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Instituto sobre Desarrollo Empresarial
Carmen Vidal Ballester.
Universidad Carlos III de Madrid
C/ Madrid, 126
28903 Getafe Madrid (Spain)
FAX (34-91)6249607

CAPM-like formulae and good deal absence with ambiguous setting and coherent risk measure

Alejandro Balbás[1]
Departamento de Economía e la Empresa
Universidad Carlos III de Madrid

Beatriz Balbás[2]
Departamento de Análisis Económico y Finanzas
Universidad de Castilla-La Mancha

Raquel Balbás[3]
Departamento Actuarial y Financiero
Universidad Complutense de Madrid

[1] C/ Madrid, 126. 28903 Getafe (Madrid, Spain).

[2] Avda. Real Fábrica de Seda, s/n. 45600 Talavera (Toledo, Spain).

[3] Somosaguas-Campus. 28223 Pozuelo de Alarcón (Madrid, Spain).

$CAPM-$like formulae and good deal absence with ambiguous setting and coherent risk measure

Alejandro Balbás,[*] Beatriz Balbás,[†] and Raquel Balbás. [‡]

Abstract. Risk measures beyond the variance have shown theoretical advantages when addressing some classical problems of Financial Economics, at least if asymmetries and/or heavy tails are involved. Nevertheless, in portfolio selection they have provoked several caveats such as the existence of good deals in most of the arbitrage free pricing models. In other words, models such as Black and Scholes or Heston allow investors to build sequences of strategies whose expected return tends to infinite and whose risk remains bounded or tends to minus infinite. This paper studies whether this drawback still holds if the investor is facing the presence of multiple priors, as well as the properties of optimal portfolios in a good deal free ambiguous framework.

With respect to the first objective, we show that there are four possible results. If the investor uncertainty is too high he/she has no incentives to buy risky assets. As the uncertainty (set of priors) decreases the interest in risky securities increases. If her/his uncertainty becomes too low then two types of good deal may arise. Consequently, there is a very important difference between the ambiguous and the non ambiguous setting. Under ambiguity the investor uncertainty may increase in such a manner that the model becomes good deal free and presents a market price of risk as close as possible to that reflected by the investor empirical evidence. Hence, ambiguity may help to overcome some meaningless findings in asset pricing.

With respect to our second objective, good deal free ambiguous models imply the existence of a benchmark generating a robust capital market line. The robust (worst-case) risk of every strategy may be divided into systemic and specific, and no robust return is paid by the specific robust risk. A couple of "betas" may be associated with every strategy, and extensions of the $CAPM$ most important formulas will be proved.

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[*]University Carlos III of Madrid. CL. Madrid 126. 28903 Getafe (Madrid, Spain). alejandro.balbas@uc3m.es

[†]University of Castilla la Mancha. Avda. Real Fábrica de Seda, s/n. 45600 Talavera (Toledo, Spain). beatriz.balbas@uclm.es

[‡]University Complutense of Madrid. Department of Actuarial and Financial Economics. Somosaguas-Campus. 28223 Pozuelo de Alarcón (Madrid, Spain). raquel.balbas@ccee.ucm.es

1. Introduction

The study of risk measures beyond the standard deviation has a long history in finance. Since Artzner *et al.* (1999) introduced the coherent measures of risk and gave a new impulse to this topic, many authors have further extended the discussion. Among many others, Goovaerts *et al.* (2004) introduced the consistent risk measures, Rockafellar *et al.* (2006) defined the expectation bounded risk measures, Zhiping and Wang (2008) defined the two-sided coherent risk measures, Brown and Sim (2009) defined the satisfying measures, and Foster and Hart (2009) provided an operational measure of riskiness. This growing interest is mainly due to two reasons: Firstly, it is not always possible to establish a clear relationship between the return variance and potential capital losses (or capital requirements), and secondly, the presence of asymmetric returns implies that the variance becomes non compatible with the second order stochastic dominance and many utility functions (Ogryczak and Ruszczynski, 2002).

Many classical financial topics have been revisited using risk measures. With respect to portfolio choice problems, among many others, Zhiping and Wang (2008) present a very general analysis, and Bali *et al.* (2011) point out that the empirical results indicate that some generalized measures of riskiness are very appropriate to rank equity portfolios based on their expected returns per unit of risk. Besides, Balbás *et al.* (2010a) prove the existence of caveats affecting the most important arbitrage free and complete pricing models of Financial Economics. For instance, for the Black and Scholes model, and for every coherent and expectation bounded measure of risk, investors can build portfolios with the intended expected return (as large as desired) whose risk is lower than zero. This pathological finding also holds for the same model and some risk measures that are not expectation bounded or coherent, such as the absolute deviation, in which case the risk level is as close to zero as desired, or the value at risk (VaR), in which case the risk level is as negative as desired, *i.e.*, the expected return tends to ∞ while the VaR tends to $-\infty$.

The presence of agents with multiple priors is usual in finance and affects many pricing and equilibrium problems as well as investment decisions. It has been also generating a growing interest, as demonstrated by the number of recent papers focusing on this issue. For instance, Epstein and Schneider (2008) study the existence of ambiguity premia, Riedel (2009) deals with the valuation of American style derivatives in an ambiguous framework, and Bossaerts *et al.* (2010) study the impact of ambiguity and ambiguity aversion on equilibrium asset prices. With regard to portfolio selection problems, Garlappi *et al.* (2007) deal with ambiguous expected returns and empirically show that, compared with portfolios from classical and Bayesian models, ambiguity aversion leads to optimal solutions that are more stable over time and deliver a higher out-of sample Sharpe ratios. In this sense, the introduction of uncertainty may be more than realistic. It could also be interesting to many traders.

Recent literature has also focused on portfolio choice with both ambiguity and risk measures beyond the variance. For instance, Calafiore (2007) provides algorithms if the risk measure is the absolute deviation, Schied (2007) establishes duality linked optimality conditions, and Zhu and Fukushima (2009) yield algorithms to minimize the worst-case (or robust) conditional value at risk ($CVaR$) if the set of priors satisfies some required assumptions.

This paper attempts to address several open problems arising from the discussion above. Following Balbás *et al.* (2010*a*), who dealt with a coherent and expectation bounded risk measure but did not consider any ambiguous framework, we analyze the existence of benchmarks generating a robust capital market line for every ambiguity level (or every set of priors) and risk measure, as well as the possibility to explain the robust (or worst-case) expected return of every available security by using this benchmark and a new notion of robust systemic risk. Furthermore, a second major objective of this paper is to investigate whether the economic meaningless equality $(risk, return) = (0, \infty)$ above may still hold when there are multiple priors. If this pathological property holds in an arbitrage free market, we will say that there are robust good deals.[1] Besides, it seems that we consider the most general case because ambiguity may affect both the set of states of nature and the probabilities of the states. Obviously, this framework contains more restricted approaches dealing with ambiguity with respect to volatilities, expected returns, price processes, etc.

The paper's outline is as follows. Section 2 is devoted to summarizing the basic notions and notations, as well as some background. We will focus on a discrete set of states of the world and the $CVaR$ as the risk measure. We selected the $CVaR$ because it has been becoming very popular among researchers, practitioners, supervisors and regulators. It is compatible with the second order stochastic dominance and the usual utility functions (Ogryczak and Ruszczynski, 2002) and appropriately overcomes several shortcomings of other risk measures when dealing with asymmetries and heavy tails (Agarwal and Naik, 2004). Choosing the $CVaR$ in our analysis, we simplify the paper's exposition. However, it is worth pointing out that the extension for a general coherent and expectation bounded risk measure is straightforward.

The portfolio selection problem is introduced in Section 3. As usual in finance, ambiguity aversion is incorporated by dealing with the worst-case principle, as justified in Gilboa and Schmeidler (1989). Though there are alternative methods also

[1] The notion of "good deal" was introduced in Cochrane and Saa-Requejo (2000). Mainly, a good deal was an investment strategy providing traders with a "very high return/risk ratio", in comparison with the market portfolio. Risk is measured with the standard deviation, and the absence of good deal is imposed in an arbitrage free model so as to price in incomplete markets.

In this paper we deal with alternative risk measures and identify the concept of good deal with a strategy yielding an infinite return/risk ratio.

consistent with the famous Ellsberg paradox (Maccheroni *et al.*, 2006), most of the authors adopt a worst-case approach; For instance, all of the ambiguity-linked papers cited above.

Our portfolio choice problem minimizes the robust (worst-case) risk for every robust expected return. Though this is not a linear problem, we extend the approach of Balbás *et al.* (2010*b*) in risk minimization so as to involve ambiguity too. Consequently, we find a linear dual problem that characterizes the primal solutions. Theorem 5 is the most important result of Section 3, since it provides Karush-Kuhn-Tucker-like necessary and sufficient optimality conditions for both the portfolio selection problem and its dual.[2]

In Section 4 we draw on the $KKT - like$ conditions in order to address our two major objectives: The existence of robust good deals and the existence of appropriate benchmarks under the good deal absence. The Remark of Theorem 6 clarifies the first point. Four disjoint and complementary results may arise. Firstly, if the investor reflects a high level of uncertainty (the set of priors is large and contains the risk neutral probability measure of the pricing model), then the market is risk neutral for this investor. This means that there are no worst-case expected returns higher than the risk free rate, and therefore the investor has no incentives to buy risky assets. This result seems to be consistent with the theoretical and empirical findings of Cao *et al.* (2005) and Bossaerts *et al.* (2010). Agents who are sufficiently ambiguity averse find open sets of prices for which they refuse to hold an ambiguous portfolio or choose not to participate in the stock market. Secondly, if the set of priors of the investor is lower, then he/she can find a benchmark that generates a robust capital market line when combined with the riskless security. This market line provides a robust market price of risk and the relationship between the worst-case expected return and the worst-case risk level, as well as the systemic and specific robust risk of every asset. Furthermore, if there is no ambiguity with respect to the states of nature, and only their probabilities are uncertain, then for every available security (or portfolio) the investor may measure two "betas" (regression coefficients) with respect to the benchmark, which allows her/him to give lower and upper bounds for the worst-case expected return. This is an obvious extension of the classical $CAPM$ formulae (Theorem 9 and Corollary 10) holding for ambiguous settings and coherent and expectation bounded risk measures. For efficient strategies both betas become identical, and so do the lower and upper bound above.

