UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 12-07
Statistics and Econometrics Series 04
March 2012

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

COMPARISONS AMONG SPACINGS FROM TWO POPULATIONS

Nuria Torrado[a] and Rosa E. Lillo[b]

Abstract

In this work, we obtain some new results in the area of stochastic comparisons of simple and normalized spacings from two heterogeneous populations. We also show some applications of our results to multiple-outlier models.

Keywords: stochastic comparisons, reliability, spacings

[a] Universidad Pública de Navarra, Departamento de Estadística e Investigación Operativa, Campus de Arrosadía, Pamplona, Navarra, Spain.
[b] Universidad Carlos III de Madrid, Departamento de Estadística, Facultad de Ciencias Sociales y Jurídicas, Campus de Getafe, Madrid, Spain.

e-mail addresses: nuria.torrado@unavarra.es (Nuria Torrado) and rosaelvira.lillo@uc3m.es (Rosa E. Lillo).

1 Introduction

Let $X_1, \ldots, X_n$ be a set of independent exponential random variables and $Y_1, \ldots, Y_n$ be another set of independent exponential random variables. If X_i and Y_i, $i = 1, \ldots, n$, are the lifetimes of the components of two $(n-i+1)$-out-of-n systems, then $X_{i:n}$ and $Y_{i:n}$ are the lifetime of the first and the second system, $D_{i:n}$ and $C_{i:n}$ are the i'th times elapsed between failures of components, respectively, which are called *simple spacings*, and $D^*_{i:n}$ and $C^*_{i:n}$ are the i'th *normalized spacings* from X_i's and Y_i's, respectively. A natural question is to examine whether the first system is better than the second one in some stochastic sense.

This problem has been previously treated in the literature but with some restrictions. For example, many researchers have considered the problem of comparing the spacings of non-identical independent exponential random variables with those corresponding to independent and identically distributed exponential random variables according to different stochastic orderings such as the usual stochastic and the likelihood ratio orderings. In particular, Pledger and Proschan [12] showed that the i'th normalized spacing of a sample of size n from heterogeneous exponential population is stochastically larger than the i'th normalized spacing of a sample of size n whose distribution is the average of the distributions in the heterogeneous case. This result give a lower bound for the survival function of normalized spacings from independent, heterogeneous exponential distributions based on the case when they are i.i.d. Kochar and Kowar [6] extended this result from stochastic ordering to likelihood ratio ordering. Recently, Kochar and Xu [7] provided necessary and sufficient conditions for stochastically comparing according to likelihood ratio ordering when $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from an exponential distribution with common hazard rate λ which can differ from $\overline{\lambda}$.

An interesting special case studied in the literature is multiple-outliers models with parameters λ and λ_*, that is, when $X_1, \ldots, X_n$ are independent exponential random variables such that X_i has hazard rate λ for $i = 1, \ldots, p$ and X_j has hazard rate λ_* for $j = p+1, \ldots, n$. Khaledi and Kochar [5] established the hazard rate ordering between successive normalized spacings from a single-outlier exponential model, that is, when $p = n-1$. This result was strengthened to likelihood ratio ordering of simple spacings from a multiple-outlier exponential model (see e.g., Wen et al. [14] and Hu et al. [8]). An important application of order statistics from multiple-outliers models is the study of the robustness of different estimators of parameters of a wide range of distributions, see e.g. Balakrishnan [2].

Not much work has been done when the two samples are nonidentical independent exponential random variables, because of the complicated nature of the problem. The objective of this article is to investigate stochastic properties between both, simple and normalized spacings, of two heterogeneous samples. Specifically, we study the likelihood ratio order between successive spacings from two samples of exponential distributions with different scale parameters and we also show some applications to multiple-outlier models.

The article is organized as follows. In Section 2, we review some stochastic orderings which are used in this article and the probability density function (pdf) of normalized and simple spacings, and give two useful lemmas which will be used in the following sections. We provide, in Section 3, some new results related to the likelihood ratio ordering of spacings of two samples from heterogeneous exponential random variables. Section 4 is devoted to stochastic comparisons of spacings in multiple-outlier models and finally we show some conclusions in Section 5.

2 Stochastic orders and preliminaries

In this article, we investigate stochastic comparisons between successive spacings based on order statistics from two samples of heterogeneous exponential random variables. Formally, if the random variables $X_1, \ldots, X_n$ are arranged in ascending order of magnitude, then the i'th smallest of X_i's, denoted by $X_{i:n}$, is the i'th order statistic, and the random variables

$$D_{i:n} = X_{i:n} - X_{i-1:n} \quad \text{and} \quad D^*_{i:n} = (n - i + 1)\, D_{i:n},$$

for $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$, are called simple spacings and normalized spacings, respectively. It is well known that the lifetime of a $(n-i+1)$-out-of-n system is usually described by the i'th order statistic, and the times between failures of components in a $(n-i+1)$-out-of-n system correspond with the spacings associated with order statistics.

Here, we give briefly a review of stochastic orders related to the *location*, the *magnitude* and the *dispersion* of random variables. Stochastic orders between probability distributions is a widely studied field, see e.g. Shaked and Shantikumar [11] as a reference in this field. Throughout, we shall use *increasing* to mean *non-decreasing* and *decreasing* to mean *non-increasing*.

Definition 2.1. For two random variables X and Y with their densities f, g and distributions functions F, G, let $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$. As the ratios in the statements bellow are well defined, X is said to be smaller than Y in the:

a) *usual stochastic order* if $\overline{F}(t) \leq \overline{G}(t)$ for all t, and in this case, we write $X \leq_{\text{st}} Y$ or $F \leq_{\text{st}} G$,

b) *hazard rate order*, denoted by $X \leq_{\text{hr}} Y$ or $F \leq_{\text{hr}} G$, if $\overline{G}(t)/\overline{F}(t)$ is increasing in t,

c) *likelihood ratio order* if $g(t)/f(t)$ is increasing in t and in this case, we write $X \leq_{\text{lr}} Y$ or $F \leq_{\text{lr}} G$.

