# Existence of competitive equilibrium in an optimal growth model with heterogeneous agents and endogenous leisure*


Aditya Goenka [1], Cuong Le Van[2] and Manh-Hung Nguyen[3]

[1] National University of Singapore

[2]CNRS, Exeter University Business School, Paris School of Economics, VCREM†

[3]LERNA-INRA, Toulouse School of Economics


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**Abstract**

This paper proves the existence of competitive equilibrium in a single-sector dynamic economy with heterogeneous agents and elastic labor supply. The method of proof relies on some recent results concerning the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence and a direct application of the Brouwer fixed point theorem.



**Keywords:** Optimal growth model, Lagrange multipliers, Competitive equilibrium, Elastic labor supply.
**JEL Classification:** *C61, D51, E13, O41*



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†VCREM: VanXuan Center of Research in Economics and Management, Vietnam

# 1 Introduction

Since the seminal work of Ramsey [12], optimal growth models have played a central role in modern macroeconomics. Classical growth theory relies on the assumption that labor is supplied in fixed amounts, although the original paper of Ramsey did include the disutility of labor as an argument in consumers' utility functions. Subsequent research in applied macroeconomics (theories of business cycles fluctuations) have reassessed the role of labor-leisure choice in the process of growth. Nowadays, intertemporal models with elastic labor continue to be the standard setting used to model many issues in applied macroeconomics.

Lagrange multiplier techniques have facilitated considerably the analysis of constrained optimization problems. The applications of those techniques in the analysis of intertemporal models inherits most of the tractability found in a finite setting. However, the passage to an infinite dimensional setting raises additional questions. These questions concern both the extension of the Lagrangean in an infinite dimensional setting as well as the representation of the Lagrange multipliers as a summable sequence.

Our purpose is to prove existence of competitive equilibrium for the basic neoclassical model with elastic labor using some recent results (see Le Van and Saglam [8]) concerning the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence and using less stringent assumptions.

Previous work addressing existence of competitive equilibrium issues in intertemporal models attacks the problem of existence from an abstract point of view. Following the early work of Peleg and Yaari [11], this approach is based on separation arguments applied to arbitrary vector spaces (see Bewley [2], Bewley [3], Aliprantis et al. [1], Dana and Le Van [4]). The advantage of this approach is that it yields general results capable of application in a wide variety of specific models but they require a high level of abstraction and some strong assumptions.

Le Van and Vailakis [9], in order to prove the existence of competitive equilibrium in a model with one representative agent and elastic labor supply impose relatively strong assumptions [1]. In this paper, the existence of equilibrium cannot be established by using marginal utilities since we may have boundary solutions.

Recently, Le Van et al [10] extend the canonical representative Ramsey model to include heterogeneous agents and elastic labor supply where supermodularity is used to establish the convergence of optimal paths. The novelty in their works is that relatively impatient consumers have their consumption and leisure converging to zero and any Pareto optimal capital path converges to a limit point as time tends towards infinity. However, if the limit points of the Pareto optimal capital paths are not bounded away from zero, then their convergence results do not ensure existence of equilibrium.

To obtain the convergence results, they impose strong assumptions which are not used in our paper [2]. Following the Negishi approach, our strategy for tackling

[1]They used assumptions $\frac{u(\epsilon,\epsilon)}{\epsilon} \to +\infty$ as $\epsilon \to 0$ for the proof $c_t > 0, l_t > 0$ and $\frac{u_{cc}}{u_c} \leq \frac{u_{cl}}{u_l}$ for the proof $k_t > 0$ for all $t$.

[2]Le Van et al [10] assumed the cross-partial derivative $u^i_{cl}$ has constant sign, $u^i_c(x,x)$ and $u^i_l(x,x)$ are non-increasing in $x$, production function $F$ is homogenous of degree $\alpha \leq 1$ and $F_{kL} \geq 0$

the question of existence relies on exploiting the link between Pareto-optima and competitive equilibria. We show that there exists a Lagrange multiplier as a price system such that together with the Pareto-optimal solution they constitute an equilibrium with transfers. These transfers depend on the individual weights involved in the social welfare function. An equilibrium exists provided that there is a set of welfare weights such that the corresponding transfers equal zero.

The organization of the paper is as follows. In section 2, we present the model and provide sufficient conditions on the objective function and the constraint functions so that Lagrangean multipliers can be presented by an $l^1_+$ sequence. We characterize some dynamic properties of the Pareto optimal paths of capital and of consumption-leisure. In particular, we prove that the optimal consumption and leisure paths of the most impatient agents will converge to zero in the long run, with a very elementary proof compared to the one in Le Van et al, [10] which uses supermodularity for lattice programming. In section 3, we prove the existence of competitive equilibrium by using the Negishi approach and the Brouwer fixed point theorem.

# 2 The model

We consider an intertemporal model with $m \geq 1$ consumers and one firm. The preferences of each consumer take additively form: $\sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i)$ where $\beta_i \in (0,1)$ is the discount factor $(i = 1, ..., m)$. At date $t$, agent $i$ consumes the quantity $c_t^i$, spends a quantity of leisure $l_t^i$ and supplies a quantity of labor $L_t^i$ and normalized as $l_t^i + L_t^i = 1$. Production possibilities are presented by the gross production function $F$ and a physical depreciation $\delta \in (0,1)$. Denote $F(k_t, \sum_{i=1}^m L_t^i) + (1-\delta)k_t = f(k_t, \sum_{i=1}^m L_t^i)$.

We next specify a set of restrictions imposed on preferences and production technology.[3] The assumptions on period utility function $u^i : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ are as follows:

**U1:** *$u^i$ is continuous, concave, increasing on $\mathbb{R}_+ \times [0,1]$ and strictly increasing, strictly concave on $\mathbb{R}_{++} \times (0,1)$.*

**U2:** $u^i(0,0) = 0$.

**U3:** *$u^i$ is twice continuously differentiable on $\mathbb{R}_{++} \times (0,1)$ with partial derivatives satisfying the Inada conditions:* $\lim_{c \to 0} u_c^i(c,l) = +\infty, \forall l \in (0,1]$ and $\lim_{l \to 0} u_l^i(c,l) = +\infty$, $\forall c > 0$.

We extend the utility functions on $\mathbb{R}^2$ by imposing $u^i(c,l) = -\infty$ if $(c,l) \in \mathbb{R}^2 \setminus \{\mathbb{R}_+ \times [0,1]\}$.

The assumptions on the production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ are as follows:

**F1:** *$F$ is continuous, concave, increasing on $\mathbb{R}_+^2$ and strictly increasing, strictly concave on $\mathbb{R}_{++}^2$.*

(assumptions U4, F4, U5, F5 )

[3] We relaxed some important assumptions in the literature. For example, Bewley [2] assumes that the production set is a convex cone (Theorem 3 , page 525). Also, he assumes the strictly positiveness of derivatives of utility functions on $\mathbb{R}_+^L$ ( Bewley [2], strictly monotonicity assumption, page 240). In our model, the utility functions may not be differentiable in $\mathbb{R}_+ \times [0,1]$ (*only differentiable on* $\mathbb{R}_+ \times (0,1)$) from which many difficulties arise when we deal with boundary points.

**F2:** $F(0,0) = 0$.

**F3:** *$F$ is twice continuously differentiable on $\mathbb{R}^2_{++}$ with partial derivatives satisfying the Inada conditions:* $\lim_{k\to 0} F_k(k,L) = +\infty$, $\forall L > 0$, $\lim_{k\to +\infty} F_k(k,m) < \delta$ and $\lim_{L\to 0} F_L(k,L) = +\infty$, $\forall k > 0$.

We extend the function $F$ over $\mathbb{R}^2$ by imposing $F(k,L) = -\infty$ if $(k,L) \notin \mathbb{R}^2_+$.