The third and fourth cases in the Remark of Theorem 6 lead to the existence of good deal. If the investor uncertainty is "too low" then the risk neutral probability of the market may be in the frontier or outside a set of probabilities constructed from the

[2] Actually, the given $KKT - like$ conditions are not exactly the same as the standard KKT conditions of the problem, and that is the reason why we say "$KKT - like$".

set of priors. If it is in the frontier then the couple $(robust\_risk, robust\_return) = (0, \infty)$ is available to the trader, and we will say that there are robust good deals of the first type. If it is beyond the frontier then $(robust\_risk, robust\_return) = (-\infty, \infty)$ is available, and there are robust good deals of the second type.

In Section 5 we attempt to overcome the presence of good deals of any type. The main results are Theorem 12 and its Remark. They show that there is a very important difference between the ambiguous and the non ambiguous setting. At least for complete markets, under ambiguity, if we increment the set of priors, then the absence of good deal is guaranteed. Furthermore, the enlargement of the degree of ambiguity may be done in such a manner that the new robust capital market line and the new robust market price of risk may be as close as possible to those reflected by the empirical evidence of the investor. In this sense, the introduction of ambiguity may overcome several caveats of many important arbitrage free pricing models of Financial Economics, such as Black and Scholes, Heston, etc. By adding ambiguity these models stop reflecting the presence of good deals when risks are measured by coherent and expectation bounded risk measures.

In Section 6 we switch from the $CVaR$ to the variance. Though this is not a suitable risk measure if there are asymmetries, it plays a crucial role in many topics of Financial Economics, so it may be worth analyzing its properties in a portfolio selection problem with ambiguity. We will see that the $CAPM-$linked properties still hold, but the absence of good deal will be now guaranteed. Nevertheless, good deals for alternative dispersion measures, such as the absolute deviation, might also arise in uncertain settings.

The last section of the paper summarizes the most important conclusions.

2. Preliminaries, notations, and the robust CVaR

Consider a finite set of states of nature $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ that may arise at a future date T, and the convex and compact set

$$\mathcal{P} = \left\{ p = (p_1, p_2, ..., p_n) \in \mathbb{R}^n ; \sum_{j=1}^{n} p_j = 1, \ p_j \geq 0 \ for \ \ j = 1, 2, ..., n \right\}$$

of probability measures on Ω. Consider $0 < \mu_0 < 1$ and $p = (p_1, p_2, ..., p_n) \in \mathcal{P}$. As usual, the conditional value at risk relative to $\{p\}$ with confidence level μ_0

$$\begin{gathered} CVaR_{(p,\mu_0)} : \mathbb{R}^n \longrightarrow \mathbb{R} \\ \mathbb{R}^n \ni y \longrightarrow CVaR_{(p,\mu_0)}(y) \end{gathered}$$

is given by

$$CVaR_{(p,\mu_0)}(y) = Max \ \left\{ -\sum_{j=1}^{n} p_j y_j z_j : z = (z_1, z_2, ..., z_n) \in \Delta_{(p,\mu_0)} \right\}, \tag{1}$$

where the sub-gradient $\Delta_{(p,\mu_0)}$ of $CVaR_{(p,\mu_0)}$ is given by

$$\Delta_{(p,\mu_0)} = \left\{ (z_1, z_2, ..., z_n) \in \mathbb{R}^n; \sum_{j=1}^{n} p_j z_j = 1,\ \frac{1}{1-\mu_0} \geq z_j \geq 0\ for\ \ j = 1, 2, ..., n \right\}. \tag{2}$$

Obviously, $\Delta_{(p,\mu_0)}$ is a convex and compact set, and therefore the maximum in (1) is attained for every $y \in \mathbb{R}^n$. Let $\mathcal{K} \subset \mathcal{P}$ be a convex closed (and therefore compact) set. $\mathcal{K}$ will represent the set of priors of a given investor. For $p \in \mathcal{K}$ we do not impose the constraint $p_j > 0,\ j = 1, 2, ..., n$, so the investor ambiguity may be beyond the probabilities of the states of nature. He/she may also reflect ambiguity with respect to the set of states Ω.

We will introduce the ambiguous or robust $CVaR$ with confidence level μ_0 relative to $\mathcal{K}$ below ($RCVaR_{(\mathcal{K},\mu_0)}$) by using the $RCVaR_{(\mathcal{K},\mu_0)}$ sub-gradient, given by

$$\tilde{\nabla}_{(\mathcal{K},\mu_0)} = \left\{ (p_1 z_1, p_2 z_2, ..., p_n z_n) \in \mathbb{R}^n; p \in \mathcal{K}\ and\ \ z \in \Delta_{(p,\mu_0)} \right\}. \tag{3}$$

We will also consider the convex hull $\nabla_{(\mathcal{K},\mu_0)} = Co\left(\tilde{\nabla}_{(\mathcal{K},\mu_0)}\right)$.

Proposition 1. *With the notations above, we have:*

a) $\tilde{\nabla}_{(\mathcal{K},\mu_0)} \subset \mathbb{R}^n$ *and its convex hull* $\nabla_{(\mathcal{K},\mu_0)}$ *are convex and compact.*

b) $\mathcal{K} \subset \tilde{\nabla}_{(\mathcal{K},\mu_0)} \subset \nabla_{(\mathcal{K},\mu_0)} \subset \mathcal{P}$.

c) The function $RCVaR_{(\mathcal{K},\mu_0)} : \mathbb{R}^n \longrightarrow \mathbb{R}$ *given by*

$$\begin{aligned} &RCVaR_{(\mathcal{K},\mu_0)}(y) = Max\ \left\{ -\sum\nolimits_{j=1}^{n} \xi_j y_j;\ \xi = (\xi_1, \xi_2, ..., \xi_n) \in \tilde{\nabla}_{(\mathcal{K},\mu_0)} \right\} \\ &= Max\ \left\{ -\sum\nolimits_{j=1}^{n} \xi_j y_j;\ \xi = (\xi_1, \xi_2, ..., \xi_n) \in \nabla_{(\mathcal{K},\mu_0)} \right\} \\ &= Max\ \left\{ CVaR_{(p,\mu_0)}(y);\ p \in \mathcal{K} \right\} \end{aligned} \tag{4}$$

is well defined for every $y \in \mathbb{R}^n$.

d) $RCVaR_{(\mathcal{K},\mu_0)}$ *is translation invariant, homogeneous, sub-additive (and therefore convex) and decreasing, i.e.,*

$$RCVaR_{(\mathcal{K},\mu_0)}(y + k(1, 1, ..., 1)) = RCVaR_{(\mathcal{K},\mu_0)}(y) - k \tag{5}$$

for every $y \in \mathbb{R}^n$ *and* $k \in \mathbb{R}$,

$$RCVaR_{(\mathcal{K},\mu_0)}(\alpha y) = \alpha RCVaR_{(\mathcal{K},\mu_0)}(y) \tag{6}$$

for every $y \in \mathbb{R}^n$ *and* $\alpha \geq 0$,

$$RCVaR_{(\mathcal{K},\mu_0)}(y_1 + y_2) \leq RCVaR_{(\mathcal{K},\mu_0)}(y_1) + RCVaR_{(\mathcal{K},\mu_0)}(y_2) \tag{7}$$

for every $y_1, y_2 \in \mathbb{R}^n$, *and*

$$RCVaR_{(\mathcal{K},\mu_0)}(y_2) \leq RCVaR_{(\mathcal{K},\mu_0)}(y_1) \tag{8}$$

for every $y_1, y_2 \in \mathbb{R}^n$ *with* $y_2 \geq y_1$.[3]

Proof. a) Let us prove that every sequence

$$\{(p_{m,1}z_{m,1}, p_{m,2}z_{m,2}, ..., p_{m,n}z_{m,n})\}_{m=1}^{\infty} \subset \tilde{\nabla}_{(\mathcal{K},\mu_0)}$$

has a convergent sub-sequence whose limit belongs to $\tilde{\nabla}_{(\mathcal{K},\mu_0)}$, and therefore this set will be compact. Since

$$\{(p_{m,1}, p_{m,2}, ..., p_{m,n})\}_{m=1}^{\infty} \subset \mathcal{K}$$

and $\mathcal{K}$ is compact, there exists a sub-sequence that converges to $p_0 = (p_{0,1}, p_{0,2}, ..., p_{0,n}) \in \mathcal{K}$. Without loss of generality we can represent the sub-sequence the same as the initial sequence. Since (2) and (3) show that

$$\{(z_{m,1}, z_{m,2}, ..., z_{m,n})\}_{m=1}^{\infty} \subset \left[0, \frac{1}{1-\mu_0}\right]^n$$

and this set is compact too, there exists a sub-sequence converging to

$$z_0 = (z_{0,1}, z_{0,2}, ..., z_{0,n}) \in \left[0, \frac{1}{1-\mu_0}\right]^n.$$

It only remains to show that

$$(p_{0,1}z_{0,1}, p_{0,2}z_{0,2}, ..., p_{0,n}z_{0,n}) \in \tilde{\nabla}_{(\mathcal{K},\mu_0)},$$

which will be obvious if $z_0 \in \Delta_{(p,\mu_0)}$. Clearly,

$$\sum_{j=1}^{n} p_{0,j}z_{0,j} = Lim_{m\to\infty}\left(\sum_{j=1}^{n} p_{m,j}z_{m,j}\right) = 1,$$

because $(z_{m,1}, z_{m,2}, ..., z_{m,n}) \in \Delta_{(p_m,\mu_0)}$ for every $m \in \mathbb{N}$.