It is well known that likelihood ratio ordering implies hazard rate ordering which, in turn, implies usual stochastic ordering. For more details, see Shaked and Shanthikumar [11].

Next, we review the dispersive order that compare the *variability* or the *dispersion* of random variables.

Definition 2.2. We say that X is smaller than Y in the *dispersive order* if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha),$$

for all $0 < \alpha < \beta < 1$, where we write $X \leq_{\text{disp}} Y$ or $F \leq_{\text{disp}} G$.

We shall also be using the concept of majorization in our discussion. Let $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

Definition 2.3. The vector $\boldsymbol{x}$ is said to be majorized by the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \leq^{\text{m}} \boldsymbol{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}, \quad \text{for } j = 1, \ldots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

Functions that preserve the ordering of majorization are said to be Schur-convex, as one can see in the following definition.

Definition 2.4. A real valued function φ defined on a set $\mathcal{A} \in \Re^n$ is said to be *Schur-convex* (*Schur-concave*) on $\mathcal{A}$ if

$$\boldsymbol{x} \leq^{\mathrm{m}} \boldsymbol{y} \Rightarrow \varphi(\boldsymbol{x}) \leq (\geq) \varphi(\boldsymbol{y}).$$

For extensive and comprehensive details on the theory of majorization orders and their applications, please refer to the excellent book of Marshall and Olkin [9].

For heterogeneous but independent exponential random variables, Kochar and Korwar [6] proved that, for $i \in \{2, \ldots, n\}$, the distribution of the i'th normalized spacing, D_i^*, is a mixture of independent exponential random variables with p.d.f.

$$f_i^*(t) = \sum_{\mathbf{r}_n} \frac{\prod_{k=1}^{n} \lambda_k}{\prod_{k=1}^{n} \left(\sum_{j=k}^{n} \lambda_{r_j} \right)} \left(\frac{\sum_{j=i}^{n} \lambda_{r_j}}{n-i+1} \right) \exp\left(-t \, \frac{\sum_{j=i}^{n} \lambda_{r_j}}{n-i+1} \right), \tag{2.1}$$

where $\mathbf{r}_n = (r_1, \ldots, r_n)$ is a permutation of $(1, \ldots, n)$. Then, following Torrado et al. [13], (2.1) can be written as

$$f_i^*(t) = \sum_{j=1}^{M_i} \Delta(\beta_{m_j}^{(i)}, n) \left(\frac{\beta_{m_j}^{(i)}}{n-i+1} \right) \exp\left(-t \, \frac{\beta_{m_j}^{(i)}}{n-i+1} \right), \tag{2.2}$$

where $M_i = \binom{n}{n-i+1}$,

$$\beta_{m_j}^{(i)} = \sum_{\ell=i}^{n} \lambda_{r_\ell}, \tag{2.3}$$

with m_j indicates a group of indices of size $n-i+1$, and

$$\Delta(\beta_{m_j}^{(i)}, n) = \sum_{\mathbf{r}_{i-1,m_j}} \left(\prod_{k \in H_{m_j}} \lambda_k \right) \left[\prod_{\ell=1}^{i-1} \left\{ \sum_{\substack{u=\ell \\ r_u \in H_{m_j}}}^{i-1} \lambda_{r_u} + \beta_{m_j}^{(i)} \right\} \right]^{-1}, \tag{2.4}$$

where $H_{m_j} = \{1, \ldots, n\} - m_j$ and the outer summation is being taken over all permutations of the elements of H_{m_j}.

The distribution of D_i is also a mixture of independent exponential random variables, with p.d.f.

$$f_i(t) = \sum_{j=1}^{M_i} \Delta(\beta_{m_j}^{(i)}, n)\, \beta_{m_j}^{(i)}\, e^{-t\beta_{m_j}^{(i)}}, \tag{2.5}$$

where M_i, $\beta_{m_j}^{(i)}$ and $\Delta(\beta_{m_j}^{(i)}, n)$ are defined as before.

Before proceeding to our main results, let us first recall two lemmas, which will be used in the following sections.

Lemma 2.5 (Lemma 3.1. in Kochar and Kowar [6]). *Let $\Delta(\beta_{m_j}^{(i)}, n)$ be as defined in (2.4). Suppose that m_1 and m_2 are two subsets of $\{1, \ldots, n\}$ of size $n-i+1$ $(1 < i \le n)$ and that they have all but one element in common. Denote the different element in m_1 by a_1 and that in m_2 by a_2. Then*

$$\lambda_{a_1}\Delta(\beta_{m_1}^{(i)}, n) \ge \lambda_{a_2}\Delta(\beta_{m_2}^{(i)}, n), \qquad \text{if} \quad \lambda_{a_2} \ge \lambda_{a_1}.$$

Lemma 2.6 (Chebyshev's inequality, Theorem 1 in Mitrinovic [10]). *Let $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_n$ be two increasing sequences of real numbers. Then*

$$n\sum_{i=1}^{n} a_i b_i \ge \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

3 Main results

Let $X_1, \ldots, X_n$ be a set of independent exponential random variables with X_i having hazard rate λ_i, for $i = 1, \ldots, n$ and $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with X_i having hazard rate θ_i, for $i = 1, \ldots, n$. Some researchers have investigated the effect on the survival function, the hazard rate function and other characteristics of the time to failure of the spacings when we switch the vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ to another vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$. Pledger and Proschan [12] proved with the help of a counterexample that, in general, the survival function of $D_{i:n}^*$ is not Schur-convex in $(\lambda_1, \ldots, \lambda_n)$. Note that, from Definition 2.4, this means that in general, if $\boldsymbol{\theta} \le^m \boldsymbol{\lambda}$ then $C_{i:n}^* \not\le_{st} D_{i:n}^*$. However, Kochar and Kowar [6] proved that the survival function of $D_{2:n}^*$ is Schur-convex in $(\lambda_1, \ldots, \lambda_n)$ and, in general, its hazard rate is not Schur-concave, although for $n = 2$, the hazard rate of the second normalized spacing is Schur-concave, i.e., if $\boldsymbol{\theta} \le^m \boldsymbol{\lambda}$ then $C_{2:2}^* \le_{\text{hr}} D_{2:2}^*$.

