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Solving Linear and Non-Linear Stiff System of Ordinary Differential Equations by Multistage Adomian Decomposition Method

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***Abstract*--In this paper, linear and non-linear stiff systems of ordinary differential equations are solved by the classical Adomian decomposition method (ADM) and the multistage Adomian decomposition method (MADM). The MADM is a technique adapted from the standard Adomian decomposition method (ADM) where standard ADM is converted into a hybrid numeric-analytic method called the multistage ADM (MADM). The MADM is tested for several examples. Comparisons with an explicit Runge-Kutta-type method (RK) and the classical ADM demonstrate the limitations of ADM and promising capability of the MADM for solving stiff initial value problems (IVPs).**

***Keywords*-- Stiff system of ODEs, Runge-Kutta-type method, Adomian decomposition method, Multistage ADM.**

I. Introduction

The mathematical equations modelling many real-world physical phenomena are often stiff equations, i.e. equations with a wide range of temporal scales. The numerical methods for solving stiff equations must have good accuracy and wide region of stability. Hojjati et al. [1] developed a multistep method for solving stiff systems of initial value problems (IVPs). Knowing that the classical explicit fourth-order Runge-Kutta method is insufficient for the solution of stiff IVPs, Ahmad et al. [2] presented an explicit Taylor-like method for solving stiff IVPs. In Ahmad and Yaacob [3], an explicit Runge-Kutta-like method is developed and shown to be efficient for the solution of stiff ODEs. Very recently, Nie et al. [4] presented a class of efficient semi-implicit schemes for stiff reaction-diffusion equations.

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A variable-step size algorithm for stiff systems has been proposed recently by Jannelli and Fazio [5]. In [6], classes of methods having properties very close to those of traditional Runge-Kutta methods were developed. Butcher and Hojjati [7] devised a class of second derivative methods possessing Runge-Kutta stability property. Hojjati et al. [8] presented a new class of second derivative multistep methods with improved stability region.

All of the methods mentioned above need some sort of discretizations. One of the papers proposing an approximate analytic method is due to Guzel and Bayram [9] who presented a power series method for stiff systems. Adomian et al. [10] first demonstrated how a power series method, called the Adomian decomposition method (ADM) [11], can be used to derive an exact solution to a specific linear stiff system of IVPs. The ADM yields, without linearization, perturbation, transformation or discretization, an analytical solution in terms of a rapidly convergent infinite power series with easily computable terms, see for example, [11]. Recently, Mahmood et al. [12] solved both linear and non-linear stiff systems of IVPs using the ADM. The ADM has been applied to a wide range of problems, [13, 14, 15, 16, 17, 18]. In [19, 20, 21, 22, 23, 24], the ADM was treated as an algorithm for approximating the solutions in a sequence of time intervals (i.e. time step). We call this approach as multi-stage ADM (MADM). In this work, we shall apply the MADM to the solutions of stiff IVPs. Comparisons will be made against the classical ADM, an explicit Runge-Kutta method and available exact solutions.

II. Solution Methods

In this section we describe the ADM and give the algorithm of the Runge-Kutta-like method of Ahmad and Yaacob [3] for solving the following initial value problem:

$$y' = f(t, y) \text{ with } y(0) = y_0, \tag{1}$$

where $f(t, y)$ may be a linear or non-linear function.

A. *Decomposition method*

Consider Eq. (1) written in the form

$$L_y + R_y + N_y = g(t), \tag{2}$$

where $L = d/dt$, R is the remainder of the linear operator and N_y represents the non-linear term. Hence, we obtain

$$y = y(0) + L^{-1}(g(t)) - L^{-1}(R_y) - L^{-1}(N_y), \tag{3}$$

where $L^{-1}(\cdot) = \int_0^t (\cdot)ds$. The ADM, [13], takes the solution y and the non-linear function N_y as infinite series, respectively,

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$$y=\sum_{k=0}^{\infty} y_k, \qquad N_y=\sum_{k=0}^{\infty} A_k . \tag{4}$$

The y_k's are determined from the recursive algorithm, [13], based on (3),

$$y_0 = y(0)+L^{-1}(g(t)), \tag{5}$$

$$y_{k+1} = -L^{-1}(R_{y_k}) - L^{-1}(A_{y_k}), \quad k \ge 0, \tag{6}$$

where the so-called Adomian polynomials A_k are given by, [13],

$$A_K = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} N\left(\sum_{j=0}^{\infty} \lambda^j y_j\right)\right]_{\lambda=0}, \quad k=0,1,2,\ldots\ldots \tag{7}$$

Convergence aspects of the ADM have been investigated in [25]. For later numerical computation, let the expression

$$\phi_n(t)=\sum_{k=0}^{n-1} y_k, \tag{8}$$

denote the $n-term$ approximation to $y(t)$.

