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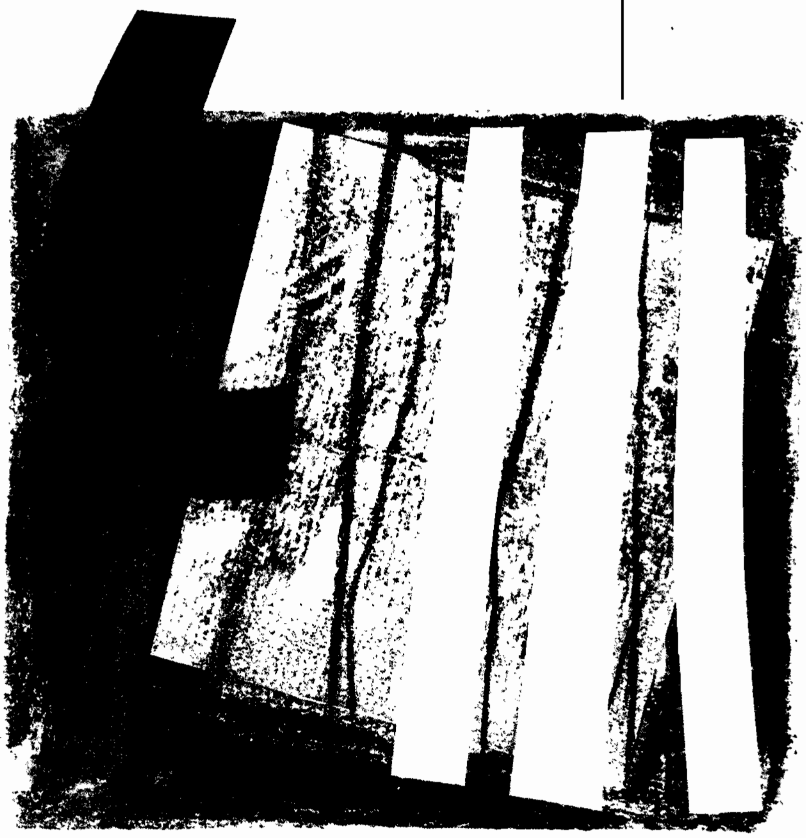

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M. Esteban-Bravo *

Abstract

In this paper we propose a general mathematical approach to existence of production equilibria in general economic model with incomplete assets markets, based on mathematical programming theory. In the first part, we demonstrate the existence of a General Equilibrium with Incomplete markets (GEI)}. In the second part, we introduce a concept of local equilibrium and we characterize such an equilibrium as the solution of a nonlinear system of equations. This system is very useful in practice since we avoid to compute the excess demand function that is difficult to obtain in large applied models. Furthermore, our characterization only requires limited short-selling and convexity assumptions at the neighborhood of the solution point. Finally, we also propose an algorithm for computing equilibria by interior point methods and we present numerical examples.

Keywords: General equilibrium; GEI equilibrium; computation of equilibria; Interior-point methods.

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1 Introduction

This paper studies a model of sequential trade for goods and assets in which all relevant information is symmetric across economic agents and trade takes place sequentially over time. A model of sequential markets is a system of reopening spot markets which are linked via a system of financial markets. The use of financial instruments allows each agent to redistribute income across the dates.

The first general model of sequential market was provided by Radner (1972). He showed the existence of equilibria in a general equilibrium model under uncertainty in a sequence of incomplete markets at successive dates. He assumed that agents forecast correctly their future environment. With perfect foresight, it is known that the Radner model is equivalent to the Arrow-Debreu model (see Arrow and Hahn (1971) and Debreu (1959)). However, in reality there are asset markets that transfer wealth across the states of the world that will revealed in the future. In other words, none of these markets is complete in Arrow-Debreu sense.

This model has evolved into the model of *general equilibrium with incomplete markets* (GEI). Surveys in this area are due to Cass (1990), Duffie (1990), Geanakoplos (1990), Magill and Shafer (1991) and Magill and Quinzii (1996), Hens (1998).

The purpose of this paper is the development of a general approach to existence problems in GEI models assuming that firms maximize a general objective, and an appropriate algorithm for the computation of equilibria in large-scale GEI models.