Next, we study conditions which are different to that of majorization under which normalized and simple spacings are ordered in the likelihood ratio ordering. First, we need an important result and a lemma that is a consequence of Lemma 2.5.

Theorem 3.1. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sequences of independent but not necessarily identically distributed random variables. Then,*

$$C_{i:n} \leq_{\mathrm{lr}} D_{i:n} \Leftrightarrow C^*_{i:n} \leq_{\mathrm{lr}} D^*_{i:n}\,,$$

for $i = 1, \ldots, n$.

Proof. It is easy to see that $D^*_{i:n} = \varphi_i(D_{i:n})$ where $\varphi_i(x) = (n-i+1)x$ is an increasing function. If $C_{i:n} \leq_{\mathrm{lr}} D_{i:n}$, then from Theorem 1.C.8. in [11] we get that $C^*_{i:n} \leq_{\mathrm{lr}} D^*_{i:n}$, and viceversa, since $\varphi^{-1}(x)$ is also an increasing function. □

Lemma 3.2. *Let $\Delta(\beta^{(i)}_{m_j}, n)$ be as defined in (2.4). Suppose that m_1 and m_2 are two subsets of $\{1, \ldots, n\}$ of size $n-i+1$ $(1 < i \leq n)$ and having all but one element in common. Denote the different element in m_1 by a_1 and that in m_2 by a_2. Then*

$$\beta^{(i)}_{m_1}\,\Delta(\beta^{(i)}_{m_1}, n) \geq \beta^{(i)}_{m_2}\,\Delta(\beta^{(i)}_{m_2}, n) \qquad \text{if} \quad \lambda_{a_2} \geq \lambda_{a_1}.$$

Proof. Let $c_1, \ldots, c_{i-1}$ be the common elements, then from Lemma 2.5, we have

$$\begin{aligned} \beta^{(i)}_{m_1}\Delta(\beta^{(i)}_{m_1}, n) = \left(\lambda_{a_1} + \sum_{j=1}^{i-1} c_i\right)\Delta(\beta^{(i)}_{m_1}, n) \geq \\ \left(\lambda_{a_2} + \sum_{j=1}^{i-1} c_i\right)\Delta(\beta^{(i)}_{m_2}, n) = \beta^{(i)}_{m_2}\Delta(\beta^{(i)}_{m_2}, n), \end{aligned}$$

and then, the proof is complete. □

Now we can establish likelihood ratio ordering between simple spacings from two heterogeneous exponential samples. First, let us define

$$\alpha^{(i)}_{\min} = \min_{1 \leq m_j \leq M_i} \alpha^{(i)}_{m_j}, \tag{3.6}$$

where $\alpha_{m_j}^{(i)}=\sum\limits_{\ell=i}^{n}\theta_{r_\ell}$. Note that

$$\alpha_{\min}^{(i)}=\sum_{j=1}^{n-i+1}\theta_{(j)}\,, \tag{3.7}$$

where $\left\{\theta_{(1)},\ldots,\theta_{(n)}\right\}$ denote the increasing arrangement of θ_i, for $i=1,\ldots,n$.

Theorem 3.3. *Let $X_1,\ldots,X_n$ be independent exponential random variables such that X_i has hazard rate λ_i, for $i=1,\ldots,n$, and $Y_1,\ldots,Y_n$ be independent exponential random variables such that Y_i has hazard rate θ_i, for $i=1,\ldots,n$. If*

$$\alpha_{\min}^{(i)}\geq(n-i+1)\overline{\lambda},$$

where $\alpha_{\min}^{(i)}$ is defined in (3.6) and $n\overline{\lambda}=\sum\limits_{i=1}^{n}\lambda_i$. Then,

$$C_{i:n}\leq_{\mathrm{lr}}D_{i:n},$$

for $i=1,\ldots,n$, where $D_{i:n}$ and $C_{i:n}$ are the i'th simple spacing from X_i's and Y_i's, respectively.

Proof. Observing equation (2.5), note that $C_{i:n}\leq_{\mathrm{lr}}D_{i:n}$ if and only if

$$\frac{f_{D_{i:n}}(t)}{f_{C_{i:n}}(t)}=\frac{\sum\limits_{k=1}^{M_i}\Delta(\beta_{m_k}^{(i)},n)\,\beta_{m_k}^{(i)}\,e^{-t\beta_{m_k}^{(i)}}}{\sum\limits_{j=1}^{M_i}\Delta(\alpha_{m_j}^{(i)},n)\,\alpha_{m_j}^{(i)}\,e^{-t\alpha_{m_j}^{(i)}}},$$

is increasing in t, where $\beta_{m_j}^{(i)}=\sum\limits_{\ell=i}^{n}\lambda_{r_\ell}$ and $\alpha_{m_j}^{(i)}=\sum\limits_{\ell=i}^{n}\theta_{r_\ell}$. Therefore, differentiating this equation with respect to t, we have to prove

$$\sum_{k=1}^{M_i}\sum_{j=1}^{M_i}\Delta(\beta_{m_k}^{(i)},n)\Delta(\alpha_{m_j}^{(i)},n)\,\beta_{m_k}^{(i)}\alpha_{m_j}^{(i)}\,e^{-t\left(\beta_{m_k}^{(i)}+\alpha_{m_j}^{(i)}\right)}\,\left(\alpha_{m_j}^{(i)}-\beta_{m_k}^{(i)}\right)\geq 0. \tag{3.8}$$