As first hinted in [26] and applied in [19, 20, 21, 22, 23], we treat the ADM as an algo-rithm for approximating the dynamical response in a sequence of time intervals (i.e. time step) $[0,t_1), [t_1,t_2), \ldots, [t_{m-1},T)$ such that the initial condition in $[t^*, t_{m+1}]$ is taken to be the condition at t^*.

B. *Runge-Kutta-like method*

Ahmad and Yaacob [3] developed composite arithmetic-harmonic, explicit Runge-Kutta-like methods for solving problem (1). The iterative formula they proposed is given as follows,

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{2k_1k_2}{k_1+k_2} + \frac{2k_2k_3}{k_2+k_3}\right), \tag{9}$$

$$k_1 = f(t_n, y_n), \tag{10}$$

$$k_2 = f\left(t_n + \frac{3}{5}h,\ y_n + \frac{3}{5}hk_1\right), \tag{11}$$

$$k_3 = f\left(t_n + \frac{4}{5}h,\ y_n + \frac{4}{5}h\left[\frac{k_1+k_2}{2}\right]\right), \tag{12}$$

where h is the time stepsize. For further details the reader is referred to [3].

III. Test Problems

In this section, we shall demonstrate how well the MADM compares with the RungeKutta-like method of [3] for the solutions of both linear and non-linear system of ordinary differential equations (ODES). The Adomian iterative algorithm is coded in the computer algebra package Maple. The Maple environment variable Digits controlling the number of significant digits is set to 16 in all the calculations done in this paper.

A. *Problem 1*

Consider linear stiff initial value problem [28]:

$$\frac{dy_1}{dt} = -0.1y_1 - 49.9y_2, \tag{13}$$

$$\frac{dy_2}{dt} = -50y_2, \tag{14}$$

$$\frac{dy_3}{dt} = 70y_2 - 120y_3, \tag{15}$$

subject to the initial conditions

$$y_1(0)=2, \qquad y_2(0)=1, \qquad y_3(0)=2. \tag{16}$$

The exact solutions of the system (13)-(15) are given by

$$y_1(t)=e^{-50t}+e^{-0.1t}, \tag{17}$$

$$y_2(t)=e^{-50t}, \tag{18}$$

$$y_3(t)=e^{-50t}+e^{120t}. \tag{19}$$

The iterative formula based on (5) and (6) for the system (13)-(15) are given by

$$y_{1,0}=2, \quad y_{2,0}=1, \quad y_{3,0}=2, \tag{20}$$

$$y_{1,\,k+1} = -0.1L^{-1}(y_{1,k}) - 49.9L^{-1}(y_{2,k}), \quad k \ge 0, \tag{21}$$

$$y_{2,k+1} = -50L^{-1}(y_{2,k}), \quad k \ge 0, \tag{22}$$

$$y_{3,\,k+1} = 70L^{-1}(y_{2,k}) - 120L^{-1}(y_{3,k}), \quad k \ge 0. \tag{23}$$

From Fig 1, it is observed that the 4-term MADM solutions agree very well with the exact solutions.

A. *Problem 2*

Now consider nonlinear stiff initial value problem [29]

$$\frac{dy_1}{dt} = -12y_1 + 10y_2^2, \tag{24}$$

$$\frac{dy_2}{dt} = y_1 - y_2 - y_1y_2, \tag{25}$$

subject to the initial conditions

$$y_1(0)=0, \qquad y_2(0)=0. \tag{26}$$

The exact solutions of the system (24)-(25) are given by

$$y_1(t)=e^{-2t}, \tag{27}$$

$$y_2(t)=e^{-t}. \tag{28}$$

The iterative formula based on (5) and (6) for the system (24)-(25) are given by

$$y_{1,0}=0, \quad y_{2,0}=0, \tag{29}$$

$$y_{1,\,k+1} = -12L^{-1}(y_{1,k}) + 10L^{-1}(A_k), \quad k \ge 0, \tag{30}$$

$$y_{2,k+1} = L^{-1}(y_{1,k}) - L^{-1}(B_k), \quad k \ge 0, \tag{31}$$

where some of the Adomian polynomials for A_k and B_k obtained from (7) are given bellow:

$$A_0 = y_{1,0}^2, \quad A_1 = 2y_{1,0}y_{2,0}, \quad A_2 = 2y_{1,0}y_{1,2} + y_{1,1}^2,$$

etc.

$$B_0 = y_{1,0}y_{2,0},\ B_1 = y_{1,0}y_{2,1} + y_{1,1}y_{2,0},\ B_2 = y_{1,0}y_{2,2} + y_{1,1}y_{2,1} + y_{1,2}y_{2,0}$$

,

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In Figure 2 we compare the 4-term decomposition solutions with the exact solutions and that obtained by the Runge-Kutta method. The MADM solutions at $h=10^{-3}$ are of comparable

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accuracy with the exact solutions and that of the Runge-Kutta method at the same step size.