The description of firms in incomplete markets is still unsatisfactory. For a survey of alternative objective functions of the firm see Duffie (1992). We consider that firms maximize a general objective subject to their *balance equations*. In reality, the objectives of the firm emerge from the objectives of those individuals who control it. For instance, a firm could consider maximizing the conditional expected profit, or a firm could be interested in the maximization of the minimum profit because of aversion risk, or the size of firm's labor force (for example, a cooperative). Hence, it is crucial to develop a general equilibrium theory that allows different decision rules. On the other hand, the balance equations plays a essential role in the theory of accounting. Then, it is really important to incorporate these effects into the firms' problem.

We present a constructive proof of existence of equilibria based on mathematical programming techniques. The convexity assumptions are crucial for the arguments used in this proof. However, in the real world we can observe the presence of non-convexities. For example, non-convexities in production can arise from indivisibilities, setup costs (the production sets do not include the origin) or (locally or globally) non-convex technologies (increasing returns of scale). Non-convexities in consumption can arise from indivisibilities or non-concave utility functions. On the other hand, we also assumes implicitly lower bounds on short sales. The limited short-selling assumption implies that agents cannot hold asset's portfolios of arbitrarily size. This condition was also assumed by Radner (1972, 1982). However, such a condition is necessary to ensure the existence of equilibria (Hart (1975) first proved that equilibria may fail to exist without bounds on short sales by a counter-example), it was considered unrealistic. If convexity or limited short-selling assumptions are relaxed, a sequential equilibrium may fail to exist due to discontinuities in the excess demand correspondence.

Approaches based on the excess demand equations were developed to overcome the problems caused by the unlimited short-selling assumption and so to prove the existence of equilibria for a generic set of economies. For example, Duffie and Shafer (1985) used algebraic and differential topology (in particular, the Grassmannian manifold) and Brown et al. (1996b) introduced an auxiliary asset. The main inconvenience of these approaches is a substantial increase in mathematical complexity.

We provide a characterization of sequential equilibria relaxing the limited short-selling and the standard convexity assumptions. Our characterization only requires to satisfy these restrictive assumptions in the neighbourhood of the solution point. The chief idea is to be locally optimal. Therefore, we first introduce a definition for what we call a local equilibrium. Then, instead of determining an equilibrium as a solution of the market excess demand function, we provide an alternative system of equations that can be solved to find equilibria. This system is very useful in practice since we avoid to compute the excess demand function that is difficult to obtain in large applied models.

We also develop an appropriate algorithm for the computation of equilibria in large-scale sequential models. Much recent work on the computation of sequential equilibria has been based on the path following or homotopy methods. There are three main approaches to compute equilibria in the GEI model. The main idea of these approaches is the use of the equivalent definition of an equilibrium known as *no-arbitrage equilibrium* given by Cass (1984).

The first method was given by DeMarzo and Eaves (1996). They consider the excess demand function defined on prices and elements of the Grassmannian manifold (see e.g. Duffie and Shafer (1985)). By applying the work of Brown et al. (1996a), they computed equilibria via homotopy algorithm. The second and alternative algorithm for computing fixed points was developed by Brown et al. (1996b). They consider the excess demand function as a function of just prices. Since this excess demand function is discontinuous, they introduce an auxiliary asset to define a family of homotopies. The third approach is due to Schmedders (1998). He computed equilibria with homotopy techniques using the first-order conditions of the agents' problem with penalties for transactions on the asset markets.

We could apply homotopy techniques to compute the equilibrium values but these algorithms are less computationally efficient. In order to improve its efficiency, we present an alternative algorithm to compute such equilibria via interior point methods.

The rest of the paper is organized in two parts. The first part is devoted to two period production models known as the basic GEI model. In section 2 we describe the basic model and stress its main properties. Section 3 is devoted to the existence of a GEI equilibria in spot-financial markets. In section 4, we characterize the local

GEI equilibrium as the solution of a nonlinear system of equations. In section 5, we also prove the uniqueness of such equilibria. In section 6, we compute such a equilibrium via interior point methods. The second part extends these results of the two periods GEI model to multiple periods.

2 The two period production model

In order to be as simple as possible, we consider only two period production economies to describe economic situations under conditions of uncertainty. Much of the theory of incomplete markets is devoted to two period exchange models. We will extend the two period production model to multiple periods in Section 7.

