this approach states that if a given increase in risk is perceived as worthwhile for a decision maker (because it yields a higher level of ex ante welfare), it should also be so for any less risk-averse decision maker.

A considerable number of papers have used Yaari's approach to define comparative risk- (or uncertainty-) aversion. This is explicitly the case in Kihlstrom and Mirman (1974), Ghirardato and Marinacci (2002) and Grant and Quiggin (2005). This is also implicit in the papers that have focused on certainty equivalents, such as Chew and Epstein (1990) and Epstein and Zin (1989), as well as in Pratt's (1964) original definitions based on risk and probability premia. In most cases, although Grant and Quiggin (2005) is a noteworthy exception, these papers (implicitly or explicitly) rely on minimalist risk orderings, where random objects are only compared to deterministic constructs. Our paper departs from this minimalist approach to provide novel insights. Instead of focusing on certainty equivalents to assess the individual's degree of risk aversion, we also account for individual preferences over marginal variations in risk. Arguably, this is a key requirement for any applicable concept, since in real life individuals never have the possibility of opting for a completely-deterministic

[1]More details on the meaning of the preferences *à la* Kihlstrom and Mirman, Quiggin, and Epstein and Zin are provided in Section 2.

life. Moreover this concurs with the general view in Economics that concepts which allow us to deal with variations "in the small and in the large" are of greater interest than those which focus exclusively on some particular "large" variations.[2]

The notion of comparative aversion that we derive when considering marginal risk variations is stronger than that focusing on certainty equivalents. In consequence, although preferences may be well ordered in terms of risk aversion when considering certainty equivalents, this may no longer hold when considering the more stringent comparison of marginal risk variations. This turns out to be the case for Epstein and Zin preferences, which would therefore appear to be ill-adapted for the analysis of risk aversion. No similar case can be made against Kihlstrom and Mirman or Quiggin's anticipated utility functions. These utility classes actually seem to be well-suited to provide insights into the impact of risk aversion, even when considering relatively broad definitions for what is meant by an increase in risk.

Abandoning these standard but somewhat restrictive frameworks, we here establish a general result allowing us to make predictions about the impact of risk aversion without assuming any particular form of rationality. This result is powerful, as it shows that it is possible to determine the impact of risk aversion under relatively weak assumptions on ordinal preferences, as long as states of the world can be ranked from bad to good independently of the agent's action. The intuition behind this result is that risk aversion enhances the willingness to redistribute from good to bad states.

We provide a direct application to savings under uncertainty. In particular we prove, under weak conditions on ordinal preferences, that risk aversion enhances precautionary savings. Moreover, we show that risk aversion has a negative (resp. positive) impact on savings when the rate of return is uncertain, as soon as the intertemporal elasticity of substitution is larger (resp. smaller) than one. Risk aversion is also found to have a negative impact on savings when the lifetime is uncertain, therefore underlining that the relation between time preference, risk aversion and mortality risk discussed in Bommier (2006) is general and is not restricted to the expected-utility framework.

The remainder of the paper is organized as follows. In Section 2, we present several classes of utility functions that have been used to analyze the role of risk aversion in intertemporal models. The main theoretical contents appear in Section 3, which is split up into several subsections. Subsection 3.1 introduces the relevant concepts, and Subsection 3.2 then focuses on the simplest random objects that we can think of: "heads or tails" gambles, which are lotteries with two equally-probable outcomes. In this case, all of the reasonably conceivable

[2]The contrast between the Arrow-Pratt coefficient of risk aversion (allowing us to consider small or large risks) and Aumann and Kurz's (2005) "Fear of ruin" coefficient (bearing exclusively on bets involving the loss of all wealth) is another example of the greater applicability of the marginal approach. Foncel and Treich (2005) discuss this issue.

definitions of comparative risk aversion coincide, and there is no possible dispute regarding the generality of the conclusions. This minimalist approach is sufficient to show that Epstein and Zin preferences are not well-ordered in terms of risk aversion. To increase applicability, the analysis is extended in Section 3.3 to continuous lotteries. We define a formal notion of comparative risk aversion and show how it can be used to obtain model-free results on the impact of risk aversion. A number of applications providing insights into the impact of risk aversion on savings behavior are then developed in Section 4.

To help the reader to grasp the paper's main message, we restrict the use of the term *Proposition* to the most significant results. The paper also includes other statements, which are useful for general understanding, or for the relation of our work to that of others, but which are admittedly less important or original. These are labeled as *Result*.

2 Popular classes of utility functions disentangling risk aversion from intertemporal substitution

Before moving on to the core of the paper we describe in this section a number of approaches that have been suggested in the literature to the analysis of risk aversion in intertemporal frameworks. This discussion is restricted to preferences over "certain $\times$ uncertain" consumption pairs as in Selden (1978) and many other papers on precautionary savings. A "certain $\times$ uncertain" consumption pair will be denoted $(c_1, \widetilde{c}_2)$, where the tilde emphasizes that the second element is random.

Kihlstrom and Mirman (1974) convincingly explain that the comparison of agents' risk aversions is possible if and only if agents have identical preferences over certain prospects. We shall therefore focus on utility classes involving different risk attitudes, while leaving preferences over certain consumption paths unchanged. This rules out the standard class of expected-utility models assuming additively-separable utility functions. Under additive separability, it is impossible to change risk preferences, without affecting ordinal preferences.[3]

We therefore consider three extensions of the standard additively-separable expected-utility model, where risk aversion can be analyzed without affecting ordinal preferences.

[3] One extremely popular representation is the following additive expected utility specification:

$$U(c_1, \widetilde{c}_2) = \frac{c_1^{1-\rho}}{1-\rho} + \mathbb{E}\left[\frac{\widetilde{c}_2^{1-\rho}}{1-\rho}\right]$$

where the parameter ρ is interpreted as reflecting the agent's risk aversion. However, changing ρ involves changing ordinal preferences (in particular, the intertemporal elasticity of substitution, which equals $\frac{1}{\rho}$) and cannot be used to analyze the impact of risk aversion. Discussions about this can be found for example in Kihlstrom and Mirman (1974) and Epstein and Zin (1989).

This is not of course an exhaustive review of what can be found in the literature, but rather focuses on the most popular specifications. The first setup, which assumes expected utility, was suggested by Kihlstrom and Mirman (1974). The second one was introduced by Selden (1978), building on the framework in Kreps and Porteus (1978), and was then extended by Epstein and Zin (1989) to deal with infinitely-long consumption streams, and appears to be a very convenient way of studying many intertemporal problems. This has now become by far the most popular approach to the analysis of risk aversion in intertemporal frameworks. The third class, based on Quiggin's (1982) anticipated-utility theory, is used in Yaari (1987), Segal, Spivak and Zeira (1988), and Bleichrodt and Eeckhoudt (2005), for example.

The initial contributions of Kihlstrom and Mirman (1974), Selden (1978) and Quiggin (1982) were very general, and were not limited to the analysis of intertemporal choices. No assumptions were made about ordinal preferences. However, applied work on savings often assumes that preferences over certain consumption paths are additively separable. The (ordinal) utility $U(c_1, c_2)$ associated with the certain consumption profile (c_1, c_2) is expressed as the sum of the utilities associated with the first-period and second-period consumptions.

$$U(c_1, c_2) = u_1(c_1) + u_2(c_2) \tag{1}$$

We include this assumption of the additive separability of ordinal preferences in our definitions of what we call "Kihlstrom and Mirman", "Selden" or "Quiggin" utility functions. At this point, it should therefore be clear that our terminology is only indicative of the general framework in which these particular specifications may be related. However, we do not aim to provide a complete account of the contributions of the corresponding papers, which consider both much broader utility classes and more complex settings.

Definition 1 (Utility classes) *A utility function $U(c_1, \widetilde{c}_2)$ is called:*

- *A Kihlstrom and Mirman utility function (denoted U_k^{KM}) if there exist continuous increasing real functions u_1, u_2 and k such that:*

$$U_k^{KM}(c_1, \widetilde{c}_2) = k^{-1}\left(\mathbb{E}\left[k\left(u_1(c_1) + u_2(\widetilde{c}_2)\right)\right]\right)$$

- *A Selden utility function (denoted U_v^S) if there exist continuous increasing real functions u_1, u_2 and v such that:*

$$U_v^S(c_1, \widetilde{c}_2) = u_1(c_1) + u_2\left(v^{-1}\left(\mathbb{E}[v(\widetilde{c}_2)]\right)\right)$$

- *A Quiggin utility function (denoted U_ϕ^Q) if there exist continuous real functions u_1, u_2*

and a continuous increasing function $\phi : [0,1] \longrightarrow [0,1]$, with $\phi(0) = 0$ and $\phi(1) = 1$, such that:

$$U_{\phi}^{Q}(c_1, \widetilde{c}_2) = u_1(c_1) + \mathbb{E}_{\phi}[u_2(\widetilde{c}_2)]$$

where $\mathbb{E}_{\phi}[\cdot]$ denotes the Choquet expectation operator associated with ϕ. For any real random variable $\widetilde{z}$ characterized by the cumulative distribution function $z \mapsto F(z)$, this operator assigns:

$$\mathbb{E}_{\phi}[\widetilde{z}] = -\int_{-\infty}^{+\infty} z\, d\left(\phi\left(1 - F(z)\right)\right)$$

The Kihlstrom and Mirman, Selden and Quiggin utility functions rank certain consumption pairs (c_1, c_2) in the same way, respecting the ordering given by the utility function U in equation (1).

One popular specification results from choosing isoelastic functions in the Selden utility function. Setting $u_1(c) = u_2(c) = \frac{c^{1-\rho}}{1-\rho}$ and $v(x) = \frac{x^{1-\gamma}}{1-\gamma}$ yields a class of utility functions assuming a constant intertemporal elasticity of substitution and homothetic preferences. Such utility functions are often called "Epstein and Zin utility functions", although they indicate only imperfectly what can be found in Epstein and Zin (1989), who consider preferences over infinitely-long consumption paths, which is a much more complex issue. Even so, this terminology has become very popular in the Economic literature, and we think it is more productive and less confusing to adhere to it, rather than introducing a new one.

Definition 2 *A utility function $U(c_1, \widetilde{c}_2)$ is called an Epstein and Zin utility function (denoted U_{γ}^{EZ}) if there exist positive scalars $\rho \neq 1$ and $\gamma \neq 1$ such that:*[4]

$$U_{\gamma}^{EZ}(c_1, \widetilde{c}_2) = \frac{c_1^{1-\rho}}{1-\rho} + \frac{1}{1-\rho}\left(\mathbb{E}\left[\widetilde{c}_2^{1-\gamma}\right]\right)^{\frac{1-\rho}{1-\gamma}}$$

Certainty-equivalent arguments have been applied to suggest that these Kihlstrom and Mirman, Selden, Epstein and Zin, and Quiggin utility functions are well-suited for the analysis of risk aversion. It can indeed easily be shown that the greater is the concavity of k (for Kihlstrom and Mirman utility functions), the greater is the concavity of v (for Selden utility functions), the greater is the scalar γ (for Epstein and Zin utility functions) and the greater is the convexity of ϕ (for Quiggin utility functions), the smaller (in terms of ordinal utility) is the certainty equivalent assigned to any random element $(c_1, \widetilde{c}_2)$. In formal terms, for any certain consumption path (C_1, C_2) and any "certain $\times$ uncertain" consumption pair $(c_1, \widetilde{c}_2)$, then the following hold:

[4]The extension to the cases where $\rho = 1$ or $\gamma = 1$ could easily be considered, but is ruled out here to avoid the systematic discussion of these particular cases.

1. $U^{KM}_{k_1}(c_1, \widetilde{c}_2) = U^{KM}_{k_1}(C_1, C_2) \Rightarrow U^{KM}_{k_2}(c_1, \widetilde{c}_2) \leq U^{KM}_{k_2}(C_1, C_2)$ for all functions k_2 which are more concave than the function k_1.[5]

2. $U^{S}_{v_1}(c_1, \widetilde{c}_2) = U^{S}_{v_1}(C_1, C_2) \Rightarrow U^{S}_{v_2}(c_1, \widetilde{c}_2) \leq U^{S}_{v_2}(C_1, C_2)$ for all functions v_2 which are more concave than v_1.

3. $U^{EZ}_{\gamma_1}(c_1, \widetilde{c}_2) = U^{EZ}_{\gamma_1}(C_1, C_2) \Rightarrow U^{EZ}_{\gamma_2}(c_1, \widetilde{c}_2) \leq U^{EZ}_{\gamma_2}(C_1, C_2)$ for all real parameters $\gamma_2 > \gamma_1$.

4. $U^{Q}_{\phi_1}(c_1, \widetilde{c}_2) = U^{Q}_{\phi_1}(C_1, C_2) \Rightarrow U^{Q}_{\phi_2}(c_1, \widetilde{c}_2) \leq U^{Q}_{\phi_2}(C_1, C_2)$ for all functions ϕ_2 which are more convex than ϕ_1.

Strict inequalities are moreover obtained as soon as $\widetilde{c}_2$ is a non degenerate random element. The fact that one agent systematically has lower certainty equivalents than another one has often been interpreted as an indication of greater risk aversion. In fact, many papers, including Pratt (1964), Yaari (1969), Kihlstrom and Mirman (1974), Chew and Epstein (1990), Epstein and Zin (1989) precisely define the notion of "more risk-averse than" by considering certainty equivalents. These utility classes are then considered by the authors to be well-ordered in terms of "risk aversion", with "risk aversion" related to the concavity (or convexity) of the functions k, v or ϕ, or the magnitude of the parameter γ.

The argument that we develop in this paper, in particular in Section 3, is that a more risk-averse agent must indeed have lower certainty equivalents, but that considerations about certainty equivalents are not sufficient to yield conclusions regarding comparative risk aversion. We argue that agent A having systematically lower certainty equivalents than agent B is a necessary condition for A being more risk-averse than B, but not a sufficient one. Considering a less-restrictive definition of comparative risk aversion implies that some of the above classes of utility functions may be well ordered in terms of risk aversion, while others may not be.