For any initial condition $k_0 \geq 0$, when a sequence $\mathbf{k} = (k_0, k_1, k_2, ..., k_t, ...)$ such that $0 \leq k_{t+1} \leq f(k_t, m)$ for all $t$, we say it is feasible from $k_0$ and we denote the class of feasible capital paths by $\Pi(k_0)$. Let $(\mathbf{c}^1, \mathbf{c}^2, ..., \mathbf{c}^i, ..., \mathbf{c}^m)$ where $\mathbf{c}^i = (c^i_0, c^i_1, ...c^i_t, ...)$ denote the vector of consumption and $(\mathbf{l}^1, \mathbf{l}^2, ..., \mathbf{l}^i, ..., \mathbf{l}^m)$ where $\mathbf{l}^i = (l^i_0, l^i_1, ...l^i_t, ...)$ denote the vector of leisure of all agents at date $t$. A pair of consumption-leisure sequences $(\mathbf{c}^i, \mathbf{l}^i) = (c^i_t, l^i_t)^\infty_{t=0}$ is feasible from $k_0 \geq 0$ if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ that satisfies $\forall t$,

$$\sum_{i=1}^m c^i_t + k_{t+1} \leq f(k_t, \sum_{i=1}^m (1 - l^i_t)) \text{ and } 0 \leq l^i_t \leq 1.$$

The set of feasible from $k_0$ consumption-leisure is denoted by $\sum(k_0)$. Assumption **F3** implies that

$$\begin{aligned} f_k(+\infty, m) &= F_k(+\infty, m) + (1-\delta) < 1, \\ f_k(0, m) &= F_k(0, m) + (1-\delta) > 1. \end{aligned}$$

It follows that there exists $\overline{k} > 0$ such that: (i) $f(\overline{k}, m) = \overline{k}$ , (ii) $k > \overline{k}$ implies $f(k,m) < k$, (iii) $k < \overline{k}$ implies $f(k,m) > k$. Therefore for any $\mathbf{k} \in \Pi(k_0)$, we have $0 \leq k_t \leq \max(k_0, \overline{k})$. Thus, a feasible sequence $\mathbf{k}$ is in $l^\infty_+$ which in turn implies that any feasible sequence $(\mathbf{c}, \mathbf{l})$ belongs to $l^\infty_+ \times [0,1]^\infty$.

In what follows, we study the Pareto optimum problem. We obtain that the Lagrange multipliers are in $l^1_+$. Then these multipliers will be used to define prices and wages systems for the equilibrium.

Let $\Delta = \{\eta_1, \eta_2, ..., \eta_m | \eta_i \geq 0 \text{ and } \sum_{i=1}^m \eta_i = 1\}$. Given a vector of welfare weights $\eta \in \Delta$, define the Pareto problem

$$\begin{aligned} \max \quad & \sum_{i=1}^m \eta_i \sum_{t=0}^\infty \beta^t_i u^i(c^i_t, l^i_t) \\ s.t. \quad & \sum_{i=1}^m c^i_t + k_{t+1} \leq f(k_t, \sum_{i=1}^m (1 - l^i_t)), \forall t \\ & c^i_t \geq 0,\ l^i_t \geq 0,\ l^i_t \leq 1,\ \forall i, \forall t \\ & k_t \geq 0,\ \forall t \text{ and } k_0 \text{ given.} \end{aligned}$$

Note that, for all $k_0 \geq 0$, $0 \leq k_t \leq \max(k_0, \overline{k})$, then $0 \leq c^i_t \leq f(\max(k_0, \overline{k}), m)$ $\forall t$, $\forall i = 1...m$. Therefore, the sequence $(u^i)_n = \sum_{i=1}^n \beta^t_i u^i(c^i_t, l^i_t)$ is increasing and bounded and will converge. Thus we can write

$$\sum_{i=1}^m \eta_i \sum_{t=0}^\infty \beta^t_i u^i(c^i_t, l^i_t) = \sum_{t=0}^\infty \sum_{i=1}^m \eta_i \beta^t_i u^i(c^i_t, l^i_t).$$

Let $\mathbf{x} = (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in (l^\infty_+)^m \times l^\infty_+ \times (l^\infty_+)^m$.

Define

$$
\begin{aligned}
\mathcal{F}(\mathbf{x}) &= -\sum_{t=0}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u^i(c_t^i, l_t^i) \\
\Phi_t^1(\mathbf{x}) &= \sum_{i=1}^{m} c_t^i + k_{t+1} - f(k_t, \sum_{i=1}^{m}(1-l_t^i)) \\
\Phi_t^{2i}(\mathbf{x}) &= -c_t^i \\
\Phi_t^{3}(\mathbf{x}) &= -k_t \\
\Phi_t^{4i}(\mathbf{x}) &= -l_t^i \\
\Phi_t^{5i}(\mathbf{x}) &= l_t^i - 1 \\
\Phi_t &= (\Phi_t^1, \Phi_t^{2i}, \Phi_{t+1}^3, \Phi_t^{4i}, \Phi_t^{5i}),\ \forall t, \forall i = 1...m
\end{aligned}
$$

The Pareto problem can be written as:

$$
\begin{aligned}
&\min \mathcal{F}(\mathbf{x}) \\
\text{s.t.}\Phi(\mathbf{x}) \quad &\leq \quad \mathbf{0}, \mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m
\end{aligned}
$$

where:

$$
\begin{aligned}
\mathcal{F} \quad &: \quad (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m \to \mathbb{R}\cup\{+\infty\} \\
\Phi \quad &= \quad (\Phi_t)_{t=0\ldots\infty} : (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m \to \mathbb{R}\cup\{+\infty\} \\
\text{Let } C \quad &= \quad dom(\mathcal{F}) = \{\mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m | \mathcal{F}(\mathbf{x}) < +\infty\} \\
\Gamma \quad &= \quad dom(\Phi) = \{\mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m | \Phi_t(\mathbf{x}) < +\infty,\ \forall t\}.
\end{aligned}
$$

The following theorem follows from Theorem1 and Theorem2 in Le Van and Saglam [8].

**Theorem 1** *Let* $\mathbf{x}$, $\mathbf{y} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$, $T \in N$.

*Define* $x_t^T(\mathbf{x},\mathbf{y}) = \begin{cases} x_t & if \quad t \leq T \\ y_t & if \quad t > T \end{cases}$

*Suppose that two following assumptions are satisfied:*

***T1****: If* $\mathbf{x} \in C$, $\mathbf{y} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$ *satisfy* $\forall T \geq T_0$, $\mathbf{x}^T(\mathbf{x},\mathbf{y}) \in C$ *then* $\mathcal{F}(\mathbf{x}^T(\mathbf{x},\mathbf{y})) \to \mathcal{F}(\mathbf{x})$ *when* $T \to \infty$.

***T2****: If* $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$ *and* $\mathbf{x}^T(\mathbf{x},\mathbf{y}) \in \Gamma$, $\forall T \geq T_0$, *then*

$$
\begin{aligned}
&a)\ \Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y})) \to \Phi_t(\mathbf{x}) as\ T \to \infty \\
&b)\ \exists M s.t. \forall T \geq T_0, \|\Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y}))\| \leq M \\
&c)\ \forall N \geq T_0, \lim_{t\to\infty}[\Phi_t(\mathbf{x}^T(\mathbf{x},\mathbf{y})) - \Phi_t(\mathbf{y})] = 0
\end{aligned}
$$

*Let* $\mathbf{x}^*$ *be a solution to (****P****) and* $\bar{\mathbf{x}} \in C$ *satisfy the Strong Slater condition:*

$$\sup_t \Phi_t(\bar{\mathbf{x}}) < 0.$$

*Suppose* $\mathbf{x}^T(\mathbf{x}^*,\bar{\mathbf{x}}) \in C \cap \Gamma$. *Then, there exists* $\mathbf{\Lambda} \in l_+^1 \backslash \{0\}$ *such that*

$$\mathcal{F}(\mathbf{x}) + \mathbf{\Lambda}\Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda\Phi(\mathbf{x}^*),\ \forall \mathbf{x} \in (l^\infty)^m \times l^\infty \times (l^\infty)^m$$

*and* $\Lambda\Phi(\mathbf{x}^*) = 0$.

Obviously, for any $\eta \in \Delta$, an optimal path will depend on $\eta$. In what follows, if possible, we will suppress $\eta$ and denote by $(\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{L}^{*i}, \mathbf{l}^{*i})$ any optimal path for each agent $i$. The following proposition characterize the Lagrange multipliers of the Pareto problem.

