Stochastic Processes and their Applications 100 (2002) 75 – 107
stochastic processes and their applications
www.elsevier.com/locate/spa

On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts

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Received 26 December 2000; received in revised form 7 February 2002; accepted 7 February 2002

Abstract

This paper provides a general framework for pricing options with a constant barrier under spectrally one-sided exponential Lévy model, and uses it to implement of Carr's approximation for the value of the American put under this model. Simple analytic approximations for the exercise boundary and option value are obtained.

Keywords: American options; Perpetual approximation; Spectrally negative exponential Lévy process

1. Introduction

This paper develops Carr's "Erlang approximation" for the price of an American put option under the spectrally negative exponential Lévy model.

There has been lots of recent interest in mathematical finance (see for example, Geman et al., 1999) towards extending results based on the exponential Brownian motion model to results based on exponential Lévy models. This is motivated by the superior fits to the data and hence improved pricing formulas and hedging strategies provided by several special classes of Lévy models—see for example the hyperbolic model of Eberlein and Keller (1995), the variance-gamma model of Madan (1999) and the inverse Gaussian model of Barndorff-Nielsen and Sheppard (2000). Following previous

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0304-4149/02/$ - see front matter
PII: S0304-4149(02)00104-7

work of Gerber and Shiu (1994) and Gerber and Landry (1998), we investigated the Lévy pricing of American puts, starting with the considerably easier case of spectrally one sided Lévy processes. The results for these processes are different depending on whether the barrier can only be crossed "continuously" or also by jumps, the first case being very similar to the classical Brownian case. We have chosen therefore to work on the latter more complicated case of pricing American puts under spectrally negative exponential Lévy models. Note that this clearly requires knowing the distribution of the "overshoot" below the barrier, and convenient formulas for this are available in the spectrally negative case which relieving one of the need to perform a Wiener Hopf factorization. The general case of Lévy processes which may jump in either direction will require further:

(1) Either obtaining Wiener–Hopf factorizations in the special cases mentioned above, or developing numerical approximations for these (as employed for example in analogue problems in queueing theory).
(2) Modifying the "early exercise decomposition" as accumulated interest on the strike price while below the barrier due to Kim (1990), Jacka (1991) and Carr et al. (1992), so as to include the effect of jumps back over the barrier identified by Pham (1997).

The pricing of American put options, with payoff

$$e^{-r\tau}(K - S_\tau)_+,$$

where $\tau = \min(\tau_*, T)$, T is the expiration time and τ_* denotes the hitting time of an optimal exercise boundary B_s, $0 \leqslant s \leqslant T$, is quite challenging due the fact that the optimal exercise boundary depends on time. More precisely, the optimal exercise point at time s is some function of the remaining time $T - s$ until expiration. An excellent overview of the problem may be found for example in the book of Kwok (1998).

To remove the dependence of the barrier on time, McKean and Samuelson (1965) proposed already in 1965 in what was maybe the first paper in mathematical finance to approximate the problem by that of pricing a "perpetual" option (with infinite expiration time $T = \infty$). For the perpetual, the dependence on the remaining time disappears, leaving us with the problem of valuation of a barrier put with fixed constant barrier B, followed by the optimization of B. By this approach, perpetual American options were recently priced under the spectrally positive/negative exponential Lévy models by Gerber and Shiu (1994) and Gerber and Landry (1998), and under the general exponential Lévy model by Boyarchenko and Levendorskii (2000); the price is simply the average of the discounted put payoff with respect to the distribution of the price after the barrier crossing. Note however that perpetual approximations are generally too crude. More refined analytic approximations have been proposed by McMillan (1986) (who also removes the boundary's dependence of time by a transformation) and Omberg (1987) (who looked for the best exponential exercise boundary).

Recently, a new method for removing the dependence on time has been proposed by Carr (1998). The first step consists in replacing the perpetual by an option with random exponential expiration time $\tilde{T}$, with $\mathbb{E}\tilde{T} = T$, which was called a "Canadian option." The exponential assumption has the virtue of making the optimal exercise boundary independent of time, just like in the classical perpetual case and so this step is as easy to implement as McKean's perpetual approximation. The Canadian approximation was then further refined by considering options whose expiration time $\tilde{T}$ is a sum of n exponentials and thus $\Gamma_{n,n/T}$ distributed (with $\mathbb{E}\,\tilde{T} = T$); this will be referred to as "n-Erlangian" options. The value of an n-Erlangian option may be computed via an iterative method with n stages. When $n \to \infty$, this approximation converges to the true value.

Carr's method retains some of the simplicity of the perpetual approximation and at the same time has the virtue of converging to the true value. Already the first stage "Canadian" approximation leads to a simple *time-dependent* analytic answer for the put exercise barrier (see (14) below) which is exact for both very short and very long expiration times (in the limit).

Our paper extends Carr's results for American put options to the spectrally negative exponential Lévy model. To achieve that, we needed to develop a general approach for pricing "Canadian" barrier options under this model.

1.1. Literature survey

Besides McMillan's, Omberg's and Carr's approaches, some other very successful recent approximation methods under the exponential Brownian motion model are due to Broadie and Detemple (1996), Ju (1998) (who extended Omberg's approach to a multistage procedure) and Ju and Zhong (1999) (who provided a second-order perturbation approximation).

In the context of American options under the exponential Lévy model, Zhang (1995, 1997) extended the approach of McMillan (1986) and Pham (1997) extended the representation of the American early exercise premium due to Kim (1990), Jacka (1991) and Carr et al. (1992).

Gerber and Shiu (1994) showed that analytical formulas for the exercise barrier of perpetual options may be obtained under certain exponential Lévy models, since they depend on knowing the Laplace transform of the value function rather than on the value function itself. Their first results were obtained under the assumption of a pure jump spectrally one sided Lévy process (the classical risk process), which allows one to use the machinery of renewal equations. Later, Gerber and Landry (1998) incorporated a Brownian component and Chan (1999b) extended the results to the case of general spectrally negative Lévy processes.

Note: Extending these results to Lévy processes with jumps of both signs is a problem of great practical interest (Madan, 1999), but considerably more complicated. Several results, like Proposition 2.1, Bingham's Theorem 6(b) (which is based on the Wiener–Hopf factorization) used in our Section 3 and Pham's representation of the American early exercise premium change substantially in the general case. Previous work in the general Lévy case was based on two special representations of Lévy processes: one as time-changed Brownian motions and the other as additive functionals of Brownian motion evaluated at the inverse of the local time at 0—see Geman et al. (1999) and Revuz and Yor (1991).

1.2. The exponential Lévy model

On a probability space $(\Omega, \{\mathscr{F}_t\}, \mathscr{F}, \mathbb{P})$, let W_t be a standard Brownian motion and J_t an independent Lévy jump process with positive jumps. We consider below assets modelled by an exponential Lévy process of the form

$$S_t = \mathrm{e}^{Y_t}, \tag{1}$$

$$Y_t = y + \mu t + \sigma W_t - J_t. \tag{2}$$

Thus, Y_t is a spectrally negative Lévy process. (As usual, we assume that Y_t is right-continuous with left limits.) We will assume that the Lévy measure of J_t v satisfies the simplifying assumption

$$\int_0^1 x\, v(\mathrm{d}x) < \infty, \tag{3}$$

which ensures that J has finite variation. [1]

If J is a compound Poisson process $\sum_{i=1}^{N_t} Z_i$, where N_t is a Poisson process with intensity λ and the jumps Z_i are i.i.d. random variables independent of N_t with a distribution $P(\mathrm{d}x)$ concentrated on the positive axis, then $v(\mathrm{d}x) = \lambda P(\mathrm{d}x)$ and (3) is automatically satisfied. [2]

Because the jumps of J are all positive, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta(Y_t - y)}]$ exists for all $\theta \geqslant 0$ and

$$\mathbb{E}[\mathrm{e}^{\theta(Y_t - y)}] = \mathrm{e}^{tc(\theta)} \tag{4}$$

for some function c, which is referred to as the cumulant generating function. Under the simplifying assumption (3), the cumulant generating function is given by

$$c(\theta) = \frac{\sigma^2\theta^2}{2} + \mu\theta + \int_0^\infty (\mathrm{e}^{-\theta x} - 1)v(\mathrm{d}x) = \frac{\sigma^2\theta^2}{2} + \mu\theta + v^*(\theta), \tag{5}$$

where we denoted by $v^*(\theta) = \int_0^\infty (\mathrm{e}^{-\theta x} - 1)v(\mathrm{d}x)$ a “Laplace transform” of the jump measure $v(\mathrm{d}x)$ appropriately modified to allow for the case when it has infinite mass around the origin (in which case $v^*(\theta)$ is still well defined by (3)).

By the fundamental theorem of asset pricing, to avoid arbitrage, it is necessary to use a risk neutral pricing measure for options. If the observed stock prices follow an exponential Lévy model, the risk neutral pricing measure is not unique and the issue of choosing between the possible risk neutral measures is quite delicate. It is not the purpose of this present paper to address this question. However, it is known (see, for example, Chan, 1999a) that under a large class of risk neutral measures which includes many of those that have been proposed and studied in the literature

[1] Condition (3) is assumed purely for convenience, to enable us to avoid excessively messy expressions. The extension of the results described in the sequel to Lévy processes with infinite variation is essentially trivial and is left to the interested reader—one must compensate for the small jumps in integrals with respect to v, resulting in rather messier algebra.

[2] The extension from compound Poisson to general Lévy processes is quite important in view of the fact that the current “favourites” for asset prices modelling, the hyperbolic and the variance gamma process have unbounded Lévy measure at 0.

(see Chan, 1999a for a list of references to these), Y_t remains a Lévy process. In this paper we shall therefore assume that a risk neutral measure has been chosen under which Y_t remains a Lévy process and simply work with this measure. In particular, we assume that the reference measure $\mathbb{P}$ is already a risk neutral measure and in the sequel all expectations $\mathbb{E}(\cdot)$ will be taken with respect to this measure; moreover, the random expiry time T will be an exponential random variable under $\mathbb{P}$. If the asset continuously pays dividends at rate q, the process $e^{-(r-q)t}S_t$ is a martingale under $\mathbb{P}$; this is tantamount to assuming the condition (see Gerber and Shiu (1994)):

$$c(1) = r - q, \tag{6}$$

where r is the discount rate and q is the dividend rate. (Although Gerber and Shiu (1994) treats one specific choice of risk neutral measure, (6) is simply the necessary and sufficient condition in terms of c for $e^{-(r-q)t}S_t$ to be a martingale and is unaffected by the choice of equivalent martingale measure used to achieve this.) Condition (6) means that the drift must satisfy

$$\mu = r - q - \sigma^2/2 - v^*(1) \tag{7}$$

so we are working only with cumulant generating functions of the form:

$$c(\theta) = \frac{\sigma^2}{2}(\theta^2 - \theta) + \int_0^\infty (e^{-\theta x} - e^{-x})v(dx) + (r-q)\theta.$$

1.3. The "Black Scholes" equation for barrier options

We consider below "down" barrier options, which may earn either a *final payoff* $f(Y_T)$ at the expiration time T or a rebate or *boundary payoff* $\pi(Y_{\tau_B})$ at the first time τ_B when the asset process Y_t drops below the exercise boundary B, which may depend on time. The example we are interested is that of the American put option, in which case the payoff and rebate have the same functional dependence given by $f = \pi = (K - S_T)_+ = (K - e^{Y_T})_+$. However, the Erlang stages approximation method we use forces us eventually to consider options with arbitrary final payoffs $f(y)$ (the boundary "put" payoff $\pi = (K - S_T)_+$ will stay fixed).

Let $v_{B,\pi,f,r}(t,y) = v(t,y)$ denote the arbitrage free value of this option under the complete exponential Brownian motion model, expressed in terms of the value y at time t of the logarithm of the price Y_t. Then

$$v_B(t,y) = \mathbb{E}_{\{Y_t=y\}}[e^{-r(\tau_B-t)}\ \pi(Y_{\tau_B})\ 1_{\{\tau_B<T\}} + e^{-r(T-t)}f(Y_T)\ 1_{\{\tau_B\geqslant T\}}].^3 \tag{8}$$

Recall that $e^{-(r-q)t}S_t$ is a martingale under the risk neutral measure. Under the unique dividends adjusted risk neutral exponential Brownian motion model, $v(t,y)$ satisfies:

$$\begin{aligned}
&\Gamma v - rv + \frac{\partial v}{\partial t} = 0 \quad \text{for } y \geqslant B_t,\\
&v(t,y) = \pi(y) \quad \text{for } y \leqslant B_t,\\
&v(T,y) = f(y),
\end{aligned} \tag{9}$$

[3] European options are the particular case in which there exists only a final payoff f and $\pi = 0$.

where Γ, the infinitesimal generator of Y is $(\Gamma f)(u) = (\sigma^2/2)f''(u) + \mu f'(u)$ and $\mu = r - q - \sigma^2/2$. (This is essentially a consequence of Itô's formula—if v is the solution to (9) then v can be expressed as (8). The detailed argument in the case of European options is a standard one and can be found in, for example, Karatzas and Shreve (1991); the argument for American options is similar except for the obvious change to the boundary conditions and one needs some additional results on the free boundary problem (9)—see, Jacka (1991) for the details.)