With respect to $\nabla_{(\mathcal{K},\mu_0)}$, this set is compact because in $\mathbb{R}^n$ the convex hull of every compact set remains compact.

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[3] According to the definition by Artzner *et al.* (1999), the fulfillment of (5), (6), (7) and (8) implies that $RCVaR_{(\mathcal{K},\mu_0)}$ is a coherent measure of risk.

b) If $p = (p_1, p_2, ..., p_n) \in \mathcal{K}$ then (2) shows that $z = (1, 1, ..., 1) \in \Delta_{(p,\mu_0)}$, so

$$p = (p_1 z_1, p_2 z_2, ..., p_n z_n) \in \tilde{\nabla}_{(\mathcal{K},\mu_0)}.$$

Besides, $\tilde{\nabla}_{(\mathbb{P},\mu_0)} \subset \mathcal{P}$ obviously follows from (2) and (3), and then $\nabla_{(\mathcal{K},\mu_0)} \subset \mathcal{P}$ because $\mathcal{P}$ is convex.

The first maximum in (4) is attainable because $\tilde{\nabla}_{(\mathcal{K},\mu_0)}$ is compact, and the remaining equalities are trivial.

d) The result trivially follows from (4) and bearing in mind that $CVaR_{(p,\mu_0)}$ is translation invariant, homogeneous and sub-additive for every $p \in \mathcal{K}$. $\square$

Suppose that $Y \subset \mathbb{R}^n$ is a linear manifold of reachable pay-offs. Every $y \in Y$ has a current price $\pi(y) \in \mathbb{R}$, and the pricing rule

$$\begin{aligned} \pi : Y &\longrightarrow \mathbb{R} \\ Y \ni y &\longrightarrow \pi(y) \end{aligned}$$

is linear. We will assume that there exists a riskless asset, so

$$(1, 1, ..., 1) \in Y$$

and its price will be denoted by

$$\pi(1, 1, ..., 1) = e^{-r_f T} \tag{9}$$

r_f denoting the risk-free rate. Being π linear, the Riesz representation theorem guarantees the existence of a unique $y_\pi \in Y$ such that

$$\pi(y) = e^{-r_f T} y_\pi y \tag{10}$$

holds for every $y \in Y$, where products in $\mathbb{R}^n$ are usual inner products. We will also assume that there are no arbitrage opportunities, so

$$\pi(y_2) \geq \pi(y_1) \tag{11}$$

for every $y_1, y_2 \in Y$ with $y_2 \geq y_1$.

Henceforth, the linear function $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by

$$L(y) = (y_1, y_2, ..., y_n)(1, 1, ..., 1) = \sum_{j=1}^{n} y_j \tag{12}$$

for every $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ will play an important role. Obviously, $L(p) = 1$ for every $p \in \mathcal{P}$, so the latter proposition guarantees that the equality also holds for $p \in \tilde{\nabla}_{(\mathcal{K},\mu_0)}$ and $p \in \nabla_{(\mathcal{K},\mu_0)}$.

As usual, the market is said to be complete if $Y = \mathbb{R}^n$, and incomplete otherwise. If the market is complete, then (10) and (11) trivially imply that the components of y_π are non negative, *i.e.*,

$$y_{\pi,j} \geq 0,\ j = 1, 2, ..., n. \tag{13}$$

There is one orthogonal projection in $\mathbb{R}^n$ endowed with the usual inner product that will be important in this paper. This is the projection on Y, and will be denoted by φ_Y. As usual, the manifold orthogonal to Y will be represented by $Y^\perp$, and similar notations will be used for the rest of the manifolds of $\mathbb{R}^n$.

Proposition 2. *a) $L(\varphi_Y(y)) = L(y)$ for every $y \in \mathbb{R}^n$. More generally, $L(\varphi_W(y)) = L(y)$ for every $y \in \mathbb{R}^n$ and for every subspace $W \subset \mathbb{R}^n$ with $(1, 1, ..., 1) \in W$.*

b) $L(y_\pi) = 1$. Moreover, if the market is complete then $y_\pi \in \mathcal{P}$.

Proof. *a*) Let $y \in \mathbb{R}^n$. $\varphi_W(y)$ is characterized by the property $y - \varphi_W(y) \in W^\perp$, so $(1, 1, ..., 1) \in W$ and (12) lead to $L(y - \varphi_W(y)) = 0$.

b) (9), (10) and (12) imply that

$$e^{-r_f T} = e^{-r_f T} y_\pi (1, 1, ..., 1) = e^{-r_f T} L(y_\pi),$$

so $L(y_\pi) = 1$. Besides, if the market is complete (13) implies that $y_\pi \in \mathcal{P}$. □

Remark 1. *Thus, if the market is complete then $y_\pi \in \mathcal{P}$, and (10) indicates that agents know the (unique) risk neutral probability measure y_π of the market. In general, regardless of the completeness of the market, (10) indicates that agents know the pricing rule, so their uncertainty is not related to prices, but to probabilities and therefore to the rest of statistical parameters (expected returns, variances, $CVaR$, etc.) associated with the future market evolution. Uncertainty related to current prices barely makes sense in practice, since real quotes are given by the market.* □

We will provide a portfolio choice problem involving the space $\mathcal{C}(\mathcal{K})$ (respectively, $\mathcal{C}\left(\nabla_{(\mathcal{K},\mu_0)}\right)$), of real valued continuous functions on the compact set $\mathcal{K}$. The dual problem will involve the dual space of $\mathcal{C}(\mathcal{K})$ (respectively, $\mathcal{C}\left(\nabla_{(\mathcal{K},\mu_0)}\right)$), which, according to the Riesz representation theorem (Luenberger, 1969), is $\mathcal{M}(\mathcal{K})$ (respectively, $\mathcal{M}\left(\nabla_{(\mathcal{K},\mu_0)}\right)$), space of real valued inner regular $\sigma-$additive measures on the Borel $\sigma-$algebra of $\mathcal{K}$ (respectively, $\nabla_{(\mathcal{K},\mu_0)}$). $\mathcal{P}(\mathcal{K})$ (respectively, $\mathcal{P}\left(\nabla_{(\mathcal{K},\mu_0)}\right)$) will denote the subset of $\mathcal{M}(\mathcal{K})$ (respectively, $\mathcal{M}\left(\nabla_{(\mathcal{K},\mu_0)}\right)$) whose elements are probability measures.

Next, let us present a main Lemma which ends this section. We will omit the proof because this is a particular case of several results regarding the representation of probability measures on convex sets by points. For instance, a more general Proposition may be found in Phelps (2001), *pp* 3.

Lemma 3. *For every $\nu \in \mathcal{P}(\mathcal{K})$ (respectively, $\nu \in \mathcal{P}\left(\nabla_{(\mathcal{K},\mu_0)}\right)$) there exists a unique $p_\nu \in \mathcal{K}$ (respectively, $p_\nu \in \nabla_{(\mathcal{K},\mu_0)}$) such that*

$$\int_{\mathcal{K}} \left(\sum_{j=1}^{n} p_j y_j \right) d\nu(p) = \sum_{j=1}^{n} p_{\nu,j} y_j$$

(respectively, $\int_{\nabla_{(\mathcal{K},\mu_0)}} \left(\sum_{j=1}^{n} p_j y_j \right) d\nu(p) = \sum_{j=1}^{n} p_{\nu,j} y_j$) for every $y \in \mathbb{R}^n$. □

3. Portfolio choice problem

Henceforth, we will fix the set $\mathcal{K}$ and the level of confidence μ_0, and we will simplify the notation by using $\rho = RCVaR_{(\mathcal{K},\mu_0)}$.

Consider the optimization problem

$$\begin{cases} Min\ \rho(y) \\ y_\pi y \le e^{r_f T} \\ \sum_{j=1}^{n} p_j y_j \ge r, \quad \forall p \in \mathcal{K} \\ y \in Y \end{cases} \tag{14}$$

The agent with uncertainty is minimizing the ambiguous or robust $CVaR$ in the set Y of attainable portfolios whose price is not higher than one dollar (first constraint) and whose expected return will be at least r regardless of the real probability that explains the market behavior (second constraint). Thus, the solution y^* of (14) will guarantee (at least) the robust (or worst-case) expected return r, and a real $CVaR$ never higher than $\rho(y^*)$. In this sense, the uncertainty with respect to the statistical parameters does not impede guaranteeing a lower bound for the expected return with an upper bound for the $CVaR$. We will assume that the investor desires at least the return of the riskless asset, so $r \ge e^{r_f T}$ must hold.