The GEI model describes an economy over two time periods ($t = 0, 1$) with uncertainty over the state of nature in period 1. At time $t = 0$ the economy is in state $s = 0$ which is known by each of the $I + J$ agents (I consumers and J firms) participating in the economy (i.e. all relevant information is symmetric across economic). But it is not known which of the S possible states at time $t = 1$ will occur (i.e. trade takes place sequentially over time).

In each state there are D goods, distinguished by their location and their physical characteristics. For each of the goods $d = 1, ..., D$ there exists a spot market in every state at spot price p_{sd}. Hence the commodity space is $\mathbb{R}^{D(S+1)}$. For any $x \in \mathbb{R}^{D(S+1)}$, x^T denotes the transpose of x, which is a $D(S+1)$-dimensional row vector. For any $x, y \in \mathbb{R}^{D(S+1)}$, $x \cdot y = x^T y$ denotes the inner product of vectors x and y.

There is a finite number C of assets traded on financial markets at asset prices q. Let θ, ξ denote the portfolio of traded assets by consumers and producers, respectively. Hence the financial asset space is $\mathbb{R}^C$. The asset θ_c can be purchased for the price q_c at date 0 and delivers a return across the states at date 1 that can be given exogenously or it can be a function of the spot market prices.

The asset structure of an asset c is described by the asset matrix $A^c = (A_1^c, ..., A_S^c)^T$ across the states at date 1, where A_s^c is the commodity bundle which asset c delivers at state s. Then the asset c delivers a nominal return $V_s^c = p_s \cdot A_s^c$ in state s. Therefore, the assets structure is summarized by the asset matrix (in units of commodities):

$$\underset{DS\times C}{A} = \begin{pmatrix} A_{11}^1 & \cdots & A_{11}^C \\ \vdots & & \vdots \\ A_{SD}^1 & \cdots & A_{SD}^C \end{pmatrix}.$$

and by the nominal return matrix:

$$\underset{S\times C}{V(p)} = \begin{pmatrix} V_1^1(p_1) & \cdots & V_1^C(p_1) \\ \vdots & & \vdots \\ V_S^1(p_S) & \cdots & V_S^C(p_S) \end{pmatrix}.$$

To be general, we consider that asset matrix $A(p, q)$ depends on spot and asset prices; i.e. $A = A(p, q)$.

We assume that $S \geq C$. The completeness condition is a very important property in the context of GEI markets. The financial markets are called *complete* if $rank\,(V\,(p,q)) = S$ for any (p,q) prices. In this situation, agents can insure themselves against any kind of contingency in period $t = 1$. When $rank\,(V\,(p,q)) < S$, the financial markets are called *incomplete.* See Magill and Shafer (1991).

We assume that the markets on which the commodities and the financial assets are traded are competitive, so that agents believe that they can buy and sell as many commodities or assets as they want without affecting their prices. —

Each consumer is described by his consumption set $\mathbb{X}_i \subset \mathbb{R}^{D(S+1)}$, the set of traded assets $\mathbb{Z} \subset \mathbb{R}^C$ and its initial endowments of the $D\,(S+1)$ goods in each state $w_i = (w_{i0}, w_{i1}, ..., w_{iS}) \in \mathbb{X}_i$. Consumers' tastes, their risk attitudes and time preferences are described by utility functions $u_i : \mathbb{X}_i \longrightarrow \mathbb{R}$. Thus, consumers face the following problem:

$$\boxed{\begin{array}{ll} \underset{s.t.}{Max} & u_i\,(x_{i0}, ..., x_{iS}) \\ & p_0 \cdot x_{i0} \leq p_0 \cdot w_{i0} - q \cdot \theta_i, \\ & p_s \cdot x_{is} \leq p_s \cdot w_{is} + p_s \cdot A_s\,(p,q) \cdot \theta_i, \ \forall s = 1, ..., S, \\ & x_i \in \mathbb{X}_i, \quad \theta_i \in \mathbb{Z}, \end{array}} \tag{1}$$

where $A_s\,(p,q)$ is the $s - th$ row of the matrix $A\,(p,q)$.