We provide two examples to illustrate that relying on different utility classes may yield different conclusions regarding the impact of risk aversion. A first example comes from the literature on precautionary savings. In a two-period consumption model, precautionary saving is the optimal amount of saving when second-period income is uncertain minus savings when income risk can be fully insured. Previous classes of utility functions in Definition 1 yield contrasting results about the possible impact of risk aversion. For Kihlstrom and Mirman utility functions, it is straightforward to conclude from Drèze and Modigliani (1972) (at least for small risks) that precautionary savings increase with the concavity of k as long as

[5] A function g_2 is said to be more concave (resp. convex) than a function g_1 if there exists a concave (resp. convex) function g such that $g_2(x) = g(g_1(x))$.

first-period consumption is a normal good. Risk aversion would then increase precautionary saving. A similar result is obtained by Bleichrodt and Eeckhoudt (2005) for Quiggin utility functions. On the contrary, Kimball and Weil (2009) prove in their Proposition 7 that the amount of precautionary savings is not monotonic in γ, for Epstein and Zin preferences. This suggests that there is no simple relationship between risk aversion and precautionary savings.

A second example concerns savings when the rate of return is random. The impact of risk aversion with Kihlstrom and Mirman preferences is actually one of the applications treated in their seminal contribution. It is proved that risk aversion increases or decreases optimal savings when the return on saving is uncertain, according to whether the intertemporal elasticity of substitution is smaller or greater than one. This finding is contradicted by Langlais (1995), who shows that no such result holds in the Selden framework. Again, this illustrates that the above classes of utility functions may lead to divergent conclusions regarding the impact of risk aversion on saving behavior. However, we shall see that for both problems – savings with uncertain incomes, and savings with uncertain returns – the role of risk aversion becomes unambiguous and particularly intuitive once a formal and general sense is given to comparative risk aversion.

3 Theory

3.1 Common features

As noted above, a number of papers, including Pratt (1964), Yaari (1969), Kihlstrom and Mirman (1974), Epstein and Zin (1989), and Chew and Epstein (1990), have focused on how agents compare lotteries to certain outcomes in order to quantify and compare risk aversion. Typically, one agent is said to be more risk-averse than another if the former systematically associates lower certainty equivalents to random objects than does the latter. Considering how agents compare random to certain outcomes is one way of measuring the agent's willingness to avoid uncertainty. However, we believe that in order to be applicable, "being more risk-averse" should mean "greater aversion to increases in risk", and not only a greater willingness to avoid all uncertainty. In other words, in order to assess risk aversion, we should not only focus on the agent's comparison of lotteries to certain outcomes, but also consider how he compares non-degenerate lotteries that are more or less risky.

In order to proceed in this way, we need a definition of "being riskier than" which is not a trivial issue. As explained in Chateauneuf, Cohen, and Meilijson (2004), the literature on monetary lotteries has not reached a consensus on what an increase in risk is. They review different notions, which are shown to yield different predictions regarding the role of risk aversion. One way of overcoming these difficulties is to focus on risk comparisons that are consensual. If it is difficult to find a convincing universal definition of an increase in risk, it is possible to restrict the analysis to cases where risk comparison is unambiguous. Yaari (1969) implicitly proceeds in such a way, with an extremely minimalist definition of "riskier than" where a lottery l_1 is said to be riskier than a lottery l_2 if and only if l_2 is a degenerate lottery providing a given outcome with certainty. Focusing on certainty equivalents as in Pratt (1964), Kihlstrom and Mirman (1974), Epstein and Zin (1989), and Chew and Epstein (1990) is equivalent to using Yaari's approach, as long as all lotteries have certainty equivalents.

Our paper takes a similar line, but departs from Yaari's minimalist approach which fails to account for many unambiguous risk comparisons. There are indeed many cases where two non-degenerate lotteries can be unambiguously compared in terms of riskiness. We account in this paper for some of these cases, which allows us to define a notion of comparative risk aversion which is stronger than that in Yaari, and which leads to interesting predictions regarding the impact of risk aversion in many concrete problems.

The interest of our approach is particularly clear when considering the simplest random objects of "heads or tails" gambles, i.e. lotteries which have only a good and a bad payoff, each equally likely. In this case, acknowledging that improving the good outcome and reducing the bad outcome corresponds to an increase in risk is sufficient to provide a unique ordering in terms of risk aversion. This ordering is stronger than that obtained from Yaari's approach, with implications for the relevance of the standard classes of utility functions for analyzing the role of risk aversion, as explained in Section 3.2.3.

The arguments in the case of continuous lotteries are somewhat more complex, but are necessary for the derivation of general results. These arguments are presented in Section 3.3.

3.1.1 The setting

This section sets out the common setting for both heads or tails gambles and continuous lotteries.

State and lottery sets. We consider an abstract space set X endowed with an ordinal preference relation $\succeq$. Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, Pr)$, where Ω is the sample space including all states of the world (it is countable in the case of heads or tails, but not for continuous lotteries), $\mathcal{F}$ is the $\sigma-$algebra of events, which are subsets of Ω, and Pr is the associated probability measure. Lotteries are measurable functions from the sample space Ω to the state space X. We denote by $L(X)$ the set of lotteries with outcomes in X. $L \in L(X)$ is a random variable, while $L(\omega) \in X$ with $\omega \in \Omega$ represents the realization of the lottery when state ω occurs. We denote by $\delta_x \in L(X)$ a degenerate lottery, which pays off $x \in X$ with certainty.

Risk preferences. We consider two agents A and B with respective preferences $\succeq^A$ and $\succeq^B$ over a subset Y of $L(X)$. This set Y may be equal to $L(X)$ but for greater generality we only assume that Y includes the set of degenerate gambles.

$$\{\delta_x | x \in X\} \subset Y \subset H(X)$$

We assume that the risk preferences $\succeq^A$ and $\succeq^B$ are consistent with ordinal preferences in the following sense:

Assumption A (Consistency with ordinal preferences) *Preferences over gambles are consistent with ordinal preferences if:*

$$x \succeq y \Leftrightarrow \delta_x \succeq^i \delta_y \text{ for all } x, y \in X \text{ and } i = A, B$$

Agents A and B should therefore rank the degenerate lotteries in exactly the same way as ordinal preferences rank outcomes. As a result, A and B agree on the ranking of degenerate lotteries. Without reproducing the discussion in Kihlstrom and Mirman (1974) and Epstein and Zin (1989), we take for granted that agents are comparable in terms of risk aversion if and only if they have the same ordinal preferences.

Another natural property when considering risk preferences is *ordinal dominance*, as formalized for example in Chew and Epstein (1990). The intuition behind this property is simple: if one lottery always provides a better outcome than another (whatever the state of the world), this lottery should be preferred. Formally:

Definition 3 (Ordinal Dominance) *Preferences over gambles $\succeq^i$ ($i = A, B$) fulfill ordinal dominance when we have for any lotteries $L, L' \in Y$:*
(i) if for all $\omega \in \Omega$, $L(\omega) \succeq L'(\omega)$ then $L \succeq^i L'$,
(ii) moreover, if there exists $\omega \in \Omega$ such that $L(\omega) \succ L'(\omega)$, then $L \succ^i L'$.

According to the definition of ordinal dominance, a first-order stochastically dominated lottery should not be preferred. Moreover, a first-order stochastically dominating lottery is strictly preferred if and only if it pays a strictly better outcome in at least one state of the world.

It can be argued that this is a reasonable requirement for defining rational risk preferences. However, as some popular preferences (such as Selden and Epstein and Zin preferences) do not satisfy this property (see for example the discussion in Chew and Epstein (1990)), we do not systematically make this assumption, but mention it whenever necessary.

3.1.2 A formal definition of comparative risk aversion

We now make clear the procedure we use to give a sense to risk-aversion comparisons, when we consider a general set of outcomes. Intuitively, an agent A will be said to be more risk-averse than an agent B, if any increase in risk that is considered to be desirable by A is also considered so by B. This procedure is general in the sense that it is valid in both the heads or tails and continuous lottery setups. However, the definition of an "increase in risk" is different across these setups, which affects the consequences of being more risk-averse.

Formally speaking, we simply suppose that there exists a binary relation defined over the lottery set Y, which we denote by R. This relation is interpreted as "riskier than" and more precisely as "at least as risky as". For example, for $L, L' \in Y$, $L\,R\,L'$ means that the lottery L is (weakly) riskier than L'. The relationship R is supposed to be reflexive and transitive, thus defining a partial preorder. Using this partial preorder R, we set out our definition of comparative risk aversion.

Definition 4 (Comparative risk aversion) *Let R be a partial preorder "riskier than" defined over the lottery set Y. A is more (weakly) risk-averse than B with respect to R if for all $L, L' \in Y$:*

$$L\,R\,L' \text{ and } L \succeq^A L' \implies L \succeq^B L'$$

This definition states that any riskier lottery, which is preferred by the more risk-averse agent is also preferred by the less risk-averse agent. This definition has of course to be completed with a reasonable notion of "riskier than". By construction the above definition is weak, in the sense that it is reflexive. Every agent is more risk-averse than himself.

We now specify the sample space Ω and the binary relation R.

3.2 Theory, Part 1: Heads or tails gambles

We give more structure to the above setting when we restrict our attention to heads or tails gambles. We then consider how standard utility classes are ordered in terms of risk aversion.

3.2.1 The setting

We suppose that the sample space is reduced to heads or tails: $\Omega = \{h, l\}$ and that both states h and l occur with the same probability of $\frac{1}{2}$. We denote by $H(X)$ the set of heads or tails gambles with outcomes in X. An element of $H(X)$ denoted $(x^l; x^h)$ is the lottery yielding $x^l \in X$ with probability 0.5 and $x^h \in X$ with probability 0.5. For sake of simplicity and without loss of generality, we suppose that the first outcome is not better than the second: $x^h \succeq x^l$. The element $(x; x)$ is the degenerate gamble which yields $x \in X$ with certainty. To clearly distinguish between these very simple binary lotteries and the more general lotteries that we will introduce in Section 3.3, we call the simple binary lotteries *gambles*, while the more general ones are called *lotteries*.

We consider two agents A and B with preferences $\succeq^i$ over gambles in Y that are consistent with ordinal preferences (Assumption A).

3.2.2 Comparative riskiness

The procedure that compares preferences in terms of risk aversion (Definition 4) presupposes the existence of a relation R allowing us to compare the riskiness of the gambles. However, we will see that we do not necessarily need to fully explicit the partial preorder R to derive non-trivial results. Minimal requirements about R may be sufficient, at least when ordinal dominance is assumed.

We take advantage of the very basic structure of gambles to discuss what could be reasonable relations R. Consider two gambles $(x_l; x_h)$ and $(y_l; y_h)$ of the set Y, and assume (without loss of generality) that the outcomes are ordered: $x_h \succeq x_l$ and $y_h \succeq y_l$. The four possible combinations are then:

Case 1: $(x_h \succ y_h$ and $x_l \succeq y_l)$ or $(x_h \succeq y_h$ and $x_l \succ y_l)$

Case 2: $(y_h \succ x_h$ and $y_l \succeq x_l)$ or $(y_h \succeq x_h$ and $y_l \succ x_l)$

Case 3: $x_h \succeq y_h \succeq y_l \succeq x_l$

Case 4: $y_h \succeq x_h \succeq x_l \succeq y_l$

In Case 1, the gamble $(x_l; x_h)$ strictly first-order dominates $(y_l; y_h)$. We can think of a number of different ways of comparing the riskiness of $(x_l; x_h)$ and $(y_l; y_h)$. However, unless we allow for risk preferences that do not satisfy ordinal dominance, the comparison of such lotteries is not of any interest for comparative risk aversion. We therefore remain agnostic about risk comparison in this case, and allow that a variety of judgments can be made. Case 2 is symmetric to Case 1.

We now consider Case 3. The lucky outcome of the gamble $(x_l; x_h)$ is better than that of $(y_l; y_h)$, but the unlucky outcome is worse; Case 4 is symmetric. We define below the notion of spread, which describes these cases:

Definition 5 (Gamble spread) *The gamble $(x_l; x_h)$ is a spread of $(y_l; y_h)$, which is denoted by $(x_l; x_h) \vdash (y_l; y_h)$, if the following relationship holds:*

$$(x_l; x_h) \vdash (y_l; y_h) \Longleftrightarrow x_h \succeq y_h \succeq y_l \succeq x_l$$

(We suppose that, in each gamble, the outcomes are ordered).

Assume that $(x_l; x_h)$ is a spread of $(y_l; y_h)$. Choosing $(x_l; x_h)$ instead of $(y_l; y_h)$ involves taking the chance of being in a better position if the odds are good, but ending up in a worse situation if the odds are bad. It is then undisputable that $(x_l; x_h)$ is riskier than $(y_l; y_h)$. Note that if the preference relation $\succeq$ is represented by a utility function, $(x_l; x_h) \vdash (y_l; y_h)$ implies that the distribution of ex-post utilities associated with the gamble $(x_l; x_h)$ is more dispersed than that in the gamble $(y_l; y_h)$ in the strong sense of Bickel and Lehman (1976), whatever the utility function chosen to represent $\succeq$. The fact that $(x_l; x_h)$ is considered to be riskier than $(y_l; y_h)$ whenever $(x_l; x_h) \vdash (y_l; y_h)$ is therefore particularly robust: it is not restricted to any particular choice of ex-post utility dispersion, and is independent of the cardinality.

While there may be disagreement about the relative riskiness of $(x_l; x_h)$ and $(y_l; y_h)$ in Cases 1 and 2, a minimal requirement for any "riskier than" relation R is that it respects the ranking of the spread relationship $\vdash$. We additionally impose that if the gamble $(x_l; x_h)$ is a spread of $(y_l; y_h)$ in a strict sense (that is with either $x_h \succ y_h$ or $y_l \succ x_l$) then $(y_l; y_h)$ cannot be considered to be riskier than $(y_l; y_h)$. We call this requirement *spread compatibility*:

Definition 6 (Spread compatibility) *A partial-order "riskier than" R is spread compatible if and only if:*

(i) $(x_l; x_h) \vdash (y_l; y_h) \Rightarrow (x_l; x_h)R(y_l; y_h)$

(ii) If $(x_l; x_h) \vdash (y_l; y_h)$ and $x_h \succ y_h$ or $y_l \succ x_l$ then it cannot be the case that $(y_l; y_h)R(x_l; x_h)$.