We suppose without loss of generality that the $\beta_{m_k}^{(i)}$'s are in increasing order. By Lemma 3.2, we know that $\beta_{m_k}^{(i)}\Delta(\beta_{m_k}^{(i)},n)$'s are in decreasing order, and it is easy to see that $e^{-t\beta_{m_k}^{(i)}}$ and $\left(\alpha_{m_j}^{(i)}-\beta_{m_k}^{(i)}\right)$ are in decreasing order also. Then, by Lemma 2.6, we have

$$\sum_{k=1}^{M_i}\sum_{j=1}^{M_i}\Delta(\beta_{m_k}^{(i)},n)\Delta(\alpha_{m_j}^{(i)},n)\,\beta_{m_k}^{(i)}\alpha_{m_j}^{(i)}\,e^{-t\left(\beta_{m_k}^{(i)}+\alpha_{m_j}^{(i)}\right)}\,\left(\alpha_{m_j}^{(i)}-\beta_{m_k}^{(i)}\right)\geq$$

$$\left(\sum_{k=1}^{M_i}\beta_{m_k}^{(i)}\Delta(\beta_{m_k}^{(i)},n)\,e^{-t\beta_{m_k}^{(i)}}\right)\,\sum_{j=1}^{M_i}\,\alpha_{m_j}^{(i)}\Delta(\alpha_{m_j}^{(i)},n)\,e^{-t\alpha_{m_j}^{(i)}}\,\sum_{k=1}^{M_i}\left(\alpha_{m_j}^{(i)}-\beta_{m_k}^{(i)}\right),$$

where

$$\sum_{k=1}^{M_i}\left(\alpha_{m_j}^{(i)}-\beta_{m_k}^{(i)}\right)=M_i\alpha_{m_j}^{(i)}-\sum_{k=1}^{M_i}\beta_{m_k}^{(i)}=\binom{n}{n-i+1}\alpha_{m_j}^{(i)}-\binom{n-1}{n-i}\sum_{i=1}^{n}\lambda_i\geq 0,$$

if and only if

$$\alpha_{m_j}^{(i)}\geq\binom{n-1}{n-i}\binom{n}{n-i+1}^{-1}\sum_{i=1}^{n}\lambda_i=(n-i+1)\overline{\lambda}.$$

Hence, the required result follows since $\alpha_{\min}^{(i)}\geq(n-i+1)\overline{\lambda}$ for $i=1,\ldots,n$. □

A natural question is to examine if the condition of Theorem 3.3 implies majorization and viceversa. The following examples show that, in general, this is not the case.

Example 3.4. If $\boldsymbol{\theta}=(40,10,1)$ and $\boldsymbol{\lambda}=(40,5.5,5.5)$, it is easy to check that $\boldsymbol{\lambda}\leq^{\mathrm{m}}\boldsymbol{\theta}$, however, for $i=2$, we have that

$$\alpha_{\min}^{(2)}=\min_{1\leq m_j\leq M_2}\alpha_{m_j}^{(2)}=11<34=(n-i+1)\overline{\lambda}=(n-i+1)\overline{\theta}.$$

Note that, in this case, the normalized spacings are not ordered in the hazard rate ordering (see example 3.2. in Kochar and Kowar[6]).

Example 3.5. If $\boldsymbol{\theta}=(40,10,1)$ and $\boldsymbol{\lambda}=(5.5,5.5,4)$, for $i=2$ we get

$$\alpha_{\min}^{(2)}=11>10=(n-i+1)\overline{\lambda},$$

and $\boldsymbol{\lambda}\not\leq^{\mathrm{m}}\boldsymbol{\theta}$.

Remark 3.6. Let $\left\{\theta_{(1)},\ldots,\theta_{(n)}\right\}$ denote the increasing arrangement of θ_i, for $i=1,\ldots,n$. It is easy to check that

$$\overline{\theta}\geq\frac{\alpha_{\min}^{(2)}}{n-1}\geq\ldots\geq\frac{\alpha_{\min}^{(i-1)}}{n-i+2}\geq\frac{\alpha_{\min}^{(i)}}{n-i+1}\geq\frac{\alpha_{\min}^{(i+1)}}{n-i}\geq\ldots\geq\frac{\alpha_{\min}^{(n-1)}}{2}\geq\theta_{(1)}. \tag{3.9}$$

Let $i=n$. Then from Theorem 3.3 we know that if $\theta_{(1)}\geq\overline{\lambda}$ then $C_{n:n}\leq_{\mathrm{lr}}D_{n:n}$. Now, by equation (3.9) we get that if $\theta_{(1)}\geq\overline{\lambda}$ then $C_{i:n}\leq_{\mathrm{lr}}D_{i:n}$, for $i=1,\ldots,n$. Even more, if we fix i, the condition $\alpha_{\min}^{(i)}\geq(n-i+1)\overline{\lambda}$ of Theorem 3.3 implies not only $C_{i:n}\leq_{\mathrm{lr}}D_{i:n}$ but also $C_{j:n}\leq_{\mathrm{lr}}D_{j:n}$, for $j=1,\ldots,i$.

Note that for $i = 1$, $X_{1:n} = D_{1:n} = D^*_{1:n}$, and from Theorem 3.3 we have

$$\sum_{i=1}^{n} \theta_i \geq \sum_{i=1}^{n} \lambda_i \Rightarrow Y_{1:n} \leq_{\text{lr}} X_{1:n},$$

which it is well known since $X_{1:n} \sim \exp(\lambda_1 + \cdots + \lambda_n)$ and $Y_{1:n} \sim \exp(\theta_1 + \cdots + \theta_n)$.