It is sometimes convenient to work with the equations in which the current time variable t is replaced by the remaining time until expiration $s = T - t$, yielding

$$\begin{aligned} &\Gamma v - rv - \frac{\partial v}{\partial s} = 0 \quad \text{for } y \geqslant B_s,\\ &v(s,y) = \pi(y) \quad \text{for } y \leqslant B_s,\\ &v(0,y) = f(y). \end{aligned} \tag{10}$$

If the stock price is modelled as an exponential Lévy model, $v(s,y)$ satisfies the same equations (9) or (10), but with the generator modified to that of Y:

$$(\Gamma f)(u) = \frac{\sigma^2}{2} f''(u) + \mu f'(u) + \int_0^\infty [f(u-z) - f(u)]\, \nu(\mathrm{d}z). \tag{11}$$

(where μ satisfies the risk neutrality assumption (7)). In this case, one has to be careful under what conditions a solution to (9) exists and whether representations (8) and (9) are equivalent. This is a question that is beyond the scope of the present paper and in any case, is not essential for the subsequent results in this paper. For now, we will simply assume that a solution to (9) satisfying certain additional regularity conditions exists, and that under such conditions, the solution is given by (8). The main idea in what follows is to consider a simpler time-homogeneous version of (9)—for which the analogous representation (8) does hold—and use this as an approximation to the function defined by (8).

1.4. Canadian options

The idea of Canadian options is to consider options with an exponentially randomized expiration time $T^{(\alpha)}$ of some arbitrary but fixed rate α. Owing to the memoryless property of the exponential, this remove the dependence on the starting time in the value of Canadian options.

Let $V_{B,\pi,f,r,\alpha}(y) = V(y)$ denote the value of the "Canadian barrier option" with payoffs π, f, defined as

$$V_{B,\pi,f,r,\alpha}(y) = \mathbb{E}_{\{Y_0=y\}}[\mathrm{e}^{-r\tau_B}\ \pi(Y_{\tau_B})\ 1_{\{\tau_B < T^{(\alpha)}\}} + \mathrm{e}^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}})\ 1_{\{\tau_B \geqslant T^{(\alpha)}\}}]. \tag{12}$$

The dependence on B, π, f, r, α will usually be suppressed. If $v(s,y)$ denotes the value of the corresponding barrier option with the same payoffs, then the Canadian value $V(y)$ is obtained simply by averaging $v(s,y)$ with an exponential distribution for the remaining time s.

$$V(y) = \int_0^\infty \alpha \mathrm{e}^{-\alpha s} v(s,y)\, \mathrm{d}s.$$

This only differs from the Laplace transform in s by the extra factor α and is also known as a Laplace–Carson transform.

Another essential simplification when valuing Canadian American options is provided by the memoryless property of the exponential distribution, which entails that their optimal exercise boundaries have to be constant. As such, valuation reduces to a one-dimensional optimization over the possible constant barriers $B \equiv a$.

Consider, therefore, the value $V_{a,\pi,f,r,\alpha}(y) = V(y)$ of a Canadian barrier option with *fixed* barrier a. Taking the Laplace–Carson transform in (10) changes the differentiation operator in t into multiplication, yielding the integro-differential equation:

$$\begin{aligned} &(\Gamma V)(y) - \delta V(y) + \alpha f(y) = 0 \quad \text{for } y \geqslant a, \\ &V(y) = \pi(y) \quad \text{for } y \leqslant a, \end{aligned} \tag{13}$$

where we put $\delta = r + \alpha$ (which may be interpreted as a combined "killing" rate). This equation is of course considerably simpler than its time-dependent counterpart. Suppose now that π and f are bounded functions and that a bounded solution to (13) exists. (The boundedness assumption is of course motivated by the fact that the price of a put option with payoff $f = \pi = (K - S_T)_+$ cannot be greater than K.) We first verify that under these assumptions, any bounded solution V to (13) admits representation (12) (and is therefore unique). Let V be any bounded solution to (13) and let $Y_0 = y > a$. An application of Itô's formula shows that

$$M_t := \mathrm{e}^{-\delta(t\wedge\tau_a)} V(Y_{t\wedge\tau_a}) - V(Y_0) + \int_0^{t\wedge\tau_a} \alpha \mathrm{e}^{-\delta s} f(Y_s)\,\mathrm{d}s,$$

where $\tau_a = \inf\{t\colon Y_t \leqslant a\}$, is a local martingale, which is bounded because of our assumption that V and f are bounded. Hence, not only is M actually a martingale, but it is also uniformly integrable and so by the optional stopping theorem $\mathbb{E}(M_{\tau_a}) = 0$, which when rearranged gives

$$\begin{aligned} V(y) &= \mathbb{E}_y\left[\mathrm{e}^{-\delta\tau_a} V(Y_{\tau_a}) + \int_0^{\tau_a} \alpha \mathrm{e}^{-\delta s} f(Y_s)\,\mathrm{d}s\right] \\ &= \mathbb{E}_y\left[\mathrm{e}^{-\delta\tau_a} \pi(Y_{\tau_a}) + \int_0^{\tau_a} \alpha \mathrm{e}^{-\delta s} f(Y_s)\,\mathrm{d}s\right] \\ &= \mathbb{E}_y[\mathrm{e}^{-r\tau_a} \pi(Y_{\tau_a})\ 1_{\{\tau_a < T^{(\alpha)}\}} + \mathrm{e}^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}})\ 1_{\{\tau_a \geqslant T^{(\alpha)}\}}]. \end{aligned}$$

This shows that, with the boundary *a fixed*, any bounded solution to (13) admits the representation (12). Of course, there is a little extra work that still needs to be done to show that the same holds for the *free boundary* version of (13): namely that the maximal solution pair (V, a^*) to the free boundary problem admits the representation

$$\begin{aligned} V(y) &= \sup_a \{\mathbb{E}_y[\mathrm{e}^{-r\tau_a} \pi(Y_{\tau_a}) \mathbf{1}_{\{\tau_a < T^{(\alpha)}\}} + \mathrm{e}^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}}) \mathbf{1}_{\{\tau_a \geqslant T^{(\alpha)}\}}]\} \\ &= \mathbb{E}_y[\mathrm{e}^{-r\tau_{a^*}} \pi(Y_{\tau_{a^*}}) \mathbf{1}_{\{\tau_{a^*} < T^{(\alpha)}\}} + \mathrm{e}^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}}) \mathbf{1}_{\{\tau_{a^*} \geqslant T^{(\alpha)}\}}]. \end{aligned}$$

However, we will avoid this slightly more delicate issue for now because in what follows we will use an alternative characterization of the optimal boundary a^*, namely that of the so-called "smooth pasting" condition which enables us to consider first the solution V_a to (13) for fixed a and then find the optimal boundary by requiring $V_a(\cdot)$ to be C^1 at the optimal boundary a^*—see Proposition 2.1 and Corollary 3.1 below.

Note: The payoff of a standard American put is given by $\pi(y) = (K - \mathrm{e}^y)_+$. Since we are interested in

$$V(y) = \sup_a \{\mathbb{E}_y[\mathrm{e}^{-r\tau_a}\pi(Y_{\tau_a})\mathbf{1}_{\{\tau_a < T^{(\alpha)}\}}] + \mathbb{E}_y[\mathrm{e}^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}})\mathbf{1}_{\{\tau_a \geqslant T^{(\alpha)}\}}]\}$$

and since $\mathbf{1}_{\{\tau_a \geqslant T^{(\alpha)}\}} \geqslant \mathbf{1}_{\{\tau_b \geqslant T^{(\alpha)}\}}$ if $a < b$, we will obviously only choose constant exercise levels $a < \log K$, otherwise $\pi(Y_{\tau_a}) = 0$ and the second term can only be smaller with a choice of a higher value of a. Therefore, the boundary condition in (13) may be simplified to $V(y) = \pi(y) = K - \mathrm{e}^y$ for $y \leqslant a$. This simplifies many of the subsequent calculations and will be repeatedly used without further explicit mention.

1.5. Outline of main results

In Section 2 we provide in Proposition 2.1 an analytic expression for the Laplace transform (with respect to the initial price) of the value of barrier options with exponential expiration. This result generalizes classical risk theory results on computing "multivariate ruin probabilities"—see Corollary 2.5 and for comparison Theorem 6(a) of Bingham (1975), Gerber and Shiu (1998), Gerber and Landry (1998) and Usabel (1999). Essentially, it reduces the risk neutral valuation of any barrier option under spectrally one-sided Lévy models to carrying out some integrations and inverting Laplace transforms.

In Section 3 we determine the optimal exercise barrier for "Canadian American" options (directly from the Laplace transform, avoiding the inversion). As an illustration of the elegance of the results obtained under the spectrally one-sided model, we highlight now the time-dependent analytic approximation of the exercise boundary (28) obtained already by the Canadian approximation. Let δ_+ denote the largest solution of the equation $c(\theta) = \delta = r + T^{-1}$, where $c(\theta)$ is the cumulant generating function of the process and T denotes the remaining time until expiration. In the absence of dividends, the boundary L^* is given by an explicit formula

$$L^* = K((\delta_+ - 1)rT)^{1/\delta_+} \tag{14}$$

requiring the solution of only one algebraic equation (this generalizes the formula obtained by Carr for the diffusion model), while in the presence of dividends an additional algebraic equation (28) is required.

While barrier (14) is only optimal for Canadian American options with exponential expiration of expected duration T, we propose to use it also as an approximate formula for the optimal exercise boundary of usual American options with remaining lifetime T (with T decreasing to 0); this provides a time-dependent approximation for the exercise boundary which is shown to be asymptotically exact when $T \to \infty$ and 0 (see Corollary 3.1(d), (e) and the Remark 3 following it).[4]

In Section 4 we identify in Theorem 4.1 and Corollary 4.2 some classical fluctuation theory identities which allow us to invert the Laplace transform in terms of the resolvent density of the Lévy process; this is available explicitly sometimes (like for compound Poisson processes with a phase-type distribution of jumps) and numerically in general. The end result, Proposition 4.3 is an explicit formula for Canadian put options in terms of either the resolvent (more convenient numerically) or the Wiener–Hopf factors of the process. Since the resolvent may be computed numerically via Fast Fourier transform–see Chan (1999b), this approach may be preferable numerically to the inversion of the Laplace transform.

In Section 5 we consider n-Erlangian options. We discuss two possible recursive implementations: one based on using Laplace transforms only, which reduces to solving algebraic equations, and the second based on computing the resolvent. We also establish in Corollary 5.2 the convergence of the n-Erlangian approximation to the finite expiration price.

Finally, in the appendix we compute the value of Canadian puts in the exponential jumps case.

We illustrate now the importance of using statistically appropriate models by displaying the approximate exercise barrier as a function of the elapsed time $0 \leqslant t \leqslant T = 10$ when the log price process is compound Poisson with exponential jumps of distribution $\beta e^{-\beta x}$; the dotted line is the exercise barrier for a Gaussian approximation of this process of equal given variance $\sigma^2 t$ (this determines the intensity of the jumps via $\lambda\beta^{-2} = \sigma^2$). We took $K = 10$ and $r = 0.1 = \sigma^2/2$ (this fixes the long time limit of the Gaussian barrier to $L = K(1/1(+\sigma^2/2r)) = 5$). We note that while for the first value $\beta=20$ (Fig. 1) the barrier is very close to its Gaussian approximation, for $\beta=1$ (Fig. 2) the difference is considerable, as both barriers approach quickly their different respective long time limits of $K(1/(1+\sigma^2/2r))1/(1+\beta^{-1})^2) = 8$ and $K(1/(1+\sigma^2/2r)) = 5$.

This example emphasizes the rather obvious point that underparametrized models like either Brownian motion or spectrally one-sided compound Poisson models may be quite inadequate for approximating more general processes and hence the desirability of developing models which are rich enough to capture market behaviour (and hopefully still admit convenient pricing and hedging formulas).

2. The Laplace transform with respect to the initial price of the value of Canadian barrier options

We will consider from now Canadian barrier options with a *fixed* barrier $B_t = a$ and value $V(y)$ (note that we are suppressing in the notation the dependence on a, π, f, r, α).

[4] The performance of this first approximation is comparable with that of the generalization of McMillan's approximation proposed by Zhang (1998), and we expect that it might bring by itself a significant impact on pricing American options in practice, since we believe that taking advantage of the superior numerical fit of the exponential Lévy model to real data may be more important than improving the accuracy of the exercise boundary.