The following result states that spread compatibility and ordinal dominance are sufficient to define an unambiguous measure of comparative risk aversion, at least for heads or tails gambles.

Result 1 *We consider two agents A and B with preferences satisfying the Definition 3 of ordinal dominance. If agent A is more risk-averse than agent B with respect to a* spread-compatible *partial order R, then he will also be with respect to any other* spread compatible *partial order R'.*

Proof. We consider two gambles $(x_l; x_h)$ and $(y_l; y_h)$, with $(x_l; x_h)R'(y_l; y_h)$ and $(x_l; x_h) \succeq^A (y_l; y_h)$. We want to prove that $(x_l; x_h) \succeq^B (y_l; y_h)$. Following our previous remarks, there are at most four possibilities:

1. $(x_l; x_h)$ strictly first-order dominates $(y_l; y_h)$. Due to the ordinal-dominance assumption (Definition 3) $(x_l; x_h) \succeq^B (y_l; y_h)$.

2. $(x_l; x_h)$ strictly first-order dominates $(y_l; y_h)$. We can rule out this possibility, because the ordinal-dominance assumption implies that $(y_l; y_h) \succ^A (x_l; x_h)$, which contradicts our initial assumption $(x_l; x_h) \succeq^A (y_l; y_h)$.

3. $(x_l; x_h) \vdash (y_l; y_h)$. This implies that $(x_l; x_h)R(y_l; y_h)$, since R is spread compatible. A is more risk-averse than B relative to R, which means that $(x_l; x_h) \succeq^B (y_l; y_h)$ by the Definition 4 of comparative risk aversion.

4. $(y_l; y_h) \vdash (x_l; x_h)$. This implies by spread compatibility that $(y_l; y_h)R'(x_l; x_h)$. Since $(x_l; x_h)R'(y_l; y_h)$ also holds, this implies that $x_l \sim y_l$ and $x_h \sim y_h$ (part (ii) of the definition of spread compatibility), and thus $(x_l; x_h) \succeq^B (y_l; y_h)$ from the ordinal-dominance property .

■

As soon as we assume ordinal dominance, it is not necessary to fully explicit the relation R in order to obtain a universal sense for being "more risk-averse than". The agreement over the statement that a spread involves an increase in risk is sufficient to define uniquely the notion of comparative risk aversion. We examine below whether the utility classes mentioned in Section 2 are well-ordered with respect to this comparative risk aversion relation. When they do not satisfy ordinal dominance, we consider whether they are well-ordered with respect to some spread-compatible relation R.

3.2.3 Application to standard classes of preferences over certain $\times$ uncertain consumption pairs

We specify our setting to ensure compatibility with the utility classes defined in Section 2. The set of outcomes X is the set of admissible two-period consumption profiles. A typical element of X is (c_1, c_2), where c_1 is first-period and c_2 second-period consumption.

We restrict our attention to binary gambles, where first-period consumption is certain. The sole source of uncertainty concerns second-period consumption, which may be either low (state l) or high (state h). The set of such binary gambles, with certain first-period consumption, will be denoted Y. An element of Y is denoted $(c_1, (c_2^l, c_2^h))$, where c_1 is certain first-period consumption, while (c_2^l, c_2^h) is the binary gamble over second-period consumption. Second-period consumption is low at c_2^l and high at c_2^h, both with probability $\frac{1}{2}$.

The preferences associated with Kihlstrom and Mirman, and Quiggin anticipated utility functions satisfy ordinal dominance. From Result 1, every spread-compatible relation R therefore yields identical conclusions about comparative risk aversion within these utility classes. The following result characterizes the comparative risk aversion ordering in both frameworks.

Result 2 (Standard utility classes and risk aversion) *The following characterization holds:*

1. *An agent with utility function $U_{k_A}^{KM}$ is more risk-averse than an agent with utility function $U_{k_B}^{KM}$ with respect to any spread-compatible relation R if and only if k_A is more concave than k_B.*

2. *An agent with utility function $U_{\phi^A}^{Q}$ is more risk-averse than agent with utility function $U_{\phi^B}^{Q}$ with respect to any spread-compatible relation R if and only if $\phi^A(\frac{1}{2}) \leq \phi^B(\frac{1}{2})$.*

Proof. See the Appendix. ■

The above result states that both Kihlstrom and Mirman and Quiggin preferences are well-ordered in terms of risk aversion. However, such a simple characterization does not hold for Epstein and Zin utility classes. On the contrary, we prove the following negative result:

Proposition 1 (Epstein and Zin utility functions and risk aversion) *We consider two agents A and B with utility functions of $U_{\gamma_A}^{EZ}$ and $U_{\gamma_B}^{EZ}$ respectively, with $\gamma_A \neq \gamma_B$. There does not exist a spread-compatible relation R, such that A is more risk-averse than B with respect to R.*

Proof. Assume that $\gamma_A > \gamma_B$ and consider a spread-compatible relation R. We show that A is not more risk-averse than B, nor is B more risk-averse than A.

Proof that A is not more risk-averse than B.

We construct two heads or tails gambles G^a and G^b, with G^a being a spread of G^b (and thus $G^a \, R \, G^b$ for every spread-compatible relation R), such that agent A is indifferent between both gambles, and agent B strictly prefers G^b, which is incompatible with A being more risk-averse than B.

With $0 < \varepsilon << 1, c_a, c_b > 0$, G^a and G^b are defined as follows:

$$G^a = \left(c_a,\ \left(3^{\frac{1}{1-\rho}}(1-\varepsilon), 3^{\frac{1}{1-\rho}}(1+\varepsilon)\right)\right)$$
$$G^b = (c_b,\ (1-2\varepsilon, 1+2\varepsilon))$$

where:[6]

$$c_a^{1-\rho} - c_b^{1-\rho} = \left[\frac{(1-2\varepsilon)^{1-\ \gamma_A} + (1+2\varepsilon)^{1-\gamma_A}}{2}\right]^{\frac{1-\rho}{1-\gamma_A}} - 3\left[\frac{(1-\varepsilon)^{1-\gamma_A} + (1+\varepsilon)^{1-\gamma_A}}{2}\right]^{\frac{1-\rho}{1-\gamma_A}} \tag{2}$$

1. Agent A is indifferent between G^a and G^b. $U^{EZ}_{\gamma_B}(G^a) = U^{EZ}_{\gamma_B}(G^b)$ is equivalent to:

$$\frac{c_a^{1-\rho}}{1-\rho} + \frac{3}{1-\rho}\left[\frac{(1-\varepsilon)^{1-\gamma_A} + (1+\varepsilon)^{1-\gamma_A}}{2}\right]^{\frac{1-\rho}{1-\gamma_A}} = \frac{c_b^{1-\rho}}{1-\rho} + \frac{1}{1-\rho}\left[\frac{1}{2}(1-2\varepsilon)^{1-\gamma_A} + \frac{1}{2}(1+2\varepsilon)^{1-\gamma_A}\right]^{\frac{1-\rho}{1-\gamma_A}} \tag{3}$$

This equality holds from the construction of c_a and c_b in (2).

2. The gamble G^a is a spread of G^b, $G^a \vdash G^b$, if:

$$\begin{aligned} \frac{c_a^{1-\rho}}{1-\rho} + \frac{3}{1-\rho}(1-\varepsilon)^{1-\rho} &< \frac{c_b^{1-\rho}}{1-\rho} + \frac{1}{1-\rho}(1-2\varepsilon)^{1-\rho} \\ &< \frac{c_b^{1-\rho}}{1-\rho} + \frac{1}{1-\rho}(1+2\varepsilon)^{1-\rho} < \frac{c_a^{1-\rho}}{1-\rho} + \frac{3}{1-\rho}(1+\varepsilon)^{1-\rho} \end{aligned} \tag{4}$$

Using Taylor expansions to express $c_a^{1-\rho} - c_b^{1-\rho}$, we show in the following that the inequality (4) holds when $0 < \varepsilon << 1$. First:

$$\left[\frac{(1-\varepsilon)^{1-\gamma_A} + (1+\varepsilon)^{1-\gamma_A}}{2}\right]^{\frac{1-\rho}{1-\gamma_A}} = \left[1 - \frac{\gamma_A(1-\gamma_A)}{2}\varepsilon^2 + O(\varepsilon^3)\right]^{\frac{1-\rho}{1-\gamma_A}} = 1 - \frac{\gamma_A(1-\rho)}{2}\varepsilon^2 + O(\varepsilon^3) \tag{5}$$

where $O(\varepsilon^3)$ denotes a function such that $\frac{O(\varepsilon^3)}{\varepsilon^3}$ is bounded as ε tends to zero.

Similarly, substituting 2ε for ε in (5) yields the following equation:

$$\left[\frac{(1-2\varepsilon)^{1-\gamma_A} + (1+2\varepsilon)^{1-\gamma_A}}{2}\right]^{\frac{1-\rho}{1-\gamma_A}} = 1 - 2\gamma_A(1-\rho)\varepsilon^2 + O(\varepsilon^3) \tag{6}$$

[6] It is always possible to find a pair of first-period consumptions (c_a, c_b) that satisfy this equality, whatever the value of ρ, since the range of $x^{1-\rho} - y^{1-\rho}$ is $\mathbb{R}$, when x and y cover $\mathbb{R}^+$.

Using both of the above first-order approximations, we simplify equality (2):

$$\frac{c_a^{1-\rho}}{1-\rho} - \frac{c_b^{1-\rho}}{1-\rho} = -\frac{2}{1-\rho} - \frac{\gamma_A}{2}\varepsilon^2 + O(\varepsilon^3) \tag{7}$$

In addition, both of the following approximations hold:

$$\begin{aligned}
\frac{3}{1-\rho}(1-\varepsilon)^{1-\rho} - \frac{1}{1-\rho}(1-2\varepsilon)^{1-\rho} &= \frac{2}{1-\rho} - \varepsilon + O(\varepsilon^2) \\
\frac{3}{1-\rho}(1+\varepsilon)^{1-\rho} - \frac{1}{1-\rho}(1+2\varepsilon)^{1-\rho} &= \frac{2}{1-\rho} + \varepsilon + O(\varepsilon^2)
\end{aligned}$$

Combining these equations, we obtain that condition (4) holds when ε is positive and small enough. G^a is then a spread of G^b. This implies that $G^a R G^b$ since R is spread compatible.

3. Agent B prefers G^b to G^a. We have:

$$\begin{aligned}
U_{\gamma_B}^{EZ}(G^b) - U_{\gamma_B}^{EZ}(G^a) &= \frac{c_b^{1-\rho}}{1-\rho} - \frac{c_a^{1-\rho}}{1-\rho} \\
&+ \frac{1}{1-\rho}\left[\frac{(1-2\varepsilon)^{1-\gamma_B} + (1+2\varepsilon)^{1-\gamma_B}}{2}\right]^{\frac{1-\rho}{1-\gamma_B}} - \frac{3}{1-\rho}\left[\frac{(1-\varepsilon)^{1-\gamma_B} + (1+\varepsilon)^{1-\gamma_B}}{2}\right]^{\frac{1-\rho}{1-\gamma_B}}
\end{aligned}$$

Using approximations (7), as well as (5) and (6), where γ_A is replaced by γ_B, we obtain:

$$\begin{aligned}
U_{\gamma_B}^{EZ}(G^b) - U_{\gamma_B}^{EZ}(G^a) &= \frac{2}{1-\rho} + \frac{1}{2}\gamma_A\varepsilon^2 - \frac{2}{1-\rho} - \frac{1}{2}\gamma_B\varepsilon^2 + O(\varepsilon^3) \\
&= \frac{1}{2}(\gamma_A - \gamma_B)\varepsilon^2 + O(\varepsilon^3)
\end{aligned}$$

which is positive when $\gamma_A > \gamma_B$ and for $\varepsilon > 0$ small enough.

Thus B strictly prefers G^b to G^a. In the case where A is more risk-averse than B, B should prefer G^a to G^b. This proves that A cannot be more risk-averse than B.

Proof that B is not more risk-averse than A.

We consider two heads or tails gambles H^a and H^b, with $c > 0$ and $0 < \varepsilon << 1$:

$$\begin{aligned}
H^a &= \left(c,\ \left((1-\varepsilon)^{\frac{1}{1-\gamma_B}}, (1+\varepsilon)^{\frac{1}{1-\gamma_B}}\right)\right) \\
H^b &= (c,\ (1,1))
\end{aligned}$$

Note that H^b is a degenerate gamble that produces the certain consumption profile $(c, 1)$ with certainty.