**Proposition 1** *If $\mathbf{x}^* = (\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i})$ is a solution to the Pareto problem:*

$$\max \sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u^i(c_t^i, l_t^i) \tag{Q}$$

$$\begin{aligned} s.t. \sum_{i=1}^{m} c_t^i + k_{t+1} &\leq f(k_t, L_t^*),\ \forall t \geq 0 \\ c_t^i &\geq 0,\ l_t^i \geq 0,\ l_t^i \leq 1,\ \forall i, \forall t \\ k_t &\geq 0,\ \forall t \text{ and } k_0 \text{ given.} \end{aligned}$$

*Then there exists, $\forall i = 1...m$, $\lambda = (\lambda^1, \lambda^{2i}, \lambda^3, \lambda^{4i}, \lambda^{5i}) \in l_+^1 \times (l_+^1)^m \times l_+^1 \times (l_+^1)^m \times (l_+^1)^m$ $\lambda \neq \mathbf{0}$ such that*

$$\begin{aligned} &\sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u^i(c_t^{*i}, l_t^{*i}) - \sum_{t=0}^{\infty} \lambda_t^1 (\sum_{i=1}^{m} c_t^{*i} + k_{t+1}^* - f(k_t^*, L_t^*)) \\ &+ \sum_{t=0}^{\infty} \sum_{i=1}^{m} \lambda_t^{2i} c_t^{*i} + \sum_{t=0}^{\infty} \lambda_t^3 k_t^* + \sum_{t=0}^{\infty} \sum_{i=1}^{m} \lambda_t^{4i} l_t^{*i} + \sum_{t=0}^{\infty} \sum_{i=1}^{m} \lambda_t^{5i} (1 - l_t^{*i}) \end{aligned}$$

$$\begin{aligned} \geq\ &\sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u^i(c_t^i, l_t^i) - \sum_{t=0}^{\infty} \lambda_t^1 (\sum_{i=1}^{m} c_t^i + k_{t+1} - f(k_t, L_t)) \\ &+ \sum_{t=0}^{\infty} \sum_{i=1}^{m} \lambda_t^{2i} c_t^i + \sum_{t=0}^{\infty} \lambda_t^3 k_t + \sum_{t=0}^{\infty} \sum_{i=1}^{m} \lambda_t^{4i} l_t^i + \sum_{t=0}^{\infty} \sum_{i=1}^{m} \lambda_t^{5i} (1 - l_t^i) \end{aligned} \tag{1}$$

$$\lambda_t^1 \left[ \sum_{i=1}^{m} c_t^{*i} + k_{t+1}^* - f(k_t^*, \sum_{i=1}^{m} L_t^{*i}) \right] = 0 \tag{2}$$

$$\lambda_t^{2i} c_t^{*i} = 0, \forall i = 1...m \tag{3}$$

$$\lambda_t^3 k_t^* = 0 \tag{4}$$

$$\lambda_t^{4i} l_t^{*i} = 0, \forall i = 1...m \tag{5}$$

$$\lambda_t^{5i} (1 - l_t^{*i}) = 0, \forall i = 1...m \tag{6}$$

$$0 \in \eta_i \beta_i^t \partial_1 u^i(c_t^{*i}, l_t^{*i}) - \{\lambda_t^1\} + \{\lambda_t^{2i}\}, \forall i = 1...m \tag{7}$$

$$0 \in \eta_i \beta_i^t \partial_2 u^i(c_t^{*i}, l_t^{*i}) - \lambda_t^1 \partial_2 f(k_t^*, L_t^*) + \{\lambda_t^{4i}\} - \{\lambda_t^{5i}\}, \forall i = 1...m \tag{8}$$

$$0 \in \lambda_t^1 \partial_1 f(k_t^*, L_t^*) + \{\lambda_t^3\} - \{\lambda_{t-1}^1\} \tag{9}$$

*where, $L_t^* = \sum_{i=1}^m L_t^{*i} = \sum_{i=1}^m (1 - l_t^{*i})$, $\partial_j u(c_t^{*i}, l_t^{*i}), \partial_j f(k_t^*, L_t^*)$ respectively denote the projection on the $j^{th}$ component of the subdifferential of function $u$ at $(c_t^{*i}, l_t^{*i})$ and the function $f$ at $(k_t^*, L_t^*)$*[4]

[4]For a concave function $f$ defined on $\mathbb{R}^n, \partial f(x)$ denotes the subdifferential of $f$ at $x$. We have to write the first-order conditions by the subgradient set since at the point $(0, 0)$, the functions $u^i$ and $f$ are not assumed to be differentiable.

**Proof**: We show that the Strong Slater condition holds. Since $f_k(0,m) > 1$,[5]for all $k_0 > 0$, there exists some $\widehat{k} \in (0,k_0)$ such that: $0 < \widehat{k} < f(\widehat{k},m)$ and $0 < \widehat{k} < f(k_0,m)$.Thus, there exists two small positive numbers $\varepsilon$, $\varepsilon_1$ such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, m - \varepsilon_1) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, m - \varepsilon_1).$$

Denote $\bar{\mathbf{x}} = (\bar{\mathbf{c}}, \bar{\mathbf{k}}, \bar{\mathbf{l}})$ where $\bar{\mathbf{c}} = (\bar{\mathbf{c}}^i)_{i=1}^m$, and

$$\bar{\mathbf{c}}^i = (\bar{c_t}^i)_{t=0,\dots\infty} = (\frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \dots)$$

$\bar{\mathbf{l}} = (\bar{\mathbf{l}}^i)_{i=1}^m$, where

$$\bar{\mathbf{l}}^i = (\bar{l_t}^i)_{t=0,\dots\infty} = (\frac{\varepsilon_1}{m}, \frac{\varepsilon_1}{m}, \frac{\varepsilon_1}{m}, \dots\dots).$$

and $\bar{\mathbf{k}} = (k_0, \widehat{k}, \widehat{k}, \dots)$. We have

$$\begin{aligned}
\Phi_0^1(\bar{\mathbf{x}}) &= \sum_{i=0}^m c_0^i + k_1 - f(k_0, \sum_{i=1}^m (1 - l_0^i)) \\
&= \varepsilon + \widehat{k} - f(k_0, m - \varepsilon_1) < 0 \\
\Phi_1^1(\bar{\mathbf{x}}) &= \sum_{i=0}^m c_1^i + k_2 - f(k_1, \sum_{i=1}^m (1 - l_1^i)) \\
&= \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon_1) < 0 \\
\Phi_t^1(\bar{\mathbf{x}}) &= \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon_1) < 0, \ \forall t \geq 2
\end{aligned}$$

$$\Phi_t^{2i}(\bar{\mathbf{x}}) = -\bar{c_t}^i = -\frac{\varepsilon}{m} < 0, \ \forall t \geq 0, \forall i = 1\dots m$$

$$\begin{aligned}
\Phi_0^3(\bar{\mathbf{x}}) &= -k_0 < 0; \\
\Phi_t^3(\bar{\mathbf{x}}) &= -\widehat{k} < 0 \ \ \forall t \geq 1.
\end{aligned}$$

$$\Phi_t^{4i}(\bar{\mathbf{x}}) = -\frac{\varepsilon_1}{m} < 0, \ \forall t \geq 0, \forall i = 1\dots m$$

$$\Phi_t^{5i}(\bar{\mathbf{x}}) = \frac{\varepsilon_1}{m} - 1 < 0, \ \forall t \geq 0, \forall i = 1\dots m$$

Therefore the Strong Slater condition is satisfied.

It is obvious that, $\forall T$, $\mathbf{x}^T(\mathbf{x}^*,\bar{\mathbf{x}})$ belongs to $(l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$.

As in Le Van and Saglam [8], Assumption **T2** is satisfied. We now check Assumption **T1**.

For any $\widetilde{\mathbf{x}} \in C, \widetilde{\widetilde{\mathbf{x}}} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$ such that for any $T$, $\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}}) \in C$ we have

$$\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) = -\sum_{t=0}^T \sum_{i=1}^m \eta_i \beta_i^t u^i(\widetilde{c_t^i}, \widetilde{l_t^i}) - \sum_{t=T+1}^\infty \sum_{i=1}^m \eta_i \beta_i^t u^i(\widetilde{\widetilde{c_t^i}}, \widetilde{\widetilde{l_t^i}}).$$

As $\widetilde{\widetilde{\mathbf{x}}} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$, $\sup_t |\widetilde{\widetilde{c_t}}| < +\infty$ , there exists $a > 0, \forall t, |\widetilde{\widetilde{c_t}}| \leq a$. Since $\beta \in (0,1)$, as $T \to \infty$ we have

$$0 \leq \sum_{t=T+1}^\infty \sum_{i=1}^m \eta_i \beta_i^t u^i(\widetilde{\widetilde{c_t^i}}, \widetilde{\widetilde{l_t^i}}) \leq u(a,1) \sum_{t=T+1}^\infty \sum_{i=1}^m \eta_i \beta_i^t = u(a,1) \sum_{i=1}^m \sum_{t=T+1}^\infty \eta_i \beta_i^t \to 0$$

[5]Assumption $f_k(0,1) > 1$ is equivalent to the Adequacy Assumption in Bewley [2], see Le Van and Dana [6] Remark 6.1.1. This assumption is crucial to have equilibrium prices in $l_+^1$ since it implies that the production set has an interior point. Subsequently, one can use a separation theorem in the infinite dimensional space to derive Lagrange multipliers.

where $u(a,1) = \max\{u_i(a,1), i = 1,...,m\}$. Hence, $\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) \to \mathcal{F}(\widetilde{\mathbf{x}})$ when $T \to \infty$. Taking account of the Theorem 1, we get (1) - (6).