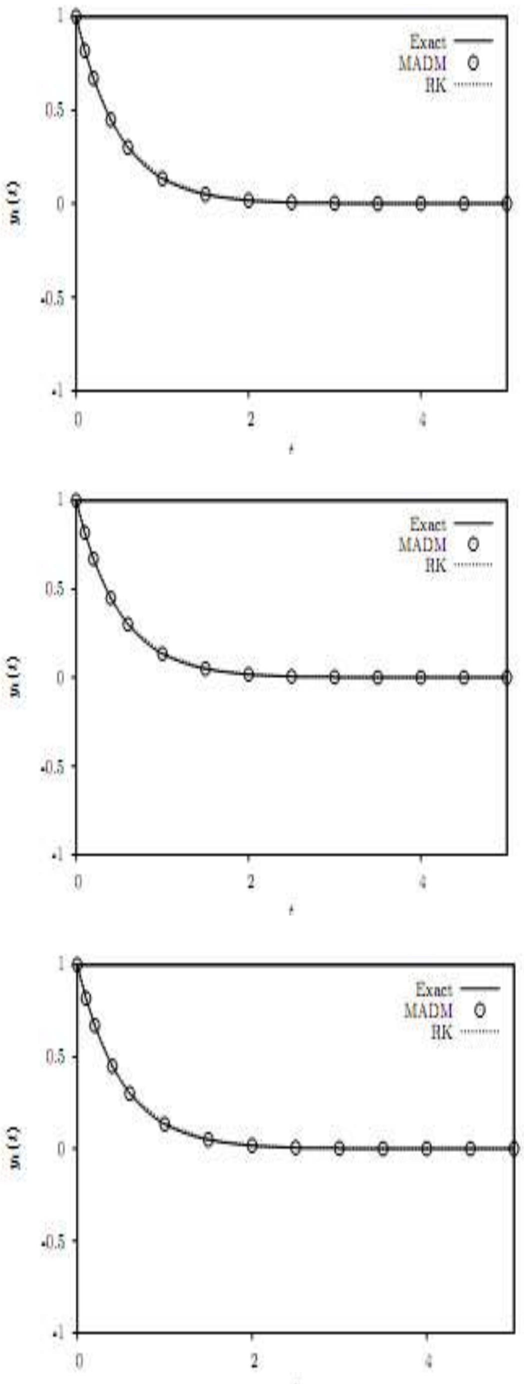

Figure 1: The MADM solutions using 4 terms at time step $h = 10^{-3}$ as compared with the exact solutions for Problem 1.

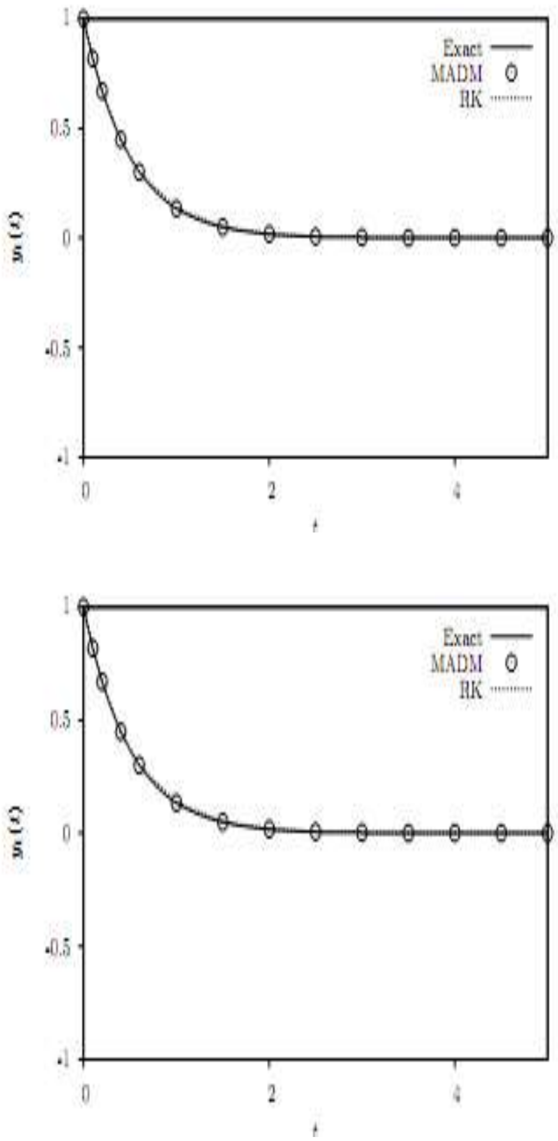

Figure 2: The MADM solutions using 4 terms as compared with the exact solutions and that obtained by the Runge-Kutta method for Problem 2.

IV. Conclusion

In this paper, we presented the multi-stage Adomian decomposition method (MADM) for solving both linear and non-linear stiff system of ODEs. Direct applications of the classical ADM can fail for stiff problems. The MADM is shown here to be a promising alternative method for stiff equations. In addition to the choice of time stepsize, the MADM has the number of terms of the series solution as an extra parameter for controlling the accuracy of solutions.

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