1. Agent B is indifferent between both gambles, since the following equality holds:

$$U_{\gamma_B}^{EZ}(H^a) = \frac{c^{1-\rho}}{1-\rho} + \frac{1}{1-\rho} = U_{\gamma_B}^{EZ}(H^b)$$

2. It is obvious that the gamble H^a is a spread of H^b. Thus $H^a R H^b$ since R is spread compatible.

3. Agent A strictly prefers H^b to H^a, since we have:

$$\begin{aligned}
U_{\gamma_A}^{EZ}(H^a) - U_{\gamma_A}^{EZ}(H^b) &= \frac{1}{1-\rho}\left[\frac{(1-\varepsilon)^{\frac{1-\gamma_A}{1-\gamma_B}} + (1+\varepsilon)^{\frac{1-\gamma_A}{1-\gamma_B}}}{2}\right]^{\frac{1-\rho}{1-\gamma_A}} - \frac{1}{1-\rho} \\
&= \frac{1}{1-\rho}\left[1 + \frac{1}{2}\frac{1-\gamma_A}{1-\gamma_B}\frac{\gamma_B-\gamma_A}{1-\gamma_B}\varepsilon^2 + O(\varepsilon^3)\right]^{\frac{1-\rho}{1-\gamma_A}} - \frac{1}{1-\rho} \\
&= \frac{1}{2}\frac{\gamma_B-\gamma_A}{(1-\gamma_B)^2}\varepsilon^2 + O(\varepsilon^3) \\
&< 0 \quad \text{since } \gamma_A > \gamma_B
\end{aligned}$$

As a conclusion, B cannot be more risk-averse than A with respect to R. ■

This latter proposition emphasizes that, unless we deny that a spread in a simple heads or tails gamble is an increase in risk, Epstein and Zin utility functions cannot be considered as appropriate tools for exploring the role of risk aversion. Changing the parameter γ in Epstein and Zin utility functions does involve changing cardinal preferences while holding ordinal preferences constant, but there is no direct relation between risk aversion and the γ parameter. An agent with a higher value of γ will exhibit greater aversion to some particular increases in risk (the second example in the proof), but also reduced aversion for some other kinds of increases in risk (the first example in the proof). Interpreting the results obtained from changes in the value of γ as reflecting the impact of risk aversion is therefore misleading. Since Epstein and Zin utility functions are a particular case of Selden utility functions, Proposition 1 *a fortiori* implies that considerations about the concavity of the function v in Selden utility functions has no direct interpretation in terms of risk aversion.

The reason for which Epstein and Zin utility functions are not well ordered in terms of risk aversion is fairly intuitive. Rewrite the Epstein and Zin utility function as:

$$U_{\gamma}^{EZ}(c_1, \widetilde{c}_2) = \frac{c_1^{1-\rho}}{1-\rho} + \frac{\mathbb{E}[\widetilde{c}_2]^{1-\rho}}{1-\rho}\left(\mathbb{E}\left[\left(\frac{\widetilde{c}_2}{\mathbb{E}[\widetilde{c}_2]}\right)^{1-\gamma}\right]\right)^{\frac{1-\rho}{1-\gamma}}$$

It is clear that a greater value of γ means greater relative risk aversion with respect to second-period consumption. But, there is no monotonic relation between relative risk over second-period consumption and aggregate risk over lifetime utility – with the latter being what matters for comparative risk aversion. A gamble may imply greater relative risk over second-period consumption than another, but at the same time less absolute risk over lifetime utility. This is actually the case when we compare the gambles G^b and G^a defined above. Even if the "relative" risk expressed as a share of average second-period consumption is larger in G^b than in G^a, the (absolute) risk embedded in G^a is greater than that in G^b. Agent B with $\gamma_B < \gamma_A$ prefers lottery G^b with the greatest second-period "relative risk" while were he to be less risk-averse he should prefer lottery G^a with less aggregate risk.

3.3 Theory, Part 2: Continuous distributions

We have so far focused on heads or tails gambles, which simplify the discussion of comparative risk aversion for two reasons. First, when considering two gambles, there are only two possibilities: either one first-order dominates the other, or one is a spread of the other. Second, in the case of gambles, the notion of spread coincides with all arguably reasonable notions of dispersion, so that there is no possible dispute of the fact that a spread is an increase in risk.

In the remainder of the paper, we consider the general case of continuous lotteries. The cost for this increase in generality is that there are now many possible definitions of greater risk, implying different meanings for being "more risk-averse than". We argue that this difficulty should be acknowledged, rather than ignored. Different definitions of what is an increase in risk provide different notions of comparative risk aversion. The more restrictive the relation over risk comparison, the less stringent the associated notion of preference comparisons. In particular, restricting risk comparisons to very particular cases (as when considering only comparisons with certain outcomes) yields a very weak notion of comparative risk aversion, with little applicability for applied topics. However, as we will see, there is no need to rely on fancy risk comparisons to obtain results of interest. In particular, we will show that the consideration of simple spreads (which are just a generalization of the spread relation introduced above for heads or tails gambles) allows us to derive a general model-free result that makes it possible to derive unambiguous conclusions regarding the impact of risk aversion in a wide variety of settings.

3.3.1 The setting

The setting is very similar to that initially described in Section 3.1.1, the only difference being that we no longer restrict the sample space Ω, which is a priori uncountable, nor the probability Pr. The set X is endowed with a preference relation $\succeq$, and $L(X)$ is the set of lotteries, defined over Ω and paying off in X.

For simplicity's sake, we suppose that the ordinal preference relationship $\succeq$ over X can be represented by a function $U : X \to \mathbb{R}$. We shall however insist on the fact that the results we derive do not depend on a particular utility representation. Any utility representation of $\succeq$, based on a different utility function would yield the same conclusions. The cumulative distribution function for a lottery $L \in L(X)$ is denoted F_L and defined over $\mathbb{R}$. For any real number u, $F_L(u)$ is simply the probability (Pr defined over the probability space) that the utility of the lottery realization (whose value is in X) is smaller than a given u:

$$\forall u \in \mathbb{R}, \quad F_L(u) = Pr\{U(L(\omega)) \leq u | \omega \in \Omega\}$$

A lottery L will be said to first-order dominate a lottery L' if and only if $F_L(u) \leq F_{L'}(u)$ for all u, and to strictly first-order dominate L', if there additionally exists v such that $F_L(v) < F_{L'}(v)$. It is clear that this notion of dominance is independent of the utility function that is chosen to represent the preference relation $\succeq$.

The preferences of agents A and B $\succeq^i$ $(i = A, B)$ over a subset $Y \subset L(X)$ of lotteries are compatible with ordinal preferences (Assumption A) and ordinal dominance (Property 3).

3.3.2 Comparative riskiness

In order to apply the general procedure for comparing risk aversion (Definition 4), we need a notion of "riskier than" that is valid for continuous lotteries. We generalize the notion of spread introduced in Definition 5 as follows:

Definition 7 ($p-$Spread) *Given a scalar $p \in]0,1[$. A lottery L is a said to be a $p-$spread of the lottery L' that we denote by $L \vdash_p L'$, if there exists an $u_0 \in \mathbb{R}$ such that:*

1. $F_L(u_0) = p$
2. *for all* $u \leq u_0$, $F_L(u) \geq F_{L'}(u)$,
3. *for all* $u \geq u_0$, $F_L(u) \leq F_{L'}(u)$.

This definition is equivalent to stating that F_L single-crosses $F_{L'}$, with the crossing occurring at the y-value of p. In Figure 1 (p. 21), lottery L is a $p-$spread of lottery L'.

It is worth noting that the above definition does not depend on the choice of the representation U of preferences, but only on ordinal preferences.[7] If a lottery L is a utility

[7]This would not be the case for other notions of dispersion, such as that suggested by Bickel and Lehman (1976), or for mean-preserving spreads, second-order stochastic dominance, etc.

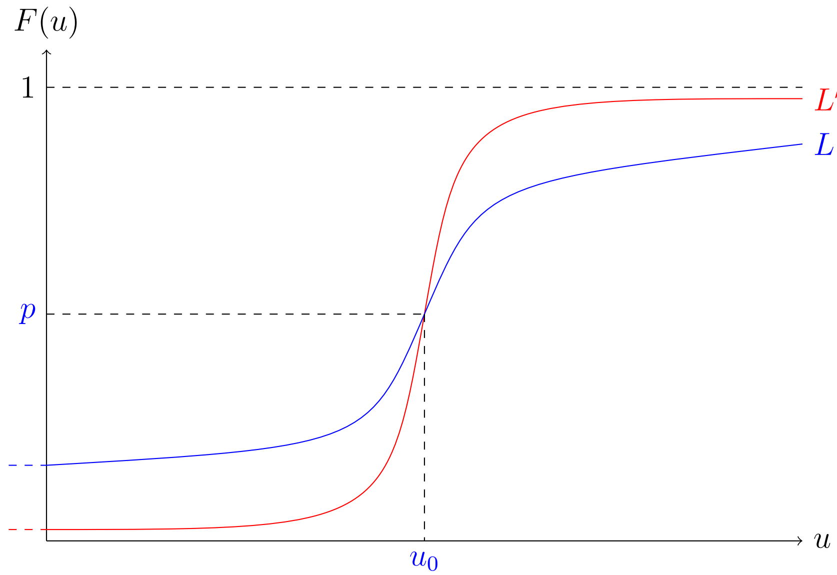

Figure 1: $p-$spread $L \vdash_p L'$

spread of another lottery L' for a given utility representation U, then it will also be so for any representation corresponding to the same ordinal preferences. The $p-$spread property is therefore an ordinal and not a cardinal concept.

We can then easily check that the $p-$spread relation is reflexive and transitive, and thus defines a partial preorder on Y. We also argue that if a lottery L is a $p-$spread of the lottery L', then L is riskier than L'. Comparing L to L', states of the world can be split up into "bad states" with measure p, and "good states" with measure $1-p$, such that: (i) the outcome of L or L' obtained in any good state of the world is preferable to that which is obtained in bad states of the world; (ii) conditional on the state being good, the lottery L first-order dominates the lottery L', while the reverse holds when states are bad. The lotteries L and L' can be seen as the result of binary gambles (determining whether the state of the world is bad, with probability p, or good, with probability $1-p$) with the good outcome of L dominating the good outcome of L', and the bad outcome of L being dominated by the bad outcome of L'. In this sense, it seems clear that L is riskier than L', since it pays off more in good states and less in bad states.

It is possible to define a notion of spread as $L \vdash L'$ if and only if $L \vdash_p L'$ for some $p \in]0,1[$. A number of papers, such as Jewitt (1987) and Johnson and Myatt (2006), have used such spreads or single-crossing properties as a criterion of greater dispersion. One particularity of single crossing is that it is not transitive and thus does not define a risk order. Taking the transitive closure of this single-crossing property is not a good option either, since the succession of two single crossings may yield something infinitely closed to an increase in risk.

Formally, we can have $L \vdash L' \vdash L''$ and L'' very close to a lottery M such that $M \vdash L$. As it may seem unappealing to have a notion of "riskier than", which is not transitive, we introduce the notion of a $p-$spread. However, the results relying on assumptions valid for all p, can equivalently be expressed using the spread relation $\vdash$.

When considering preferences over certain $\times$ uncertain consumption pairs, and the standard utility classes mentioned in Definition 1, both the Kihlstrom and Mirman and Quiggin utility functions can easily be ordered in terms of aversion for $p-$spread increases in risk.

Result 3 (Comparative risk aversion and standard utility classes) *The following results hold for standard utility classes:*

- *An agent with a utility function $U^{KM}_{k^A}$ is more risk-averse than an agent with a utility function $U^{KM}_{k^B}$ with respect to all $p-$spread relations if and only if k^A is more concave than k^B.*

- *An agent with a utility function $U^{Q}_{\phi^A}$ is more risk-averse than an agent with a utility function $U^{Q}_{\phi^B}$ with respect to all $p-$spread relations if and only if ϕ^A is more convex than ϕ^B.*

Proof. See the Appendix. ■

This result extends the characterization of comparative risk aversion obtained in Result 2.

Regarding the Epstein and Zin class, we already know from Proposition 1 that Epstein and Zin utility functions are not properly ranked with respect to aversion for $1/2-$spread increases in risk, so that there is no chance of reaching a conclusion similar to those of Result 3. Furthermore, we can easily slightly modify the proof of Proposition 1 to show that for any two different Epstein and Zin utility functions $U^{EZ}_{\gamma^A}$ and $U^{EZ}_{\gamma^B}$ with $\gamma_A \neq \gamma_B$ and any $p \in]0,1[$, it cannot be the case that $U^{EZ}_{\gamma^A}$ is more risk-averse than $U^{EZ}_{\gamma^B}$ with respect to $p-$spread increases in risk.

3.3.3 A model-free result

Having provided a formal meaning of comparative risk aversion, we are now interested in deriving results for the impact of risk aversion on agents' behaviors. We suppose that agents may chose an action $t \in I \subset \mathbb{R}$, which modifies the payoff of a lottery. Such a lottery is noted $L_t \in Y$ and its realization when state $\omega \in \Omega$ occurs is $L_t(\omega) \in X$. With minimal assumptions, which are detailed below, we prove a very general result stating that the optimal action under uncertainty covaries monotonically with risk aversion.

Our first assumption is that the action t has a true effect on lotteries.

Assumption B (Non-Constant) *Consider two actions $t_1 \in I$ and $t_2 \in I$. If $L_{t_1}(\omega) \sim L_{t_2}(\omega)$ for all $\omega \in \Omega$, then $t_1 = t_2$.*

The above assumption is obviously a necessary condition for our model-free result. In the extreme case, when t does not have any influence on the lottery L_t, we would obviously be silent about the impact of the risk aversion on the choice of the action.

Second, we make an assumption of single peakedness. For each $\omega \in \Omega$, the application $t \mapsto L_t(\omega)$ is supposed to be single-peaked, which implies that in a given state of the world ω: (i) there exists a best action t_ω and (ii) an action is all the more preferred the closer it is to t_ω.

Assumption C (Single-Peakedness) *For all $\omega \in \Omega$:*

$$\exists t_\omega \in \Omega \text{ such that } \forall t \in I,\ L_{t_\omega}(\omega) \succeq L_t(\omega)$$

$$t_1 \leq t_2 \leq t_\omega \leq t_3 \leq t_4 \ (\in I),\ \Rightarrow \begin{cases} L_{t_\omega}(\omega) \succeq L_{t_2}(\omega) \succeq L_{t_1}(\omega) \\ L_{t_\omega}(\omega) \succeq L_{t_3}(\omega) \succeq L_{t_4}(\omega) \end{cases}$$

Third, we assume that actions do not modify the initial order of the lottery outcomes. Whatever the action chosen by the agent, the ranking of lottery payoffs remain unaffected. In other words, for any pair of actions t and t', the lotteries L_t and $L_{t'}$ are comonotonic.