Obviously, $\cap_{i=1}^m ri(dom(u^i)) \neq \emptyset$ where $ri(dom(u^i))$ is the relative interior of $dom(u^i)$. It follows from the Proposition 6.5.5 in Florenzano and Le Van [5], we have

$$\partial \sum_{i=1}^m \eta_i \beta_i^t u^i(c_t^{*i}, l_t^{*i}) = \eta_i \beta_i^t \sum_{i=1}^m \partial u^i(c_t^{*i}, l_t^{*i})$$

We then get (7) - (9) as the Kuhn-Tucker first-order conditions. ∎

**Remark 1** *1. 1. It is easy to prove that $\eta_i = 0 \Rightarrow c_t^{*i} = 0,\ l_t^{*i} = 0,\ \forall t$.*

*2. For any optimal solution $(\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i})$, we have for any $t$, any $i$, $\partial_1 u^i(c_t^{*i}, l_t^{*i}) \neq \emptyset$, $\partial_2 u^i(c_t^{*i}, l_t^{*i}) \neq \emptyset$, $\partial_1 f(k_t^*, L_t^*) \neq \emptyset$, $\partial_2 f(k_t^*, L_t^*) \neq \emptyset$, where $L_t^* = m - \sum_i l_t^{*i}$.*

*3. We have $c_t^{*i} > 0$ iff $l_t^{*i} > 0$. In this case, $\partial_1 u^i(c_t^{*i}, l_t^{*i}) = \{u_c^i(c_t^{*i}, l_t^{*i})\}, \partial_2 u^i(c_t^{*i}, l_t^{*i}) = \{u_l^i(c_t^{*i}, l_t^{*i})\}$.*

*4. For any $k_0 > 0$, there exists $t$ with $\sum_i c_t^{*i} > 0$ and hence $\sum_i l_t^{*i} > 0$ (if not, the value of the Pareto problem is null which is a contradiction).*

Let us denote $I = \{i\ |\eta_i > 0\}$, $\beta = \max\{\beta_i | i \in I\}$, $I_1 = \{i \in I \mid \beta_i = \beta\}$ and $I_2 = \{i \in I \mid \beta_i < \beta\}$.

In the following proposition, we will prove the positiveness of the optimal capital path.

**Proposition 2** *If $k_0 > 0$, the optimal capital path satisfies $k_t^* > 0, \forall t$.*

**Proof**: Let $k_0 > 0$ but assume that $k_1^* = 0$. From (9), $L_1^* = 0$. This implies $\sum_i c_1^{*i} = 0$ and $l_1^{*i} = 1, \forall i$: a contradiction with (7). Hence $k_1^* > 0$. By induction, $k_t^* > 0, \forall t > 0$.

∎

**Remark 2** *From (9) and Proposition 2, if $k_0 > 0$, we have $L_t^* > 0$ for any $t \geq 0$. Hence, for any $t \geq 0$, $\partial_1 f(k_t^*, L_t^*) = \{f_k(k_t^*, L_t^*)\}$, $\partial_2 f(k_t^*, L_t^*) = \{f_L(k_t^*, L_t^*)\}$.*

**Proposition 3** *Let $k_0 > 0$. (a) With any $\eta \in \Delta$, there exists a unique solution to the Pareto problem $\big((\mathbf{c}^{*i}), (\mathbf{l}^{*i}), \mathbf{k}^*\big)$. We have:*

*For any $t \geq 0$,*

$$\lambda_t^1(\eta) \in \cap_{i \in I}\ \eta_i \beta_i^t \partial_1 u^i(c_t^{*i}, l_t^{*i}) \tag{10}$$

$$\lambda_t^1(\eta) f_L(k_t^*, L_t^*) \in \cap_{i \in I}\ \eta_i \beta_i^t \partial_2 u^i(c_t^{*i}, l_t^{*i}) \tag{11}$$

*and for any $t \geq 1$,*

$$0 \in \lambda_t^1(\eta) \partial_1 f(k_t^*, L_t^*) - \lambda_{t-1}^1(\eta) \tag{12}$$

*(b) Conversely, if the sequences* $\mathbf{c}^{*i}, \mathbf{l}^{*i}, \mathbf{k}^*, \mathbf{L}^*$ *satisfy*

$$\begin{aligned} \forall t \geq 0,\ L_t^* &= \sum_i (1 - l_t^{*i}) \\ \sum_i c_t^{*i} &= f(k_t^*, L_t^*) - k_{t+1}^* \\ k_0^* &= k_0 \end{aligned}$$

*and if there exists* $\lambda^1 \in l_+^1$ *which satisfies (10), (11) and (12), then* $\mathbf{c}^{*i}, \mathbf{l}^{*i}, \mathbf{k}^*$ *solve the Pareto problem with weights* $\eta$ *and* $\lambda^1$ *is an associated multiplier.*

**Proof**: (a) For any $c_t \geq 0$, we have

$$\begin{aligned} \eta_i \beta_i^t u^i(c_t^{*i}, l_t^{*i}) - \eta_i \beta_i^t u^i(c_t, l_t^{*i}) &\geq (\lambda_t^1 - \lambda_t^{2i})(c_t^{*i} - c_t) \\ &\geq \lambda_t^1 (c_t^{*i} - c_t) + \lambda_t^{2i} c_t \geq \lambda_t^1 (c_t^{*i} - c_t) \end{aligned}$$

If $c_t < 0$,, then $u^i(c_t, l_t^{*i}) = -\infty$, and the inequality still holds. Thus, $\lambda_t^1(\eta) \in \eta_i \beta_i^t \partial_1 u^i(c_t^{*i}, l_t^{*i})$, $\forall i$.
Similarly, we can prove $\lambda_t^1(\eta) f_L(k_t^*, L_t^*) \in \cap_{i \in I}\ \eta_i \beta_i^t \partial_2 u^i(c_t^{*i}, l_t^{*i})$.

We have from (9),

$$\begin{aligned} \lambda_t^1(\eta) \left[ f(k_t^*, L_t^*) - f(k, L_t^*) \right] &\geq [\lambda_{t-1}^1 - \lambda_t^3](k_t^* - k) \\ &\geq \lambda_{t-1}^1 (k_t^* - k) + \lambda_t^3 k \geq \lambda_{t-1}^1 (k_t^* - k), \text{ if } k \geq 0. \end{aligned}$$

If $k < 0$, then $f(k, L) = -\infty$ and the inequality still holds.

(b) The proof is easy. ■

When $m > 1$ we introduce an additional assumption which, together with **U1, U2, U3, F1, F2, F3** , will ensures positiveness of the total optimal consumptions for any period.

**U4:** There exists an agent $i$ with a utility function which satisfies $\lim_{c \to 0_+} \frac{u^i(c,0)}{c} = +\infty$.

**Proposition 4** *Add* ***U4****. Assume* $k_0 > 0$*. Then* $\sum_i c_t^{*i} > 0$ *(and hence* $\sum_i l_t^{*i} > 0$*) for any* $t$*.*

**Proof**: Assume the contrary. There exist $t$ with $c_t^{*i} = l_t^{*i} = 0$ for every $i \in I$. Take $i$ which satisfies **U4**. Then for any $c > 0$,

$$\eta_i \beta_i^t \left[ u^i(0,0) - u^i(c,0) \right] \geq -\lambda_t^1 c$$

since $\lambda^1 \in \eta_i \beta_i^t \partial_1 u^i(0,0)$ from Proposition 3. We then obtain

$$\begin{aligned} \lambda_t^1 &\geq \eta_i \beta_i^t \frac{u^i(c,0)}{c},\ \forall c > 0 \\ \text{and } \lambda_t^1 &\geq \eta_i \beta_i^t \lim_{c \to +\infty} \frac{u^i(c,0)}{c} = +\infty \end{aligned}$$

contradicting $\boldsymbol{\lambda}^{\mathbf{1}} \in l^1$. ■

We now show that the consumption and leisure paths of all agents with a discount factor less than the maximum one converge to zero. The proof is very simple compared to the one in Le Van et al [10] which uses the supermodular structure inspired by lattice programming.

**Proposition 5** *If* $(\mathbf{k}^*, \mathbf{c}^{*i}, \mathbf{l}^{*i})$ *denotes the optimal path starting from* $k_0$, *then* $\forall i \in I_2$, $c_t^{*i} \longrightarrow 0$ *and* $l_t^{*i} \longrightarrow 0$.