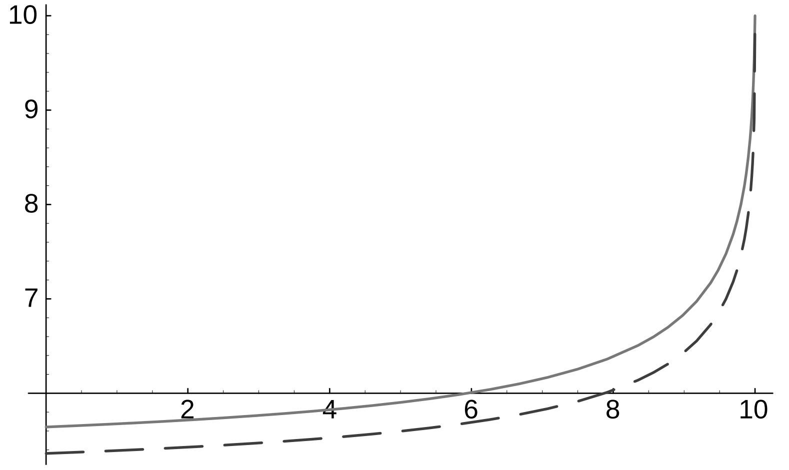

Fig. 1. Exercise Barrier approximations, Lévy and Gaussian approx., $\beta = 20$.

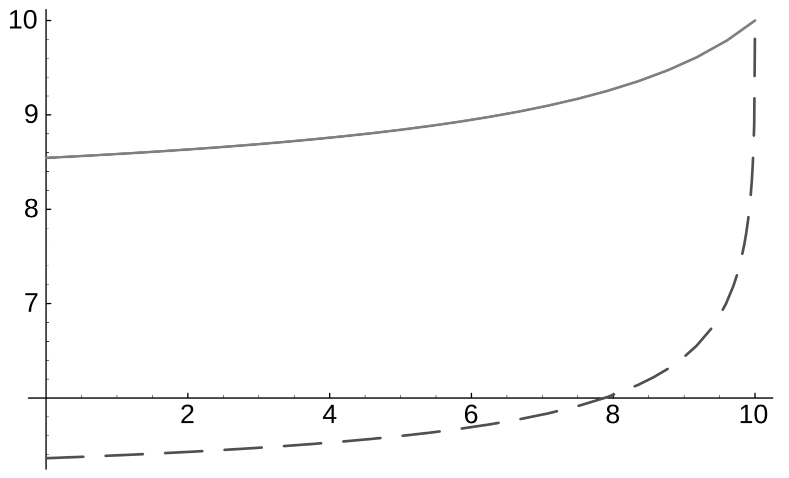

Fig. 2. Exercise Barrier approximations, Lévy and Gaussian approx., $\beta = 1$.

It will be convenient here to work with the "surplus over the barrier" variable $u = y - a$ which moves the barrier to $u = 0$. We denote by $V_a(u)$ the function defined by the "shift" $V_a(u) = V(a+u) = V(y)$ (similarly, the "shifted" payoffs f, π are $f_a(u) = f(u + a), \pi_a(u) = \pi(u + a)$).

We obtain now the Laplace transform of the value function above the barrier, by which we mean the usual Laplace transform with respect to the surplus variable

$$V_a^*(\theta) = \int_a^\infty e^{-\theta(y-a)} V(y)\, dy = \int_0^\infty e^{-\theta u} V_a(u)\, du \tag{15}$$

for functions satisfying Eq. (13) (or (59), which is a particular case of 13).

Let $f_a^*(\theta) = \int_0^\infty e^{-\theta u} f(u + a))\,du = e^{\theta a} \int_a^\infty e^{-\theta y} f(y)\,dy$ and $v_x^*(\theta) = e^{\theta x} \int_x^\infty e^{-\theta y} v(dy)$ denote the "over the barrier Laplace transforms" of $f_a(y)$ and $v_x(dy)$.

Taking Laplace transforms in (13), we obtain the following key result, similar to a result of Usabel (1999).

Proposition 2.1. (a) *The Laplace transform* $V_a^*(\theta)$ *is given by*

$$V_a^*(\theta) = \frac{\alpha(f_a^*(\theta) - f_a^*(\delta_+)) + \int_0^\infty \pi(a-x)(v_x^*(\theta) - v_x^*(\delta_+))\,dx - (\sigma^2/2)\pi(a)(\theta - \delta_+)}{\delta - c(\theta)}, \tag{16}$$

where δ_+ *denotes the largest (necessarily non-negative) root of* $c(\theta) - \delta = 0$.

(b) *In the pure-jump case, the value of a Canadian barrier option with* 0 *initial surplus is*

$$\mu V_a(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x) v_x^*(\delta_+)\,dx. \tag{17}$$

(c) *In diffusion-jump case* $V_a(0+) = \pi(a)$ *and*

$$\frac{\sigma^2}{2} V_a'(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x) v_x^*(\delta_+)\,dx - \left(\frac{\sigma^2}{2}\delta_+ + \mu\right)\pi(a). \tag{18}$$

Note: The equation $c(\theta) - \delta = 0$ (which generalizes McMillan's quadratic equation obtained in the pure diffusion case) has always at most 2 solutions, since the Laplace exponent c is convex. Note that when $\delta = 0$, $\theta = 0$ is always a solution; let θ_0 denote the largest solution in this case. Note that $\theta_0 > 0$ if and only if $c'(0) = \mathbb{E}(Y_1) < 0$. Also, $c(\theta) \leqslant 0$ for $\theta \in [0, \theta_0]$ while c is increasing over the interval $[\theta_0, \infty)$; therefore, $c(\theta)$ has a unique continuous inverse $\gamma(\delta) \geqslant \theta_0$ which is defined for $\delta \geqslant 0$ and satisfies $c(\gamma(\delta)) = \delta$ for all $\delta > 0$ and $\gamma(c(\theta)) = \theta$ for $\theta \geqslant \theta_0$. Hence, $c(\theta) - \delta = 0$ has always a unique positive solution $\delta_+ = \gamma(\delta)$.

Proof. Noting that $V_a(u - x) = \pi(a + u - x)$ for $x \geqslant u$, we take the Laplace transform with respect to the surplus variable u in (13) to obtain

$$0 = \frac{\sigma^2}{2}(\theta^2 V_a^*(\theta) - (V_a'(0+) + \theta V_a(0+))) + \mu(\theta V_a^*(\theta) - V_a(0+)) - \delta V_a^*(\theta)$$

$$+ \alpha f_a^*(\theta) + \int_{x=0}^\infty \int_{u=0}^\infty e^{-\theta u}[V_a(u-x) - V_a(u)]\,du\,v(dx)$$

$$= \frac{\sigma^2}{2}(\theta^2 V_a^*(\theta) - (V_a'(0+) + \theta V_a(0+))) + \mu(\theta V_a^*(\theta) - V_a(0+)) - \delta V_a^*(\theta)$$

$$+ \alpha f_a^*(\theta) + \int_0^\infty \left\{ \int_0^x e^{-\theta u} \pi(a + u - x)\,du \right.$$

$$+ \int_x^\infty e^{-\theta u} V_a(u-x)\,du - V_a^*(\theta)\Big\} v(dx)$$

$$= \frac{\sigma^2}{2}(\theta^2 V_a^*(\theta) - (V_a'(0+) + \theta V_a(0+))) + \mu(\theta V_a^*(\theta) - V_a(0+)) - \delta V_a^*(\theta)$$

$$+ \alpha f_a^*(\theta) + \int_0^\infty \left\{ e^{-\theta x} \int_{-x}^0 e^{-\theta z} \pi(a+z)\,dz + (e^{-\theta x} - 1) V_a^*(\theta) \right\} v(dx)$$

$$= c(\theta) V_a^*(\theta) - \delta V_a^*(\theta) + \alpha f_a^*(\theta) - \frac{\sigma^2}{2}(V_a'(0+) + \theta V_a(0+)) - \mu V_a(0+)$$

$$+ \int_0^\infty e^{-\theta x} \int_{-x}^0 e^{-\theta z} \pi(a+z)\,dz\, v(dx).$$

Rearranging the above gives

$$V_a^*(\theta) = (\delta - c(\theta))^{-1} \left\{ -\mu V_a(0+) - \frac{\sigma^2}{2}(V_a'(0+) + \theta V_a(0+)) \right.$$

$$\left. + \ \alpha f_a^*(\theta) + \int_0^\infty e^{-\theta x} \int_{-x}^0 e^{-\theta z} \pi(z+a)\,dz\, v(dx) \right\}. \tag{19}$$

Changing the order of integration and the integral variable to $x=z-u, z$ with $0 \leqslant x \leqslant \infty$, $z \geqslant x$ we find

$$\int_0^\infty e^{-\theta x} \int_{-x}^0 e^{-\theta z} \pi(z+a)\,dz\, v(dx) = \int_0^\infty \pi(a-x) v_x^*(\theta)\,dx. \tag{20}$$

The right-hand side of (19) still involves certain unknown quantities, namely $V_a'(0+)$ and $V_a(0+)$. At this point, we need to consider two different cases separately.

Pure-jump model: Suppose the logarithmic stock price is pure jump and has no Brownian component, so that $\sigma = 0$. In this case, we need only find $V_a(0+)$. Consider the value of the right-hand side of (19) at the largest (necessarily non-negative) root δ_+ of $c(\theta) - \delta = 0$. Since the Laplace transform is well-defined for all $\theta \geqslant 0$, it must in particular be finite at δ_+, implying thus Eq. (17): $\mu V_a(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x) v_x^*(\delta_+)\,dx$.

Substituting the above and (20) into (19) establishes (16).

Diffusion-jump model: [5] If a Brownian component is present, so that $\sigma \neq 0$, $u \mapsto V_a(u)$ is always continuous at 0 (see Chan, 1999b), so we know that $V_a(0+) = \pi(a)$. To find an expression for $V_a'(0+)$, consider again the value of the right-hand side of (19) at δ_+; we obtain now (18)

$$\frac{\sigma^2}{2} V_a'(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x) v_x^*(\delta_+))\,dx - \left(\frac{\sigma^2}{2}\delta_+ + \mu\right)\pi(a).$$

Substituting this and (20) into (19) establishes (16). □

[5] The key difference between this case and the pure-jump case actually centres on whether Y has infinite or finite variation; a similar result with the diffusion case below is also found for the pure-jump model, if the jump process X has infinite variation.

We consider now the put boundary payoff $\pi(y) = (K - e^y)_+ = K - e^y$, starting with the preliminary case $\pi(x) = e^{\eta x}$ (to be applied with $\eta = 0$ and 1).

Lemma 2.2. *If* $\pi(x) = e^{\eta x}$, *then*

$$\int_0^\infty \pi(-x) v_x^*(\theta)\, dx = \int_0^\infty e^{-\eta x} v_x^*(\theta)\, dx = \frac{v^*(\eta) - v^*(\theta)}{\theta - \eta}, \tag{21}$$

where $v^*(\theta) = \int_0^\infty (e^{-\theta x} - 1)\, v(dx)$.

Proof.

$$\int_0^\infty \pi(-x) v_x^*(\theta)\, dx = \int_0^\infty e^{-\eta x} v_x^*(\theta)\, dx = \int_0^\infty e^{-\eta x} \int_x^\infty e^{\theta(x-y)} v(dy)$$

$$= \int_0^\infty e^{-\theta y} \frac{e^{(\theta-\eta)y - 1}}{\theta - \eta} v(dy) = \int_0^\infty \frac{e^{-\eta y} - e^{-\theta y}}{\theta - \eta} v(dy) = \frac{v^*(\eta) - v^*(\theta)}{\theta - \eta}. \qquad \square$$

Applying formula (21) of Lemma 2.2 with $\eta = 0$ and 1 we find

Lemma 2.3. *In the case of the put boundary payoff* $\pi(y) = (K - e^y)_+ = K - e^y$,

$$\int_0^\infty \pi(a - x) v_x^*(\theta)\, dx = K \left(\frac{v^*(0) - v^*(\theta)}{\theta} \right) - L \left(\frac{v^*(1) - v^*(\theta)}{\theta - 1} \right) \tag{22}$$

$$= -c(\theta) z(\theta) - \frac{\sigma^2 L}{2} - \frac{(r - q)L}{\theta - 1} + \left(\frac{\sigma^2 \theta}{2} + \mu \right) \pi(a), \tag{23}$$

where $z(\theta) = K/\theta - L/(\theta - 1)$, *and* $L = e^a$ *(*z *is the transform of* $K - Le^u$*).*

Proof. The second expression (needed to establish 3.1b) is obtained by using expression (5) connecting the functions $v_x^*(\theta)$, $c(\theta)$ and the equalities $c(1) = r - q$, $c(0) = 0$.

The following lemma follows readily from elementary calculations which are left to the reader.

Lemma 2.4. *The Laplace transform in the surplus variable* u *of the final put payoff* $(K - Le^u)_+$ *is*

$$f_a^*(\theta) = \frac{K}{\theta} (1 - (L/K)^\theta) - \frac{L}{\theta - 1} (1 - (L/K)^{\theta - 1})$$

$$= z(\theta) + \frac{K}{\theta(\theta - 1)} (L/K)^\theta, \tag{24}$$

where $L = e^a$ *and* $z(\theta) = K/\theta - L/(\theta - 1)$.