Assumption D (Comonotonicity) *Consider two states $\omega_1, \omega_2 \in \Omega$. Lottery outcomes satisfy the following:*

$$(L_t(\omega_1) \succeq L_t(\omega_2) \text{ for some } t \in I) \Rightarrow\ (L_{t'}(\omega_1) \succeq L_{t'}(\omega_2) \text{ for all } t' \in I)$$

When Assumption D holds, the states of the world may be ranked from good to bad, independently of agents' actions. This assumption holds whenever it is possible to tell what constitutes good news, without knowing agents' actions. This is for example the case when considering random income, random returns, provided that we assume that agents' well-being increases with wealth. This assumption may not however hold in other circumstances, for example when action t involves betting on a particular horse, since in this case the action determines which outcome is preferred. When comonotonicity holds, we will write $\omega_1 \geq \omega_2$ if $L_t(\omega_1) \succeq L_t(\omega_2)$ for all $t \in I$.

The last assumption we make for practical purposes is that the sequence of optimal actions $(t_\omega)_{\omega\in\Omega}$ is ordered according to the states of the world ω. The better the state of the world ω, the greater is the optimal action t_ω.

Assumption E (Action Order) *For any two states $\omega_1, \omega_2 \in \Omega$.*

$$\omega_1 \geq \omega_2 \implies t_{\omega_1} \geq t_{\omega_2}$$

This last assumption is simply technical. Up to a modification of the action, this assumption always holds. It is always possible to define a bijection $\psi : I \to I$ such that $\psi(t)$ is well-ordered. Without loss of generality, we assume that the better the state of the world, the greater the optimal action contingent on that state.

It is now possible to formalize a general result about the role of risk aversion:

Proposition 2 (A general model-free result) *Consider two agents A and B who have to choose an action t providing them with a lottery satisfying assumptions B, C, D, and E. We assume in addition that the preferences of agents A and B satisfy ordinal dominance (Property 3) and define the respective single optimal actions t^A and t^B. We then have the following implication:*

If agent A is more risk-averse than agent B with respect to any $p-$spread relation then $t^A \leq t^B$.

Proof. We assume that $t^A > t^B$. In order to obtain a contradiction, we prove that lottery L_{t^A} is a $p-$spread of L_{t^B} for some $p \in]0;1[$. In this case we have that $L_{t^A} \succeq^A L_{t^B}$ and $L_{t^A} \vdash_p L_{t^B}$, which would imply that $L_{t^A} \succeq^B L_{t^B}$ because A is more risk-averse than B, contradicting the optimality of t^B for agent B.

Proving that lottery L_{t^A} is a $p-$spread of L_{t^B} involves showing that there exists $u_0 \in \mathbb{R}$ and $p \in]0;1[$, such that $F_{L_{t^A}}(u_0) = F_{L_{t^B}}(u_0) = p$, $F_{L_{t^A}}(u) \geq F_{L_{t^B}}(u)$ for $u \leq u_0$ and $F_{L_{t^A}}(u) \leq F_{L_{t^B}}(u)$ for $u \geq u_0$.

We define ξ^- as the subset of $\mathbb{R}$, where the cumulative distribution of L_{t^B} is larger than that of L_{t^A}. Conversely, ξ^+ is the subset, where the cumulative distribution of L_{t^A} is larger than that of L_{t^B}.

$$\xi^- = \left\{u \in \mathbb{R}, F_{L_{t^B}}(u) \geq F_{L_{t^A}}(u)\right\} \quad \text{and} \quad \xi^+ = \left\{u \in \mathbb{R}, F_{L_{t^A}}(u) \geq F_{L_{t^B}}(u)\right\}$$

First, note that each $u \in \mathbb{R}$ belongs either to ξ^+ or ξ^-: $\xi^+ \cup \xi^- = \mathbb{R}$. We then distinguish four cases, depending on whether the sets ξ^+ and ξ^- are included in each other or not.

1. Suppose that $\xi^+ = \xi^- = \mathbb{R}$. This means that for all $u \in \mathbb{R}$, $F_{L_{t^A}}(u) = F_{L_{t^B}}(u)$, which implies that lotteries pay off the same outcomes in all states of the world. Assumption B implies that $t^A = t^B$, which contradicts the assumption that $t^B < t^A$.

2. Suppose that $\xi^+ \subsetneq \xi^-$ (this means that ξ^+ is either empty or contains only elements u such that $F_{L_{t^A}}(u) = F_{L_{t^B}}(u)$). The c.d.f. of lottery L_{t^B} is always larger than that of L_{t^A}, and is strictly larger at least once: L_{t^A} strictly first-order dominates the lottery L_{t^B}. Since preferences satisfy ordinal dominance (Property 3), agent B strictly prefers L_{t^A} to L_{t^B}, which contradicts the optimality of t^B.

3. Suppose that $\xi^- \subsetneq \xi^+$. Analogously to the case above, this contradicts the optimality of t^A for agent A.

4. We now necessarily have that $\xi^- \not\subset \xi^+$ and $\xi^+ \not\subset \xi^-$. There exists at least one element in each set, not belonging to the other one, which we denote $u^+ \in \xi^+$ (and $u^+ \notin \xi^-$) and $u^- \in \xi^-$ (and $u^- \notin \xi^+$).

   We first focus on u^-. By definition, $1 - F_{L_{t^B}}(u^-) < 1 - F_{L_{t^A}}(u^-)$, or equivalently $\{\omega \in \Omega | U(L_{t^B}(\omega)) \geq u^-\} \subsetneq \{\omega \in \Omega | U(L_{t^A}(\omega)) \geq u^-\}$. There exists $\omega_1 \in \Omega$ in the second set but not in the first: $U(L_{t^B}(\omega_1)) \leq u^- \leq U(L_{t^A}(\omega_1))$. Single-peakedness (Assumption C) implies that there exists t_{ω_1} such that: $t_{\omega_1} \geq t^A \geq t^B$.

   We consider $u \geq u^-$ and would like to show that $\{\omega \in \Omega | U(L_{t^B}(\omega)) \geq u\} \subset \{\omega \in \Omega | U(L_{t^A}(\omega)) \geq u\}$. Let $\omega^u \in \{\omega \in \Omega | U(L_{t^B}(\omega)) \geq u\}$. Since $U(L_{t^B}(\omega^u)) \geq u \geq u^- \geq U(L_{t^B}(\omega_1))$, we deduce, from Assumption D of comonotonicity, that $\omega^u \geq \omega_1$. From Assumption E, we deduce that $t_{\omega^u} \geq t_{\omega_1} \geq t^A > t^B$. Single-peakedness allows us to conclude that $U(L_{t^A}(\omega^u)) \geq U(L_{t^B}(\omega^u)) \geq u$ and $\omega^u \in \{\omega \in \Omega | U(L_{t^A}(\omega)) \geq u\}$.

   We have therefore proved that $[u^-, +\infty[\subset \xi^-$. We can show analogously that $]-\infty, u^+] \subset \xi^+$. u^+ (resp. u^-) is a lower (resp. upper) bound for ξ^- (resp. ξ^+) (otherwise we can show that $u^+ \in \xi^-$, which is contradictory). We thus define $\overline{u} = \inf \xi^-$ and $\underline{u} = \sup \xi^+$, which satisfy $\underline{u} \leq \overline{u}$ (otherwise $\xi^+ \cup \xi^- \neq \mathbb{R}$). We define u_0 as an element of the non-empty segment $[\underline{u}; \overline{u}]$ and $p = F_{L_{t^B}}(u_0) = F_{L_{t^A}}(u_0)$. The cumulative distribution functions satisfy:

$$\forall u \leq u_0, \quad F_{L_{t^B}}(u) \leq F_{L_{t^A}}(u)$$
$$\forall u \geq u_0, \quad F_{L_{t^B}}(u) \geq F_{L_{t^A}}(u)$$

   According to Definition 7, this therefore shows that lottery L_{t^A} is a utility spread of L_{t^B}, which terminates the proof.

■

This model-free result shows that, under some mild assumptions, the more risk-averse is the agent, the smaller is his optimal action. We can summarize the intuition as follows.

Consider the optimal action t^B. We can group the states of the worlds into two subsets. The first consists of the optimal actions t_ω, which are smaller than t^B, while the second consists of the optimal actions that are larger than t^B. Since, without uncertainty, optimal actions are assumed to be larger when the state of the world is better, we can qualify the former as "bad" states of the world and the latter as "good" states. Due to single-peakedness, choosing an action t smaller than t^B involves increasing the agent's welfare in bad states and reducing it in good states. Opting for a smaller action is thus one way of redistributing welfare from good to bad states, and a way of reducing risk regarding agent welfare, which strategy is preferred by more risk-averse agents.

4 Applications

We use the results from Proposition 2 to analyze in a very simple two-period framework the savings behavior of an agent facing uncertainty. We consider in turns three types of uncertainty: (i) second-period income is random; (ii) the savings interest rate is uncertain; and (iii) the agent faces a mortality risk, i.e. a risk of dying at the end of the first period.

4.1 Application to precautionary savings

We consider the case of agents who live for two periods, have random second-period incomes, and have to decide how much to save. This very simple problem has been the object of number of inspirational contributions, including Leland (1968), Sandmo (1970), Drèze and Modigliani (1972), Caperaa and Eeckhoudt (1975), Kimball (1990), and Kimball and Weil (2009). These led to the development of the notion of prudence, whose link to risk aversion has not been clarified despite some impressive efforts (Kimball and Weil, (2009)). We will see, however, that our general approach does lead to clear and simple conclusions.

To apply our general result, we specify the setting as in Section 3.2.3. The set X is $\mathbb{R}^{+2}$. A typical element of X is (c_1, c_2), where c_1 is first-period and c_2 second-period consumption. This set X is endowed with an ordinal preference relationship $\succeq$ represented by a utility function u. The set of lotteries with outcomes in X is denoted $H(X)$, and $Y \subset H(X)$ is the set of lotteries with deterministic first-period consumption. We consider two agents A and B with preferences $\succeq^i$ $(i = A, B)$ defined over Y. We assume that these preferences satisfy the consistency assumption and ordinal dominance.

We now introduce two assumptions regarding ordinal preferences.

Assumption F (Convexity of ordinal preferences) *For all* (c_1, c_2), (c_1', c_2'), *and* (c_1'', c_2'')

in X, and all $\lambda \in [0,1]$:

$$(c_1', c_2') \succeq (c_1, c_2) \text{ and } (c_1'', c_2'') \succeq (c_1, c_2) \implies (\lambda c_1' + (1-\lambda)c_1'', \lambda c_2' + (1-\lambda)c_2'') \succeq (c_1, c_2)$$

Assumption G (Normality of first-period consumption) *Consider the agent's optimization problem $\max_{c_1,c_2} u(c_1,c_2)$ subject to the budget constraint $c_1 + \frac{1}{(1+R)}c_2 = Y$, where $Y \geq 0$ is the discounted total certain income and $R > -1$ the gross certain interest rate. The ordinal preference relationship $\succeq$ is such that this problem has a unique solution denoted $(c_1(Y,R), c_2(Y,R))$, where, additionally, first-period consumption $c_1(Y,R)$ increases with total income Y.*

The assumption of preference convexity is fairly standard in the analysis of consumer behavior.[8] This implies Assumption C of single-peakedness which is required for our model-free result.[9] Assumption G of good normality is also very standard. In cases where the preference relation $\succeq$ can be represented by a differentiable utility function $u(c_1,c_2)$ over first-period c_1 and second-period c_2 consumptions, this assumption concerns the derivative of the marginal rate of substitution between consumption in both periods relative to second-period consumption (namely $\frac{\partial}{\partial c_2}\left(\frac{\frac{\partial u}{\partial c_1}}{\frac{\partial u}{\partial c_2}}\right) > 0$). However, we believe that greater insight is gained by emphasizing that the requirement is good normality.

We can now express our finding with respect to precautionary savings:

Proposition 3 (Precautionary savings) *Consider two agents A and B, who choose first-period consumption c_1 providing them with a certain $\times$ uncertain income profile denoted $(c_1, \widetilde{y}_2 + (1+R)(y_1 - c_1))$, where $y_1 > 0$ is certain first-period income, $\widetilde{y}_2$ random second-period income, and $R > -1$ the certain interest rate. If:*

1. *The ordinal preference relationship $\succeq$ satisfies Assumptions F and G.*
2. *Risk preferences $\succeq^A$ and $\succeq^B$ satisfy the ordinal dominance Property 3 and define optimal first-period consumption levels of c_1^A and c_1^B.*

Then, the following holds:

$$\text{Agent } A \text{ is more risk-averse than agent } B \implies c_1^A \leq c_1^B$$

Proof. This proposition comes via an application of the model-free result formulated in Proposition 2. We simply need to check that the required assumptions hold in this setting.

[8] See for example the discussion in Mas Colell, Whinston and Green (1995), page 44.

[9] This is formally shown below, in the proof of Proposition 3. In fact, single-peakedness and convexity are equivalent if we restrict ourselves to the case of continuous preferences.

- The action chosen by the agent is the first-period consumption c_1.

- Assumptions B and D hold by construction.

- The normality of first-period consumption (Assumption G) ensures that Assumption E regarding optimal-action ordering holds. Indeed, by definition, the better the state of the world (i.e., the larger is second-period income $\widetilde{y}_2(\omega)$), the greater is optimal first-period consumption.

- The convexity of the relation $\succeq$ implies the single peakedness of preferences. Let $s^\star$ be the solution of

$$\max_s u\left(y_1 - s, y_2 + (1+R)s\right)$$

and consider, for example, $s' < s'' < s^\star$. By the definition of $s^\star$, we have first that $(y_1 - s^\star, y_2 + (1+R)s^\star) \succeq (y_1 - s', y_2 + (1+R)s')$ and also $(y_1 - s', y_2 + (1+R)s') \succeq (y_1 - s', y_2 + (1+R)s')$. Convexity then implies that for all $\lambda \in [0,1]$ we can deduce $(y_1 - (\lambda\, s^\star + (1-\lambda)\, s'), y_2 + (1+R)(\lambda\, s^\star + (1-\lambda)\, s')) \succeq (y_1 - s', y_2 + (1+R)s')$. As $s' < s'' < s^\star$, we can choose $\lambda \in [0,1]$, such that $s'' = \lambda\, s^\star + (1-\lambda)\, s'$, which proves single-peakedness.