**Proof**: Consider problem $\mathcal{R}_t$

$$\begin{aligned} V_t(k_t, k_{t+1}) &= \max \sum_{i=1}^{m} \eta_i \beta_i^t u^i(c_t^i, l_t^i) \\ \text{s.t. } \sum_{i=1}^{m} c_t^i + k_{t+1} &\leq F(k_t, \sum_{i=1}^{m}(1 - l_t^i)) + (1-\delta)k_t \ . \end{aligned}$$

It is easy to see that the Pareto problem is equivalent to

$$\begin{aligned} & \max \sum_{t=0}^{\infty} V_t(k_t, k_{t+1}) \\ \text{s.t. } 0 \leq & \ k_{t+1} \leq F(k_t, m) + (1-\delta)k_t, \ \forall t \geq 0 \\ & k_0 \text{ is given.} \end{aligned}$$

Observe that

$$\begin{aligned} V_t(k_t, k_{t+1}) &= \beta^t \max \sum_{i=1}^{m} \eta_i \left(\frac{\beta_i}{\beta}\right)^t u^i(c_t^i, l_t^i) \\ \text{s.t. } \sum_{i=1}^{m} c_t^i + k_{t+1} &\leq F(k_t, \sum_{i=1}^{m}(1 - l_t^i)) + (1-\delta)k_t \ . \end{aligned}$$

Denote $Z^t = \left(\eta_i(\frac{\beta_i}{\beta})^t\right)$. From Berge Theorem, the strict concavity and the increasingness of the utility functions, the optimal $c^{*i}$, $l^{*i}$ are continuous with respect to $(Z^t, k_t, k_{t+1})$. Denote these functions by $\left(\Gamma^i(Z^t, k_t^*, k_{t+1}^*), \Lambda^i(Z^t, k_t^*, k_{t+1}^*)\right)_i$. Let $\kappa^*, \xi^*$ denote the limit points of $k_t^*, k_{t+1}^*$ when $t \to +\infty$. Then, for $i \in I_2$, $\Gamma^i(Z^t, k_t^*, k_{t+1}^*)$ converges to $\Gamma^i(0_{I_2}, (\eta_i)_{i \in I_1}, \kappa^*, \xi^*) = 0$, and $\Lambda^i(Z^t, k_t^*, k_{t+1}^*)$ converges to $\Lambda^i(0_{I_2}, (\eta_i)_{i \in I_1}, \kappa^*, \xi^*) = 0$. ■

# 3 Existence of competitive equilibrium

Let us give the characterization of equilibrium. For each consumer $i$, let $\alpha^i > 0$ denote the share of the profit of the firm which is owned by consumer $i$. We have $\sum_{i=1}^m \alpha^i = 1$. Let $\vartheta^i > 0$ be the share of the initial endowment owned by consumer $i$. Obviously, $\sum_{i=1}^m \vartheta^i = 1$. Clearly, $\vartheta^i \ k_0$ is the endowment of consumer $i$.

**Definition 1** *Let* $k_0 > 0$. *A competitive equilibrium for this model consists of a sequence of price* $\mathbf{p}^* = (p_t^*)_{t=0}^{\infty}$ *for the consumption good, a wage sequence* $\mathbf{w}^* = (w_t^*)_{t=0}^{\infty}$ *for the labor, a price* $r$ *for the initial capital stock* $k_0$ *and an allocation* $\{\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i}, \mathbf{L}^{*i}\}$ *such that*

*i)*

$$\begin{aligned} \mathbf{c}^* &\in l_+^{\infty}, \mathbf{l}^{*i} \in l_+^{\infty}, \mathbf{L}^{*i} \in l_+^{\infty}, \mathbf{k}^* \in l_+^{\infty}, \\ \mathbf{p}^* &\in l_+^1 \backslash \{0\}, \mathbf{w}^* \in l_+^1 \backslash \{0\}, r > 0. \end{aligned}$$

*ii)For every $i$, $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ is a solution to the problem*

$$\begin{array}{rl} & \max \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i) \\ s.t & \sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \ \leq \sum_{t=0}^{\infty} w_t^* + \vartheta^i r k_0 + \alpha^i \pi^* \end{array}$$

*where $\pi^*$ is the maximum profit of the single firm.*

*iii) $(\mathbf{k}^*, \mathbf{L}^*)$ is a solution to the firm's problem*

$$\begin{aligned} \pi^* &= \max \sum_{t=0}^{\infty} p_t^* [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \\ st \quad 0 &\leq k_{t+1} \leq f(k_t, L_t), 0 \leq L_t, \forall t \end{aligned}$$

*iv) Markets clear: $\forall t$,*

$$\sum_{t=1}^{m} c_t^{*i} + k_{t+1}^* = f(k_t^*, \sum_{t=1}^{m} L_t^{*i}),$$

$$l_t^{*i} + L_t^{*i} = 1, L_t^* = \sum_{t=1}^{m} L_t^{i^*} \textit{ and } k_0^* = k_0.$$

Now we define an equilibrium with transfers.

**Definition 2** *A given allocation $\{\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i}, \mathbf{L}^{*i}\}$, together with a price sequence $\mathbf{p}^*$ for consumption good, a wage sequence $\mathbf{w}^*$ for labor and a price $r$ for the initial capital stock $k_0$ constitute an equilibrium with transfers if*

*i)*

$$\begin{aligned} \mathbf{c}^* &\in (l_+^{\infty})^m, \mathbf{l}^* \in (l_+^{\infty})^m, \mathbf{L}^* \in (l_+^{\infty})^m, \mathbf{k}^* \in l_+^{\infty}, \\ \mathbf{p}^* &\in l_+^1 \backslash \{0\}, \mathbf{w}^* \in l_+^1 \backslash \{0\}, \ r > 0 \end{aligned}$$

*ii) For every $i = 1...m$, $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ is a solution to the problem*

$$\begin{array}{rl} & \max \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i) \\ st & \sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \leq \sum_{t=0}^{\infty} p_t^* c_t^{*i} + \sum_{t=0}^{\infty} w_t^* l_t^{*i} \end{array}$$

*iii) $(\mathbf{k}^*, \mathbf{L}^*)$ is a solution to the firm's problem:*

$$\begin{aligned} \pi^* &= \max \sum_{t=0}^{\infty} p_t^* [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - r k_0 \\ st \quad 0 &\leq k_{t+1} \leq f(k_t, L_t), 0 \leq L_t, \forall t \end{aligned}$$

*iv) Markets clear*

$$\begin{array}{rl} \forall t, \ \sum_{i=1}^{m} c_t^{*i} + k_{t+1}^* = & f(k_t^*, \sum_{i=1}^{m} L_t^{*i}), \\ L_t^* = \sum_{i=1}^{m} L_t^{*i}, \ l_t^{*i} = 1 - L_t^{*i} & \textit{and } k_0^* = k_0 \end{array}$$

Before proving existence of an equilibrium, we will first prove that any $\mathbf{x}^* = (\mathbf{c}^{*i}, \mathbf{k}^*, \mathbf{l}^{*i})$, *solution to the Pareto problem* associated with $k_0 > 0$ and $\eta \in \Delta$ is an equilibrium with transfers, with some appropriate prices $(p_t^*) \in l_+^1 \setminus \{0\}$ and wages $(w_t^*) \in l_+^1 \setminus \{0\}$.

The following result is required.

**Proposition 6** *Let $k_0 > 0$. 1. For any $\varepsilon > 0$, there exists $T$ such that, for any $\eta \in \Delta$,*

$$\sum_T^{+\infty} {\lambda^1}_t(\eta) \sum_i c_t^{*i} \leq \varepsilon$$

$$\sum_T^{+\infty} {\lambda^1}_t(\eta) f_L(k_t^*, L_t^*) \sum_i l_t^{*i} \leq \varepsilon$$

$$\sum_T^{+\infty} {\lambda^1}_t(\eta) f_L(k_t^*, L_t^*) \leq \varepsilon$$

*2. There exists $M$ such that, for any $\eta \in \Delta$,*

$$\begin{aligned} \sum_{t=0}^{+\infty} {\lambda^1}_t(\eta) \sum_i c_t^{*i} &\leq M \\ \sum_{t=0}^{+\infty} {\lambda^1}_t(\eta) f_L(k_t^*, L_t^*) \sum_i l_t^{*i} &\leq M \\ \sum_{t=0}^{+\infty} {\lambda^1}_t(\eta) f_L(k_t^*, L_t^*) &\leq M \end{aligned}$$

**Proof**: 1. We know that there exists $A$ such that $c_t^{*i}(\eta) \leq A$, $\forall t$, $\forall i$, $\forall \eta \in \Delta$. Therefore

$$\begin{aligned} \frac{\beta^T}{1-\beta} \sum_i u^i(A, 1) &\geq \sum_T^{+\infty} \sum_i \eta_i \beta_i^t [u^i(c_t^{*i}, l_t^{*i}) - u^i(0,0)] \\ &\geq \sum_T^{+\infty} \lambda_t^1 \sum_i c_t^{*i} + \sum_T^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \sum_i l_t^{*i} \end{aligned}$$

Let $\varepsilon > 0$. There exists $T$ such that $\frac{\beta^T}{1-\beta} \leq \varepsilon$. Hence, $\sum_T^{+\infty} {\lambda^1}_t(\eta) \sum_i c_t^{*i} \leq \varepsilon$, $\sum_T^{+\infty} {\lambda^1}_t(\eta) f_L(k_t^*, L_t^*) \sum_i l_t^{*i} \leq \varepsilon$, for any $\eta$.