Similarly to Balbás *et al.* (2010*b*), and bearing in mind (4), we can add a new decision variable θ and consider the equivalent problem

$$\begin{cases} Min\ \theta \\ \theta + \sum_{j=1}^{n} \xi_j y_j \ge 0, \quad \forall \left(\xi_j\right)_{j=1}^{n} \in \nabla_{(\mathcal{K},\mu_0)} \\ y_\pi y \le e^{r_f T} \\ \sum_{j=1}^{n} p_j y_j \ge r, \qquad \forall p \in \mathcal{K} \\ \theta \in \mathbb{R},\ y \in Y \end{cases} \tag{15}$$

Notice that the first and third constraints are $\mathcal{C}\left(\nabla_{(\mathcal{K},\mu_0)}\right)$ and $\mathcal{C}(\mathcal{K})$ valued, respectively. Thus, the Lagrangian function is (Luenberger, 1969, or Balbás *et al.*, 2010*b*)

$$\mathcal{L}(\theta, y, \nu_1, \lambda, \nu_2) =$$

$$\theta\left(1-\int_{\nabla_{(\mathcal{K},\mu_0)}} d\nu_1(\xi)\right) - \int_{\nabla_{(\mathcal{K},\mu_0)}} \left(\sum_{j=1}^{n} \xi_j y_j\right) d\nu_1(\xi)$$

$$+\lambda\left(y_\pi y - e^{r_f T}\right) + r\int_{\mathcal{K}} d\nu_2(p) - \int_{\mathcal{K}} \left(\sum_{j=1}^{n} p_j y_j\right) d\nu_2(p),$$

for every $(\theta, y, \nu_1, \lambda, \nu_2) \in \mathbb{R} \times Y \times \mathcal{M}\left(\nabla_{(\mathcal{K},\mu_0)}\right) \times \mathbb{R} \times \mathcal{M}(\mathcal{K})$. Moreover, (ν_1, λ, ν_2) is dual-feasible if and only if $\nu_1, \lambda, \nu_2 \geq 0$ and the infimum of $\mathcal{L}(\theta, y, \nu_1, \lambda, \nu_2)$ in $(\theta, y) \in \mathbb{R}\times Y$ is bounded. Therefore, ν_1 must become a probability $\left(\nu_1 \in \mathcal{P}\left(\nabla_{(\mathcal{K},\mu_0)}\right)\right)$ and so must ν_2 $(\nu_2 \in \mathcal{P}(\mathcal{K}))$ if we add the variable $\Lambda = \int_{\mathcal{K}} d\nu_2(p) \geq 0$. Then,

$$\mathcal{L}(y, \nu_1, \lambda, \nu_2, \Lambda) =$$

$$-\int_{\nabla_{(\mathcal{K},\mu_0)}} \left(\sum_{j=1}^{n} \xi_j y_j\right) d\nu_1(\xi)$$

$$+\lambda\left(y_\pi y - e^{r_f T}\right) + \Lambda r - \Lambda\int_{\mathcal{K}} \left(\sum_{j=1}^{n} p_j y_j\right) d\nu_2(p).$$

The dual objective function at $(\nu_1, \lambda, \nu_2, \Lambda)$ is the infimum of $\mathcal{L}(y, \nu_1, \lambda, \nu_2, \Lambda)$ for $y \in Y$. Hence, bearing in mind Lemma 3, $(\nu_1, \lambda, \nu_2, \Lambda)$ may be replaced by

$$(\xi, \lambda, p, \Lambda) \in \nabla_{(\mathcal{K},\mu_0)} \times \mathbb{R} \times \mathcal{K} \times \mathbb{R},$$

the Lagrangian function becomes

$$\mathcal{L}(y, \xi, \lambda, p, \Lambda) =$$

$$\left(\sum_{j=1}^{n} y_j\left(-\xi_j + \lambda y_{\pi,j} - \Lambda p_j\right)\right) - \lambda e^{r_f T} + \Lambda r,$$

and the dual problem is

$$\begin{cases} Max \ -\lambda e^{r_f T} + \Lambda r \\ -\xi + \lambda y_\pi - \Lambda p \in Y^{\perp} \\ \xi \in \nabla_{(\mathcal{K},\mu_0)}, \ p \in \mathcal{K}, \ \lambda, \Lambda \in \mathbb{R}, \ \lambda, \Lambda \geq 0 \end{cases} \tag{16}$$

Applying the function $L = L \circ \varphi_Y$ of (12) in the first constraint of (16), and bearing in mind Propositions 1 and 2, we get $-1 + \lambda - \Lambda = 0$, which leads to

$$\Lambda = \lambda - 1. \tag{17}$$

On the other hand, constraint $-\xi + \lambda y_\pi - \Lambda p \in Y^\perp$ is equivalent to

$$\varphi_Y \left(-\xi + \lambda y_\pi - \Lambda p\right) = 0.$$

Thus, the dual problem simplifies to

$$\left\{ \begin{array}{l} Max\ \lambda \left(r - e^{r_f T}\right) - r \\ \lambda y_\pi = \varphi_Y \left(\xi + \lambda p - p\right) \\ \xi \in \nabla_{(\mathcal{K},\mu_0)},\ p \in \mathcal{K},\ \lambda \in \mathrm{I\!R},\ \lambda \geq 1 \end{array} \right. \tag{18}$$

Proposition 4. *If (14) is feasible for some $r_0 > e^{r_f T}$ then for every $r > e^{r_f T}$ (14) is also feasible and (15) satisfies the Slater condition, i.e., its three constraints hold as strict inequalities for some feasible solution.*

Proof. There exists a portfolio $y_0 \in Y$ such that $\pi\left(y_0\right) \leq 1$ and $\sum_{j=1}^n p_j y_{0,j} > e^{r_f T}$ for every $p \in \mathcal{K}$. Then, consider βy_0 such that $0 < \beta < 1$, $\pi\left(\beta y_0\right) < 1$ and $\sum_{j=1}^n \beta p_j y_{0,j} > e^{r_f T}$ for every $p \in \mathcal{K}$. For every $\alpha > 0$ we have that

$$\begin{array}{l} \pi\left(\left(1+\alpha\right)\beta y_0 - \alpha e^{r_f T}\left(1,1,...,1\right)\right) = \\ \left(1+\alpha\right)\pi\left(\beta y_0\right) - \alpha = \pi\left(\beta y_0\right) + \alpha\left(\pi\left(\beta y_0\right) - 1\right) \leq \pi\left(\beta y_0\right) < 1, \end{array}$$

and

$$\begin{array}{l} p\left(\left(1+\alpha\right)\beta y_0 - \alpha e^{r_f T}\left(1,1,...,1\right)\right) = \\ \left(1+\alpha\right)\beta p y_0 - \alpha e^{r_f T} = \alpha\left(\beta p y_0 - e^{r_f T}\right) + \beta p y_0 \longrightarrow \infty \end{array}$$

as $\alpha \longrightarrow \infty$. Thus, $\left(1+\alpha\right)\beta y_0 - \alpha e^{r_f T}\left(1,1,...,1\right)$ is a portfolio satisfying the second and third constraints in (15) as strict inequalities. Besides, the first inequality may be satisfied as a strict one if θ is large enough. Indeed, since $\nabla_{(\mathcal{K},\mu_0)}$ is compact, it is sufficient to consider

$$\theta > Max\ \left\{-\sum_{j=1}^n \xi_j \beta y_{0,j};\ \ \left(\xi_j\right)_{j=1}^n \in \nabla_{(\mathcal{K},\mu_0)}\right\},$$

and (15) satisfies the Slater condition. □

Due to the fulfillment of the Slater condition there is no duality gap between (15) and (16). Thus, proceeding as in Balbás *et al.* (2010*b*), there is no duality gap between (14) and (18) either.

Theorem 5. *Suppose that (14) is feasible. (14) is bounded if and only if (18) is feasible, in which case the following conditions hold:*

a) The infimum of (14) equals the (attainable) maximum of (18).

b) $y^* \in Y$ and $(\xi^*, \lambda^*, p^*) \in \nabla_{(\mathcal{K},\mu_0)} \times \mathrm{I\!R} \times \mathcal{K}$ solve (14) and (18) if and only if the following Karush-Kuhn-Tucker-like conditions

$$\begin{cases} \sum_{j=1}^{n} \xi_j y_j^* \geq \sum_{j=1}^{n} \xi_j^* y_j^*, & \forall \left(\xi_j\right)_{j=1}^{n} \in \nabla_{(\mathcal{K},\mu_0)} \\ (\lambda^* - 1)\left(\sum_{j=1}^{n} p_j^* y_j^* - r\right) = 0 & \\ y_\pi y^* = e^{r_f T} & \\ \sum_{j=1}^{n} p_j y_j^* \geq r, & \forall p \in \mathcal{K} \\ \lambda^* y_\pi = \varphi_Y\left(\xi^* + \lambda^* p^* - p^*\right) & \\ \lambda^* \geq 1 & \end{cases} \tag{19}$$

hold. □

Remark 2. *Notice that the first constraint in (18) may be represented by*

$$y_\pi = \varphi_Y\left(\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p\right). \tag{20}$$

Since $\lambda \geq 1$ then $\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p$ is a linear convex combination of $\xi \in \nabla_{(\mathcal{K},\mu_0)}$ and $p \in \mathcal{K}$, and therefore,

$$\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p \in \nabla_{(\mathcal{K},\mu_0)} \subset \mathcal{P}$$

(see Proposition 1). Hence, (10) and (20) imply that

$$\pi(y) = e^{-r_f T} y_\pi y = e^{-r_f T}\left(\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p\right)y, \tag{21}$$

and $\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p$ may be interpreted as a risk neutral probability measure of this market. Thus, (18) is feasible (or (14) is bounded) if and only if there are risk neutral probabilities that may be represented as linear convex combinations of an element belonging to $\nabla_{(\mathcal{K},\mu_0)}$ and a second element belonging to $\mathcal{K}$.

Besides, the feasible set of (18) does not depend on r nor does its solution, since problem

$$\begin{cases} Max\ \lambda \\ y_\pi = \varphi_Y\left(\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p\right) \\ \xi \in \nabla_{(\mathcal{K},\mu_0)},\ p \in \mathcal{K},\ \lambda \in \mathrm{I\!R},\ \lambda \geq 1 \end{cases} \tag{22}$$

provides us with the optimal dual solution (ξ^*, λ^*, p^*) for every $r > e^{r_f T}$. □

4. Capital market line and CAPM-like formulae

First of all, let us characterize those cases when (14) is feasible.

Theorem 6. *(14) is feasible if and only if $y_\pi \notin \varphi_Y(\mathcal{K})$.*

Proof. Suppose that $y_\pi \in \varphi_Y(\mathcal{K})$. In such a case, take $p \in \mathcal{K} \subset \nabla_{(\mathcal{K},\mu_0)}$ with $y_\pi = \varphi_Y(p)$ and we obviously have

$$y_\pi = \varphi_Y\left(\frac{1}{\lambda}p + \left(1 - \frac{1}{\lambda}\right)p\right)$$

for every $\lambda \geq 1$. Thus, (18) and (22) are unbounded and Theorem 5 shows that (14) is not feasible for every $r > e^{r_f T}$.