From Lemma 2.2, we obtain also a convenient form for the well-known classical formula of the joint moment generating function $\Psi_{\delta,\eta}(u) = \mathbb{E}_u e^{-\delta\tau_0 + \eta Y_{\tau_0}}$—see Bingham (1975), Theorem 6a, where τ_0 is the first time when 0 is overshot by a spectrally negative Lévy process.

Corollary 2.5. *The Laplace transform* $\Psi^*_{\delta,\eta}(\theta) = \int_0^\infty e^{-\theta u}\Psi_{\delta,\eta}(u)\,du$ *of the joint moment generating function* $\Psi_{\delta,\eta}(u) = \mathbb{E}_u e^{-\delta\tau_0+\eta Y_{\tau_0}}$ *is given by*

$$\Psi^*_{\delta,\eta}(\theta) = (\delta - c(\theta))^{-1}\left(\frac{c(\theta)-c(\eta)}{\eta-\theta} - \frac{c(\delta_+)-c(\eta)}{\eta-\delta_+}\right). \tag{25}$$

Proof. By Lemma 2.2 and (5), we get $\int_0^\infty \pi(-x)v^*_x(\theta)\,dx - (\sigma^2/2)\theta = (c(\eta)-c(\theta))/(\theta-\eta) + (\sigma^2/2)\eta + \mu$. The result follows from Proposition 2.1(a) with $f=0, a=0$, $\pi(x) = e^{\eta x}$ and $\delta = \alpha + r$.

Note: It would be quite useful to extend somehow this approach to the case of Lévy processes with two-sided jumps, which is the one of main interest in mathematical finance. In that case however we need to incorporate in the generator equation the contribution of the negative jumps as well, and taking Laplace transform ceases to yield an algebraic equation for the transform. The two-sided case requires considerably more work. For example, the joint moment generating function presented in Corollary 2.5 ceases to be a function of the Laplace exponent $c(\theta)$ only; instead (see Bingham, 1975, Theorem 1(e)) it requires obtaining the two Wiener–Hopf factors of the function $\delta/(\delta - c(\theta))$, which are known explicitly only in some particular cases although many of these cases are of interest in financial models (see Boyarchenko and Levendorskii, 2000).

3. The optimal barrier for Canadian–American options

Proposition 2.1 yielded over barrier Laplace transform for the value $V_a(y)$ of a Canadian option. A further explicit formula for the value $V_a(y)$ itself will be provided in Proposition 4.3. By differentiating that, one might determine in principle the optimal barrier a^* for exercising a Canadian–American option. However, a more direct determination is obtained by using "smooth fit" continuity conditions for the values $V_a(0+), V'_a(0+)$ at the barrier (these were already obtained in (17), (18)).

More precisely, it is known that in the pure-jump case the optimal value of a is given by the continuity condition $V_a(0+)=\pi(a)$ (see Chan, 1999b or Gerber and Shiu, 1998) and in the jump-diffusion case it is given by the smoothness condition $V'_a(0+)=\pi'(a)$ (see Chan, 1999b or Gerber and Landry, 1998). Although the smooth fit condition does not always yield the optimal barrier in the case of all Lévy processes, it has been shown in Chan (1999b) and Gerber and Landry (1998) that for the spectrally one-sided Lévy processes considered here, the smooth fit condition is equivalent to finding the optimal barrier by differentiating with respect to a.

We summarize in the Corollary below the results obtained from the smooth fit equations (parts (b) and (c)) are needed to determine the optimal barriers of n-Erlangian and Canadian puts, and (d) and (e) are needed to establish the long/short-term asymptotics of the approximate barrier).

Corollary 3.1. *The optimal exercise barrier for Canadian–American options*

(a) *The optimal exercise barrier* a^* *for a "stationary" option which solves equation* (13) *must satisfy the nonlinear equation*:

$$\alpha f_a^*(\delta_+) = -\int_0^\infty \pi(a-x)v_x^*(\delta_+)\,\mathrm{d}x + \left(\frac{\sigma^2}{2}\delta_+ + \mu\right)\pi(a) + \frac{\sigma^2}{2}\pi'(a). \quad (26)$$

(b) *For an arbitrary final payoff* f *and "put type" boundary payoff* $\pi(y) = (K - \mathrm{e}^y)_+$, *the optimal exercise boundary* a *satisfies*

$$\alpha(f_a^*(\delta_+) - z(\delta_+)) = \frac{rK}{\delta_+} - \frac{qL}{\delta_+ - 1}, \quad (27)$$

where $z(\theta) = K/\theta - L/(\theta - 1)$, *and* $L = \mathrm{e}^a$.

(c) *In the case of both final and boundary payoff* $f(y) = (K - \mathrm{e}^y)_+$, *the optimal exercise boundary* $L = \mathrm{e}^a$ *satisfies*

$$\alpha\left(\frac{L}{K}\right)^{\delta_+} = (\delta_+ - 1)r - \delta_+ q\left(\frac{L}{K}\right). \quad (28)$$

(d) *In the particular case* $\alpha = 0$ *we get*

$$\frac{L}{K} = \frac{r(\delta_r - 1)}{q\delta_r} = \frac{r(\delta_r - 1)}{\delta_r(c(\delta_r) - c(1))}, \quad (29)$$

where δ_r *is the positive root of the equation* $c(\theta) = r$. *If furthermore the dividend is* 0, *then* $\delta_r = 1$ *and the equation becomes*

$$\frac{L}{K} = \frac{r}{c'(1)} = \frac{r}{\sigma^2 + \mu - \int_0^\infty x\mathrm{e}^{-x}\nu(\mathrm{d}x)}.$$

(*This last formula has also been obtained in* Chan (1999b).)

(e) *In the limit* $\alpha \to \infty$, *the solution of* (28) *becomes*

$$\frac{L}{K} = \min\left(\frac{r}{q}, 1\right).$$

Remarks. (1) In the pure-jump case (26) simplifies to

$$\alpha f_a^*(\delta_+) = -\int_a^\infty \pi(a-x)v_x^*(\delta_+))\,\mathrm{d}x + \mu\pi(a). \quad (30)$$

This generalizes the formula before the references in Zhang's (1998) discussion of Gerber and Shiu's paper (Zhang, 1998).

(2) Formula (28) generalizes formula (18) of Carr (1998).

(3) Since we plan to use (28) as an approximation for options with fixed finite expiry T, it is of interest to compare the approximations' limits as $T \to \infty$ and 0 (obtained in parts (d), (e) by letting $\alpha \to 0$ and ∞, respectively) with the correct asymptotics of the exact optimal exercise barrier as $T \to \infty$ and 0. We see that at least in the Brownian case (in which they are known) they do coincide with the well-known exact asymptotics of the exercise barrier (see Kwok, 1998). The limit as $T \to \infty$ (the $\alpha = 0$ case) in the pure Brownian case is usually written in the form

$$\frac{L}{K} = \frac{r}{r + \sigma^2\delta_r/2}$$

or in equivalent form

$$\frac{L}{K} = \frac{\tilde{\theta}_r}{\tilde{\theta}_r - 1},$$

where $\tilde{\theta}_r$ is the negative root of the quadratic equation $\sigma^2\theta^2/2+(r-q-\sigma^2/2)\theta-r=0$. We may easily check that these coincide in the Brownian case with our expression $r(\delta_r - 1)/q\delta_r$ provided in (29).

Proof (*Corollary 3.1*).

(a) The optimal exercise barrier is known to make the value function continuous at the barrier in the pure-jump case $V_a(0+) = \pi(a)$, and to make the derivative of the value function continuous at the barrier in the diffusion-jump case $V_a'(0+)=\pi'(a)$ (see Chan, 1999b; Gerber and Shiu, 1998 and Gerber and Landry, 1998).

The equation $V_a'(0+)=\pi'(a)$ yields (26) after using the formula for $V_a'(0+)$ provided in Proposition 2.1(c). In the case $\sigma=0$, (26) becomes (30) which by Proposition 2.1(b) is tantamount to $V_a(0+) = \pi(a)$.

(b) Using the second expression in Lemma 2.3 and the expression of the put's payoff $\pi(a) = K - e^a$, we rewrite (26) as

$$\begin{aligned}
\alpha f_a^*(\delta_+) &= \delta z(\delta_+) + \frac{\sigma^2 L}{2} + \frac{(r-q)L}{\delta_+ - 1} + \frac{\sigma^2\pi'(a)}{2} \\
&= \alpha z(\delta_+) + \frac{rK}{\delta_+} + \frac{\sigma^2 L}{2} - \frac{qL}{\delta_+ - 1} - \frac{\sigma^2 L}{2} \\
&= \alpha z(\delta_+) + \frac{rK}{\delta_+} - \frac{qL}{\delta_+ - 1},
\end{aligned}$$

which establishes (27).

To obtain (c) we substitute the expression $f_a^*(\theta)-z(\theta)=(K/\theta(\theta-1))(L/K)^\theta$ provided in Lemma 2.4 for the transform of the put's final payoff into (b).

(d) To obtain (d) we note that the L.H.S. in (28) goes to 0 and that $q=c(\delta_r)-c(1)$.

(e) It is known (see Chan, 1999b) that $\lim_{\theta\to\infty} c(\theta)/\theta^2 = \sigma^2/2$, so that $\delta_+(\alpha) \sim O(\sqrt{\alpha})$ as $\alpha \to \infty$. Dividing (28) by $q\delta_+$ and replacing δ_+ by its large α asymptotic, we get

$$q^{-1}\sqrt{\alpha}\exp\{\sqrt{\alpha}\log(L(\alpha)/K)\} + \exp\{\log(L(\alpha)/K)\} = \frac{(\sqrt{\alpha}-1)r}{q\sqrt{\alpha}} \to \frac{r}{q}.$$

Since $L(\alpha)/K \leqslant 1$, if $L(\alpha)/K \nrightarrow 1$, the first term above converges to 0 while the second term is never bigger than 1. Therefore, if $r/q > 1$, the only way the left-hand side above can converge to a limit large than 1 is for the first term to converge to a non-zero limit, which implies $L(\alpha)/K \to 1$. Similarly, if $r/q \leqslant 1$, the first term above must converge to 0, so that $L(\alpha)/K \to r/q$. Therefore, the $t \to 0$ asymptotic is

$$\frac{L}{K} = \min\left(\frac{r}{q}, 1\right)$$

which also agrees with the known asymptotic of the optimal boundary in the pure Brownian case. The same is also true in the pure-jump case ($\sigma = 0$), only this time $\lim_{\theta\to\infty} c(\theta)/\theta = \mu$ so $\delta_+(\alpha) \sim \mathrm{O}(\alpha)$. □

Corollary 3.2. (a) *For an arbitrary final payoff f and "put type" boundary payoff $\pi(y) = (K - \mathrm{e}^y)_+$, the Laplace transform in the surplus variable u of the value of a Canadian–American option $V_a(u)$ is given by*

$$V^*(\theta) = \frac{b(\theta) - b(\delta_+)}{\delta - c(\theta)}, \tag{31}$$

where

$$b(\theta) = \alpha f_a^*(\theta) - c(\theta)z(\theta) - (r-q)L/(\theta - 1), \quad L = \mathrm{e}^a \text{ and } z(\theta) = \frac{K}{\theta} - \frac{L}{\theta - 1}.$$

(b) *After the barrier a^* is optimized, this becomes*

$$V^*(\theta) = z(\theta) + \frac{\alpha(f_{a^*}^*(\theta) - z(\theta)) - (rK/\theta - qL/(\theta - 1))}{\delta - c(\theta)}. \tag{32}$$

Note: The expression above is well defined as $\theta \to \delta_+$ by the barrier optimality condition of Corollary 3.1(b).

Proof. (a) Substituting expression (23) of the transform of the put's boundary payment into (16) yields (a) immediately.

(b) The result follows upon noticing that

$$\begin{aligned} b(\delta_+) &= \alpha f_a^*(\delta_+) - \delta z(\delta_+) - (r-q)L/(\delta_+ - 1) \\ &= \alpha(f_a^*(\delta_+) - z(\delta_+)) - rK/\delta_+ + qL/(\delta_+ - 1) = 0 \end{aligned}$$

(by the smooth fit condition (27) provided in Corollary 3.1(b)) and that $b(\theta)$ can also be written as

$$\begin{aligned} b(\theta) &= \alpha f_a^*(\theta) - c(\theta)z(\theta) - (r-q)L/(\theta - 1) \\ &= z(\theta)(\delta - c(\theta)) + \alpha f_a^*(\theta) - \delta z(\theta) - (r-q)L/(\theta - 1) \\ &= z(\theta)(\delta - c(\theta)) + \alpha(f_a^*(\theta) - z(\theta)) - rK/\theta + qL/(\theta - 1). \qquad \square \end{aligned}$$

4. The value of "put type" Canadian barrier options

In Sections 2 and 3 we took advantage of the Laplace transform approach available for spectrally negative Lévy processes to obtain (in Proposition 2.1) a general formula for the Laplace transform in the initial capital for general Canadian barrier options. In this section we exploit yet another simplification of "fluctuation theory" available in the spectrally negative (positive) case: an explicit formula for the joint moment generating function of the time and place of exit from a one-sided interval in terms of the negative (positive) part of the resolvent density. Note that in the general case, the same task requires performing a Wiener–Hopf factorization—see Theorem 1(e) of Bingham (1975). We quote now the spectrally one-sided result, following Theorem 6(b) of Bingham (1975).