We now prove that for $T$ large enough, $\sum_T^{+\infty} {\lambda^1}_t(\eta) f_L(k_t^*, L_t^*) \leq \varepsilon$ for any $\eta$. We have

$$\sum_i c_t^{*i} = f(k_t^*, L_t^*) - k_{t+1}^*$$

Since

$$f(k_t^*, L_t^*) = f(k_t^*, L_t^*) - f(0,0) \geq f_k(k_t^*, L_t^*) k_t^* + f_L(k_t^*, L_t^*) L_t^*$$

we obtain by using (9):

$$\sum_{t=T}^{T+\tau} \lambda_t^1 \sum_i c_t^{*i} \geq \lambda_T^1 f_k(k_T^*, L_T^*) k_T^* - \lambda_{T+\tau}^1 k_{T+\tau+1}^* + \sum_{t=T}^{T+\tau} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^*$$

Let $\tau \to +\infty$. Since $\lambda^1 \in l^1$, and $k_t^* \le \max\{k_0, \bar{k}\}, \forall t$, we have

$$\begin{aligned}\sum_{t=T}^{+\infty} \lambda_t^1 \sum_i c_t^{*i} &\ge \lambda_T^1 f_k(k_T^*, L_T^*) k_T^* + \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* \\ &\ge \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^* = \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*)(m - \sum_i l_t^{*i})\end{aligned}$$

Hence, for $T$ large enough,

$$m \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \le \sum_{t=T}^{+\infty} \lambda_t^1 \sum_i c_t^{*i} + \sum_{t=T}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \sum_i l_t^{*i} \le \varepsilon$$

for any $\eta$.
2. Obviously:

$$\begin{aligned}\sum_0^{+\infty} \lambda_t^1 \sum_i c_t^{*i} + \sum_0^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) \sum_i l_t^{*i} &\le M_1 = \frac{1}{1-\beta} \sum_i u^i(A, 1) \\ \sum_{t=0}^{+\infty} \lambda_t^1 f_L(k_t^*, L_t^*) &\le M_2 = \frac{2}{m} \times \frac{1}{1-\beta} \sum_i u^i(A, 1)\end{aligned}$$

■

**Proposition 7** *Let $k_0 > 0$. Let $(\mathbf{k}^*, \mathbf{c}^*, \mathbf{L}^*, \mathbf{l}^*)$ solve the Pareto problem associated with $\eta \in \Delta$. Take*

$$\begin{aligned} p_t^* &= \lambda_t^1, \ w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*) \text{ for any } t \\ \text{and } r &= \lambda_0^1 [F_k(k_0, 0) + 1 - \delta].\end{aligned}$$

*Then $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ is an equilibrium with transfers .*

**Proof**: i) We have

$$\mathbf{c}^* \in (l_+^\infty)^m, \mathbf{l}^* \in (l_+^\infty)^m, \mathbf{k}^* \in l_+^\infty, \mathbf{p}^* \in l_+^1, \mathbf{w}^* \in l_+^1.$$

From Remark 1 statement 4, $\mathbf{p}^* \neq \mathbf{0}$, and together with Remark 2, $\mathbf{w}^* \neq \mathbf{0}$.
ii) We now show that $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ solves the consumer's problem. Let $(\mathbf{c}^i, \mathbf{l}^i)$ satisfies

$$\sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \le \sum_{t=0}^{\infty} p_t^* c_t^{*i} + \sum_{t=0}^{\infty} w_t^* l_t^{*i}.$$

By the concavity of $u^i$, we have:

$$\Delta = \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^{*i}, l_t^{*i}) - \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i)$$

$$\ge \sum_{t=0}^{\infty} \beta_i^t u_c^i(c_t^{*i}, l_t^{*i})(c_t^{*i} - c_t^i) + \sum_{t=0}^{\infty} \beta_i^t u_l^i(c_t^{*i}, l_t^{*i}) \ (l_t^{*i} - l_t^i).$$

Combining (3 ),(6) yields

$$\Delta \ge \sum_{t=0}^{\infty} \frac{(\lambda_t^1 - \lambda_t^{2i})}{\eta_i}(c_t^{*i} - c_t^i) + \sum_{t=0}^{\infty} \frac{(\lambda_t^1 f_L(k_t^*, L_t^*) - \lambda_t^{4i} + \lambda_t^{5i})}{\eta_i}(l_t^{*i} - l_t^i)$$

$$\begin{aligned}
&\geq \sum_{t=0}^{\infty} \frac{\lambda_t^1}{\eta_i}(c_t^{*i} - c_t^i) + \sum_{t=0}^{\infty} \frac{\lambda_t^1 f_L(k_t^*, L_t^*)}{\eta_i}(l_t^{*i} - l_t^i) + \sum_{t=0}^{\infty} \frac{\lambda_t^{5i}(1 - l_t^i)}{\eta_i} \\
&\geq \sum_{t=0}^{\infty} \frac{\lambda_t^1}{\eta_i}(c_t^{*i} - c_t^i) + \sum_{t=0}^{\infty} \frac{\lambda_t^1 f_L(k_t^*, L_t^*)}{\eta_i}(l_t^{*i} - l_t^i) \\
&= \sum_{t=0}^{\infty} \frac{p_t^*}{\eta_i}(c_t^{*i} - c_t^i) + \sum_{t=0}^{\infty} \frac{w_t^*}{\eta_i}(l_t^{*i} - l_t^i) \geq 0.
\end{aligned}$$

This means $(\mathbf{c}^{*i}, \mathbf{l}^{*i})$ solves the consumer's problem.

iii) We now show that $(\mathbf{k}^*, \mathbf{L}^*)$ is solution to the firm's problem. Since $p_t^* = \lambda_t^1$, $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$, we have

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - r k_0.$$

Let :

$$\begin{aligned}
\Delta_T \ &= \sum_{t=0}^{T} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - r k_0 \\
&\quad - \left( \sum_{t=0}^{T} \lambda_t^1 [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t - r k_0 \right)
\end{aligned}$$

By the concavity of $f$ , we get

$$\begin{aligned}
\Delta_T \ &\geq \sum_{t=1}^{T} \lambda_t^1 f_k(k_t^*, L_t^*)(k_t^* - k_t) - \sum_{t=0}^{T} \lambda_t^1 (k_{t+1}^* - k_{t+1}) \\
&= [\lambda_1^1 f_k(k_1^*, L_1^*) - \lambda_0^1](k_1^* - k_1) + ... \\
&\quad + [\lambda_T^1 f_k(k_T^*, L_T^*) - \lambda_{T-1}^1](k_T^* - k_T) - \lambda_T^1 (k_{T+1}^* - k_{T+1}).
\end{aligned}$$

By (4) and (9), we have: $\forall t = 1, 2, ..., T$

$$\begin{aligned}
&[\lambda_t^1 f_k(k_t^*, L_t^*) - \lambda_{t-1}^1](k_t^* - k_t) \\
&= -\lambda_t^3 (k_t^* - k_t) = \lambda_t^3 k_t \geq 0.
\end{aligned}$$

Thus,

$$\Delta_T \geq -\lambda_T^1 (k_{T+1}^* - k_{T+1}) = -\lambda_T^1 k_{T+1}^* + \lambda_T^1 k_{T+1} \geq -\lambda_T^1 k_{T+1}^*.$$

Since $\lambda^1 \in l_+^1$, $\sup_T k_{T+1}^* < +\infty$, we have

$$\lim_{T \to +\infty} \Delta_T \geq \lim_{T \to +\infty} - \lambda_T^1 k_{T+1}^* = 0.$$

We have proved that the sequences $(\mathbf{k}^*, \mathbf{L}^*)$ maximize the profit of the firm.