Conversely, suppose that $y_\pi \notin \varphi_Y(\mathcal{K})$. Since $y_\pi \in Y$ and $\varphi_Y(\mathcal{K}) \subset Y$ is obviously convex and compact, the Hahn Banach separation theorem (Luenberger, 1969) implies that existence of $y_0 \in Y$ such that

$$y_\pi y_0 < Min\ \{y_0 y;\ y \in \varphi_Y(\mathcal{K})\}. \tag{23}$$

We can also assume that $y_\pi y_0 > 0$ because otherwise y_0 could be substituted by $y_0 + \alpha(1,1,...,1)$ with a large enough α. Indeed, notice that

$$(y_0 + \alpha(1,1,...,1))\varphi_Y(p) = y_0\varphi_Y(p) + \alpha(1,1,...,1)p = y_0\varphi_Y(p) + \alpha$$

for every $p \in \mathcal{K}$, whereas

$$(y_0 + \alpha(1,1,...,1))y_\pi = y_\pi y_0 + \alpha.$$

Thus, considering $e^{r_f T}y_0/(y_\pi y_0)$ if necessary, we can assume that $y_\pi y_0 = e^{r_f T}$. Then, (23) implies that (14) is feasible for every r lying within the spread

$$(y_\pi y_0, Min\ \{y_0 y;\ y \in \varphi_Y(\mathcal{K})\}),$$

and Proposition 4 implies that (14) is feasible for every $r > e^{r_f T}$. $\square$

According to Theorem 5, Remark 2, and Theorem 6, we can consider four disjoint and complementary situations:

Remark 3. *H1. $y_\pi \in \varphi_Y(\mathcal{K})$. In such a case (14) is not feasible for every $r > e^{r_f T}$. We will say that the market is $\mathcal{K}-$risk-neutral.*

H2A. $y_\pi \in \varphi_Y\left(\nabla_{(\mathcal{K},\mu_0)}\right)$ and $y_\pi \notin \varphi_Y(\mathcal{K})$. In such a case (14) is feasible and, bearing in mind that

$$y_\pi = \varphi_Y\left(\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p\right)$$

for some $\xi \in \varphi_Y\left(\nabla_{(\mathcal{K},\mu_0)}\right)$, $\lambda = 1$, *and every* $p \in \mathcal{K}$, *(18) and (22) are also feasible. Theorem* 5 *shows that (18) and (22) attain a maximum value. Let* λ^* *be the maximum value of (22). If* $\lambda^* > 1$ *then, bearing in mind the objective function of (18), the optimal robust* $CVaR$ *is*

$$\rho = \lambda^* \left(r - e^{r_f T}\right) - r, \tag{24}$$

or, equivalently,

$$r = \frac{\rho + \lambda^* e^{r_f T}}{\lambda^* - 1} \tag{25}$$

yields the relationship between the optimal robust $CVaR$ *and the guaranteed (or robust) expected return. We will say that* $\dfrac{1}{\lambda^* - 1}$ *is the robust market price of risk for the investor whose ambiguity is given by the set of priors* $\mathcal{K}$, *and (25) will be called the* $\mathcal{K}-$*capital market line (*$\mathcal{K} - CML$*).*

H2B. $y_\pi \in \varphi_Y\left(\nabla_{(\mathcal{K},\mu_0)}\right)$ *and* $y_\pi \notin \varphi_Y(\mathcal{K})$. *(14) (18) and (22) are feasible, but the optimal solution of (22) satisfies* $\lambda^* = 1$. *Then, bearing in mind the objective function of (18), the optimal robust* $CVaR$ *is*

$$\rho = \lambda^* \left(r - e^{r_f T}\right) - r = -e^{r_f T}$$

and it does not depend on r. *In fact, one can construct sequences of portfolios whose robust expected return is as large as desired (tends to* $+\infty$*) whereas their risk is as close to* $-e^{r_f T}$ *as desired. There is no market price of risk because every guaranteed expected return is reached with a similar risk level. We will say that there are* $\mathcal{K}-$*good deals of the first type.*

*H*3. $y_\pi \notin \varphi_Y\left(\nabla_{(\mathcal{K},\mu_0)}\right)$. *(14) is feasible, but the first constraint in (22) cannot hold because Proposition* 1 *guarantees that*

$$\varphi_Y\left(\frac{1}{\lambda}\xi + \left(1 - \frac{1}{\lambda}\right)p\right) \in \varphi_Y\left(\nabla_{(\mathcal{K},\mu_0)}\right)$$

for every $\lambda \geq 1$, *every* $\xi \in \nabla_{(\mathcal{K},\mu_0)}$ *and every* $p \in \mathcal{K}$. *In this case (14) is unbounded for every* $r > e^{r_f T}$ *(Theorem* 5*), i.e., one can construct a sequence of portfolios guaranteeing an expected return as large as desired (tends to* $+\infty$*) and whose risk, given by the robust* $CVaR$, *is as negative as desired (tends to* $-\infty$*). We will say that there are* $\mathcal{K}-$*good deals of the second type.* □

We will devote the rest of this section to dealing with Case $H2A$, *i.e.*, the natural situation where (14) and (18) are feasible and bounded, $\lambda^* > 1$, $\dfrac{1}{\lambda^* - 1}$ is the market price of risk, and the $\mathcal{K} - CML$ of (24) and (25) gives the relationship between the optimal robust risk and its associated robust expected return. The remaining cases will be analyzed in Section 5.

Fix $r^* > e^{r_f T}$ and consider the corresponding primal and dual solutions, y^* and (ξ^*, λ^*, p^*). y^* is a priced one portfolio due to the third condition in (19). Obviously, the dual solution will remain the same if we modify the guaranteed return. However, the optimal portfolio will be a combination of y^* and the riskless asset. In fact, we have:

Proposition 7. *If $r > e^{r_f T}$ replaces r^* then the primal solution becomes*

$$\alpha y^* + (1-\alpha)\left(e^{r_f T}, e^{r_f T}, ..., e^{r_f T}\right)$$

with

$$\alpha = \frac{r - e^{r_f T}}{r^* - e^{r_f T}}.$$

Proof. It is sufficient to see that $\alpha y^* + (1-\alpha)\left(e^{r_f T}, e^{r_f T}, ..., e^{r_f T}\right)$ and (ξ^*, λ^*, p^*) satisfy (19). With respect to the first inequality of (19), for $\left(\xi_j\right)_{j=1}^n \in \nabla_{(\mathcal{K},\mu_0)}$ we have that

$$\sum\nolimits_{j=1}^n \xi_j \left(\alpha y_j^* + (1-\alpha) e^{r_f T}\right) = \alpha \sum\nolimits_{j=1}^n \xi_j y_j^* + (1-\alpha) e^{r_f T}$$

$$\geq \alpha \sum\nolimits_{j=1}^n \xi_j^* y_j^* + (1-\alpha) e^{r_f T} = \sum\nolimits_{j=1}^n \xi_j^* \left(\alpha y_j^* + (1-\alpha) e^{r_f T}\right).$$

The second condition in (19) and $\lambda^* > 1$ lead to $\sum\nolimits_{j=1}^n p_j^* y_j^* = r^*$. Then,

$$\sum\nolimits_{j=1}^n p_j^* \left(\frac{r - e^{r_f T}}{r^* - e^{r_f T}} y_j^* + \frac{r^* - r}{r^* - e^{r_f T}} e^{r_f T}\right) =$$

$$\frac{r - e^{r_f T}}{r^* - e^{r_f T}} \sum\nolimits_{j=1}^n p_j^* y_j^* + \frac{r^* - r}{r^* - e^{r_f T}} e^{r_f T} \sum\nolimits_{j=1}^n p_j^*$$

$$= \frac{r - e^{r_f T}}{r^* - e^{r_f T}} r_0 + \frac{r^* - r}{r^* - e^{r_f T}} e^{r_f T} = r.$$

The third condition in (19) is

$$y_\pi \left(\alpha y^* + (1-\alpha)\left(e^{r_f T}, e^{r_f T}, ..., e^{r_f T}\right)\right) =$$

$$\alpha y_\pi y^* + (1-\alpha) y_\pi \left(e^{r_f T}, e^{r_f T}, ..., e^{r_f T}\right) = e^{r_f T}.$$

Finally, the fourth condition in (19) leads to

$$\sum_{j=1}^{n} p_j \left(\alpha y_j^* + (1-\alpha)\, e^{r_f T}\right) \geq \alpha r^* + (1-\alpha)\, e^{r_f T} = \frac{r - e^{r_f T}}{r^* - e^{r_f T}} r^*_+ \frac{r^* - r}{r^* - e^{r_f T}} e^{r_f T} = r$$

for every $p \in \mathcal{K}$. $\square$

Remark 4. *(Systemic and specific risk). Henceforth, we will consider the benchmark y^* generating the set of efficient portfolios by combinations with the riskless asset. Notice that (19) implies that y^* is a priced one strategy whose robust expected return is r^*. According to the latter proposition and its proof, for every $r > e^{r_f T}$*

$$\frac{r - e^{r_f T}}{r^* - e^{r_f T}} y^* + \frac{r^* - r}{r^* - e^{r_f T}} \left(e^{r_f T}, e^{r_f T}, ..., e^{r_f T}\right) \tag{26}$$

is a priced one efficient strategy whose robust expected return is r.