Theorem 4.1. *Suppose that X is a Lévy process whose jumps are positive but which is not non-decreasing, with $X_0 = 0$.*

(a) *Then the resolvent measure $r_\delta(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-\delta t}\mathbb{P}(X_t \in \mathrm{d}x)\,\mathrm{d}t$ is absolutely continuous with respect to Lebesgue measure and the Laplace transform of its density $r_\delta(x)$, for $x > 0$ is given by*

$$r_\delta^*(\theta) = \int_0^\infty \mathrm{e}^{-\theta x} r_\delta(x)\,\mathrm{d}x = \frac{1}{\delta - c(\theta)} - \frac{1}{c'(\delta_+)(\delta_+ - \theta)}. \tag{33}$$

(b) *For $u > 0$, let $\tau_u = \inf\{t \geqslant 0: X_t > u\}$. The joint law of τ_u and of the overshoot $X(\tau_u) - u$ is*

$$\begin{aligned}\Psi_{\delta,\eta}(u) &= \mathbb{E}[\mathrm{e}^{-\delta\tau_u - \eta(X(\tau_u)-u)}]\\ &= (\delta - c(\eta))\left(\frac{r_\delta(u)}{\delta_+ - \eta} + \int_u^\infty \mathrm{e}^{-\eta(z-u)} r_\delta(z)\,\mathrm{d}z\right)\end{aligned} \tag{34}$$

for $\delta, \eta \geqslant 0$.

Notes: (1) While the resolvent function is rarely available analytically, efficient numerical approaches for computing it exist, and so this provides an alternative to inverting the Laplace transform. For example, in Chan (1999b) the fast Fourier transform algorithm is used to compute $r_\delta(x)$.

(2) Relation (34) may be also obtained from Corollary 2.5. Indeed, starting from the end by taking Laplace transform in u of (34), we get

$$\begin{aligned}\Psi_{\delta,\eta}^*(\theta) &= (\delta - c(\eta))\left(\frac{r_\delta^*(\theta)}{\delta_+ - \eta} - \frac{r_\delta^*(\theta) - r_\delta^*(\eta)}{\theta - \eta}\right)\\ &= \frac{\delta - c(\eta)}{\delta_+ - \eta}\left(\frac{1}{\delta - c(\theta)} - \frac{1}{c'(\delta_+)(\delta_+ - \theta)}\right) - \frac{\delta - c(\eta)}{\theta - \eta}\\ &\quad\times\left(\frac{1}{\delta - c(\theta)} - \frac{1}{c'(\delta_+)(\delta_+ - \theta)} - \frac{1}{\delta - c(\eta)} + \frac{1}{c'(\delta_+)(\delta_+ - \eta)}\right)\end{aligned}$$

in which the three terms involving $c'(\delta_+)$ cancel out and the remaining terms yield the result of Corollary 2.5 for the process $Y(t) = u - X(t)$.

(3) The first relation (33) is a consequence of a simple identity

$$\frac{1}{\delta - c(\theta)} = r_\delta^*(\theta) + n_\delta^*(\theta), \tag{35}$$

where $n_\delta^*(\theta) = \int_{-\infty}^0 \mathrm{e}^{-\theta u} r_\delta(u)\,\mathrm{d}u$ denotes the Laplace transform of the negative part of the resolvent density, and of the fact that the latter is the exponential function $r_\delta(u) = \mathrm{e}^{\delta_+ u}/c'(\delta_+)$ (for $u < 0$) as shown for example in Bingham (1975). Let now $W(u)$ denote the inverse Laplace transform of the function $1/(\delta - c(\theta))$. For general two-sided Lévy processes, we have

$$W(u) = \int_0^\infty \mathrm{e}^{-\delta t}\mathbb{P}(X_t \in \mathrm{d}u)\,\mathrm{d}t + \int_0^\infty \mathrm{e}^{-\delta t}\mathbb{P}(X_t \in -\,\mathrm{d}u)\,\mathrm{d}t \tag{36}$$

(by (35)). We provide now an alternative expression for $W(u)$ available in the spectrally negative case

Corollary 4.2. *The inverse Laplace transform of the function* $1/(\delta - c(\theta))$ *for spectrally negative Lévy processes is given by*

$$W(u) = r_\delta(u) - \frac{\mathrm{e}^{\delta_+ u}}{c'(\delta_+)} = \frac{\Psi_{\delta,\delta_+}(u) - \mathrm{e}^{\delta_+ u}}{c'(\delta_+)}. \tag{37}$$

Proof. The first equality is immediate from Theorem 4.1(a). The second follows by letting $\eta \to \delta_+$ in (34), which yields the resolvent density in terms of the joint Laplace transform of τ_u and $X(\tau_u)$:

$$r_\delta(u) = \frac{\mathrm{e}^{\delta_+ u}}{c'(\delta_+)}\,\mathbb{E}[\mathrm{e}^{-(\delta\tau_u + \delta_+ X(\tau_u))}] = \frac{\Psi_{\delta,\delta_+}(u)}{c'(\delta_+)}, \quad u > 0. \qquad \square \tag{38}$$

Note: Formula (38) shows that knowing the resolvent is equivalent to knowing the joint Laplace transform of the hitting time τ_u and of the overshoot $X(\tau_u)$, evaluated at the particular values which make the "Wald exponential" a martingale. In some special cases, like when X is a compound Poisson process with exponential jumps—see Gerber and Shiu, 1998, or more generally, when the jumps have a phase-type distribution—see Asmussen et al., 2000, the joint Laplace transform can be computed analytically, which makes (37) preferable to (36).

Using this corollary, we give now two explicit representations for the value of Canadian barrier options with put boundary payoff and arbitrary final payoff.

Proposition 4.3. *Let* $L = \mathrm{e}^a$ *denote a fixed barrier and* $u = y - a$ *denote the initial log-surplus over the barrier* a. *If* $X_t = y - Y_t$ *satisfies the assumptions of Theorem* 4.1, *and* $\pi(y) = (K - \mathrm{e}^y)_+ = (K - L\mathrm{e}^u)_+$ *is the put barrier payoff, then*

$$V_{a,\pi,f,r,\alpha}(y) = V_a(u) = K\Psi_{\delta,0}(u) - L\Psi_{\delta,1}(u) + \int_0^u \alpha W(u-x) f_a(x)\,\mathrm{d}x - \alpha W(u) f_a^*(\delta_+) \tag{39}$$

$$= K\delta\left(\frac{r_\delta(u)}{\delta_+} + \int_u^\infty r_\delta(z)\,\mathrm{d}z\right) - (\alpha + q)L\left(\frac{r_\delta(u)}{\delta_+ - 1} + \int_u^\infty \mathrm{e}^{u-z} r_\delta(z)\,\mathrm{d}z\right) + \int_0^u \alpha r_\delta(u-x) f_a(x)\,\mathrm{d}x - \alpha f_a^*(\delta_+) r_\delta(u) + \int_u^\infty \frac{\mathrm{e}^{\delta_+(u-x)}}{c'(\delta_+)} f_a(x)\,\mathrm{d}x. \tag{40}$$

Notes: (1) The first formula is more convenient for examples which allow an explicit Wiener–Hopf factorization, since both $\Psi_{\delta,\eta}(u)$ and $W(u)$ are expressible in terms of the Wiener–Hopf factors. The second formula is preferable if the resolvent is to be computed numerically.

(2) Even though formula (39) makes perfect sense for spectrally two-sided Lévy processes, we conjecture that it does *not* provide the value of Canadian–American puts in that case.

Proof. It is enough to establish separately the two particular cases A: $f \equiv 0$, $\pi = (K - Le^u)_+$ (the payoff from overshooting the boundary) and B: $\pi \equiv 0$ (the payoff from expiration, if the boundary is not crossed).

Case A: When $f \equiv 0$, observe that the solution of (13) can be expressed probabilistically as

$$V_a(u) = \mathbb{E}_y[e^{-\delta\tau_u}\pi(Y(\tau_u))],$$

where the subscript in τ_u refers to the barrier of the process $X_t = y - Y_t$: $\tau_u = \inf\{t\colon X_t > u\} = \inf\{t\colon Y_t < a\}$. The function $V_a(u)$ may therefore be obtained directly by applying (34) to the spectrally positive process $X_t = y - Y_t$ with $\eta = 0$ and 1 and using $c(1) = r - q$, yielding

$$\begin{aligned} V_a(u) &= \mathbb{E}[e^{-\delta\tau_u}(K - Le^{-(X(\tau_u)-u)})] = K\Psi_{\delta,0}(u) - L\Psi_{\delta,1}(u) \\ &= K\delta\left(\frac{r_\delta(u)}{\delta_+} + \int_u^\infty r_\delta(z)\,dz\right) - L(\delta - c(1)) \\ &\quad\times\left(\frac{r_\delta(u)}{\delta_+ - 1} + \int_u^\infty e^{-(z-u)} r_\delta(z)\,dz\right) \\ &= K\delta\left(\frac{r_\delta(u)}{\delta_+} + \int_u^\infty r_\delta(z)\,dz\right) - (\alpha + q) \\ &\quad\times L\left(\frac{r_\delta(u)}{\delta_+ - 1} + \int_u^\infty e^{u-z} r_\delta(z)\,dz\right). \end{aligned} \tag{41}$$

Case B: When $\pi \equiv 0$, we recognize that the inverse Laplace transform of

$$\frac{\alpha(f_a^*(\theta) - f_a^*(\delta_+))}{\delta - c(\theta)} \tag{42}$$

is a combination of the inverse Laplace transform $W(u)$ of $1/(\delta - c(\theta))$ provided in Corollary 4.2 and of the convolution of $W(u)$ with $f_a(x)$:

$$\int_0^u \alpha\left[r_\delta(u-x) - \frac{e^{\delta_+(u-x)}}{c'(\delta_+)}\right] f(x+a)\,dx - \alpha f_a^*(\delta_+)\left[r_\delta(u) - \frac{e^{\delta_+(u)}}{c'(\delta_+)}\right]. \tag{43}$$

Finally, simplifying (43) and putting it together with (41) gives the desired result (40). □

Note: The joint Laplace transform $\Psi_{\delta,\eta}$ may be computed explicitly for the compound Poisson model with phase-type jumps (see for example Asmussen et al., 2000), leading to an explicit formula for the resolvent density $r_\delta(u)$ via (38). In that case, (40) provides an explicit formula for the American put requiring only root solving and matrix exponentiation.

5. The recursive algorithm

For a fixed positive integer n, let S_n be a random variable with $\Gamma(n, n/T)$ distribution, independent of Y. Since $\mathbb{E}(S_n)=T$ and $\mathrm{Var}(S_n)=T/n$, $S_n \to T$ in distribution as $n \to \infty$. The idea is to use

$$V_n(y) := \mathbb{E}_y[\mathrm{e}^{-r(\tau_* \wedge S_n)}\pi(Y_{\tau_* \wedge S_n})]$$

for large n as an approximation to the price of an American put. Since $S_n = T_1 + T_2 + \cdots + T_n$ where T_i are independent exponential rate n/T random variables, the Markov property enables us to compute V_n using the following recursive algorithm:

$$V_0(y) := \pi(y),$$

$$V_i(y) := \mathbb{E}_y[\mathrm{e}^{-r\tau_i}\pi(Y_{\tau_i})\mathbf{1}_{\{\tau_i \leqslant T_i\}}] + \mathbb{E}_y[\mathrm{e}^{-rT_i}V_{i-1}(Y_{T_i})\mathbf{1}_{\{\tau_i > T_i\}}], \quad i = 1, \ldots, n, \tag{44}$$

where the stopping times τ_i are now the optimal exercise times of American type options with random expiry time T_i and final payoff given by V_{i-1}. The memoryless property of the exponential distribution means that the optimal exercise levels at each stage are now constants, so that the optimal exercise times τ_i are of form

$$\tau_i = \inf\{t \geqslant 0\colon\ Y_t \leqslant a_i\} \tag{45}$$

for some constant optimal levels a_i, which can be thought of as a piecewise constant approximation to the time-dependent optimal exercise curve for the fixed expiry option. Note that some care is needed when computing the final payoffs V_{i-1}, since these can come either from immediate stopping (in which case $V_{i-1}(y) = (K - \mathrm{e}^y)_+$), or from continuing in which case $V_{i-1}(y)$ is given by Proposition 4.3.