Finally, the market is cleared by the strict increasingness of the utility functions.

■

Let $k_0 > 0$. We define:

$$\phi_i(\eta) = \{\sum_{t=0}^{\infty} p_t^*(\eta) c_t^{*i}(\eta)\} + \{\sum_{t=0}^{\infty} w_t^*(\eta) l_t^{*i}(\eta)\} - \{\sum_{t=0}^{\infty} w_t^*(\eta)\} - \{\vartheta^i r k_0\} - \{\alpha^i \pi^*(\eta)\}$$

where

$$\begin{aligned}\forall t,\ p_t^* &\in \{\lambda_t^1\},\ w_t^* \in \{\lambda_t^1 f_L(k_t^*, L_t^*)\}\\ \pi^*(\eta) &= \sum_{t=0}^{\infty} p_t^*(\eta)[f(k_t^*(\eta), L_t^*(\eta)) - k_{t+1}^*(\eta)] - \sum_{t=0}^{\infty} w_t^*(\eta)L_t^*(\eta) - rk_0.\end{aligned}$$

The correspondence $\phi_i$ is convex valued. Indeed, we know that $c_t^{*i}, l_t^{*i}, k_t^*, L_t^*$ are single-valued. If $\sum_i c_t^{*i} > 0$, then from Remark 1, we also have $\sum_i l_t^{*i} > 0$, and $\lambda_t^1$ is also single-valued from (7). In this case, $p_t^*, w_t^*$ are single-valued. If $\sum_i c_t^{*i} = 0$, then $p_t^* c_t^{*i} = 0, w_t^* l_t^{*i} = 0$. Let $\mathcal{T} = \{t : \sum_i c_t^{*i} > 0\}$. We have

$$\phi_i(\eta) = \sum_{t\in\mathcal{T}} \{p_t^*(\eta)c_t^{*i}(\eta) + \sum_{t\in\mathcal{T}} w_t^*(\eta)l_t^{*i}(\eta)\} - \sum_{t=0}^{+\infty}\{w_t^*(\eta)\} - \{\vartheta^i rk_0\} - \alpha^i\{\pi^*(\eta)\}$$

where

$$\begin{aligned}\forall t,\ p_t^* &= \lambda_t^1,\ w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)\\ \pi^*(\eta) &= \sum_{t\in\mathcal{T}} p_t^*(\eta)[f(k_t^*(\eta), L_t^*(\eta)) - k_{t+1}^*(\eta)] - \sum_{t\in\mathcal{T}} w_t^*(\eta)L_t^*(\eta) - rk_0.\end{aligned}$$

The correspondence $\phi_i$ is uniformly bounded (see Proposition 6, statement 3).

**Proposition 8** *i) Let $k_0 > 0$. Then for any $\eta \in \Delta$, $\pi^*(\eta) \geq 0$.*
*ii) If $\eta_i = 0$ then $\forall t$, $c_t^{*i} = 0$, $l_t^{*i} = 0$ and $\phi_i(\eta) < 0$.*

**Proof**: i) Let $(k_0, 0, 0, ...) \in \Pi(k_0)$. Then

$$\begin{aligned}\pi^*(\eta) &\geq \lambda_0^1(\eta)[F(k_0, 0) + (1-\delta)k_0] - rk_0\\ &= \lambda_0^1(\eta)[F(k_0, 0) + (1-\delta)k_0] - \lambda_0^1(\eta)[F_k(k_0, 0) + 1 - \delta]k_0\\ &\geq 0.\end{aligned}$$

ii) Let $\eta_i = 0$. From Remark 1, $c_t^{*i} = l_t^{*i} = 0$, $\forall t$. Now, we have

$$\begin{aligned}\phi_i(\eta) &= \sum_{t=0}^{\infty} p_t^*(\eta)c_t^{*i}(\eta) + \sum_{t=0}^{\infty} w_t^*(\eta)l_t^{*i}(\eta) - \sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i rk_0 - \alpha^i\pi^*(\eta)\\ &= -\sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i rk_0 - \alpha^i\pi^*(\eta) \leq -\sum_{t=0}^{\infty} w_t^*(\eta) < 0,\ \text{since } \mathbf{w}^* \in l_+^1 \setminus \{0\}.\end{aligned}$$

■

We can now state our main result.

**Theorem 2** *Assume **U1, U2, U3, F1, F2, F3**. If $m > 1$, then add **U4**. Let $k_0 > 0$. Then there exists $\bar{\eta} \in \Delta, \bar{\eta} >> 0$, such that $\phi_i(\bar{\eta}) = 0, \forall i$ . That means there exists an equilibrium.*

**Proof**: When $m = 1$, the result follows Proposition 7. When $m > 1$, from Proposition 3, for any $t$, there exists $i$ with $c_t^{*i} > 0$, $l_t^{*i} > 0$. Obviously, $\eta_i > 0$. Then $p_t^* = \lambda_t^1 = \eta_i \beta_i^t u_c c_t^{*i}, l_t^{*i}$. The correspondence $\phi_i$ is single-valued. We now prove it is continuous.

Let $\varepsilon > 0$. Observe that

$$\pi^*(\eta) = \sum_{t=0}^{+\infty} p_t^*(\eta) \sum_i c_t^{*i} - \sum_{t=0}^{+\infty} w_t^*(\eta)(m - \sum_i l_t^{*i})(\eta) - rk_0.$$

From Proposition 6, there exists $T$ such that, for any $n$,

$$\begin{aligned}
&|\sum_{t\geq T} p_t^*(\eta^n) c_t^{*i}(\eta^n) + \sum_{t\geq T} w_t^*(\eta^n) l_t^{*i}(\eta^n) \\
&- \sum_{t\geq T} w_t^*(\eta^n) - \vartheta^i rk_0 - \alpha^i \sum_{t\geq T} p_t^*(\eta^n) \sum_i c_t^{*i}(\eta^n) \\
&- \sum_{t\geq T} w_t^*(\eta^n)(m - \sum_i l_t^{*i}(\eta^n)) - rk_0| \leq \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&|\sum_{t\geq T} p_t^*(\eta) c_t^{*i}(\eta) + \sum_{t\geq T} w_t^*(\eta) l_t^{*i}(\eta) \\
&- \sum_{t\geq T} w_t^*(\eta) - \vartheta^i rk_0 - \alpha^i \sum_{t\geq T} p_t^*(\eta) \sum_i c_t^{*i}(\eta) \\
&- \sum_{t\geq T} w_t^*(\eta)(m - \sum_i l_t^{*i}(\eta)) - rk_0| \leq \varepsilon
\end{aligned}$$

Consider $t \in \{0, \ldots, T-1\}$. If $c_t^{*i}(\eta) > 0$, then

$$c_t^{*i}(\eta^n) \to c_t^{*i}(\eta), l_t^{*i}(\eta^n) \to l_t^{*i}(\eta), k_t^*(\eta^n) \to k_t^*(\eta), p_t^*(\eta^n) \to p_t^*(\eta), w_t^*(\eta^n) \to w_t^*(\eta)$$

Since for any $t$, there exists $i$ with $c_t^{*i} > 0$, we have $p_t^*(\eta^n) \to p_t^*(\eta)$ and $w_t^*(\eta^n) \to w_t^*(\eta)$

We have $k_t^*(\eta^n) \to k_t^*(\eta) > 0$, $L_t^*(\eta^n) \to L_t^*(\eta) > 0$.

The proof of the continuity of $\phi_i$ is complete.