■

This proposition makes it clear that the greater is risk aversion, the more the agent saves. The intuition behind this result is very simple. Take an agent who decides to save $s(\widetilde{y}_2)$ anticipating a random second-period income of $\widetilde{y}_2$. For simplicity, we assume that this random income can take two values, $\underline{y}_2$ and $\overline{y}_2$. The amount $s(\widetilde{y}_2)$ is an intermediate value between what he would have saved knowing that he would receive $\overline{y}_2$ and what he would have saved knowing that he would earn $\underline{y}_2$.

$$s(\overline{y}_2) < s(\widetilde{y}_2) < s\ (\underline{y}_2)$$

By saving more than $s(\widetilde{y}_2)$ he increases his welfare in the bad state of the world, but reduces it in the good state. As this reduces the degree of risk regarding his welfare, it will therefore be preferred by more risk-averse agents.

Precautionary savings, which are usually defined as the difference between savings with an uninsurable uncertain second-period income and savings in a full-insurance context, therefore rise with risk aversion. While our results say nothing about the sign of any precautionary savings (which may be negative), they do establish that precautionary savings rise with risk aversion.

4.2 Application to optimal savings with interest-rate uncertainty

We now raise the question of the relationship between optimal savings and risk aversion, not in the face of income uncertainty, but rather interest-rate uncertainty. This question was addressed by Kihlstrom and Mirman (1974) in the expected-utility framework, and Langlais (1995) for Selden utility functions, with diverging conclusions.

The formal setting of this question (the structure of X, etc.) is exactly the same as in the previous section. However, the ordinal properties that are required to obtain results regarding risk aversion are different.

In a deterministic setting, increasing the interest rate is equivalent to changing the price of second-period consumption. As with any price change, a movement in the interest rate implies both income and substitution effects. A higher interest rate means a lower price for second-period consumption, with a positive income effect yielding higher first-period consumption and lower savings. The opposing substitution effect reduces first-period consumption, and therefore increases savings. The income and substitution effects thus have opposing effects on optimal savings, and the overall effect may be either positive or negative. If the income effect dominates, higher interest rates yield lower savings; on the other hand, if the substitution effect dominates, higher interest rates imply greater savings. For the sake of clarity, we define the optimal savings function $s(y_1, y_2, R)$.

$$s(y_1, y_2, R) = \arg\max_s (y_1 - s, y_2 + (1+R)s)$$

This function may either rise or fall with respect to R, depending on ordinal preferences. This sign is key for the determination of the effect of risk aversion on savings when interest rates are non-deterministic.

Proposition 4 *Consider two agents A and B, who choose first-period consumption c_1 providing them with a certain $\times$ uncertain income profile $\left(c_1, y_2 + (1+\widetilde{R})(y_1 - c_1)\right)$, where $y_1 > 0$ ($y_2 > 0$) is certain first- (second-) period income, and $\widetilde{R}$ is the random interest rate. If:*

1. *The ordinal preference relationship $\succeq$ satisfies Assumption F.*
2. *Risk preferences $\succeq^A$ and $\succeq^B$ satisfy the ordinal dominance Property 3 and define optimal first-period consumptions c_1^A and c_1^B.*

Then, the following holds:

$$\text{Agent } A \text{ is more risk-averse than agent } B \implies \begin{cases} c_1^A \leq c_1^B \text{ if } R \mapsto s(y_1, y_2, R) \text{ is decreasing} \\ \text{or} \\ c_1^B \leq c_1^A \text{ if } R \mapsto s(y_1, y_2, R) \text{ is increasing} \end{cases}$$

Proof. The proof is straightforward and is very similar to that in Proposition 3. When the substitution effect dominates, in order to use the result of Proposition 2 directly, we may consider that the action is not c_1, but rather $s = y_1 - c_1$. ■

If preferences can be represented by a twice continuously-differentiable utility function, the derivative $\frac{\partial s}{\partial R}$ is positive (negative) if the intertemporal elasticity of substitution is greater (less) than one (Kihlstrom and Mirman, (1974)). The preceding proposition could then be expressed by refering to the value of the intertemporal elasticity of substitution, although this would be slightly less general (as differentiability would then be required).

Our findings extend those in Kihlstrom and Mirman, which were restricted to the expected-utility framework, with differentiable utility functions. They contradict those in Langlais (1995), who considers Selden utility functions, the explanation being that these latter functions are not well-ordered in terms of risk aversion.

4.3 Application to optimal savings with lifetime uncertainty

In our last application, we consider the effect of an uncertain lifetime on optimal savings. The traditional view in Economics is that risk aversion and time preference are orthogonal aspects of preferences. However, Bommier (2006) and (2008) has underlined that as soon as we take lifetime uncertainty into account, there is a strong direct relationship between risk aversion and time discounting, with significant implications for savings behavior. Bommier's results were however derived in an expected-utility framework, omitting some aspects of preferences such as bequests. We here show how the impact of risk aversion on savings with lifetime uncertainty can be addressed without assuming expected utility and allowing for bequests.

We consider an agent who has an initial endowment W and who may live for one or two time periods. This agent chooses his consumption c_1 in the first period. In the second period, either he survives and consumes his wealth, or dies, and his wealth is transmitted to his heirs, for whom he may care. To account for the potential existence of annuities, we assume that the return to saving may depend on whether the agent survives or not. More precisely, if we denote by c_2^a second-period consumption in the case where the agent is alive, and by c_2^d the amount transmitted to his heirs if he dies, we have:

$$\begin{aligned} c_2^a &= (1 + R_a)(W - c_1) \\ c_2^d &= (1 + R_d)(W - c_1) \end{aligned}$$

where R_a and R_d are the returns obtained respectively in the case of survival and death. We assume that $R_a > -1$ and $R_d \geq -1$, but make no assumptions about the relative values of

R_a and R_d. When there are no annuities or taxes on bequests we have $R_a = R_d$, while with perfect annuities we have $R_a > R_d = -1$. There are also many intermediary situations (e.g. when there are taxes on bequests) or contracts (death insurance) with R_d greater than R_a.

We now apply our model-free result to show that risk aversion produces an unambiguous result. Formally, the set X has to be defined to reflect the specificity of the context. The second-period outcome can no longer be described by a scalar variable c_2, but by a pair (c_2, σ) of a scalar c_2 and a binary variable $\sigma \in \{a, d\}$ indicating whether the individual is dead or alive. This means that $X = (\mathbb{R}^+)^2 \times \{a, d\}$. The notation $(c_1, c_2)_a$ and $(c_1, c_2)_d$ will however be used instead of the cumbersome (c_1, c_2, a) and (c_1, c_2, d). The index a or d thus indicates whether the individual is alive or not in the second period.

We make three assumptions regarding death. First, we suppose that the agent is always better off when alive. This simply means that the agent prefers to live in the second period and to consume, rather than to die and bequeath his wealth. Second, we suppose that optimal saving conditional on living for two periods is greater than optimal saving conditional on living one period. In other words, in a deterministic setting, the propensity to consume falls with life duration. This seems a very natural assumption in this setting, where agents have no second-period income. Last, we introduce a convexity assumption similar to Assumption F. Note however that X is not a convex set, and is not even connected. The assumption of convexity is only meaningful when considering convex subsets of X such as $(\mathbb{R}^+)^2 \times \{a\}$ and $(\mathbb{R}^+)^2 \times \{d\}$. Our assumptions can be formalized as follows:

Assumption H (Assumptions regarding ordinal preferences) *Ordinal preferences satisfy the following:*

- *The agent is always better off when alive:* $\forall c_1,\ (c_1, (1 + R_a)(W - c_1))_a \succeq (c_1, (1 + R_d)(W - c_1))_d$
- *For* $i = a, d$*, all* $W > 0$ *and* $R_a, R_d \geq -1$*, the problem* $\max_s u(W - s, (1 + R_i)s)_i$ *has a unique solution denoted by* s_i*.*
- *Optimal saving when surviving is always greater than that when dying:* $s_a > s_d$*.*
- *Ordinal preferences are convex over both* $(\mathbb{R}^+)^2 \times \{a\}$ *and* $(\mathbb{R}^+)^2 \times \{d\}$*.*

Given Assumption H, Proposition 2 allows us to determine how the savings of an agent facing an uncertain lifetime depend on risk aversion.

Proposition 5 (Saving when lifetime is uncertain) *We consider two agents A and B, who face an (identical) exogenous risk of dying after the first period. They have to choose* *a saving level of s providing them with a consumption profile of $(W - s, (1 + R_a)s)_a$ if they survive and a consumption-bequest profile of $(W - s, (1 + R_d)s)_d$ if they die. If:*

1. *The ordinal preference relationship $\succeq$ satisfies Assumption H.*

2. *Risk preferences $\succeq^A$ and $\succeq^B$ satisfy the ordinal dominance Property 3 and define optimal savings s^A and s^B.*

Then the following holds:

$$\text{Agent } A \text{ is more risk-averse than agent } B \implies s^A \leq s^B$$

Proof. Assumptions B and C hold by construction of the consumption profile and the convexity of preferences. Assumptions D and E directly stem from Assumption H. The result is then straightforward from Proposition 2. ■

The more risk-averse agent saves less. In other words, when mortality is taken into account, there is a positive relationship between risk aversion and impatience. We shall however emphasize that Proposition 5 assumes that A and B have the same probability of dying. The relation between risk aversion and impatience holds when comparing agents with identical mortality, but can not be applied to any correlations obtained from individuals with different mortality risks.[10]

5 Conclusion

In everyday life agents take actions which may enhance or reduce the uncertainty concerning the future. It is however difficult to imagine actions that eliminate uncertainty entirely. As a result, in order to have some applicability, the concept of risk aversion should reflect the willingness to reduce risks, and not only the willingness to avoid all sources of risk by choosing a certain outcome. Nonetheless, one popular approach to quantifying (and comparing) agents' risk aversion involves focusing on how individuals compare lotteries with certain outcomes. All of the definitions which are based on risk premia or certainty equivalents proceed in this way. It is then implicitly taken for granted that a greater willingness to avoid all sources of risk implies a greater willingness to marginally decrease risk. This turns out to be true when a number of additional assumptions are introduced. It is for example the case when assuming that agents are expected-utility maximizers. However, there is

[10]In particular, the fact that women might seem to be more patient and less risk-averse than men should not be interpreted as contradicting this proposition. This could in fact follow from gender differences in mortality.

no reason to believe that this always holds. In particular, considering extremely simple random objects (heads or tails gambles), for which the definition of an increase in risk is undisputable, preferences in Selden's framework, and in particular those associated with Epstein and Zin utility functions, have been shown to be not properly ordered in terms of aversion for (marginal) increases in risk. These preferences are well-ordered in terms of the willingness to opt for certain outcomes, but not for the willingness to (marginally) reduce the degree of risk taking. These classes of utility functions are therefore inadequate for the analysis of risk aversion in situations where it is not possible (or simply not optimal) to choose risk-free outcomes. Considering savings behavior shows that this inadequacy is not only an intellectual curiosity, but that it can also prevent us from reaching the correct conclusions about the role of risk aversion, as illustrated by Kimball and Weil (2009).

A better approach to risk aversion may then consist in focusing on utility functions that are correctly ordered in terms of aversion to increases in risk, as in Kihlstrom and Mirman utility functions, or Quiggin utility functions. This may be sufficient to establish negative results about risk aversion, for example showing that risk aversion has an ambiguous effect on a given behavior. However, in the case where a positive result is obtained, we cannot be sure whether this illustrates a general consequence of risk aversion or whether it is a consequence of the particular model of rationality under consideration. Moreover, analytical results often make use of unnecessary regularity conditions, tend to focus on small risks, and may be more difficult to understand. It is not always obvious to infer from an analytical expression the assumptions that generated a particular result.

The approach that we develop in this paper is model-free. We have shown that in many cases it is possible to derive unambiguous results about the impact of risk aversion without assuming a particular form of rationality. The basic intuition is that increasing risk aversion should increase the willingness to redistribute from good states to bad states of the world, in order to reduce the dispersion of ex-post utilities. This statement obviously presupposes that states of the world can be ordered from bad to good, independently of the actions that are taken, and can therefore not be applied in all cases. However, as our formal result shows, it applies under very minimal conditions, and allows us to deal with a number of interesting problems, such as precautionary savings, savings with uncertain lifetimes, and savings with uncertain returns.

The applications that we develop in this paper are of interest in their own right. First, we have shown that risk aversion increases precautionary savings. We believe that this clarifies the link between prudence and risk aversion. Agents may be prudent or imprudent in the sense that they may react positively or negatively to an increase in income uncertainty. Drèze and Modigliani (1972) and Kimball (1990) have established the conditions for prudence to occur in the expected utility framework. Our results complement their findings by showing that, for a given level of income uncertainty, increasing risk aversion leads to increased savings. As such, the precautionary savings associated with a given income uncertainty increases in risk aversion.

Second, when the returns on saving are uncertain, our straightforward prediction is that risk aversion has either a positive or a negative impact, depending on the intertemporal elasticity of substitution. This finding extends those of Kihlstrom and Mirman (1974) under expected utility, but differs from those of Langlais (1995) who considered Selden utility functions, which are not well ordered in terms of risk aversion.

Last, considering savings with an uncertain lifetime, we have shown that independently of the existence of an annuity market, and of whether agents derive utility from bequests, the greater is risk aversion, the less the agent saves. This underlines that, once mortality is taken into account, there is a positive relation between risk aversion and impatience.