With τ_i as in (45) and continuing with the previous notation $y = u + a_i$, consider the value of a put with the exercise level at each stage fixed at a_i:

$$V_i(u, a_i) = \mathbb{E}_{u+a_i}[\mathrm{e}^{-r\tau_i}\pi(Y_{\tau_i})\mathbf{1}_{\{\tau_i \leqslant T_i\}}] + \mathbb{E}_{u+a_i}[\mathrm{e}^{-rT_i}V_{i-1}(Y_{T_i} - a_{i-1}, a_{i-1})\mathbf{1}_{\{\tau_i > T_i\}}]$$

and its Laplace transform

$$V_i^*(\theta, a_i) = \int_0^\infty \mathrm{e}^{-\theta u}V_i(u, a_i)\,\mathrm{d}u.$$

(We already know that $V_i(x, a) = \Pi(a + x)$ for $x \leqslant 0$, by definition.) Of course, the $V_i(u, a)$ and $V_i^*(u, a)$ are particular cases of the $V_a(u)$ and $V_a^*(u)$ considered in the previous section. At each stage we need to maximize $V_i(u, a)$ over a to obtain $V_i(y) = \sup_a V_i(y - a, a) = V_i(y - a_i, a_i)$. Note that the optimal exercise levels a_i do not depend on the starting point y (hence nor the initial excess u).

Before we can implement such a recursive algorithm, we need to verify that $\mathbb{E}[\mathrm{e}^{-r(\tau_* \wedge S_n)}\ \pi(Y_{\tau_* \wedge S_n})]$ can indeed be used as an approximation for $\mathbb{E}[\mathrm{e}^{-r(\tau_* \wedge T)}$ $\pi(Y_{\tau_* \wedge T})]$—more precisely, that

$$\mathbb{E}[\mathrm{e}^{-r(\tau_* \wedge S_n)}\pi(Y_{\tau_* \wedge S_n})] \to \mathbb{E}[\mathrm{e}^{-r(\tau_* \wedge T)}\pi(Y_{\tau_* \wedge T})]$$

as $n \to \infty$. Recall that $S_n \to T$ in distribution means that $\mathbb{E}[f(S_n)] \to \mathbb{E}[f(T)]$ for every bounded *continuous* function f. In the Brownian model considered in Carr (1998), Y is continuous, so $\mathbb{E}[e^{-r(\tau_* \wedge S_n)}\pi(Y_{\tau_* \wedge S_n})] \to \mathbb{E}[e^{-r(\tau_* \wedge T)}\pi(Y_{\tau_* \wedge T})]$ essentially follows by definition. When Y is discontinuous, a little more work is required.

Lemma 5.1. *Let Y be a spectrally negative Lévy process and for fixed T let S_n be a sequence of random variables with $\Gamma(n, n/T)$ distribution, independent of Y. Let f be a bounded uniformly Lipschitz function satisfying*

$$|f(t,x) - f(s,y)| < \min(K, C(|s-t| + |x-y|)) \quad \forall s,t,x,y \tag{46}$$

for some constants K and C. Let $\varepsilon > 0$ be sufficiently small but otherwise arbitrary. Then for any stopping time τ we have

$$|\mathbb{E}[f(\tau \wedge S_n, Y_{\tau \wedge S_n})] - \mathbb{E}[f(\tau \wedge T, Y_{\tau \wedge T})]| \leqslant \varepsilon$$

whenever

$$n > -18T^2K^2 \frac{\nu(\varepsilon/(6C), \infty)\log\varepsilon}{\varepsilon^2} \tag{47}$$

(*where ν is the Lévy measure of $-Y$.*)

Proof. Choose δ_1 and $\delta_2 \sim o(\delta_1)$ such that $|f(s,y_1) - f(t,y_2)| < \varepsilon/3$ for all $|y_1 - y_2| < \delta_1$ and $|s-t| < \delta_2$. From the central limit theorem we have (using a well-known asymptotic for $\bar{\Phi}$)

$$\mathbb{P}(|S_n - T| > \delta_2) \sim 2\bar{\Phi}(\delta_2\sqrt{n}/T) \sim \frac{2T}{\delta_2\sqrt{2\pi n}} e^{-\delta_2^2 n/(2T^2)} \tag{48}$$

for arbitrary $\delta_2 \ll 1$. Consider first the case that $Y_t = \sigma W_t$ has only the Brownian component. We can choose δ_2 such that whenever $|s-t| < \delta_2$,

$$\mathbb{P}|Y_s - Y_t| > \delta_1) \sim \frac{\sigma\sqrt{|t-s|}}{\sqrt{2\pi}\delta_1} e^{-\delta_1^2/2\sigma^2|t-s|} \leqslant \frac{\sigma\sqrt{\delta_2}}{\sqrt{2\pi}\delta_1} e^{-\delta_1^2/2\sigma^2\delta_2}. \tag{49}$$

(We have used essentially the same estimate as in (48).) On the other hand, if Y_t has no Brownian component, we can choose δ_2 such that whenever $|s-t| < \delta_2$,

$$\mathbb{P}|Y_s - Y_t| > \delta_1) \sim 1 - e^{-\delta_2 \nu(\delta_1, \infty)} \tag{50}$$

because for $\delta_2 \ll \delta_1$ the dominant term in $\mathbb{P}|Y_s - Y_t| > \delta_1)$ is the probability of making a big jump of size δ_1 in the time interval (s,t). When Y_t has both the Brownian component and the Lévy jump component

$$|Y_t - Y_s| \leqslant |\mu(t-s)| + \sigma|W_t - W_s| + |J_t - J_s|$$

hence

$$\begin{aligned}
\mathbb{P}|Y_s - Y_t| > \delta_1) &\leqslant \mathbb{P}(|\mu(t-s)| + \sigma|W_t - W_s| + |J_t - J_s| > \delta_1) \\
&\leqslant \mathbb{P}(\sigma|W_t - W_s| > \delta_1/2 \text{ or } |\mu(t-s)| + |J_t - J_s| > \delta_1/2) \\
&= \mathbb{P}(\sigma|W_t - W_s| > \delta_1/2) + \mathbb{P}(|\mu(t-s)| + |J_t - J_s| > \delta_1/2) \\
&\quad - \mathbb{P}(\sigma|W_t - W_s| > \delta_1/2)\mathbb{P}(|\mu(t-s)| + |J_t - J_s| > \delta_1/2).
\end{aligned}$$

Since the estimate (50) is much bigger than (49) we have

$$\mathbb{P}|Y_s - Y_t| > \delta_1) \leqslant \mathbb{P}(|\mu(t-s)| + |J_t - J_s| > \delta_1/2) \sim 1 - \mathrm{e}^{-\delta_2 v(\delta_1/2,\infty)}. \tag{51}$$

Recall that δ_1 and $\delta_2 = \mathrm{o}(\delta_1)$ have been chosen so that $|f(s,y_1) - f(t,y_2)| < \varepsilon/3$ whenever $|y_1 - y_2| < \delta_1$ and $|s-t| < \delta_2$. Because of the Lipschitz assumption (46) and $\delta_2 = \mathrm{o}(\delta_1)$, choosing δ_1 to be as large as possible we may take $\delta_1 \sim \varepsilon/(3C)$ and the asymptotic behaviour of (51) is then $1 - \mathrm{e}^{-\delta_2 v(\varepsilon/(6C),\infty)}$. We now choose δ_2 so small that whenever $|s-t| < \delta_2$,

$$\mathbb{P}|Y_s - Y_t| > \delta_1) \leqslant 1 - \mathrm{e}^{-\delta_2 v(\varepsilon/(6C),\infty)} < \varepsilon/(3K),$$

which requires

$$\delta_2 < -\frac{\log(1-\varepsilon/(3K))}{v(\varepsilon/(6C),\infty)}. \tag{52}$$

Finally, choose n so large that (from (48))

$$\mathbb{P}|S_n - T| > \delta_2) \sim \frac{2T}{\delta_2\sqrt{2\pi n}}\,\mathrm{e}^{-\delta_2^2 n/(2T^2)} < \varepsilon/(3K). \tag{53}$$

A sufficient condition for this is

$$\mathrm{e}^{-\delta_2^2 n/(2T^2)} < \varepsilon/(3K)$$

and because of (52) we need

$$n > -\frac{2T^2 v(\varepsilon/(6C),\infty)^2 \log\varepsilon}{[\log(1-\varepsilon/(3K))]^2} \sim -18T^2K^2\,\frac{v(\varepsilon/(6C),\infty)^2\log\varepsilon}{\varepsilon^2}.$$

Then

$$\begin{aligned}
&|\mathbb{E}[f(\tau\wedge S_n, Y_{\tau\wedge S_n})] - \mathbb{E}[f(\tau\wedge T, Y_{\tau\wedge T})]|\\
&\quad = \mathbb{E}[|f(\tau\wedge S_n, Y_{\tau\wedge S_n}) - f(\tau\wedge T, Y_{\tau\wedge T}))|\mathbf{1}_{\{|S_n - T| > \delta_2\}}]\\
&\qquad + \mathbb{E}[|f(\tau\wedge S_n, Y_{\tau\wedge S_n}) - f(\tau\wedge T, Y_{\tau\wedge T})|\mathbf{1}_{\{|S_n - T| \leqslant \delta_2\}}\mathbf{1}_{\{|Y(\tau\wedge S_n) - Y(\tau\wedge T)| > \delta_1\}}]\\
&\qquad + \mathbb{E}[|f(\tau\wedge S_n, Y_{\tau\wedge S_n}) - f(\tau\wedge T, Y_{\tau\wedge T})|\mathbf{1}_{\{|S_n - T| \leqslant \delta_2\}}\mathbf{1}_{\{|Y(\tau\wedge S_n) - Y(\tau\wedge T)| \leqslant \delta_1\}}]\\
&\quad \leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3. \qquad \square
\end{aligned}$$

Corollary 5.2. *Let $f(x)$ be a bounded uniformly continuous function. For an arbitrary piecewise continuous function $h(t)$, let $\tau_h = \inf\{t\colon Y_t \geqslant h(t)\}$. Then for any fixed $r > 0$,*

$$\left|\sup_h \mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge S_n)} f(Y_{\tau_h\wedge S_n})] - \sup_h \mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge T)} f(Y_{\tau_h\wedge T})]\right| < \varepsilon$$

whenever (47) *holds* (*where the suprema are taken over all piecewise continuous functions $h(t)$*.

Proof. From Lemma 5.1 $\mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge S_n)} f(Y_{\tau_h\wedge S_n})] \to \mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge)T} f(Y_{\tau_h\wedge T})]$ uniformly in h as $n\to\infty$—that is, $\sup_h |\mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge S_n)} f(Y_{\tau_h\wedge S_n})] - \mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge T)} f(Y_{\tau_h\wedge T})]| \to 0$ at the same rate as (47). The result now follows from the fact that

$$\left|\sup_h \mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge S_n)} f(Y_{\tau_h\wedge S_n})] - \sup_h \mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge T)} f(Y_{\tau_h\wedge T})]\right|$$

$$\leqslant \sup_h \left|\mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge S_n)} f(Y_{\tau_h\wedge S_n})] - \mathbb{E}[\mathrm{e}^{-r(\tau_h\wedge T)} f(Y_{\tau_h\wedge T})]\right|. \qquad \square$$

Remarks. (1) The rate of convergence given by (47) depends critically on the behaviour of v near 0. For a compound Poisson process, v is a finite measure whose total masses $v(0,\infty)$ is the jump rate. In this case (47) becomes $-18T^2K^2v(0,\infty)^2\varepsilon^{-2}\log\varepsilon$ and so the rate of convergence is essentially $1/\sqrt{n}$ but with a logarithmic correction. In the case of a general Lévy process with finite variation satisfying (3), an integration by parts shows that

$$\int_\varepsilon^1 xv(\mathrm{d}x) = \varepsilon v(\varepsilon,1) + \int_\varepsilon^1 v(x,1)\,\mathrm{d}x.$$

Since the left-hand side above has a finite limit as $\varepsilon \to 0$ and both terms on the right-hand side are positive, both must have a finite limit as $\varepsilon \to 0$. Therefore, the worst-case scenario in (47) can be obtained from the integrability condition $\int_{0+} v(x,\infty)\mathrm{d}x < \infty$. For example, we must have in particular that $v(\varepsilon,\infty) \sim$ o $((\varepsilon\log\varepsilon)^{-1})$ which gives a rate of convergence better than $-\varepsilon^{-4}(\log\varepsilon)^{-1}$. On the other hand, if $v(\varepsilon,\infty) \sim -\log\varepsilon$ (which is the case for a Gamma process, for example), the rate of convergence is $-18T^2K^2\varepsilon^{-2}(\log\varepsilon)^3$—in other words, it is still essentially $1/\sqrt{n}$ but with a different logarithmic correction. If we do not make the finite variation assumption (3), we would only have $\int_0^1 x^2v(\mathrm{d}x) < \infty$ and the same integration by parts argument would give $\int_{0+} xv(x,\infty)\,\mathrm{d}x < \infty$. Thus different types of Lévy processes can give rise to very different rates of convergence. The recursive algorithm performs best with Lévy processes which have a "small" Lévy measure.

(2) Note that the rate of convergence would be much faster if Y_t did not have jumps but only a Brownian component with drift, for then instead of (51) we would be able to use (49) which would then give an exponential rate of convergence.