On the other hand, each firm is described by its technology $\mathbb{Y}_j \subset \mathbb{R}^{D(S+1)}$, and the set of traded assets $\mathbb{Z} \subset \mathbb{R}^C$. To model the behavior of the firm, we consider a general objective $o_j : \mathbb{Y}_j \times \mathbb{Z} \times \mathbb{R}^{D(S+1)} \times \mathbb{R}^C \longrightarrow \mathbb{R}$. For example, if the firms seek to maximize their expected profits, then the firms' objective is given by

$$o\,(y, \xi, p, q) = E\left[p_0^T y_0 + p_s^T y_s\right] - q\xi = p_0^T y_0 + \sum_{s=1}^{S} \rho_s p_s^T y_s - q\xi.$$

For a survey of alternative objective functions of the firm see Duffie (1992). Thus, firms face the following problem:

$$\boxed{\begin{array}{ll} \underset{s.t.}{Max} & o_j\left(y_j, \xi_j, p, q\right) \\ & p_0 \cdot y_{j0} + q \cdot \xi_j = 0 \\ & p_s \cdot y_{js} - p_s \cdot A_s\,(p,q) \cdot \xi_j = 0, \ \forall s = 1, ..., S, \\ & y_j \in \mathbb{Y}_j, \quad \xi_j \in \mathbb{Z}. \end{array}} \tag{2}$$

For all $j = 1, ..., J$, the equations

$$\left\{ \begin{array}{l} p_0 \cdot y_{j0} + q \cdot \xi_j = 0, \\ p_s \cdot y_{js} - p_s \cdot A_s \cdot \xi_j = 0, \forall s = 1, ..., S, \end{array} \right.$$

are known as *balance equations* for the J firms. These equations play a crucial role in the theory of accounting. Note that if we consider inequalities such as ≥ 0 instead of equalities, there exist profits that will not be distributed. This has no economic interpretation (in equilibrium, the market clearing conditions will not be satisfied at positive prices). We will prove by contradiction. Suppose that, in equilibrium, for all $j = 1, ..., J$,

$$\begin{cases} p_0^* \cdot y_{j0}^* + q^* \cdot \xi_j^* > 0, \\ p_s^* \cdot y_{js}^* - p_s^* \cdot A_s \cdot \xi_j^* > 0, \ \forall s = 1, ..., S, \end{cases}$$

adding and using the market clearing condition, we have

$$\begin{aligned} p_0^* \cdot \sum_{j=1}^{J} y_{j0}^* + q^* \cdot \sum_{j=1}^{J} \xi_i^* &= p_0^* \cdot \left(\sum_{i=1}^{I} x_{i0}^* - \sum_{i=1}^{I} w_{i0} \right) + q^* \cdot \sum_{i=1}^{I} \theta_i^* > 0, \\ p_s^* \cdot \sum_{j=1}^{J} y_{js}^* - p_s^* \cdot A_s \cdot \sum_{j=1}^{J} \xi_i^* &= p_s^* \cdot \left(\sum_{i=1}^{I} x_{is}^* - \sum_{i=1}^{I} w_{is} \right) - p_s^* \cdot A_s \cdot \sum_{i=1}^{I} \theta_i^* > 0, \forall s. \end{aligned}$$

On the other hand, we have that the GEI equilibrium satisfies the budget constraint of each consumer; i.e. for all $i = 1, ..., I$,

$$\begin{aligned} p_0^* \cdot (x_{i0}^* - w_{i0}) + q^* \cdot \theta_i^* &\leq 0, \\ p_s^* \cdot (x_i^* - w_i) - p_s^* \cdot A_s \cdot \theta_i^* &\leq 0, \forall s = 1, ..., S \end{aligned}$$

that contradicts the previous result.

A similar problem arises in the context of general Arrow-Debreu equilibrium models when we assume that firms are not owned by consumers.

We now introduce the concept of a GEI equilibrium.