For a general priced one portfolio y we will consider its robust expected return

$$\mathbb{E}_{\mathcal{K}}(y) = Min \left\{ \sum_{j=1}^{n} p_j y_j;\ p \in \mathcal{K} \right\}$$

and its guaranteed (or robust) $CVaR$, $\rho(y)$. Moreover, the role of $\mathcal{K}$ may be also played by every alternative closed convex subset of $\mathcal{P}$. If $\mathbb{E}_{\mathcal{K}}(y) = r > e^{r_f T}$ then one obviously has

$$\rho(y) \geq \rho\left(\frac{r - e^{r_f T}}{r^* - e^{r_f T}} y^* + \frac{r^* - r}{r^* - e^{r_f T}} \left(e^{r_f T}, e^{r_f T}, ..., e^{r_f T}\right) \right),$$

and (5) and (6) lead to

$$\rho(y) \geq \frac{r - e^{r_f T}}{r^* - e^{r_f T}} \rho(y^*) + \frac{r - r^*}{r^* - e^{r_f T}} e^{r_f T}, \tag{27}$$

i.e., the risk of y is never lower than the risk of the efficient portfolio (26) with the same guaranteed return as y. The right hand side in (27) will be said to be the systemic risk of y, while

$$\rho(y) - \frac{r - e^{r_f T}}{r^* - e^{r_f T}} \rho(y^*) - \frac{r - r^*}{r^* - e^{r_f T}} e^{r_f T} \geq 0 \tag{28}$$

will be said to be its specific risk. Obviously, the inequality in (27) shows that "the market does not pay anything" for the specific risk, since the efficient portfolio of Proposition 7 guarantees the same return r and it only reflects the systemic risk of y. $\square$

The Remark above justifies that y^* describes the set of efficient strategies and yields the proportion of the risk of every portfolio that is related to its robust return. Next, let us see that some kind of "beta" between an arbitrary portfolio y and the benchmark y^* may explain the worst-case return guaranteed by y.

Suppose now that $p = (p_1 p_2, ..., p_n) \in \mathcal{K}$ and $p_j > 0$, $j = 1, 2, ..., n$. Then, it is known that $\mathbb{R}^n$ may be endowed with the alternative inner product

$$\mathbb{E}_p \left(y_1 z_2, y_1 z_2, ..., y_n z_n \right) = \sum_{j=1}^{n} p_j y_j z_j, \tag{29}$$

for every y, $z \in \mathbb{R}^n$. This different inner product does not modify the topological properties, but geometrical ones will become different. So, the orthogonal projection on Y will be denoted now by $\varphi_{(Y,p)}$, and, obviously, in general $\varphi_{(Y,p)} \neq \varphi_Y$. This also applies if Y is replaced by another manifold of $\mathbb{R}^n$.

Besides, the Riesz representation theorem also ensures the existence of a unique $z_p \in Y$ such that

$$\pi \left(y \right) = e^{-r_f T} \sum_{j=1}^{n} p_j y_j z_{p,j} \tag{30}$$

for every $y \in Y$. We will denote by φ_p the orthogonal projection $\varphi_p : \mathbb{R}^n \longrightarrow \mathcal{L} \left\{ (1, 1, ..., 1), z_p \right\}$ with the inner product (29), and by $y_p^* = \varphi_p \left(y^* \right)$.

Proposition 8. *If $p = (p_1 p_2, ..., p_n) \in \mathcal{K}$ and $p_j > 0$, $j = 1, 2, ..., n$, then $(1, 1, ..., 1)$ and y_p^* are not proportional.*

Proof. $y^* - y_p^*$ must be orthogonal to $(1, 1, ..., 1)$ and z_p in the inner product (29), so we have that

$$\mathbb{E}_p \left(y_p^* \right) = \mathbb{E}_p \left(y^* \right) \geq \mathbb{E}_{\mathcal{K}} \left(y^* \right) = r^* > e^{r_f T}$$

and $\pi \left(y_p^* \right) = \pi \left(y^* \right) = 1$. If there existed a with $y_p^* = a \left(1, 1, ..., 1 \right)$ then (9) and $\pi \left(y_p^* \right) = 1$ would imply $a = e^{r_f T}$, which would lead to $\mathbb{E}_p \left(y_p^* \right) = e^{r_f T}$. $\square$

The previous proposition implies that the variance $Var_p \left(y_p^* \right)$ under p of y_p^* is strictly positive, so we can consider the regression coefficient (under p) between every $y \in Y$ and y_p^* given by

$$\beta_p \left(y \right) = \frac{Cv_p (y, y_p^*)}{Var_p \left(y_p^* \right)},$$

Cv_p denoting covariance under p. If $p = p^*$ the subscript "p" will be omitted. Similarly, if there is no confusion, Portfolio y may be omitted too.

Theorem 9. *($CAPM-$like formulae). Let $y \in Y$ be a priced one portfolio with a positive worst-case risk premium $\mathbb{E}_{\mathcal{K}}(y) - e^{r_f T}$.*[4] *Suppose that both p^* and p have strictly positive components, $p \in \mathcal{K}$ being such that $\mathbb{E}_{\mathcal{K}}(y) = \mathbb{E}_p(y)$. Then,*

$$\beta_p \left(\mathbb{E}_{\mathcal{K}}(y^*) - e^{r_f T} \right) \leq \mathbb{E}_{\mathcal{K}}(y) - e^{r_f T} \leq \beta \left(\mathbb{E}_{\mathcal{K}}(y^*) - e^{r_f T} \right). \quad (31)$$

Furthermore, if $p = p^*$ (which always holds for an efficient portfolio, see (26)) then

$$\mathbb{E}_{\mathcal{K}}(y) - e^{r_f T} = \beta \left(\mathbb{E}_{\mathcal{K}}(y^*) - e^{r_f T} \right). \quad (32)$$

Proof. Consider the linear manifold $\mathcal{L}\{(1,1,...,1), z_p\}$. Proposition 8 shows that $\mathcal{L}\{(1,1,...,1), z_p\} = \mathcal{L}\left\{(1,1,...,1), y_p^*\right\}$. Since $(1,1,...,1)$ and $y_p^* - \mathbb{E}_p\left(y_p^*\right)(1,1,...,1)$ are obviously orthogonal under p, the projection lemma of Hilbert spaces (Maurin, 1967) shows that

$$y = \mathbb{E}_p(y)(1,1,...,1) + \beta_p \left(y_p^* - \mathbb{E}_p\left(y_p^*\right)(1,1,...,1) \right) + \varepsilon,$$

where $\varepsilon \in Y$ satisfies $\mathbb{E}_p(\varepsilon) = 0$ and $\pi(\varepsilon) = 0$. Multiplying by y_π we have

$$e^{r_f T} = \mathbb{E}_p(y) + \beta_p \left(\pi(y_p^*) e^{r_f T} - \mathbb{E}_p\left(y_p^*\right) \right).$$

As in the proof of the previous proposition, $\mathbb{E}_p\left(y_p^*\right) = \mathbb{E}_p(y^*)$ and $\pi(y_p^*) = \pi(y^*) = 1$, so straightforward manipulations lead to

$$\mathbb{E}_p(y) - e^{r_f T} = \beta_p \left(\mathbb{E}_p(y^*) - e^{r_f T} \right).$$

Since $\mathbb{E}_p(y) - e^{r_f T} = \mathbb{E}_{\mathcal{K}}(y) - e^{r_f T} > 0$, and $\mathbb{E}_p(y^*) - e^{r_f T} \geq \mathbb{E}_{\mathcal{K}}(y^*) - e^{r_f T} > 0$, we have that $\beta_p > 0$. Then, the first inequality in (31) trivially follows from $\mathbb{E}_{\mathcal{K}}(y) = \mathbb{E}_p(y)$ and $\mathbb{E}_p(y^*) \geq \mathbb{E}_{\mathcal{K}}(y^*)$.

Proceeding in a similar manner, we can show that

$$\mathbb{E}_{p^*}(y) - e^{r_f T} = \beta \left(\mathbb{E}_{p^*}(y^*) - e^{r_f T} \right),$$

so the second inequality in (31) becomes obvious if one bears in mind that $\mathbb{E}_{\mathcal{K}}(y^*) = \mathbb{E}_{p^*}(y^*)$ (second and fourth conditions in (19), along with $\lambda^* > 1$) and $\mathbb{E}_{p^*}(y) \geq \mathbb{E}_{\mathcal{K}}(y)$. $\square$

[4] Notice that there are portfolios with a negative risk premium, at least if the benchmark is sold. For instance,

$$y = 2e^{r_f T}(1,1,...,1) - y^*$$

is priced one and for every $p \in \mathcal{K}$ we have that

$$\mathbb{E}_p(y) - e^{r_f T} = e^{r_f T} - \mathbb{E}_p(y^*) \leq e^{r_f T} - r^* < 0.$$

If $p \in \mathcal{K}$ and $p_j = 0$ for some $j = 1, 2, ..., n$, then one can interpret that there may be less than n states of nature, *i.e.*, the ambiguity affects both the states of nature and their probabilities. The statement of Theorem 9 may be slightly simplified if there is no ambiguity with respect to the states of nature.

Corollary 10. *Suppose that every probability $p \in \mathcal{K}$ has strictly positive components. Let $y \in Y$ be a priced one portfolio with a positive worst-case risk premium $\mathbb{E}_{\mathcal{K}}(y) - e^{r_f T}$. Let $p \in \mathcal{K}$ such that $\mathbb{E}_{\mathcal{K}}(y) = \mathbb{E}_p(y)$. Then (31) holds, and so does (32) if $p = p^*$. In particular, (32) holds for every efficient portfolio.* □

Expression (31) is an obvious extension of the well-known one in the classical $CAPM$. Indeed, if there is no ambiguity and $\mathcal{K} = \{p\}$ is a singleton then both inequalities in (31) lead to the equality

$$\mathbb{E}_{\mathcal{K}}(y) - e^{r_f T} = \beta \left(\mathbb{E}_{\mathcal{K}}(y^*) - e^{r_f T} \right).$$

Nevertheless, it is worth remarking that y^* does not optimize the variance but the $CVaR$, which is relevant in the presence of asymmetric returns or fat tails because $CVaR$ is still compatible with the second order stochastic dominance (Ogryczak and Ruszczynski, 2002, Agarwal and Naik, 2004, etc.), whereas the variance may present incompatibility.