Corollary 3.7. *Let $X_1, \ldots, X_n$ be independent exponential random variables such that X_i has hazard rate λ_i for $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate θ. Then,*

a) $C_{i:n} \leq_{\text{lr}} D_{i:n}$, if $\overline{\lambda} \leq \theta$,

b) $D_{i:n} \leq_{\text{lr}} C_{i:n}$, if $\theta \leq \frac{\beta^{(i)}_{min}}{n-i+1}$,

for $i = 1, \ldots, n$.

Proof. a) It is easy to see that $\alpha^{(i)}_{m_j} = (n-i+1)\theta$ for all m_j, since $Y_1, \ldots, Y_n$ have the same hazard rate. Then, (3.8) holds since $\theta \geq \overline{\lambda} \Leftrightarrow \alpha^{(i)}_{m_j} = (n-i+1)\theta \geq (n-i+1)\overline{\lambda}$, which is true.

b) Replacing $C_{i:n}$ by $D_{i:n}$ in Theorem 3.3, it is easy to see that $D_{i:n} \leq_{\text{lr}} C_{i:n}$ if $\beta^{(i)}_{min} \geq (n-i+1)\overline{\theta}$ which is equivalent to $\beta^{(i)}_{min} \geq (n-i+1)\theta$, since $Y_1, \ldots, Y_n$ have the same hazard rate.

□

Note that Theorem 3.5. in Kochar and Kowar [6] can be seen as a particular case of Corollary 3.7a), when $\theta = \overline{\lambda}$. In order to illustrate the performance of the above result, we present here some interesting special cases. Let $X_1, \ldots, X_n$ be independent exponential random variables such that X_i has hazard rate λ_i for $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate θ. Suppose that $\lambda_1 = \ldots = \lambda_n = \lambda$, it follows from Corollary 3.7 that $C_{i:n} \leq_{\text{lr}} D_{i:n} \Leftrightarrow \theta \geq \lambda$, which is a well known result in the literature. Another interesting special case is the following.

Proposition 3.8. *Let $X_1, \ldots, X_n$ be independent exponential random variables such that X_i has hazard rate λ_i, for $i = 1, \ldots, n$, $Y_1, \ldots, Y_n$ be a random sample of size n from an* *exponential distribution with common hazard rate* $\lambda_{(n)} = \max\{\lambda_1, \ldots, \lambda_n\}$*, and* $Z_1, \ldots, Z_n$ *be a random sample of size* n *from an exponential distribution with common hazard rate* $\lambda_{(1)} = \min\{\lambda_1, \ldots, \lambda_n\}$*. Then*

$$C_{i:n} \leq_{\text{lr}} D_{i:n} \leq_{\text{lr}} H_{i:n},$$

for $i = 1, \ldots, n$ *where* $C_{i:n}$*,* $D_{i:n}$*,* $H_{i:n}$ *denote the* i*'th simple spacings of* Y_i*'s,* X_i*'s and* Z_i*'s, respectively.*

Proof. It is easy to check that $\lambda_{(n)} \geq \overline{\lambda}$. Then due to Corollary 3.7a), it follows that $C_{i:n} \leq_{\text{lr}} D_{i:n}$, for $i = 1, \ldots, n$. By (3.9), we know that $\beta_{min}^{(i)} \geq (n-i+1)\lambda_{(1)}$ for all i, and applying again Corollary 3.7b) we get $D_{i:n} \leq_{\text{lr}} H_{i:n}$, for $i = 1, \ldots, n$. □

This result is of interest because it provides upper and lower bounds for the survival and the hazard rate functions since the likelihood ratio order implies the usual stochastic and the hazard rate orders. To illustrate this result, we provide the following example.

Example 3.9. Assume that $(\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}) = (0.9, 1.0, 4.0)$. Figure 1 shows the consequences of Proposition 3.8, where one can see the survival function of the second simple spacing from a heterogeneous exponential random sample with hazard rate $(\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}) = (0.9, 1.0, 4.0)$. This survival function is bounded by the survival function of the second simple spacing from an exponential random sample with hazard rate $\lambda_{(1)} = 0.9$ and by the survival function of the second simple spacing from an exponential random sample with hazard rate $\lambda_{(3)} = 4$. Even more, we can consider as the lower bound the survival function of the second simple spacing from an exponential random sample with hazard rate $\overline{\lambda} = 1.967$.

Another interesting upper bound for the i'th simple spacing of X_i's when $X_1, \ldots, X_n$ are independent heterogeneous exponential random variables is the following.

Proposition 3.10. *Let* $X_1, \ldots, X_n$ *be independent exponential random variables such that* X_i *has hazard rate* λ_i*, for* $i = 1, \ldots, n$*, and* $Y_1, \ldots, Y_n$ *be a random sample of size* n *from an exponential distribution with common hazard rate* $\lambda_{min}^{(i+1)}/(n-i)$*, Then*

$$D_{i:n} \leq_{\text{lr}} C_{i:n},$$

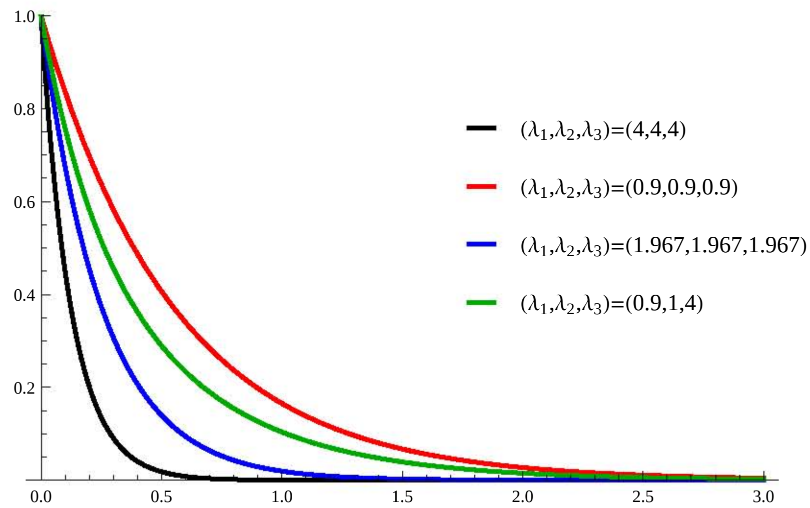

Figure 1: The survival curves with different parameters.

for $i = 1, \ldots, n$ *where* $C_{i:n}$ *and* $D_{i:n}$ *denote the* i *'th simple spacings of* Y_i*'s and* X_i*'s, respectively.*

The proof is straightforward from Corollary 3.7b) and (3.9).