Definition 1 ***GEI Equilibrium.*** *The vector prices $(p^*, q^*) \in \mathbb{R}^{D(S+1)} \times \mathbb{R}^C$, with $(p^*, q^*) \neq 0$, and the allocation*

$$((x^*, \theta^*), (y^*, \xi^*)) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) \times \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z}),$$

is a GEI equilibrium for the economy E *if:*

CP (x_i^*, θ_i^*) *is an optimal solution to Problem* (1), $\forall i = 1, ..., I$,

FP (y_j^*, ξ_j^*) *is an optimal solution to Problem* (2), $\forall j = 1, ..., J$,

MC *Market clearing:* $\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} y_j^* + \sum_{i=1}^{I} w_i$ *and* $\sum_{i=1}^{I} \theta_i^* = \sum_{j=1}^{J} \xi_i^*$.

The next step is to introduce the definition of a GEI equilibrium with excess demand. This differs from the GEI equilibrium in the market clearing condition.

Definition 2 ***GEI Equilibrium with free disposability.*** *The vector prices* $(p^*, q^*) \in \mathbb{R}^{D(S+1)} \times \mathbb{R}^C$, *with* $(p^*, q^*) \neq 0$, *and the allocation*

$$((x^*, \theta^*), (y^*, \xi^*)) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) \times \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z}),$$

is a GEI equilibrium with free disposal for the economy E *if the following hold: (FP), (CP), and*

MC' *Feasibility:*

$$\begin{aligned} \sum_{i=1}^{I} x_i^* &\leq \sum_{j=1}^{J} y_j^* + \sum_{i=1}^{I} w_i, \qquad p^* \cdot \left(\sum_{j=1}^{J} y_j^* + \sum_{i=1}^{I} w_i - \sum_{i=1}^{I} x_i^* \right) = 0, \\ \sum_{i=1}^{I} \theta_i^* &= \sum_{j=1}^{J} \xi_i^*. \end{aligned}$$

Without loss of generality, we assume that prices (p, q) are defined on

$$\Delta_+ = \left\{ (p, q) \in \mathbb{R}_+^{D(S+1)} \times \mathbb{R}^C : \sum_{d=1}^{D(S+1)} p_d + \sum_{d=1}^{C} |q_d| \leq 1 \right\},$$

whenever spot prices are assumed to be nonnegative, or, in the general case, prices (p, q) are defined on

$$\Delta = \left\{ (p, q) \in \mathbb{R}^{D(S+1)} \times \mathbb{R}^C : \sum_{d=1}^{D(S+1)} |p_d| + \sum_{d=1}^{C} |q_d| \leq 1 \right\}.$$

Thus, a GEI economy can thus be described by a set

$$E = \left(\{\mathbb{X}_i, u_i, w_i\}_{i=1}^{I}, \{\mathbb{Y}_j, o_j\}_{j=1}^{J}, (\mathbb{Z}, A) \right),$$

whose elements satisfy the following conditions:

H.1. The ith consumer's consumption set $\mathbb{X}_i \subset \mathbb{R}^{D(S+1)}$ is closed and $w_i \in \mathbb{X}_i$.

H.2. The utility function $u_i(\cdot)$ that represents the ith consumer's preference relation $\succeq_i$ is continuous.

H.3. $\mathbb{X}_i$ is convex, $\forall i = 1, ..., I$.

H.4. $u_i(\cdot)$ is concave in $\mathbb{X}_i$, $\forall i = 1, ..., I$.

H.5. The utility function $u_i(\cdot)$ is strictly monotonous and increasing.

H.6. Survival assumption : For every consumer, there exists $\underline{x}_i \in \mathbb{X}_i$ such that $\underline{x}_i \ll w_i$.

H.7. The production set for the $j-th$ firm, $\mathbb{Y}_j \subset \mathbb{R}^{D(S+1)}$, is closed, bounded and $0 \in \mathbb{Y}_j$.

H.8. The objective function for the $j-th$ firm $o_j : \mathbb{Y}_j \times \mathbb{Z} \times \mathbb{R}^{D(S+1)} \times \mathbb{R}^C \to \mathbb{R}$ is continuous.

H.9. $\mathbb{Y}_j$ is convex, $\forall j = 1, ..., J$.

H.10. $o_j\left(y_j, \xi_j, p, q\right)$ is concave in $\mathbb{Y}_j$ and $\mathbb{Z}$, $\forall (p,q) \in \mathbb{R}^{D(S+1)} \times \mathbb{R}^C$, $\forall j = 1, ..., J$.