Observe that $\sum_i \phi_i(\eta) = 0$ for any $\eta$ by Walras Law. Let us define $T : \Delta \to \Delta$, $T(\eta) = (T_1(\eta), T_2(\eta), ..., T_m(\eta))$ where $T_i(\eta)$ defined as

$$T_i(\eta) = \frac{\eta_i + \phi_i^{'}(\eta)}{1 + \sum_{i=1}^m \phi_i^{'}(\eta)}$$

with $\phi_i^{'}(\eta) = -\phi_i(\eta)$ if $\phi_i(\eta) < 0$, and $\phi_i^{'}(\eta) = 0$ if $\phi_i(\eta) \geq 0$. $T$ is a continuous mapping from the simplex into itself. By the Brouwer fixed point theorem, there exists $\bar{\eta} \in \Delta$ such that $T(\bar{\eta}) = \bar{\eta}$. We have

$$\bar{\eta_i} = \frac{\overline{\lambda}_i + \phi_i^{'}(\bar{\eta})}{1 + \sum_{i=1}^m \phi_i^{'}(\bar{\eta})} \Leftrightarrow \bar{\eta_i} \sum_{i=1}^m \phi_i^{'}(\bar{\eta}) = \phi_i^{'}(\bar{\eta}) \tag{13}$$

If $\bar{\eta_i} = 0$, Proposition 8 (ii) implies that $\phi_i(\bar{\eta_i}) < 0$ and $\phi_i^{'}(\bar{\eta}) > 0$ :a contradiction with (13). Thus, $\bar{\eta_i} > 0$, $\forall i$. If $\sum\limits_{i=1}^m \phi_i^{'}(\bar{\eta}) > 0$, then $\phi_i^{'}(\bar{\eta}) > 0$, $\forall i$. From the definition of $\phi_i^{'}(\eta)$ this implies $\phi_i(\eta) < 0$, $\forall i$. But this contradicts Walras Law which says $\sum\limits_{i=1}^m \phi_i(\bar{\eta}) = 0$. Thus, $\sum\limits_{i=1}^m \phi_i^{'}(\bar{\eta}) = 0$ which implies $\phi_i^{'}(\bar{\eta}) = 0$, $\forall i$. But in this case we have $\phi_i(\bar{\eta}) \geq 0$, $\forall i$. From Walras Law we have $\phi_i(\bar{\eta}) = 0$, $\forall i$. ■

**Remark 3** *Existence of equilibrium can be obtained without **U4** by assuming, as in Bewley [2], that any consumer i has at each t an endowment* $\omega_t^i \geq 0$ *which satisfies* ***O1:*** $\sum_{i=1}^m \omega_t^i \in int\ l_+^\infty$

*The feasible constraints become:*

$$\forall t \geq 0,\ \sum_i c_t^i + k_{t+1} \leq \sum_i \omega_t^i + f(k_t, L_t)$$

*The proof of existence of equilibrium is briefly given Appendix*

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# 4 Appendix

**Proof of the claim in Remark 3**
Let $\alpha = \inf_t \sum_i \omega_t^i$.
(1) We can prove as in the proof of Proposition 6 that for any $\eta \in \Delta$, any $T$

$$\frac{\beta^T}{1-\beta}\sum_i u^i(A,1) \geq \sum_{t=T}^{+\infty}\lambda_t^1 \sum_i c_t^{*i} \geq \alpha \sum_T^{+\infty}\lambda_t^1 + \sum_T^{+\infty}\lambda_t^1 f_L(k_t^*, L_t^*)L_t^*$$

or

$$\frac{2\beta^T}{1-\beta}\sum_i u^i(A,1) \geq \sum_{t=T}^{+\infty}\lambda_t^1 \sum_i \left(c_t^{*i} + f_L(k_t^*, L_t^*)l_t^{*i}\right) \geq \alpha \sum_T^{+\infty}\lambda_t^1 + m\sum_T^{+\infty}\lambda_t^1 f_L(k_t^*, L_t^*) \quad (14)$$

Define $p_t^*(\eta) = \lambda_t^1,\ w_t^*(\eta) = \lambda_t^1 f_L(k_t^*, L_t^*)$. Inequality (14) shows that the sets $\{p_t^*(\eta)\}_\eta,\ \{w_t^*(\eta)\}_\eta$ are relatively weakly compact in $l^1$.
(2) To prove that an equilibrium exists, it remains to prove that the correspondences $\phi_i$ are usc. Since they are in fixed compact sets, we have just to check their closedness.
Let $\eta^n \to \eta$. Let $z_i^n \in \phi_i(\eta^n)$. There exists $\mathbf{p}^*(\eta^n), \mathbf{w}^*(\eta^n), \mathbf{c}^{*i}(\eta^n), \mathbf{l}^{*i}(\eta^n), \mathbf{k}^*(\eta^n)$ such that

$$z_i^n = \sum_{t=0}^{\infty} p_t^*(\eta)c_t^{*i}(\eta^n) + \sum_{t=0}^{\infty} w_t^*(\eta^n)l_t^{*i}(\eta^n) - \sum_{t=0}^{\infty} w_t^*(\eta^n) - \vartheta^i r(\eta^n)k_0 - \alpha^i\pi^*(\eta^n)$$

where $r(\eta^n) = p_0^*(\eta^n)[F_k(k_0, 0) + 1 - \delta]$.

We first have

$$c_t^{*i}(\eta^n) \to c^{*i}(\eta),\ l_t^{*i}(\eta^n) \to l_t^{*i}(\eta),\ k_t^*(\eta^n) \to k_t^*(\eta)$$

Since the sets $\{p_t^*(\eta)\}_\eta,\ \{w_t^*(\eta)\}_\eta$ are relatively weakly compact in $l^1$, we can assume that $\mathbf{p}^*(\eta^n),\ \mathbf{w}^*(\eta^n)$ converge weakly to $\bar{\mathbf{p}} \in l^1, \bar{\mathbf{w}} \in l^1$ and $\bar{w}_t = \bar{p}_t f_L\left(k_t^*(\eta), L_t^*(\eta)\right)$ for any $t$. We can easily check that, for any $t$

$$0 \in \eta_i\beta_i^t\partial_1 u^i(c_t^{*i}(\eta), l_t^{*i}(\eta)) - \bar{p}_t,\ \forall i = 1...m$$

$$0 \in \eta_i\beta_i^t\partial_2 u^i(c_t^{*i}(\eta), l_t^{*i}(\eta)) - \bar{p}_t\partial_2 f(k_t^*(\eta), L_t^*(\eta)),\ \forall i = 1...m$$

$$0 \in \bar{p}_t\partial_1 f(k_t^*(\eta), L_t^*(\eta)) - \bar{p}_{t-1}$$

That means, from Proposition 3, that $\bar{\mathbf{p}}$ is a multiplier associated with $\mathbf{c}^{*i}(\eta), \mathbf{l}^{*i}(\eta), \mathbf{k}^*(\eta)$ and $\bar{p}_t = p_t^*(\eta), \bar{w}_t = w_t^*(\eta),\ \forall t$.
Define

$$z_i = \sum_{t=0}^{\infty}\bar{p}_t c_t^{*i}(\eta) + \sum_{t=0}^{\infty}\bar{w}_t l_t^{*i}(\eta) - \sum_{t=0}^{\infty}\bar{w}_t^* - \vartheta^i\bar{r}k_0 - \alpha^i\pi^*(\eta)$$

where

$$\pi^*(\eta) = \sum_{t=0}^{\infty}\bar{p}_t\left[f(k_t^*(\eta), L_t^*(\eta)) - k_{t+1}^*(\eta)\right] - \sum_{t=0}^{\infty}\bar{w}_t L_t^*(\eta) - \bar{r}k_0$$

and $\bar{r} = \bar{p}_0[F_k(k_0, 0) + 1 - \delta]$. Obviously, $z_i \in \phi_i(\eta)$.

Let $\varepsilon > 0$ be given. From inequality (14), there exists $T$ such that for any $n$ we have:

$$\begin{aligned}
&|\sum_{t\geq T} p_t^*(\eta^n)c_t^{*i}(\eta^n) + \sum_{t\geq T} w_t^*(\eta^n)l_t^{*i}(\eta^n) \\
&-\sum_{t\geq T} w_t^*(\eta^n) - \vartheta^i r k_0 - \alpha^i \sum_{t\geq T} p_t^*(\eta^n) \sum_i c_t^{*i}(\eta^n) \\
&-\sum_{t\geq T} w_t^*(\eta^n)(m - \sum_i l_t^{*i}(\eta^n)) - rk_0| \leq \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&|\sum_{t\geq T} \bar{p}_t c_t^{*i}(\eta) + \sum_{t\geq T} \bar{w}_t l_t^{*i}(\eta) \\
&-\sum_{t\geq T} \bar{w}_t - \vartheta^i \bar{r} k_0 - \alpha^i \sum_{t\geq T} \bar{p}_t \sum_i c_t^{*i}(\eta) \\
&-\sum_{t\geq T} \bar{w}_t(m - \sum_i l_t^{*i}(\eta)) - \bar{r}k_0| \leq \varepsilon
\end{aligned}$$

Consider $t \in \{0, \dots, T-1\}$. One has: $p_t^*(\eta^n) \to \bar{p}_t$, $w_t^*(\eta^n) \to \bar{w}_t$, $c_t^{*i}(\eta^n) \to c_t^{*i}(\eta)$, $l_t^{*i}(\eta^n) \to l_t^{*i}(\eta)$, $k_t^*(\eta^n) \to k_t^*(\eta)$. Thus, for $n$ large enough, we have $|z_i^n - z_i| \leq 3\varepsilon$. That means $z_i^n \to z_i$. The proof is complete.