Bagai and Kochar [1] proved that if $X \leq_{\text{hr}} Y$ and either F or G is DFR (decreasing failure rate), then $X \leq_{\text{disp}} Y$. It is known that spacings of independent heterogeneous exponential random variables have DFR distributions (cf. Kochar and Kowar [6]) and that the likelihood ratio order implies the hazard rate order. Combining these observations, we have proved the following corollary.

Corollary 3.11. *Under the same assumptions as those in Theorem 3.3,*

$$C_{i:n} \leq_{\text{disp}} D_{i:n},$$

for $i = 1, \ldots, n$.

Consequences of Corollary 3.11 are that $\text{Var}(C_{i:n}) \leq \text{Var}(D_{i:n})$ for $i = 1, \ldots, n$.

4 Applications to multiple-outlier models

Motivated by robustness issues, studies of order statistics and spacings from (single and multiple) outlier models have been developed during the past fifty years or so. These results have enabled useful and interesting discussions on the robustness of different estimators of parameters of a wide range of distributions. In particular, detailed robustness examination has been carried out for the normal distribution in David and Shu [4], for the Laplace distribution in the presence of a single outlier in Balakrishnan and Ambagaspitiya [3], and for logistic and exponential distributions in Balakrishnan [2].

In this section, we consider the special case when $X_1, \ldots, X_n$ are independent exponential random variables such that X_i has hazard rate λ for $i = 1, \ldots, p$ and X_j has hazard rate λ_* for $j = p+1, \ldots, n$, where two samples are independent. The simple spacings and normalized spacings from a multiple-outlier exponential model are, respectively, defined by

$$D_{i:n}\left(p, q; \lambda, \lambda_*\right) = X_{i:n} - X_{i-1:n} \quad \text{and} \quad D^*_{i:n}\left(p, q; \lambda, \lambda_*\right) = \left(n - i + 1\right) D_{i:n}\left(p, q; \lambda, \lambda_*\right),$$

for $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$, $q = n - p \geq 1$ and $p \geq 1$.

Khaledi and Kochar [5] proved that

$$D^*_{i:n}\left(n-1, 1; \lambda, \lambda_*\right) \leq_{\mathrm{hr}} D^*_{i+1:n}\left(n-1, 1; \lambda, \lambda_*\right), \quad \text{for } i = 1, \ldots, n-1,$$

in a single-outlier exponential model. Wen et al. [14] established the likelihood ratio ordering of simple spacings from a multiple-outlier exponential model, that is,

$$D_{i:n}\left(p, q; \lambda, \lambda_*\right) \leq_{\mathrm{lr}} D_{i+1:n}\left(p, q; \lambda, \lambda_*\right), \quad \text{for } p \geq 1, q \geq 1 \text{ and } i = 1, \ldots, n-1.$$

Hu et al. [8] also investigated stochastic comparisons of simple spacings from a multiple-outlier exponential model. They proved, for $\lambda_1 \leq \lambda_* \leq \lambda_2$,

$$\Big(D_{1:n}(p, q; \lambda_2, \lambda_*), \ldots, D_{n:n}(p, q; \lambda_2, \lambda_*)\Big) \leq_{\mathrm{lr}} \Big(D_{1:n}(p, q; \lambda_1, \lambda_*), \ldots, D_{n:n}(p, q; \lambda_1, \lambda_*)\Big),$$

with $p, q \geq 2$. Since the multivariate likelihood ratio order is closed under marginalization (see Shaked and Shanthikumar[11]), it holds that, for $\lambda_1 \leq \lambda_* \leq \lambda_2$,

$$D_{i:n}(p, q; \lambda_2, \lambda_*) \leq_{\mathrm{lr}} D_{i:n}(p, q; \lambda_1, \lambda_*), \qquad \text{for } i = 1, \ldots, n. \tag{4.10}$$

By considering the multiple-outlier model as a special case in the independent and non identically distributed framework, we present results on simple spacings from multiple-outlier exponential models. Applying Theorem 3.1, we obtain also stochastic comparisons between successive normalized spacings from multiple-outlier exponential models. In the following example, we show that (4.10) is a special case of Theorem 3.3.

Example 4.1. Suppose that $\lambda_1 \le \lambda_* \le \lambda_2$. Then, if $\alpha_{min}^{(i)} \ge (n-i+1)\overline{\lambda}$, where

$$\overline{\lambda} = \frac{p\lambda_1 + (n-p)\lambda_*}{n}, \quad \text{and}$$

$$\alpha_{min}^{(i)} = \begin{cases} (n-i+1)\lambda_*, & \text{if } i \ge p+1, \\ (n-p)\lambda_* + (p-i+1)\lambda_2, & \text{if } i < p+1, \end{cases} \tag{4.11}$$

we get that (4.10) holds from Theorem 3.3. Thus, if $i \ge p+1$, we have that

$$\alpha_{min}^{(i)} \ge (n-i+1)\overline{\lambda} \Leftrightarrow n\lambda_* \ge p\lambda_1 + (n-p)\lambda_* \Leftrightarrow \lambda_* \ge \lambda_1.$$

And, when $i < p+1$, we get that

$$\alpha_{min}^{(i)} \ge (n-i+1)\overline{\lambda} \Leftrightarrow (n-p)\lambda_* + (p-i+1)\lambda_2 \ge (n-i+1)\overline{\lambda}.$$

As $\lambda_2 \ge \lambda_*$, it is easy to see that, if $i < p+1$,

$$(n-p)\lambda_* + (p-i+1)\lambda_2 = (n-i+1)\lambda_* \ge (n-i+1)\overline{\lambda} \Leftrightarrow \lambda_* \ge \lambda_1.$$

Hence, $D_{i:n}(p,q;\lambda_2,\lambda_*) \le_{\text{lr}} D_{i:n}(p,q;\lambda_1,\lambda_*)$, for $i = 1,\ldots,n$.