H.11. $o_j\left(y_j, \xi_j, p, q\right)$ is homogeneous of degree α in (p,q) for some $\alpha > 0$, $\forall j = 1, ..., J$.

H.12. The set of feasible financial assets $\mathbb{Z} \subset \mathbb{R}^C$ is closed, bounded and $0 \in \mathbb{Z}$.

H.13. $\mathbb{Z} \subset \mathbb{R}^C$ is convex.

H.14. The return functions $A_s(p,q)$ are nonnegative, continuous, and homogeneous of degree zero in (p,q).

H.15. Financial asset survival assumption: For all $(p,q) \in \mathbb{R}^{D(S+1)} \times \mathbb{R}^C$, $\exists\ \theta \in \mathbb{Z}$ such that $A_s(p,q)\,\theta \gg 0$ (this assumption ensures that consumers will never be satiated in their asset demand).

H.16. The set of feasible allocations

$$\mathbb{A} = \left\{ ((x,\theta),(y,\xi)) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) \times \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z}) : \sum_{i=1}^{I} \theta_i = \sum_{j=1}^{J} \xi_j, \right.$$
$$\left. \sum_{i=1}^{I} x_{i0} \leq \sum_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} y_{j0},\ \sum_{i=1}^{I} x_{is} \leq \sum_{i=1}^{I} w_{is} + \sum_{j=1}^{J} y_{js},\ \forall s = 1, ..., S \right\}$$

is nonempty, closed, and bounded.

H.17. For each $(p,q) \in \Delta_+$, the set

$$\mathbb{F}(p,q) = \left\{ (x,\theta) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) :\ \exists (y,\xi) \in \left(\widehat{y}(p,q), \widehat{\xi}(p,q)\right),\ ((x,\theta),(y,\xi)) \in \mathbb{A}, \right.$$
$$\left. p_s \cdot x_{is} \leq p_s \cdot w_{is} + p_s \cdot A_s(p,q) \cdot \theta_i,\ \forall s = 1, ..., S, \forall i = 1, ..., I \right\}$$

is nonempty, where $\widehat{y}(p,q) = \sum_{j=1}^{J} \widehat{y}_j(p,q)$ and $\widehat{\xi}(p,q) = \sum_{j=1}^{J} \widehat{\xi}_j(p,q)$ are defined as

$$\boxed{\left(\widehat{y}_j(p,q), \widehat{\xi}_j(p,q)\right) = \underset{y_j \in \mathbb{Y}_j,\ \xi_j \in \mathbb{Z}}{arg\ max} \left\{ o_j\left(y_j, \xi_j, p, q\right) : p_0 \cdot y_{j0} + q \cdot \xi_j = 0 \quad p_s \cdot y_{js} - p_s \cdot A_s(p,q) \cdot \xi_j = 0,\ \forall s \right\}}$$

We have considered the assumptions on the characteristics of the economy under which the general equilibrium model works. Most of them are standard in the theory of incomplete markets except Assumptions $(H.12)$, $(H.14)$ and $(H.17)$.

Assumption $(H.12)$, known as the limited short-selling assumption, is not too restrictive condition. Note that if the consumption sets $\mathbb{X}_i$ and the production sets $\mathbb{Y}_j$ are assumed to be bounded, we have implicitly assumed lower bounds on short sales.

The limited short-selling assumption implies that agents cannot hold ~~asset~~'s portfolios of arbitrarily size. This condition was also assumed by Radner (1972, 1982). However, such a condition is necessary to ensure the existence of equilibria (Hart (1975) first proved that equilibria may fail to exist without bounds on short sales by a counter-example), it was considered unrealistic. Nevertheless, there exist exogenous reasons to assume that assets are bounded. In the real world, there is a limitation on the number of financial contracts that can be traded by financial markets or financial intermediaries. The number of brokers or agents that make contractual commitments is finite and moreover, there is a limitation on the number of operations to buy-sell. In other words, there exist technological constraints that prevent agents from holding an arbitrarily large number of asset's portfolios. As a consequence, from the economic point of view, it seen reasonable to assume that $\mathbb{Z}$ is bounded. For example, we can consider a number l large enough such that $\|\theta_i\| \leq l$, $\|\xi_j\| \leq l$.