5. The no good deal condition

Remark 3 implies the existence of four disjoint and complementary scenarios. As stated in Theorem 9 and Corollary 10, the most natural one ($H2A$) implies some consequences quite parallel to those of the classical $CAPM$. Besides, $H1$ may be also natural. If the investor reflected high uncertainty, then it could be impossible to guarantee expected returns larger than the risk free rate. Theorem 6 implies that this situation clearly holds, for instance, if the market is complete and $\mathcal{K} = \mathcal{P}$ (highest level of ambiguity). Actually, the absence of worst-case expected returns higher than r_f may imply that the investor has no incentives to buy risky assets. This result seems to be consistent with the theoretical and empirical findings of Cao *et al.* (2005) and Bossaerts *et al.* (2010). Agents who are sufficiently ambiguity averse find open sets of prices for which they refuse to hold an ambiguous portfolio or choose not to participate in the stock market.

On the contrary, $H2B$ or $H3$ imply the presence of good deals, which allows investors to construct portfolios whose guaranteed expected return is as large as desired and whose guaranteed risk is bounded from below (first type good deal) or as close to $-\infty$ as desired (second type good deal). This pathological finding has been already pointed out by Balbás *et al.* (2010*a*). These authors do not consider any sort of ambiguity but they show that very important pricing models (Black and Scholes, Heston etc.) imply the presence of good deals for every coherent and expectation bounded risk measure. It seems to be a serious shortcoming non compatible with any kind of equilibrium, since agents should try to create good deals to become as rich as intended, regardless of the risk measure they are dealing with. Those securities composing the good deal should be requested with the appropriate sign (long or short), which would change the evolution of prices.

A major objective of this paper is to prove that the presence of ambiguity may help to overcome this caveat, at least for complete markets. To this purpose, suppose that $H2B$ or $H3$ hold (*i.e.*, there are good deals of the first or second type) and assume the fulfillment of Condition $C1$ below.

$C1$. $y_\pi \in \varphi_Y(\mathcal{P})$.

Proposition 11. *If the market is complete then $C1$ holds.*

Proof. Since φ_Y becomes the identity map, this result is an obvious consequence of Proposition 2b. □

Consider the set

$$\bigcup = \left\{\mathcal{H};\ \mathcal{K} \subset \mathcal{H} \subset \mathcal{P},\ \mathcal{H}\ is\ convex\ and\ compact,\ H1\ or\ H2A\ hold\ for\ \mathcal{H}\right\}.$$

Obviously, $\bigcup$ is non void because $\mathcal{P} \in \bigcup$ (Condition $C1$ and Theorem 6). For every $\mathcal{H} \in \bigcup$ we can consider the optimal value $\lambda^*_{\mathcal{H}}$ of (22) with $\mathcal{H}$ replacing $\mathcal{K}$. Obviously,

$$1 < \lambda^*_{\mathcal{H}} \leq +\infty.$$

Furthermore, for $\mathcal{H} = \mathcal{P}$, we have that

$$\lambda^*_{\mathcal{P}} = +\infty,$$

since for $y_\pi = \varphi_Y(p)$ with $p \in \mathcal{P} = \nabla_{(\mathcal{P},\mu_0)}$ we have that

$$y_\pi = \varphi_Y\left(\frac{1}{\lambda}p + \left(1 - \frac{1}{\lambda}\right)p\right)$$

and (p, λ, p) is (22)-feasible for every $\lambda \geq 1$. In other words, given $1 < \lambda^* \leq \infty$,

$$\bigcup\nolimits_{\lambda^*} = \left\{\mathcal{H} \in \bigcup;\ \lambda^* \leq \lambda^*_{\mathcal{H}}\right\} \neq \varnothing$$

because $\mathcal{P} \in \bigcup_{\lambda^*}$.

Theorem 12. *If there are $\mathcal{K}-$good deals of the first or second type and $C1$ holds then given $1 < \lambda^* < \infty$ there exists $\mathcal{H}_0 \in \bigcup_{\lambda^*}$ such that $\lambda^* \leq \lambda^*_{\mathcal{H}_0}$ and $\lambda^*_{\mathcal{H}_0} \leq \lambda^*_{\mathcal{H}}$ for every $\mathcal{H} \in \bigcup_{\lambda^*}$.*

Proof. Let us consider the (obviously convex a compact) set

$$\mathcal{H}_0 = \bigcap_{\mathcal{H} \in \bigcup_{\lambda^*}} \mathcal{H}.$$

Let us prove that $\mathcal{H}_0$ is the set we are looking for. The inclusion $\mathcal{K} \subset \mathcal{H}_0$ is obvious, so let us see the remaining properties guaranteeing that $\mathcal{H}_0 \in \bigcup_{\lambda^*}$.

At the moment let us assume that $\lambda^*_{\mathcal{H}} < \infty$ if $\mathcal{H}$ is small enough. The theorem of Caratheodory ensures that for every $\mathcal{H} \in \bigcup_{\lambda^*}$ there exist $\left(h_1^{\mathcal{H}}, ..., h_{n+1}^{\mathcal{H}}\right) \in \tilde{\nabla}^{n+1}_{(\mathcal{H},\mu_0)}$, $\left(t_1^{\mathcal{H}}, ..., t_{n+1}^{\mathcal{H}}\right) \in [0,1]^{n+1}$ with $\sum_{j=1}^{n+1} t_j^{\mathcal{H}} = 1$, and $h^{\mathcal{H}} \in \mathcal{H}$ such that

$$y_\pi = \varphi_Y \left(\frac{1}{\lambda^*_{\mathcal{H}}} \left(\sum_{j=1}^{n+1} t_j^{\mathcal{H}} h_j^{\mathcal{H}} \right) + \left(1 - \frac{1}{\lambda^*_{\mathcal{H}}} \right) h^{\mathcal{H}} \right) \tag{33}$$

The compactness of the involved sets guarantees the existence of

$$(h_1, ..., h_{n+1,} t_1, ..., t_{n+1}, h, \nu^*)$$

agglomeration point of the net

$$\left(h_1^{\mathcal{H}}, ..., h_{n+1}^{\mathcal{H}}, t_1^{\mathcal{H}}, ..., t_{n+1}^{\mathcal{H}}, h^{\mathcal{H}}, \lambda^*_{\mathcal{H}}\right)_{\mathcal{H} \in \bigcup_{\lambda^*}}.$$

Furthermore, since $\lambda^*_{\mathcal{H}}$ obviously decreases with $\mathcal{H}$,

$$\nu^* = Lim\ \lambda^*_{\mathcal{H}} = Inf\ \lambda^*_{\mathcal{H}} \geq \lambda^* > 1. \tag{34}$$

Thus, (33) leads to

$$y_\pi = \varphi_Y \left(\frac{1}{\nu^*} \left(\sum_{j=1}^{n+1} t_j h_j \right) + \left(1 - \frac{1}{\nu^*} \right) h \right)$$

and $\left(\sum_{j=1}^{n+1} t_j h_j, \nu^*, h \right)$ will be (22)-feasible for $\mathcal{H}_0$ instead of $\mathcal{K}$ if we see that $h \in \mathcal{H}_0$ and $h_j \in \tilde{\nabla}_{(\mathcal{H}_0,\mu_0)}$ for $j = 1, ..., n+1$. $h \in \mathcal{H}_0$ is clear because every $\mathcal{H}$ is compact and therefore $h \in \mathcal{H}$, so let us focus on the second property. We will fix $j = 1$ since the remaining cases are similar. For every $\mathcal{H} \in \bigcup_{\lambda^*}$ there exist $q^{\mathcal{H}} \in \mathcal{H}$ and $z^{\mathcal{H}} \in \Delta_{(q,\mu_0)}$ such that

$$\left(h_{1,1}^{\mathcal{H}}, ..., h_{1,n}^{\mathcal{H}}\right) = \left(q_{1,1}^{\mathcal{H}} z_{1,1}^{\mathcal{H}}, ..., q_{1,n}^{\mathcal{H}} z_{1,n}^{\mathcal{H}}\right).$$

As in the proof of Proposition 1a, there is an agglomeration point $\left(q^{\mathcal{H}_0}, z^{\mathcal{H}_0}\right)$ of $\left(q^{\mathcal{H}}, z^{\mathcal{H}}\right)_{\mathcal{H}}$, and h_1 being an agglomeration point of $\left(h_1^{\mathcal{H}}\right)_{\mathcal{H}}$ we can get

$$(h_{1,1}, ..., h_{1,n}) = \left(q_{1,1}^{\mathcal{H}_0} z_{1,1}^{\mathcal{H}_0}, ..., q_{1,n}^{\mathcal{H}_0} z_{1,n}^{\mathcal{H}_0}\right) \in \tilde{\nabla}_{(\mathcal{H}_0, \mu_0)}.$$

Once we have that $\left(\sum_{j=1}^{n+1} t_j h_j, \nu^*, h\right)$ is (22)-feasible for $\mathcal{H}_0$ instead of $\mathcal{K}$, and it is obvious that (see (34))

$$1 < \lambda^* \leq \nu^* \leq \lambda_{\mathcal{H}_0}^* \leq \lambda_{\mathcal{H}}^*$$

for every $\mathcal{H} \in \bigcup_{\lambda^*}$.