Using again Theorem 3.3, we give below a similar result to (4.10) when the number of exponential random variables with hazard rate λ_1 and λ_* can be changed.

Theorem 4.2. *Let $X_1,\ldots,X_n$ follow the multiple-outlier model with parameters λ_1 and λ_* and let $Y_1,\ldots,Y_n$ follow the multiple-outlier model with parameters λ_2 and λ_*. If $\lambda_1 \le \lambda_* \le \lambda_2$, then*

i) $D_{i:n}(p,q;\lambda_2,\lambda_*) \le_{\text{lr}} D_{i:n}(p+k_1, q-k_1;\lambda_1,\lambda_*)$*, with* $1 \le k_1 \le q$ *and*

ii) $D_{i:n}(p,q;\lambda_2,\lambda_*) \le_{\text{lr}} D_{i:n}(p-k_2, q+k_2;\lambda_1,\lambda_*)$*, with* $1 \le k_2 \le p$*,*

where $q = n-p \ge 1$*,* $p \ge 1$*.*

Proof. We have to show that $\alpha_{min}^{(i)} \ge (n-i+1)\overline{\lambda}$ and then, from Theorem 3.3 we will conclude that the result follows. It is easy to see that (4.11) holds.

i) In this case, $n\overline{\lambda} = (p + k_1)\lambda_1 + (q - k_1)\lambda_*$, with $1 \leq k_1 \leq q$. When $i \geq p + 1$, we get that

$$\alpha^{(i)}_{min} \geq (n-i+1)\overline{\lambda} \Leftrightarrow (n-i+1)\lambda_* \geq (n-i+1)\overline{\lambda} \Leftrightarrow (\lambda_* - \lambda_1)(p+k_1) \geq 0 \Leftrightarrow \lambda_* \geq \lambda_1.$$

And when $i < p + 1$, we have that

$$\alpha^{(i)}_{min} \geq (n - i + 1)\overline{\lambda} \Leftrightarrow (n - p)\lambda_* + (p - i + 1)\lambda_2 \geq (n - i + 1)\overline{\lambda}.$$

As $\lambda_2 \geq \lambda_*$, then

$$\alpha^{(i)}_{min} \geq (n - i + 1)\overline{\lambda} \Leftrightarrow (n - i + 1)\lambda_* \geq (n - i + 1)\overline{\lambda} \Leftrightarrow \lambda_* \geq \lambda_1.$$

ii) In this case, $n\overline{\lambda} = (p - k_2)\lambda_1 + (q + k_2)\lambda_*$, where $1 \leq k_2 \leq p$. As before, it is easy to check that

$$\alpha^{(i)}_{min} \geq (n-i+1)\overline{\lambda} \Leftrightarrow (n-i+1)\lambda_* \geq (n-i+1)\overline{\lambda} \Leftrightarrow (\lambda_* - \lambda_1)(p-k_2) \geq 0 \Leftrightarrow \lambda_* \geq \lambda_1,$$

for $i = 1, \ldots, n$. □

Wen et al. [14] obtained the following result.

Theorem 4.3 (Wen et al. [14])**.** *Let $X_1, \ldots, X_n$ follow the multiple-outlier model with parameters λ and λ_*. If $\lambda \leq \lambda_*$, $p \geq 1$ and $q \geq 1$, then*

$$D_{i:n}(p, q; \lambda, \lambda_*) \leq_{\mathrm{lr}} D_{i:n}(p + 1, q - 1; \lambda, \lambda_*), \quad \textit{for } i = 1, \ldots, n.$$

We now state the analogue of this last result as a special case of Theorem 3.3, when $\lambda \geq \lambda_*$.

Theorem 4.4. *Let $X_1, \ldots, X_n$ follow the multiple-outlier model with parameters λ and λ_*. If $\lambda \geq \lambda_*$, $p \geq 1$ and $q \geq 1$, then*

$$D_{i:n}(p - k_2, q + k_2; \lambda, \lambda_*) \geq_{\mathrm{lr}} D_{i:n}(p, q; \lambda, \lambda_*) \geq_{\mathrm{lr}} D_{i:n}(p + k_1, q - k_1; \lambda, \lambda_*),$$

where $1 \leq k_1 \leq q$, $1 \leq k_2 \leq p$ and $i = 1, \ldots, n$.