Assumption $(H.14)$ cannot be applied to some particular set of assets as options.

Our basic tool to prove the existence of GEI equilibria, and consequently, to characterize GEI equilibria is to consider a mathematical program. Assumption $(H.17)$ assure that this program will have a solution and cannot be relaxed. If the set $\mathbb{F}(p, q)$ is empty for some $(p, q) \in \Delta_+$, the excess demand correspondence would be empty for such (p, q) and, consequently, there may exist no equilibria.

Now we states sufficient conditions under which a GEI equilibrium exists.

3 Existence of a two-period GEI equilibrium

The purpose of this section is to establish the existence of a two-period GEI equilibrium. First we prove the existence of an equilibrium with $(p^*, q^*) \in \Delta_+$; i.e. with nonnegative spot prices, $p^* \geq 0$. Then, we prove the existence of an equilibrium with $(p^*, q^*) \in \Delta$; i.e. with nonzero spot prices, $p^* \neq 0$.

Theorem 3 ***Existence of a GEI equilibrium.*** *Let* E *be an economy satisfying conditions* $(H.1)$ *to* $(H.17)$. *Then there exists a GEI equilibrium.*

Proof.

The proof will be decomposed into 6 parts.

Step 0:

For all $(p,q) \in \Delta_+$, let define the set of feasible production as

$$\overline{\mathbb{Y}}_j(p,q) = \{(y,\xi) \in \mathbb{Y}_j \times \mathbb{Z} : p_0 \cdot y_0 + q \cdot \xi = 0,\ p_s \cdot y_s - p_s \cdot A_s(p,q) \cdot \xi = 0,\ \forall s.\}.$$

Trivially, this set is a nonempty (since $0 \in \overline{\mathbb{Y}}_j(p,q)$) and compact set for all $(p,q) \in \Delta_+$. Then, for all $j = 1,\dots,J$, the solutions $\left(\widehat{y}_j(p,q), \widehat{\xi}_j(p,q)\right)$ of the firms' problem

$$\boxed{\begin{array}{lll} \underset{s.t.}{Max} & o_j\left(y_j,\xi_j,p,q\right) & \\ & p_0 \cdot y_{j0} + q \cdot \xi_j = 0 & \\ & p_s \cdot y_{js} - p_s \cdot A_s(p,q) \cdot \xi_j = 0, & \forall s \\ & y_{js} \in \mathbb{Y}_{js}, \quad \xi_j \in \mathbb{Z}, & \end{array}}$$

are non-empty, compact-valued, convex-valued, usc correspondences on $\overline{\mathbb{Y}}_j(p,q)$, for all $(p,q) \in \Delta_+$. by applying the Maximum Theorem under convexity. Note that $\left(\widehat{y}_j(p,q), \widehat{\xi}_j(p,q)\right)$ are homogeneous of degree zero by Assumption $(H.11)$ and $(H.14)$, and are well-defined at $(p,q) = 0$.

On the other hand, let consider the set

$$\widehat{\mathbb{Y}} = \left\{(y,\xi) \in \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z}) :\ \exists (x,\theta) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}),\ ((x,\theta),(y,\xi)) \in \mathbb{A}\right\}.$$

Trivially, this set is a nonempty and compact.

Now, consider $(p,q) \in \Delta_+$, $(y,\xi) \in \widehat{\mathbb{Y}}$ and $\delta \in \Lambda_\delta$ such that define the following problem,

$$\boxed{\begin{array}{l} \underset{s.t.}{Max} \sum\limits_{i=1}^{I} \delta_i u_i(x_{i0},\dots,x_{iS}) \\ \sum\limits_{i=1}^{I} x_{i0} \leq \sum\limits_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} \widehat{y}_{j0}(p,q), \\ \sum\limits_{i=1}^{I} x_{is} + \sum_{j=1}^{J} A_s(p,q)\widehat{\xi}_j(p,q) \leq \sum\limits_{i=1}^{I} w_{is} + \sum_{j=1}^{J} \widehat{y}_{js}(p,q) + \sum\limits_{i=1}^{I} A_s(p,q) \cdot \theta_i,\ \forall s = 1,\dots,S, \\ \sum\limits_{i=1}^{I} \theta_i = \sum_{j=1}^{J} \widehat{\xi}_j(p,q), \\ p_s \cdot x_{is} \leq p_s \cdot w_{is} + p_s \cdot A_s \cdot \theta_i,\ \forall s = 1,\dots,S,\ \forall i = 1,\dots,I, \\ x_i \in \mathbb{X}_i,\ \theta_i \in \mathbb{Z}. \end{array}} \tag{3}$$

Note also that the Slater's constraint qualification is satisfied since the constraints of Problem (3) are linear in x and θ.