One strength of these results is that they have been established without assuming any given model of rationality (e.g.: expected utility), and without considering small risks. We should also emphasize the simplicity with which they have been derived. Thus far, we have illustrated our work by looking at simple problems (savings in a two-period framework), as this allows us to highlight the interest of our formal approach in a simple way. The advantage of using a model-free approach, instead of a parametric model, may be even greater when looking at more complex issues, such as savings in N-period models, where it is simple to extend our results. On the contrary the increase in complexity appears to be much greater when using parametric approaches – as illustrated by the lack of results in the economic literature. This is because the key assumptions that are required, such as good normality, are relatively cumbersome to express in terms of utility once we abandon the two-period framework (unless very strong assumptions on the structure of the utility function are introduced). Opting for a model-free approach makes it possible to emphasize the true economic assumptions that are required (e.g. good normality), without paying the cost of translating these assumptions into complex properties of utility functions.

Appendix

A Proof of Result 2

We consider each utility class in turn.

1. Kihlstrom and Mirman utility functions.

We define $k_A = k \circ k_B$ where $\circ$ denotes the composition operator. The function k is increasing and continuous.

The utility associated to a gamble $(x_l; x_h)$ for agent A is:

$$U_{k_A}^{KM}(x_l; x_h) = k_A^{-1}\left(\frac{k_A(x_l) + k_A(x_h)}{2}\right)$$

- We first show that if k concave then A is more risk-averse than B.

  Assume that k is concave and consider two gambles $(x_l; x_h)$ and $(y_l; y_h)$. We need to show that if $(x_l; x_h) \vdash (y_l; y_h)$ and A prefers $(x_l; x_h)$ to $(y_l; y_h)$ then B prefers $(x_l; x_h)$ to $(y_l; y_h)$. By definition:

  The agent A prefers $(x_l; x_h)$ to $(y_l; y_h)$ iff: $$\frac{k(k_B(x_l)) + k(k_B(x_h))}{2} \geq \frac{k(k_B(y_l)) + k(k_B(y_h))}{2} \quad (8)$$

  The agent B prefers $(x_l; x_h)$ to $(y_l; y_h)$ iff: $$\frac{k_B(x_l) + k_B(x_h)}{2} \geq \frac{k_B(y_l) + k_B(y_h)}{2} \quad (9)$$

  Since $x_l < y_l \leq y_h < x_h$ and k_B is increasing, the inequality in (8) becomes:

  $$\frac{k(k_B(x_h)) - k(k_B(y_h))}{k_B(x_h) - k_B(y_h)} \geq \frac{k(k_B(y_l)) - k(k_B(x_l))}{k_B(y_l) - k_B(x_l)} \frac{k_B(y_l) - k_B(x_l)}{k_B(x_h) - k_B(y_h)}$$

  (if $k_B(x_h) = k_B(y_h)$ or $k_B(x_l) = k_B(y_l)$, the result is straightforward)

  k is concave, which implies that $0 \leq \frac{k(k_B(x_h)) - k(k_B(y_h))}{k_B(x_h) - k_B(y_h)} \leq \frac{k(k_B(y_l)) - k(k_B(x_l))}{k_B(y_l) - k_B(x_l)}$. We then deduce that:

  $$1 \geq \frac{k_B(y_l) - k_B(x_l)}{k_B(x_h) - k_B(y_h)} \quad \text{or} \quad k_B(x_l) + k_B(x_h) \geq k_B(y_l) + k_B(y_h)$$

  which implies that the inequality in (9) holds, and thus B prefers $(x_l; x_h)$ to $(y_l; y_h)$.

- We now show that the concavity of k is a necessary condition for A to be more risk-averse than B. Suppose that the inequality in (8) and $(x_l; x_h) \vdash (y_l; y_h)$ imply the inequality in (9). We choose a level of income $y = y_l = y_h$ such that the inequality in (8) holds with equality for a given pair x_l, x_h. The inequality in (9) then implies that $k(\frac{k_B(x_l) + k_B(x_h)}{2}) \geq k(k_B(y)) \geq \frac{k(k_B(x_l)) + k(k_B(x_h))}{2}$, or that k is concave. Indeed, a continuous function f is concave iff for all x_1, x_2, $f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2}$.

2. Quiggin anticipated utility function.

We consider two different gambles $(x_l; x_h) \vdash (y_l; y_h)$ ($x_l < y_l \leq y_h < x_h$; as before, the result is straightforward to prove if there is equality), such that agent A prefers $(x_l; x_h)$ to $(y_l; y_h)$. The utility associated with the gamble $(x_l; x_h)$ for A is:

$$U^Q_{\phi^A}(x_l; x_h) = x_l + (x_h - x_l)\phi^A(\frac{1}{2})$$

Agent A prefers $(x_l; x_h)$ to $(y_l; y_h)$ iff $U^Q_{\phi^A}(x_l; x_h) \geq U^Q_{\phi^A}(y_l; y_h)$ or:

$$\phi^A(\frac{1}{2}) \geq \frac{y_l - x_l}{x_h - y_h + y_l - x_l}$$

It is then straightforward that agent B prefers $(x_l; x_h)$ to $(y_l; y_h)$ iff $\phi^B(\frac{1}{2}) \geq \phi^A(\frac{1}{2})$.

B Proof of Result 3

We prove the result for each utility class.

1. Kihlstrom and Mirman utility functions.

1. a. The first implication stems directly from our previous Result 2.

1. b. We prove the reverse implication. We suppose that k^A is more concave than k^B. Thus, there exists a continuous, increasing, concave function k such that $k^A = k \circ k^B$. We consider two lotteries L_1 and L_2, such that: $L_1 \vdash_p L_2$ and $L_1 \succeq^A L_2$. As agent A prefers L_1 to L_2, we have:

$$\int_{-\infty}^{\infty} k\left(k^B(u)\right) dF_{L_1}(u) \geq \int_{-\infty}^{\infty} k\left(k^B(u)\right) dF_{L_2}(u)$$

Since $L_1 \vdash_p L_2$ there exists $u_0 \in \mathbb{R}$, such that $F_{L_1}(u_0) = F_{L_2}(u_0) = p$. Additionally, $F_{L_1}(u) \geq F_{L_2}(u)$ for $u \leq u_0$ and $F_{L_1}(u) \leq F_{L_2}(u)$ for $u \geq u_0$. Splitting both of the preceding integrals in u_0, we deduce:

$$\int_{-\infty}^{u_0} k\left(k^B(u)\right) dG^-(u) \geq \int_{u_0}^{\infty} k\left(k^B(u)\right) dG^+(u) \tag{10}$$

$$\text{where:} \quad G^+(u) = F_{L_2}(u) - F_{L_1}(u) \text{ and } G^-(u) = F_{L_1}(u) - F_{L_2}(u) \tag{11}$$

Functions G^+ and G^- satisfy $G^+(+\infty) = G^+(u_0) = 0 = G^-(u_0) = G^-(-\infty)$, as well as $G^+(u) \geq 0$ for $u \geq u_0$ and $G^-(u) \geq 0$ for $u \leq u_0$.

A concave (or convex) function defined over an open set admits left and right derivatives everywhere. Both are equal to each other and the function is differentiable, except on a countable set.[11] In consequence, we deduce that there exists a (countable) partition $\{s_j, j \in$

[11] This result stems for example from Theorem 1.3.7 p. 23 in Nicolescu and Persson (2006)

$\mathbb{N}\}$ of $]u_0,\infty[$, such that k and k^B are differentiable on every interval $]s_j,s_{j+1}[$. We deduce:

$$\begin{aligned}\int_{u_0}^{\infty} k\left(k^B(u)\right)dG^+(u) &= \sum_{j=0}^{\infty}\int_{s_j}^{s_{j+1}} k\left(k^B(u)\right)dG^+(u)\\ &= \sum_{j=0}^{\infty}\left(k\left(k^B(s_{j+1})\right)G^+(s_{j+1}) - k\left(k^B(s_j)\right)G^+(s_j)\right)\\ &- \sum_{j=0}^{\infty}\int_{s_j}^{s_{j+1}} k^{B\prime}(u)\,k'\left(k^B(u)\right)G^+(u)du\end{aligned}$$

We first have $G^+(+\infty) = G^+(u_0) = 0$ and $G^+(u) \geq 0$ for $u \geq u_0$. Since k is increasing, concave and admits left and right derivatives everywhere, we also have for $u \geq u_0$, $0 \leq k'\left(k^B(u)\right) \leq k^{\prime,r}\left(k^B(u_0)\right)$, where $k^{\prime,r}\left(k^B(u_0)\right)$ is the right derivative of k in $k^B(u_0)$. We therefore deduce that:

$$\int_{u_0}^{\infty} k\left(k^B(u)\right)dG^+(u) = -\sum_{j=0}^{\infty}\int_{s_j}^{s_{j+1}} k^{B\prime}(u)\,k'\left(k^B(u)\right)G^+(u)du \leq 0$$

$$\text{and}\quad \int_{u_0}^{\infty} k\left(k^B(u)\right)dG^+(u) \geq -\,k^{\prime,r}\left(k^B(u_0)\right)\sum_{j=0}^{\infty}\int_{s_j}^{s_{j+1}} k^{B\prime}(u)G^+(u)du$$

We proceed in a reverse way to integrate by parts $u \mapsto k^{B\prime}(u)G^+(u)$ and deduce:

$$0 \geq \int_{u_0}^{\infty} k\left(k^B(u)\right)dG^+(u) \geq k^{\prime,r}\left(k^B(u_0)\right)\int_{u_0}^{\infty} k^B(u)dG^+(u)$$

Analogously, we obtain the following inequality for the left-hand side of (10):

$$\int_{-\infty}^{u_0} k\left(k^B(u)\right)\,dG^-(u) \leq k^{\prime,l}(k^B(u_0))\int_{-\infty}^{u_0} k^B(u)\,dG^-(u) \leq 0$$

where $k^{\prime,l}$ is the left derivative of k

Plugging these two inequalities into (10) yields:

$$0 \geq k^{\prime,l}(k^B(u_0))\int_{-\infty}^{u_0} k^B(u)\,dG^-(u) \geq k^{\prime,r}(k^B(u_0))\int_{u_0}^{\infty} k^B(u)\,dG^+(u)$$

Since k is increasing and concave, $k^{\prime,l}(k^B(u_0)) \geq k^{\prime,r}(k^B(u_0)) \geq 0$. Moreover, both integrals $\int_{-\infty}^{u_0} k^B(u)dG^-(u)$ and $\int_{u_0}^{\infty} k^B(u)dG^+(u)$ are negative. We deduce:

$$0 \geq \int_{-\infty}^{u_0} k^B(u)\,dG^-(u) \geq \int_{u_0}^{\infty} k^B(u)\,dG^+(u)$$

Expressing G^- and G^+ as functions of F_{L^1} and F_{L^2} and bringing together the integrals gives us:

$$\int_{-\infty}^{\infty} k^B(u)F'_{L_1}(u)du \geq \int_{-\infty}^{\infty} k^B(u)F'_{L_2}(u)du$$

This inequality states that agent B prefers lottery L_1 to L_2, proving the result.

2. Quiggin anticipated utility functions.

2.a. First implication.

We use a proof strategy here which is very similar to Chateauneuf, Cohen, and Meilijson (2004). Assume that agent A is more risk-averse than agent B. We denote by $U^Q_{\phi^i}$, $i = A, B$ their respective Quiggin anticipated utility functions.

We consider lotteries with four possible outcomes. L_1 is a lottery paying $x_1 < x_2 < x_3 < x_4$ with respective probabilities p_1, p_2, p_3, and $p_4 = 1 - p_1 - p_2 - p_3$. L_2 is a lottery which pays the outcomes x_1, $x_2 - \varepsilon_2$, $x_3 + \varepsilon_3$, x_4 with the same probabilities, with ε_2, $\varepsilon_3 > 0$ and small enough to respect the initial outcome order. Unambiguously, L_2 is a $(p_1 + p_2)-$spread of L_1.

The utility of A associated with L_1 is written as:

$$\begin{aligned}
U^Q_{\phi^A}(L_1) &= -x_1(\phi^A(1-p_1) - \phi^A(1)) - x_2(\phi^A(1-p_1-p_2) - \phi^A(1-p_1)) \\
&\quad - x_3(\phi^A(1-p_1-p_2-p_3) - \phi^A(1-p_1-p_2)) - x_4(\phi^A(0) - \phi^A(1-p_1-p_2-p_3)) \\
&= x_1 + (x_2 - x_1)\phi^A(p_2+p_3+p_4) + (x_3 - x_2)\phi^A(p_3+p_4) + (x_4 - x_3)\phi^A(p_4) \\
&= x_1 + (x_2 - x_1)\phi^A(q_2) + (x_3 - x_2)\phi^A(q_3) + (x_4 - x_3)\phi^A(q_4)
\end{aligned}$$

where: $\quad p_j = q_j - q_{j+1}$ with $1 = q_1 \geq q_2 \geq q_3 \geq q_4 \geq q_5 = 0$

We choose ε_3 such that agent A is indifferent between L_2 and L_1. Agent A being more risk-averse than B, B prefers L_2 to L_1. Noting $\phi^A = \phi \circ \phi^B$ (which implies that ϕ is increasing and continuous), we have the following two relationships:

$$\begin{aligned}
\varepsilon_3 \left(\phi \circ \phi^B(q_3) - \phi \circ \phi^B(q_4)\right) &= \varepsilon_2 \left(\phi \circ \phi^B(q_2) - \phi \circ \phi^B(q_3)\right) \\
\varepsilon_3 \left(\phi^B(q_3) - \phi^B(q_4)\right) &\geq \varepsilon_2 \left(\phi^B(q_2) - \phi^B(q_3)\right)
\end{aligned}$$

Substituting the first equality ($\phi^B(q_2) \geq \phi^B(q_3) \geq \phi^B(q_4)$ since ϕ^B is increasing) yields:

$$\frac{\phi \circ \phi^B(q_2) - \phi \circ \phi^B(q_3)}{\phi^B(q_2) - \phi^B(q_3)} \geq \frac{\phi \circ \phi^B(q_3) - \phi \circ \phi^B(q_4)}{\phi^B(q_3) - \phi^B(q_4)}$$

$$\frac{\phi \circ \phi^B(q_2)}{\phi^B(q_2) - \phi^B(q_3)} + \frac{\phi \circ \phi^B(q_4)}{\phi^B(q_3) - \phi^B(q_4)} \geq \phi \circ \phi^B(q_3) \frac{\phi^B(q_2) - \phi^B(q_4)}{(\phi^B(q_3) - \phi^B(q_4))(\phi^B(q_2) - \phi^B(q_3))}$$

$$\frac{\phi^B(q_3) - \phi^B(q_4)}{\phi^B(q_2) - \phi^B(q_4)} \phi \circ \phi^B(q_2) + \frac{\phi^B(q_2) - \phi^B(q_3)}{\phi^B(q_2) - \phi^B(q_4)} \phi \circ \phi^B(q_4) \geq \phi \circ \phi^B(q_3)$$

Since $\frac{\phi^B(q_3)-\phi^B(q_4)}{\phi^B(q_2)-\phi^B(q_4)} > 0$ and $\frac{\phi^B(q_2)-\phi^B(q_3)}{\phi^B(q_2)-\phi^B(q_4)} > 0$, the last inequality states that ϕ is a convex function.