Proof. First, we will see that $D_{i:n}(p-k_2, q+k_2; \lambda, \lambda_*) \geq_{\rm lr} D_{i:n}(p, q; \lambda, \lambda_*)$, where $1 \leq k_2 \leq p$. A trivial verification shows that

$$\overline{\lambda} = \frac{(p-k_2)\lambda + (q+k_2)\lambda_*}{n}, \qquad \text{and}$$

$$\alpha_{min}^{(i)} = \begin{cases} (n-i+1)\lambda, & \text{if } i \geq q+1, \\ p\lambda + (n-i+1-p)\lambda_*, & \text{if } i < q+1. \end{cases}$$

It follows immediately that, if $i \geq q+1$,

$$\alpha_{min}^{(i)} \geq (n-i+1)\overline{\lambda} \Leftrightarrow n\lambda \geq (p-k_2)\lambda + (q+k_2)\lambda_*$$
$$\Leftrightarrow (\lambda - \lambda_*)(q+k_2) \geq 0 \Leftrightarrow \lambda \geq \lambda_*.$$

And, if $i < q+1$, then,

$$\alpha_{min}^{(i)} \geq (n-i+1)\overline{\lambda} \Leftrightarrow p\lambda + (n-i+1-p)\lambda_* \geq \frac{n-i+1}{n}\Big((p-k_2)\lambda + (q+k_2)\lambda_*\Big)$$
$$\Leftrightarrow (\lambda - \lambda_*)\Big(nk_2 + (i-1)(p-k_2)\Big) \geq 0 \Leftrightarrow \lambda \geq \lambda_*.$$

To prove that $D_{i:n}(p, q; \lambda, \lambda_*) \geq_{\rm lr} D_{i:n}(p+k_1, q-k_1; \lambda, \lambda_*)$ where $1 \leq k_1 \leq q$, we get

$$\overline{\lambda} = \frac{p\lambda + q\lambda_*}{n}, \qquad \text{and}$$

$$\alpha_{min}^{(i)} = \begin{cases} (n-i+1)\lambda, & \text{if } n-i+1 \leq p+k_1, \\ (p+k_1)\lambda + (n-i+1-p-k_1)\lambda_*, & \text{if } n-i+1 > p+k_1. \end{cases}$$

Clearly, when $n-i+1 \leq p+k_1$,

$$\alpha_{min}^{(i)} \geq (n-i+1)\overline{\lambda} \Leftrightarrow n\lambda \geq p\lambda + q\lambda_* \Leftrightarrow q(\lambda - \lambda_*) \geq 0 \Leftrightarrow \lambda \geq \lambda_*.$$

And when $n-i+1 > p+k_1$, we have

$$\alpha_{min}^{(i)} \geq (n-i+1)\overline{\lambda} \Leftrightarrow (p+k_1)\lambda + (n-i+1-p-k_1)\lambda_* \geq \frac{n-i+1}{n}(p\lambda + q\lambda_*)$$
$$\Leftrightarrow (\lambda - \lambda_*)\Big(nk_1 + p(i-1)\Big) \geq 0 \Leftrightarrow \lambda \geq \lambda_*.$$

Hence, we have proved that $\alpha_{min}^{(i)} \geq (n-i+1)\overline{\lambda} \Leftrightarrow \lambda \geq \lambda_*$, and from Theorem 3.3 we get the desired result. □

5 Conclusions

This article is devoted to establishing stochastic comparisons of spacings from two samples of heterogeneous exponential random variables. In particular, we have provided sufficient conditions under which the simple and normalized spacings are ordered according to the likelihood ratio ordering. We also have obtained lower and upper bounds for the survival and the hazard rate functions of simple and normalized spacings from a sample of exponential random variables with different scale parameters.

As multiple-outlier models are a special case in the independent and non identically distributed framework, we have applied our main results to compare simple and normalized spacings from multiple-outlier exponential models.

Acknowledgements

We would like to professor S. Kochar for very useful comments and value references to this paper. The research of the authors was supported by projects of Comunidad de Madrid (CCG10-UC3M/HUM-5114) and the Spanish Ministry of Education and Science (SEJ2007-64500).

References

[1] I. Bagai, S.C. Kochar, On tail ordering and comparison of failure rates, Communications in Statistics: Theory and Method 15(1986)13771388.

[2] N. Balakrishnan, Permanents, order statistics, outliers, and robustness, Revista Matemática Complutense 20 (2007) 7-107.

[3] N. Balakrishnan and R. S. Ambagaspitiya, Relationships among moments of order statistics in samples from two related outlier models and some applications, Comm. Statist. Theory Methods 17(1988)2327-2341.

[4] H. A. David and V. S. Shu, Robustness of location estimators in the presence of an outlier, Contributions to survey sampling and applied statistics: Papers in honour of H. O. Hartley (H. A. David, ed.), Academic Press, New York, 1978, pp. 235-250.

[5] B. Khaledi and S.C. Kochar, Stochastic properties of spacings in a single-outlier exponential model, Probability in Engineering and Information Sciences 15(2001)401–408.

[6] S.C. Kochar, R. Korwar, Stochastic orders for spacings of heterogeneous exponential random variables, J. Multivariate Anal. 57(1996)69-83.

[7] S.C. Kochar and M. Xu, Stochastic comparisons of spacings from heterogeneous samples in Advances, In M. Wells and A. Sengupta, editors, Advances in Directional and Linear Statistics, pp. 113–129. Festschrift Volume for J.S. Rao, springer, 2011.

[8] T. Hu, F. Wan, Z. Zhu, Stochastic comparisons and dependence of spacings from two samples of exponential random variables, Comm. Statist. Theory Methods 35 (2006) 979-988.

[9] A.W. Marshall and I. Olkin, Inequalities : Theory of Majorization and Its Applications, Academic Press, New York, 1979.

[10] D.S. Mitrinovic, Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[11] M. Shaked, J.G. Shanthikumar, Stochastic Orders and their applications, Springer, New York, 2007.

[12] G. Pledger, F. Proschan, Comparisons of order statistics from heterogeneous populations, with applications in reliability, in: J.S. Rustagi Ed., Optimizing Methods in Statistics, Academic Press, New York, 1971, p.p. 89-113.

[13] N. Torrado, R.E. Lillo, and M.P. Wiper, On the conjecture of Kochar and Kowar, J. Multivariate Anal. 101(2010), 1274-1283.

[14] S. Wen, Q. Lu, and T. Hu, Likelihood ratio orderings of spacings of heterogeneous exponential random variables, J. Multivariate Anal. 98(2007)743-756.