(3) The rate of convergence in Corollary 5.2 is stated in terms of boundary crossing times of the form described because this is the context in which Corollary 5.2 is applied to obtain convergence of the recursive algorithm presented in this section. However, the same rate of convergence holds for arbitrary stopping times.

(4) The sizes of the constants in (47) also have practical implications for the performance of the recursive algorithm. In the case of a standard American put with strike price K we have $f(t,x) = \mathrm{e}^{-rt}(K - \mathrm{e}^x)_+$ and it is easily checked that the K in (47) is the strike price and we may take $C = K(1 \wedge r)$.

(5) The assumption that f is uniformly Lipschitz is not strictly necessary; the same proof will work for a bounded uniformly continuous function. The only difference is that the rate of convergence will involve $v(\delta_1(\varepsilon),\infty)$ for some function $\delta_1(\varepsilon)$ determined by the modulus of continuity of f.

5.1. A recursive algorithm for the value of n-Erlang American puts

In the context of the recursive algorithm (44), at the ith stage, we know the optimal exercise level a_{i-1} for the $(i-1)$th stage, $V_{i-1}(u,a_{i-1})$ and its Laplace transform $V^*_{i-1}(\theta,a_{i-1})$ (start with $a_0=\log K$, $V_0(u)=\pi(u)=(K-\mathrm{e}^u)_+$ and $V_0^*(\theta,a_0)=\int_0^{a_0}\mathrm{e}^{-\theta u}\pi(u)\,\mathrm{d}u$). We take as the final payoff at the end of each stage i $f_i(z)=V_{i-1}(z-a_{i-1},a_{i-1})$ (note that this depends on a_{i-1}), so that

$$f_i^*(\theta,a)=\int_0^\infty \mathrm{e}^{-\theta u}V_{i-1}(u+a-a_{i-1},a_{i-1})\,\mathrm{d}u.$$

But at the ith stage we are looking for an exercise level $a\leqslant a_{i-1}$, so we have

$$f_i^*(\theta,a)=\int_0^{a_{i-1}-a}\mathrm{e}^{-\theta u}\pi(u+a)\,\mathrm{d}u+\mathrm{e}^{-\theta(a_{i-1}-a)}V^*_{i-1}(\theta,a_{i-1}). \tag{54}$$

To find the optimal exercise level a_i for the ith stage and $V_i^*(\theta,a_i)$, we take $f^*(\theta,a)$ as given by (54) and solve (27) to find the optimal exercise level a_i, at which point (32) gives an expression for $V_i^*(\theta,a_i)$ in terms of quantities which are all known by the ith stage. Formula (40) (with $f(z)=f_i(z)=V_{i-1}(z-a_{i-1},a_{i-1})$ and $f_i^*(\theta,a_i)$ as given by (54)) gives an expression for $V_i(u,a_i)$. We now have $V_i^*(\theta,a_i)$ and $V_i(u,a_i)$ which are needed at the $(i+1)$th stage. Of course, in order to use (40), we need to compute the resolvent density $r_\delta(x)$ for $x>0$. Although there is no explicit formula for this except in the case of only a few Lévy processes, it has been pointed out in Chan (1999b) that $r_\delta(x)$ can be readily computed using the fast Fourier transform algorithm, at least when $\sigma\neq 0$; when $\sigma=0$ the convergence of the relevant Fourier sums may be rather slow.

Alternatively, we can use a similar recursive algorithm to compute the Laplace transforms V_i^* at each stage (these are expressed solely in terms of V^*_{i-1} and π) and once V_n^* has been obtained, a single numerical Laplace inversion can be performed to obtain $V_n(u,a_n)=\mathbb{E}_{u+a_n}[\mathrm{e}^{-r(\tau_*\wedge S_n)}\pi(Y_{\tau_*\wedge S_n})]$. However, inverting a Laplace transform numerically is usually more computationally demanding than the fast Fourier transform which is needed to compute $V_n(u,a_n)$ by means of Proposition 4.3. (It should be emphasised that the fast Fourier transform is only suggested here as a suitable means of computing the resolvent density r_δ; since the Fourier transform of r_δ has a simple formula—(see Bertoin, 1996) this involves a straightforward Fourier inversion. We are not proposing here to use FFT methods to compute $V_n(u,a_n)$ directly.) On the other hand, since Corollary 3.1 (especially part(b)) gives the optimal exercise level in terms of Laplace transforms f_i^* (and hence, by virtue of (54), V^*_{i-1}), it is possible to give more explicit formulae for the Laplace transforms V_i^* at each stage, at least when $\pi(a)=(K-\mathrm{e}^a)_+$. Also, Proposition 4.3 only holds in the case $\pi(a)=(K-\mathrm{e}^a)_+$, whereas (16) and (26) hold for general π.

The Erlang put presented above essentially uses a (random) piecewise constant approximation to the optimal exercise boundary of an American put with fixed expiry. The associated approximate optimal exercise policy can therefore be described as follows. At the earliest stage (which in our notation above is actually labelled stage n), we exercise as soon as the log price Y falls below the optimal level a_n. If this does not happen before the first exponential time T_n expires, then at time T_n, we re-adjust the current exercise level to the new optimal level a_{n-1} for the next stage and exercise as soon as Y crosses this new level, and so on. Of course, because of the jumps in the piecewise constant exercise boundary, it may well happen that as soon as we switch to a new exercise level, we exercise immediately because we find that although Y_{T_i} was above the old level a_i, it is below the new level a_{i-1}.

5.2. An algorithm for the value of n-Erlang American puts via Laplace inversion

In this section, we present an algorithm for computing the value $V_n(u, a_n)$ of n-Erlang American puts via Laplace inversion. It must be emphasised that the algorithm does not need to invert a Laplace transform at each stage. Instead it computes the Laplace transform $V_n^*(\theta, a_n)$ explicitly and ends performing a single Laplace inversion.

In this section, we will denote by $L_k = \mathrm{e}^{a_k}$ the successive barriers, by $f_k^* = f_{a_k}^*(\theta)$, $V_k^* = V_k^*(\theta, a_k)$ the successive Laplace transforms of the final payoffs and value functions of the various stages and by $z_k(\theta)$ the expressions $K/\theta - L_k/(\theta - 1)$.

Since Corollary 3.2(b) yields the Laplace transform $V_k^* - z_k$ in terms of $f_k^* - z_k$, and Corollary 3.1(b) expresses the optimal level L also in terms of $f_k^* - z_k$, it will be enough to obtain an explicit formula for $f_k^* - z_k$, and this is achieved in Proposition 5.3.

Proposition 5.3. *Let* $\pi(a) = (K - \mathrm{e}^a)_+$. (a) *For* $k \geqslant 2$, *the transforms in u of the final payoff at the end of stage k satisfy the recursion*

$$f_k^*(\theta) - z_k(\theta) = \left(\frac{L_k}{L_{k-1}}\right)^{\theta} \frac{\alpha}{\delta - c(\theta)} \left(f_{k-1}^*(\theta) - z_{k-1}(\theta) - \left(\frac{rK}{\alpha\theta} - \frac{qL_{k-1}}{\alpha(\theta - 1)}\right)\right), \tag{55}$$

where L_k *are defined recursively as solutions of*

$$\frac{rK}{\alpha\delta_+} - \frac{qL_k}{\alpha(\delta_+ - 1)} = \alpha \left(\frac{L_k}{L_{k-1}}\right)^{\delta_+} \lim_{\theta \to \delta_+} \frac{(f_{k-1}^*(\theta) - z_{k-1}(\theta) - (rK/\alpha\theta - (qL_{k-1})/(\alpha(\theta - 1))))'}{\delta - c'(\theta)}. \tag{56}$$

(b) *The transform in u of the final payoff at the end of stage k is given by*

$$f_k^* - z_k = \left(\frac{\alpha}{\delta - c(\theta)}\right)^{k-1} \left(\frac{L_k}{K}\right)^{\theta} \frac{K}{\theta(\theta - 1)} - \sum_{i=1}^{k-1} \left(\frac{\alpha}{\delta - c(\theta)}\right)^{k-i} \left(\frac{L_k}{L_i}\right)^{\theta} \left(\frac{rK}{\alpha\theta} - \frac{qL_i}{\alpha(\theta - 1)}\right). \tag{57}$$

Notes: (1) The indeterminate limit in (56) yielding the optimal levels L_k requires using L'Hôpital's rule $k-1$ times. For example, for $k=2$ we find the optimality condition for the second barrier in the no dividends case to be

$$\left(\frac{L_2}{L_1}\right)^{\delta_+} = \frac{c'(\delta_+)}{\alpha(1/(\delta_+ - 1) - \log(L_1/K))}.$$

(2) It is possible to show using the smooth junction equation of Corollary 3.1(b) that expressions (56), (57) are well defined as $\theta \to \delta_+$.

Proof. (a) At the end of stage k, the barrier is switched from L_k to a higher value $L_{k-1} = L_k e^a$ where $a = \log(L_{k-1}/L_k)$. Thus, the payoff will either be an immediate exercise payoff $K - S_t$ if S_t is between the two barriers, or a continuation Canadian payoff given by (32). Thus, the Laplace transform at the k'th stage is:

$$\begin{aligned} f_k^*(\theta) &= \int_0^a e^{-\theta u}(K - L_k e^u) + e^{-\theta a} V_{k-1}^*(\theta) \\ &= \frac{K}{\theta}\left(\left(1 - \frac{L_k}{L_{k-1}}\right)\right)^{\theta} - \frac{L_k}{\theta - 1}\left(1 - \left(\frac{L_k}{L_{k-1}}\right)^{\theta-1}\right) + \left(\frac{L_k}{L_{k-1}}\right)^{\theta} V_{k-1}^*(\theta) \\ &= z_k + \left(\frac{L_k}{L_{k-1}}\right)^{\theta}(V_{k-1}^* - z_{k-1}). \end{aligned}$$

Recursion (55) follows now by plugging in $f_k^*(\theta) - z_k(\theta) = (L_k/L_{k-1})^{\theta}(V_{k-1}^*(\theta) - z_{k-1}(\theta))$ the formula for V_{k-1}^* provided in Corollary 3.2(b).

Recursion (56) for the barriers is obtained from Corollary 3.1(b) by applying (55) and L'Hôpital's limit rule.

(b) Follows by iterating the recursion of (a), starting with the expression $f_1 - z_1 = (L_1/K)^{\theta}(K/\theta(\theta-1))$ provided in Lemma 2.4. □

6. Concluding remarks

As demonstrated by this paper, the value of American options with one exercise barrier may be approximated analytically under a spectrally one-sided (negative or positive) exponential Lévy model. It seems quite interesting to investigate whether this is still possible for the various spectrally two-sided Lévy models recently proposed by Eberlein, Madan and Barndorf Nielsen, and for problems with two barriers.

Also, since the first step, the Canadian approximation was found to already have an exercise boundary which is exact asymptotically, it seems interesting to compare it with other similar one step approximations, based on "the early exercise decomposition" approach.

This approach replaces the "nonhomogeneous" equation (13) by an "homogeneous" equation with no final payoff f, by subtracting out of V the "Canadian European" value of the final payoff

$$\tilde{f}(y) = \mathbb{E}_y[e^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}})] \tag{58}$$

(which satisfies the same nonhomogeneous equation as V). Thus, the difference $\tilde{V} = V - \tilde{f}$, called *early exercise premium*, satisfies the homogeneous equation:

$$(\Gamma\tilde{V})(y) - \delta\tilde{V}(y) = 0,$$
$$\tilde{V}(y) = \tilde{\pi}(y) \quad \text{for } y \leqslant a,$$
$$\tilde{V}(\infty) = 0, \tag{59}$$

where $\tilde{\pi}(y) = \pi(y) - \tilde{f}(y)$. In the final approximation we would replace back the "Canadian European" value $\tilde{f}(y)$ by the exact European value, and add to it the approximate early exercise premium $\tilde{V}$. This idea has appeared in several variations in the literature: for example, the approach taken in McMillan (1986), Zhang (1998), Chan (1999b) and Gerber and Shiu (1998) all result in the same type of equation as (59). McMillan (1986) proposed approximating the "early exercise" premium as $\tilde{v}(t,y) \approx h(t)\tilde{v}(y)$, where $h(t) = 1 - \mathrm{e}^{-r(T-t)}$ was judiciously chosen to combine the terms $-rh + h'$ in the resulting equation and to equal 0 at expiration. This leads again to equation (59) where $\delta = r/(1 - \mathrm{e}^{-rT})$ and $\tilde{f}$ is again the initial European value given by $\tilde{f}(T,y) = \mathbb{E}_y[\mathrm{e}^{-rT} f(Y_T)]$. We note that in the classical Brownian case studied by McMillan, the ODE (59) has for fixed a the explicit solution $\tilde{V} = \tilde{\pi}(a)\mathrm{e}^{\theta(y-a)}$, where θ is the negative root of the quadratic equation $\sigma^2/2\theta^2 + \mu\theta - \delta = 0$. Hence, the optimal exercise barrier is found quite easily by maximization with respect to a, leading to the nonlinear equation $\tilde{\pi}'(a) = \theta\tilde{\pi}(a)$. Zhang (1998) proposed the "one-step discretization" approximation for the "early exercise" premium. For an interval of size T, this leads to Eq. (59) with $\delta = r + 1/T$ and $\tilde{f}$ being the European value given by $\tilde{f}(T,y) = \mathbb{E}_y[\mathrm{e}^{-rT} f(Y_T)]$. It may be interesting to compare in further work the Canadian approximation with these methods which use the exact European value and only approximate the early exercise part.