2.b. Second implication.

We suppose that ϕ^A is more convex than ϕ^B. There thus exists a continuous, increasing, convex function ϕ, such that $\phi^A = \phi \circ \phi^B$ and $\phi(0) = 0$ and $\phi(1) = 1$. We consider two lotteries L_1 and L_2, such that: $L_1 \vdash_p L_2$ and $L_1 \succeq^A L_2$. Stating that agent A prefers L_1 to L_2 yields, using Quiggin's functional forms:

$$-\int_{-\infty}^{\infty} u\, d\left(\phi\left(\phi^B(1 - F_{L_1}(u))\right)\right) \geq -\int_{-\infty}^{\infty} u\, d\left(\phi\left(\phi^B(1 - F_{L_2}(u))\right)\right)$$

Since $L_1 \vdash_p L_2$ there exists $u_0 \in \mathbb{R}$, such that $F_{L_1}(u_0) = F_{L_2}(u_0) = p$. Additionally, $1 - F_{L_2}(u) \geq 1 - F_{L_1}(u)$ for $u \leq u_0$ and $1 - F_{L_2}(u) \leq 1 - F_{L_1}(u)$ for $u \geq u_0$. We deduce:

$$\begin{aligned}
-\int_{-\infty}^{u_0} u\, d\left(\phi\left(\phi^B(1 - F_{L_1}(u))\right)\right) - \int_{u_0}^{\infty} u\, d\left(\phi\left(\phi^B(1 - F_{L_1}(u))\right)\right) \geq& \\
-\int_{-\infty}^{u_0} u\, d\left(\phi\left(\phi^B(1 - F_{L_2}(u))\right)\right) - \int_{u_0}^{\infty} u\, d\left(\phi\left(\phi^B(1 - F_{L_2}(u))\right)\right)& \\
u_0 \left(1 - \phi\left(\phi^B(1 - F_{L_1}(u_0))\right)\right) - \int_{-\infty}^{u_0} \left(1 - \phi\left(\phi^B(1 - F_{L_1}(u))\right)\right) du& \\
+ u_0 \phi\left(\phi^B(1 - F_{L_1}(u_0))\right) + \int_{u_0}^{\infty} \phi\left(\phi^B(1 - F_{L_1}(u))\right) du \geq& \\
u_0 \left(1 - \phi\left(\phi^B(1 - F_{L_2}(u_0))\right)\right) - \int_{-\infty}^{u_0} \left(1 - \phi\left(\phi^B(1 - F_{L_2}(u))\right)\right) du& \\
+ u_0 \phi\left(\phi^B(1 - F_{L_2}(u_0))\right) + \int_{u_0}^{\infty} \phi\left(\phi^B(1 - F_{L_2}(u))\right) du& \\
-\int_{-\infty}^{u_0} \left(1 - \phi\left(\phi^B(1 - F_{L_1}(u))\right)\right) du + \int_{u_0}^{\infty} \phi\left(\phi^B(1 - F_{L_1}(u))\right) du \geq& \\
-\int_{-\infty}^{u_0} \left(1 - \phi\left(\phi^B(1 - F_{L_2}(u))\right)\right) du + \int_{u_0}^{\infty} \phi\left(\phi^B(1 - F_{L_2}(u))\right) du&
\end{aligned}$$

Finally, we obtain:

$$\int_{-\infty}^{u_0} \left[\phi\left(\phi^B(1-F_{L_2}(u))\right) - \phi\left(\phi^B(1-F_{L_1}(u))\right)\right] du \leq \\ \int_{u_0}^{\infty} \left[\phi\left(\phi^B(1-F_{L_1}(u))\right) - \phi\left(\phi^B(1-F_{L_2}(u))\right)\right] du \tag{12}$$

Focusing on the left-hand side, we deduce:

$$\int_{-\infty}^{u_0} \left[\phi\left(\phi^B(1-F_{L_2}(u))\right) - \phi\left(\phi^B(1-F_{L_1}(u))\right)\right] du = \\ \int_{-\infty}^{u_0} \left(\phi^B(1-F_{L_2}(u)) - \phi^B(1-F_{L_1}(u))\right) \frac{\phi\left(\phi^B(1-F_{L_2}(u))\right) - \phi\left(\phi^B(1-F_{L_1}(u))\right)}{\phi^B(1-F_{L_2}(u)) - \phi^B(1-F_{L_1}(u))} du$$

We use a similar argument as in the Kihlstrom and Mirman case. We denote $\phi^B(1-F_{L_i}(u)) = 1-t_i$ (for $i=1,2$). Since ϕ^B is increasing, $0 \leq t_2 \leq t_1 \leq 1-\phi^B(1-p)$ for $u \leq u_0$. We then focus on $\frac{\psi(t_1)-\psi(t_2)}{t_1-t_2}$, where $\psi(t) = -\phi(1-t)$ is increasing and concave. We can find a lower bound for this expression, which is $\psi'^{,l}(1-\phi^B(1-p)) = \phi'^{,r}(\phi^B(1-p))$. Since $\phi^B(1-F_{L_2}(u)) \geq \phi^B(1-F_{L_1}(u))$ for $u \leq u_0$, we deduce:

$$\int_{-\infty}^{u_0} \left[\phi\left(\phi^B(1-F_{L_2}(u))\right) - \phi\left(\phi^B(1-F_{L_1}(u))\right)\right] du \geq \\ \phi'^{,r}(\phi^B(1-p)) \int_{-\infty}^{u_0} \left[\phi^B(1-F_{L_2}(u)) - \phi^B(1-F_{L_1}(u))\right] du$$

Focusing on the right-hand side of (12), we similarly obtain:

$$\int_{u_0}^{\infty} \left[\phi\left(\phi^B(1-F_{L_1}(u))\right) - \phi\left(\phi^B(1-F_{L_2}(u))\right)\right] du = \\ \int_{u_0}^{\infty} \left(\phi^B(1-F_{L_1}(u)) - \phi^B(1-F_{L_2}(u))\right) \frac{\phi\left(\phi^B(1-F_{L_1}(u))\right) - \phi\left(\phi^B(1-F_{L_2}(u))\right)}{\phi^B(1-F_{L_1}(u)) - \phi^B(1-F_{L_2}(u))} du$$

We similarly denote $\phi^B(1-F_{L_i}(u)) = 1-x_i$ (for $i=1,2$), with $1-\phi^B(1-p) \leq x_1 \leq x_2 \leq 1$. We focus on $\frac{\psi(x_2)-\psi(x_1)}{x_2-x_1}$, where $\psi(t) = -\phi(1-t)$ is increasing and concave. We can find a upper bound for this expression, which is $\psi'^{,r}(1-\phi^B(1-p)) = \phi'^{,l}(\phi^B(1-p))$. We deduce because $\phi^B(1-F_{L_1}(u)) \geq \phi^B(1-F_{L_2}(u))$ for $u \geq u_0$:

$$\int_{u_0}^{\infty} \left[\phi\left(\phi^B(1-F_{L_1}(u))\right) - \phi\left(\phi^B(1-F_{L_2}(u))\right)\right] du \leq \\ \phi'^{,l}(\phi^B(1-p)) \int_{u_0}^{\infty} \left(\phi^B(1-F_{L_1}(u)) - \phi^B(1-F_{L_2}(u))\right) du$$

The inequality (12) becomes:

$$\phi^{\prime,r}(\phi(1-p))\int_{-\infty}^{u_0}\left[\phi^B(1-F_{L_2}(u))-\phi^B(1-F_{L_1}(u))\right]du\leq$$
$$\phi^{\prime,l}(\phi^B(1-p))\int_{u_0}^{\infty}\left(\phi^B(1-F_{L_1}(u))-\phi^B(1-F_{L_2}(u))\right)du$$

Since ϕ is convex and increasing, we have $0\leq\phi^{\prime,l}(1-\phi^B(p))\leq\phi^{\prime,r}(1-\phi^B(p))$. Because both integrals are positive, this yields:

$$\int_{-\infty}^{u_0}\left[\phi^B(1-F_{L_2}(u))-\phi^B(1-F_{L_1}(u))\right]du\leq\int_{u_0}^{\infty}\left(\phi^B(1-F_{L_1}(u))-\phi^B(1-F_{L_2}(u))\right)du$$

We carry out the same manipulations in reverse order and deduce:

$$-\int_{-\infty}^{\infty}u\,d\left(\phi^B\left(1-F_{L_1}(u)\right)\right)\geq-\int_{-\infty}^{\infty}u\,d\left(\phi^B\left(1-F_{L_2}(u)\right)\right)$$

Agent B therefore prefers lottery L_1 to L_2, which completes the proof.

References

Aumann, R. J., and M. Kurz (2005): "Power and Taxes," *Econometrica*, 58(5), 1137–1161.

Bickel, P. J., and E. L. Lehmann (1976): "Descriptive Statistics for Nonparametric Models. III. Dispersion," *The Annals of Statistics*, 4(6), 1139–1158.

Bleichrodt, H., and L. Eeckhoudt (2005): "Saving under Rank-Dependent Utility," *Economic Theory*, 25(2), 505–511.

Bommier, A. (2006): "Uncertain Lifetime and Intertemporal Choice: Risk Aversion as a Rationale for Time Discounting," *International Economic Review*, 47(4), 1223–1246.

——— (2008): "Rational Impatience?," *Working Paper*, http://hal.archives-ouvertes.fr/hal-00441880/en/.

Caperaa, P., and L. Eeckhoudt (1975): "Delayed Risks and Risk Premiums," *Journal of Financial Economics*, 2(3), 309–320.

Chateauneuf, A., M. Cohen, and I. Meilijson (2004): "Four Notions of Mean Preserving Increase in Risk, Risk Attitudes and Applications to the Rank-Dependent Expected Utility Model," *Journal of Mathematical Economics*, 40(5), 547–571.

Chew, S. H., and L. G. Epstein (1990): "Nonexpected Utility Preferences in a Temporal Framework with an Application to Consumption-Savings Behaviour," *Journal of Economic Theory*, 50(1), 54–81.

Drèze, J. H., and F. Modigliani (1972): "Consumption Decisions under Uncertainty," *Journal of Economic Theory*, 5(3), 308–335.

Epstein, L. G., and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal

Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4), 937–969.
Foncel, J., and N. Treich (2005): "Fear of Ruin," *Journal of Risk and Uncertainty*, 31(3), 289–300.
Ghirardato, P., and M. Marinacci (2002): "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102(2), 251–289.
Grant, S., and J. C. Quiggin (2005): "Increasing Uncertainty: A Definition," *Mathematical Social Sciences*, 49(2), 117–141.
Jewitt, I. (1987): "Risk Aversion and the Choice Between Risky Prospects: The Preservation of Static Results," *Review of Economic Studies*, 54(1), 73–85.
Johnson, J. P., and D. P. Myatt (2006): "On the Simple Economics of Advertising, Marketing, and Product Design," *American Economic Review*, 96(3), 756–784.
Kihlstrom, R. E., and L. J. Mirman (1974): "Risk Aversion with many Commodities," *Journal of Economic Theory*, 8(3), 361–388.
Kimball, M. S. (1990): "Precautionary Savings in the Small and in the Large," *Econometrica*, 58(1), 53–73.
Kimball, M. S., and P. Weil (2009): "Precautionary Saving and Consumption Smoothing Across Time and Possibilities," *Journal of Money, Credit and Banking*, 41(2-3), 245–284.
Kreps, D. M., and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46(1), 185–200.
Langlais, E. (1995): "A Measure of the Sensitivity of Saving to Interest Rate Uncertainty with Non-Expected Preferences," *Economic Letters*, 48(3-4), 325–330.
Leland, H. E. (1968): "Saving and Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics*, 82(3), 465–473.
Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford University Press.
Nicolescu, C., and L.-E. Persson (2006): *Convex Functions and their Applications: A Contemporary Approach*. Canadian Mathematical Society, Springer.
Pratt, J. W. (1964): "Risk Aversion in the Small and in the Large," *Econometrica*, 32(1/2), 122–136.
Quiggin, J. C. (1982): "A Theory of Anticipated Utility," *Journal of Economic Behavior and Organization*, 3(4), 323–343.
Sandmo, A. (1970): "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies*, 37(3), 353–360.
Segal, U., A. Spivak, and J. Zeira (1988): "Precautionary Saving and Risk Aversion: An Anticipated Utility Approach," *Economic Letters*, 27(3), 223–227.
Selden, L. (1978): "A New Representation of Preferences over "Certain x Uncertain" Consumption Pairs: The "Ordinal Certainty Equivalent" Hypothesis," *Econometrica*, 46(5), 1045–1060.
Yaari, M. E. (1969): "Some Remarks on Measures of Risk Aversion and their Uses," *Journal of Economic Theory*, 1(3), 315–329.
——— (1987): "The Dual Theory of Choice under Risk," *Econometrica*, 55(1), 95–105.