To complete the proof, let us assume that $\lambda_{\mathcal{H}}^* = \infty$ for every $\mathcal{H} \in \bigcup_{\lambda^*}$. Fix $m > 0$ and consider the net satisfying (33) such that $m \leq \lambda_{\mathcal{H}}^*$ for every $\mathcal{H} \in \bigcup_{\lambda^*}$. Consider the agglomeration point and $\left(\sum_{j=1}^{n+1} t_j h_j, \nu^*, h\right)$ as above with $\nu^* \leq \infty$. If $\nu^* < \infty$ then proceed as in the previous case and, otherwise, (33) leads to $y_\pi = \varphi_Y(h)$ with $h \in \mathcal{H}_0$, so Theorem 6 implies that $\lambda_{\mathcal{H}_0}^* = \infty$ satisfies the required properties. □

Remark 5. *(Interpretation of Theorem 12). Theorem 12 above shows that the presence of $\mathcal{K}-$good deals may be overcome if one enlarges the degree of ambiguity (the set of priors). Indeed, the presence of $\mathcal{K}-$good deals of the first or second type (i.e., the fulfillment of $H2B$ or $H3$) implies that the investor may become as rich as desired with a risk level bounded from above by zero. Since it is not realistic, an investor with uncertainty may estimate the robust market price of risk $\dfrac{1}{\lambda^* - 1}$ according to different criteria. For instance, the empirical evidence could be adequate. Obviously, $\dfrac{1}{\lambda^* - 1} < \infty$, so the robust good deal would not exist with this estimation. If*

$$0 < \frac{1}{\lambda^* - 1} < \infty,$$

then

$$1 < \lambda^* < \infty,$$

and Theorem 12 guarantees the existence of $\mathcal{H}_0 \in \bigcup_{\lambda^*}$ such that $\lambda^* \leq \lambda_{\mathcal{H}_0}^*$ and $\lambda_{\mathcal{H}_0}^* \leq \lambda_{\mathcal{H}}^*$. In other words,

$$\frac{1}{\lambda_{\mathcal{H}}^* - 1} \leq \frac{1}{\lambda_{\mathcal{H}_0}^* - 1} \leq \frac{1}{\lambda^* - 1} < \infty,$$

which implies that the investor can enlarge the degree of ambiguity (the set of priors) by replacing $\mathcal{K}$ with $\mathcal{H}_0$. If so, the market price of risk $\dfrac{1}{\lambda_{\mathcal{H}_0}^* - 1}$ will be finite (so there is no $\mathcal{H}_0-$good deal) and it will be as close as possible to the estimation $\dfrac{1}{\lambda^* - 1}$. Furthermore, the $\mathcal{H}_0 - CML$ will be also as close as possible to that indicated by the empirical evidence. □

6. Switching to the standard deviation

Let us consider the standard deviation instead of the $CVaR$. Although asymmetries and fat tails could make it rather unsuitable to use this risk measure,[5] it plays a central role in financial literature, and it may be worthwhile analyzing its properties in a portfolio selection problem with ambiguity. Most of the proofs are similar to those above, so we will merely summarize the main analogies and differences between both the standard deviation and the $CVaR$.

Consider the set of states $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ and the compact and convex set of priors $\mathcal{K} \subset \mathcal{P}$. Suppose that for every $p \in \mathcal{K}$

$$\sigma_p : \mathbb{R}^n \longrightarrow \mathbb{R}$$

is a deviation measure in the sense of Rockafellar *et al.* (2006). Suppose finally that a parallel result to Proposition 1a may be proved, in the sense that the union of the sets with elements $(p_1 z_1, p_2 z_2, ..., p_n z_n) \in \mathbb{R}^n$, $p \in \mathcal{K}$ and $(z_1, z_2, ..., z_n)$ in the sub-gradient of σ_p, is compact. For instance, similarly to Proposition 1a, we can prove that this property holds for the standard deviation if every $p \in \mathcal{K}$ has strictly positive coordinates, since the sub-gradient of the standard deviation is

$$\Delta_{\sigma_p} = \left\{ (z_1, z_2, ..., z_n) \in \mathbb{R}^n; \sum_{j=1}^{n} p_j z_j = 0, \ \sum_{j=1}^{n} p_j z_j^2 \leq 1 \right\} \tag{35}$$

for every $p \in \mathcal{K}$ (Rockafellar *et al.*, 2006). Denote by ∇_σ the convex and compact convex hull of the union above and by

$$\sigma(y) = Max \ \{\sigma_p(y); p \in \mathcal{K}\}.$$

Due to the differences between expectation bounded risk measures and deviations, $\nabla_{(\mathcal{K},\mu_0)} \subset \mathcal{P}$ in Proposition 1b must be replaced by

$$L(\xi) = 0 \tag{36}$$

for every $\xi \in \nabla_\sigma$,[6] and then (17) becomes

$$\Lambda = \lambda. \tag{37}$$

Let us now consider Problem (14) with σ rather than ρ. Bearing in mind (35), (36), (37) and the comments above, and proceeding as in Section 3, the dual problem becomes

$$\begin{cases} Max \ \left(r - e^{r_f T}\right) \lambda \\ \lambda y_\pi = \varphi_Y(\xi + \lambda p) \\ \xi \in \nabla_\sigma, \ p \in \mathcal{K}, \ \lambda \in \mathbb{R}, \ \lambda \geq 0 \end{cases} \tag{38}$$

[5] obviously, symmetry can hardly be guaranteed by an investor reflecting multiple priors.

[6] See the first constraint in (35).

Proposition 4 still applies with the same proof, and so does Theorem 5 properly modified. Since the existence of (14)-feasible elements does not depend on the selected risk measure, Theorem 6 also applies for σ. As in Remark 3, let us consider disjoint and complementary situations related to (14). We will assume that σ_p is the standard deviation and that therefore σ is a worst-case standard deviation. We will also assume that every $p \in \mathcal{K}$ has strictly positive coordinates.

Remark 6. *H*1. $y_\pi \in \varphi_Y(\mathcal{K})$. *In such a case (14) is not feasible for every* $r > e^{r_f T}$. *We will say that the market is* $\mathcal{K}-$*risk-neutral.*

*H*2. $y_\pi \notin \varphi_Y(\mathcal{K})$. *In such a case Theorem* 6 *guarantees that (14) is feasible and Theorem* 5 *guarantees that (38) achieves its optimal value (if feasible) and there is no duality gap. Suppose that we prove that (38) is feasible with some* $\lambda > 0$. *Then cases* $H2B$ *or* $H3$ *in Remark* 3 *cannot hold and we will be under the conditions of* $H2A$. *Proceeding as in Section* 4 *one can prove the existence of a robust* CML, *a robust market price of risk, systemic and specific risk for every strategy, and adapted versions of Theorem* 9.

Let us prove now that (38) is feasible with some $\lambda > 0$.[7] *Indeed, the first constraint in (38) is equivalent to* $\lambda(y_\pi - \varphi_Y(p)) = \varphi_Y(\xi)$, *which will obviously hold if* $\lambda(y_\pi - p) = \xi$. *Take* $p \in \mathcal{K}$ *and*

$$z_{\pi,j} = \frac{y_{\pi,j}}{p_j},\ j = 1, 2, ..., n.$$

Then, it is sufficient to see that

$$\lambda(z_{\pi,j} - 1)\, p_j = \xi_j,\ j = 1, 2, ..., n. \tag{39}$$

Since $L(y_\pi) = 1$ *(Proposition* 2*b),* $\sum_{j=1}^{n} p_j z_{\pi,j} = 1$, *so*

$$\sum_{j=1}^{n} p_j (z_{\pi,j} - 1) = 0.$$

Hence,

$$\lambda \sum_{j=1}^{n} p_j (z_{\pi,j} - 1) = 0$$

for every $\lambda > 0$, *and*

$$\lambda^2 \sum_{j=1}^{n} p_j (z_{\pi,j} - 1)^2 \leq 1$$

[7]For a general deviation measure this assertion cannot be proved, so the presence of first type good deals might hold. Second type good deals cannot exist because every deviation is bounded from below by zero, and the risk cannot tend to $-\infty$.

if $\lambda > 0$ is small enough. Whence, (35) implies that $(\lambda(z_{\pi,j}-1))_{j=1}^{n} \in \Delta_{\sigma_p}$, so $\xi = (\lambda p_j(z_{\pi,j}-1))_{j=1}^{n} \in \nabla_{\sigma}$, and (39) holds. $\square$

7. Conclusions

The presence of multiple priors is a common property affecting many pricing problems and investment decisions. Recent literature has addressed this problem from different perspectives. This paper considers the most general case because ambiguity may affect both the set of states of nature and the probabilities of the states. Obviously, this framework contains more restricted approaches dealing with ambiguity with respect to volatilities, expected returns, price processes, etc.

In this wide setting a general portfolio choice problem has been studied. We minimize the guaranteed (ambiguous, or robust) $CVaR$ for a given guaranteed (or robust) expected return. We have chosen the $CVaR$ as the risk measure because it presents some advantages when facing asymmetric returns and/or heavy tails, but the analysis may be extended to other coherent and expectation bounded risk measures in a straightforward manner.

Depending on how large the investor uncertainty is, we have shown the existence of four disjoint and complementary possible results of the portfolio choice problem. Under the most realistic one there is a benchmark that creates a robust CML when combined with the riskless asset. Consequently, there is a robust market price of risk for the investor with uncertainty, the global risk (robust $CVaR$) of every portfolio may be divided into systematic and specific, and no robust expected return is paid by the specific risk that can be diversified. Moreover, if there is no ambiguity with respect to the states of nature (only their probabilities are uncertain), then the classical $CAPM-$formulae may be extended to our general context. In this sense, some correlations between every strategy and the benchmark allow us to explain this strategy robust expected return.

Two of the four complementary results above lead to the existence of good deal, *i.e.*, the agent with uncertainty can guarantee every expected return with a maximum (or robust) $CVaR$ bounded from above by zero. This drawback has already been pointed out for very important arbitrage free non ambiguous pricing models. For instance, the Black and Scholes and the Heston model lead to this pathological situation. However, there is a very important difference between the ambiguous and the non ambiguous setting. If we increment the degree of ambiguity, *i.e.*, if we increment the set of priors, then the absence of good deal is guaranteed. Furthermore, the enlargement of the degree of ambiguity may be done in such a manner that the new robust CML and the new robust market price of risk may be as close as possible to those reflected by the empirical evidence. In this sense, the introduction of ambiguity may overcome several caveats of many important pricing models.

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The usual caveat applies.

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