Acknowledgements

We thank S. Asmussen and M. Pistorius for valuable comments. We are also grateful to the anonymous referee whose comments have helped to eliminate many inaccuracies and points of confusion from earlier drafts of this paper.

Appendix A. The value of Canadian put options under the compound Poisson model with exponential jumps

In this section we assume that Y_t is compound Poisson with exponential jumps, i.e. we assume that $\sigma = 0$ and $J_t = \sum_{i=1}^{N_t} Z_i$, where N_t is a Poisson process with rate λ and Z_i have exponential density $\beta\mathrm{e}^{-\beta x}$.

Let now $a = \log L$ and $k = \log K$ denote the exercise barrier and strike price of a put option on the logarithmic scale, with $a < k$. Since $V(y) = K - \mathrm{e}^y$ for $y \leqslant a$, we find that the differential equation for the value of Canadian options (13) becomes in the put case

$$(GV)(y) - \delta V(y) + \lambda \int_{y-a}^{\infty} (K - \mathrm{e}^{y-z})p(z)\,\mathrm{d}z + \alpha(K - \mathrm{e}^{y})_{+}$$

$$= (GV)(y) - \delta V(y) + \lambda(K\bar{P}(y-a) - Lp^{*}_{y-a}(1)) + \alpha(K - \mathrm{e}^{y})_{+} = 0 \quad \text{for } y \geqslant a,$$

$$V(y) = K - \mathrm{e}^{y} \quad \text{for } y \leqslant a, \tag{A.1}$$

where $(GV)(y) = \mu V'(y) + \lambda \int_0^{y-a} [V(y-z)]\, p(z)\,\mathrm{d}z - \lambda V(y)$ and V is differentiable over (a,k) and (k,∞) and continuous at k.

In the exponential case we have $p^{*}_{u}(\theta) = p(u)/(\theta+\beta)$, and so (A.1) becomes

$$(\bar{G}\bar{V})(y) - \delta\bar{V}(y) + \lambda \int_{y-k}^{y-a} V(y-z)p(z)\,\mathrm{d}z$$

$$+ \lambda p(y-a)\left(\frac{K}{\beta} - \frac{L}{\beta+1}\right) = 0 \quad \text{for } y \geqslant k, \tag{A.2}$$

$$(GV)(y) - \delta V(y) + \lambda p(y-a)\left(\frac{K}{\beta} - \frac{L}{\beta+1}\right)$$

$$+ \alpha(K - \mathrm{e}^{y}) = 0 \quad \text{for } a \leqslant y \leqslant k \tag{A.3}$$

$$\tilde{V}(y) = K - \mathrm{e}^{y} \quad \text{for } y \leqslant a,$$

where $\bar{V}, V, \tilde{V}$ denote the value of the option in the three ranges and $\bar{G}$ is the operator $(\bar{G}f)(y) = \mu f'(y) + \lambda \int_0^{y-k} [f(y-z)]p(z)\,\mathrm{d}z - \lambda f(y)$.

Using the results of the previous section, we may check that the solution of (A.1) is a combination of exponential functions. However, once the right form of the solution is known it becomes simpler to determine it directly, using the easily checked:

Lemma A.1. *The operator G acts on exponential functions as*

$$G\mathrm{e}^{\theta y} = c(\theta)\mathrm{e}^{\theta y} - \mathrm{e}^{\theta a}\lambda p^{*}_{y-a}(\theta). \tag{A.4}$$

The equation $c(\theta) = \mu\theta + \lambda\beta/(\theta+\beta) - \lambda = \mu\theta - \lambda\theta/(\theta+\beta) = \delta$ becomes in this case: $\mu\theta^{2} + \theta(\mu\beta - (\lambda+\delta)) - \delta\beta$. Let δ_{+}, δ_{-} denote the positive and negative roots of this equation. [6]

The structure of the Eqs. (A.2) forces us to start with the middle case $a \leqslant y \leqslant k$, for which we guess a solution of the form $V(y) = k_1\mathrm{e}^{\theta_1 y} + k_2\mathrm{e}^{\theta_1 y} + A_1 + A_2\mathrm{e}^{y}$ (the two undetermined exponents end up being the two roots of the equation $c(\theta) = \delta$). Plugging in Eq. (A.3) and using the fact that

$$G\mathrm{e}^{\theta y} = \mathrm{e}^{\theta y}c(\theta) - \frac{\lambda p(y-a)\mathrm{e}^{\theta a}}{\beta+\theta} \tag{A.5}$$

[6] Including the diffusion term leads to a similar analysis, the only difference being that the equation $c(\theta) - \delta$ will be of third order.

yields

$$k_1\left(e^{\theta_1 y}(c(\theta_1)-\delta)-\frac{\lambda p(y-a)e^{\theta_1 a}}{\beta+\theta_1}\right)+k_2\left(e^{\theta_2 y}(c(\theta_2)-\delta)-\frac{\lambda p(y-a)e^{\theta_2 a}}{\beta+\theta_2}\right)$$
$$+A_1\left((c(0)-\delta)-\frac{\lambda p(y-a)}{\beta}\right)+A_2\left(e^{y}(c(1)-\delta)-\frac{\lambda p(y-a)e^{a}}{\beta+1}\right)$$
$$+\lambda p(y-a)\left(\frac{K}{\beta}-\frac{L}{\beta+1}\right)+\alpha(K-e^{y})=0.$$

Using $c(1)=r-q$ and $c(0)=0$ and putting $\theta_1=\delta_+,\ \theta_2=\delta_-$ we find that $A_1=K(\alpha/(\alpha+r))$ and $A_2=-\alpha/(\alpha+q)$ and k_1,k_2 satisfy

$$\frac{k_1e^{\delta_+a}}{\beta+\delta_+}+\frac{k_2e^{\delta_-a}}{\beta+\delta_-}=\left(\frac{Kr}{\beta(\alpha+r)}-\frac{Lq}{(\beta+1)(\alpha+q)}\right).$$

For the case $y\geqslant k$, we guess a solution of the form $\bar V(y)=\bar k e^{\theta y}$ with $\theta=\delta_-$ (since this domain is unbounded and the solution has to be bounded as $y\to\infty$, θ must be negative). Substituting this into (A.2) and using (A.5) and $\int_{y-k}^{y-a}Ce^{\theta(y-z)}\beta e^{-\beta z}\,dz=\frac{C}{\beta+\theta}(e^{\theta k}p(y-k)-e^{\theta a}p(y-a))$ yields

$$\bar k\left(e^{\theta y}(c(\theta)-\delta)-\frac{\lambda p(y-k)e^{\theta k}}{\beta+\theta}\right)+\lambda\int_{y-k}^{y-a}V(y-z)p(z)\,dz$$
$$+\lambda p(y-a)\left(\frac{K}{\beta}-\frac{L}{\beta+1}\right)$$
$$=-\bar k\frac{\lambda p(y-k)e^{\delta_-k}}{\beta+\delta_-}+\lambda\beta\int_{y-k}^{y-a}(k_1e^{\delta_+y-(\delta_++\beta)z}$$
$$+(k_2e^{\delta_-y-(\delta_-+\beta)z}+A_1e^{-\beta z}+A_2e^{y-(\beta+1)z})\,dz+\lambda p(y-a)\left(\frac{K}{\beta}-\frac{L}{\beta+1}\right)$$
$$=\lambda\left\{-\bar k\frac{p(y-k)e^{\delta_-k}}{\beta+\delta_-}+\frac{k_1}{\delta_++\beta}(p(y-k)e^{\delta_+k}-p(y-a)e^{\delta_+a})\right.$$
$$+\frac{k_2}{\delta_-+\beta}(p(y-k)e^{\delta_-k}-p(y-a)e^{\delta_-a})+\frac{A_1}{\beta}(p(y-k)-p(y-a))$$
$$\left.+\frac{A_2}{\beta+1}(p(y-k)e^{k}-p(y-a)e^{a})+p(y-a)\left(\frac{K}{\beta}-\frac{L}{\beta+1}\right)\right\}=0.$$

The constants k_1 and k_2 have been chosen so that all the terms involving $p(y-a)$ in the above expression cancel. We must therefore choose $\bar k$ so that

$$\frac{\bar k e^{\delta_-k}}{\beta+\delta_-}=\frac{k_1e^{\delta_+k}}{\beta+\delta_+}+\frac{k_2e^{\delta_-k}}{\beta+\delta_-}+\frac{A_1}{\beta}+\frac{A_2e^{k}}{\beta+1}$$
$$=\frac{k_1e^{\delta_+k}}{\beta+\delta_+}+\frac{k_2e^{\delta_-k}}{\beta+\delta_-}+\frac{K\alpha}{\beta(\alpha+r)}-\frac{K\alpha}{(\beta+1)(\alpha+q)}.$$

Finally, using the continuity at k we find

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} + \frac{K\alpha(q-r)}{(\alpha+r)(\alpha+q)} = \bar{k} e^{\delta_- k}.$$

We have now proved the following result.

Proposition A.2. *The value of a Canadian put option on a compound Poisson process with exponential jumps is given by*

$$V(y) = \begin{cases} k_1 e^{\delta_+ y} + k_2 e^{\delta_- y} + \frac{K\alpha}{\alpha+r} - \frac{\alpha}{\alpha+q} e^y & \text{for } a \leqslant y \leqslant k, \\ \bar{k} e^{\delta_- y} & \text{for } y \geqslant k, \end{cases} \tag{65}$$

where

$$\frac{k_1 e^{\delta_+ a}}{\beta+\delta_+} + \frac{k_2 e^{\delta_- a}}{\beta+\delta_-} = \left(\frac{Kr}{\beta(\alpha+r)} - \frac{Lq}{(\beta+1)(\alpha+q)} \right),$$

$$\frac{k_1 e^{\delta_+ k}}{\beta+\delta_+} + \frac{k_2 e^{\delta_- k}}{\beta+\delta_-} = \frac{\bar{k} e^{\delta_- k}}{\beta+\delta_-} - \frac{K\alpha}{\beta(\alpha+r)} + \frac{K\alpha}{(\beta+1)(\alpha+q)},$$

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k} e^{\delta_- k} + \frac{K\alpha(r-q)}{(\alpha+r)(\alpha+q)}.$$

We compare now the results for exponential jumps with those obtained by approximating our Lévy process by a Brownian motion of equal variance $\sigma^2 = 2\lambda\beta^{-2}$ (and $\mu = \frac{\sigma^2}{2}(\beta/(\beta+1))^2(\beta+1)+r-q)$, where $\beta \to \infty$. In the limit, δ_+, δ_- become the positive and negative root of the quadratic equation $(\sigma^2/2)\theta^2 + (r-q-\sigma^2/2)\theta - (r+\alpha) = 0$. We multiply the first equation by β and replace the second equation in the system above by the third minus β times the second

$$\frac{k_1 e^{\delta_+ a}\beta}{\beta+\delta_+} + \frac{k_2 e^{\delta_- a}\beta}{\beta+\delta_-} = \left(\frac{Kr}{(\alpha+r)} - \frac{Lq\beta}{(\beta+1)(\alpha+q)} \right),$$

$$\frac{k_1 e^{\delta_+ k}\delta_+}{\beta+\delta_+} + \frac{k_2 e^{\delta_- k}\delta_-}{\beta+\delta_-} = \frac{\bar{k} e^{\delta_- k}\delta_-}{\beta+\delta_-} + \frac{K\alpha}{(\beta+1)(\alpha+q)},$$

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k} e^{\delta_- k} + \frac{K\alpha(r-q)}{(\alpha+r)(\alpha+q)}.$$

Finally, multiplying the second equation by β and taking the limit $\beta \to \infty$ yields

$$k_1 e^{\delta_+ a} + k_2 e^{\delta_- a} = \left(\frac{Kr}{(\alpha+r)} - \frac{Lq}{\alpha+q} \right),$$

$$k_1 e^{\delta_+ k}\delta_+ + k_2 e^{\delta_- k}\delta_- = \bar{k} e^{\delta_- k}\delta_- + \frac{K\alpha}{\alpha+q},$$

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k} e^{\delta_- k} + \frac{K\alpha(r-q)}{(\alpha+r)(\alpha+q)}$$

If $q = 0$, this becomes

$$k_1 e^{\delta_+ a} + k_2 e^{\delta_- a} = \frac{Kr}{\alpha + r},$$

$$k_1 \delta_+ e^{\delta_+ k} + k_2 \delta_- e^{\delta_- k} - K = \bar{k} \delta_- e^{\delta_- k},$$

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k} e^{\delta_- k} + \frac{Kr}{\alpha + r}$$

which agrees with the results of Carr (1998) when the boundary L is optimized.

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