To prove the existence of a equilibrium, we first prove that a solution $\left(\widehat{x},\widehat{\theta},\widehat{\lambda},\widehat{\phi}\right)$ of Problem (3) exists. In step II, we show that $\left(\widehat{x},\widehat{\theta},y,\xi,\widehat{\lambda},\widehat{\phi}\right)$ could be a GEI equilibrium with free disposability, under certain conditions with $\delta \gg 0$. In step III, we prove the existence of $(\delta^*,x^*,\theta^*,y^*,\xi^*,p^*,q^*)$ that satisfies the conditions required in step II, by means of a fixed point theorem. Finally, we verify that $(x^*, \theta^*, y^*, \xi^*, p^*, q^*)$ also satisfies the market clearing conditions and $\delta^* \gg 0$.

Step I:

Note that $\mathbb{F}(p,q)$ is a compact set by Assumption $(H.16)$. By Weierstrass' theorem, for each $(p,q) \in \Delta_+$, $(y,\xi) \in \widehat{\mathbb{Y}}$ and $\delta \in \Lambda$, we can guarantee the existence of a solution for Problem (3). Let us denote this solution as

$$\left(\widehat{x}(\delta,p,q,y,\xi), \widehat{\theta}(\delta,p,q,y,\xi)\right) \in \prod_{i=1}^{I}(\mathbb{X}_i \times \mathbb{Z}).$$

Note that these correspondences are nonempty.

Step II:

Given the primal Problem (3), define the Lagrangian function

$$\Phi(x,\theta,\lambda,\phi,\gamma) = \sum_{i=1}^{I} \delta_i\, u_i(x_{i0},...,x_{iS}) + \lambda_0 \cdot \left(\sum_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} y_{j0} - \sum_{i=1}^{I} x_{i0} \right) +$$
$$+ \sum_{s=1}^{S} \lambda_s \cdot \left(\sum_{i=1}^{I} w_{is} + \sum_{j=1}^{J} y_{js} + \sum_{i=1}^{I} A_s(p,q)\cdot\theta_i - \sum_{i=1}^{I} x_{is} - \sum_{j=1}^{J} A_s(p,q)\,\xi_j \right)$$
$$+ \phi \cdot \left(\sum_{j=1}^{J} \xi_j - \sum_{i=1}^{I} \theta_i \right) + \sum_{i=1}^{I}\sum_{s=1}^{S} \gamma_{is} p_s \cdot (w_{is} + A_s \cdot \theta_i - x_{is})$$

By Lagrange's duality theorem[1], for all $\left(\widehat{x},\widehat{\theta}\right)$ solution of Problem (3), there exist $\widehat{\lambda} = \widehat{\lambda}(\delta,p,q,y,\xi) \geq 0$, $\widehat{\phi} = \widehat{\phi}(\delta,p,q,y,\xi) \neq 0$, $\widehat{\gamma}(\delta,p,q,y,\xi) \neq 0$, with $\left(\widehat{\lambda},\widehat{\phi},\widehat{\gamma}\right) \in \mathbb{R}_+^{D(S+1)} \times \mathbb{R}^C \times \mathbb{R}_+^{IS}$, such that:

1.

$$\Phi\left(\left(\widehat{x},\widehat{\theta}\right),\widehat{\lambda},\widehat{\phi},\widehat{\gamma}\right) = \max\left\{ \Phi\left((x,\theta),\widehat{\lambda},\widehat{\phi},\widehat{\gamma}\right) : (x,\theta) \in \prod_{i=1}^{I}(\mathbb{X}_i \times \mathbb{Z}) \right\} \tag{4}$$

2.

$$\boxed{\begin{array}{l}
\sum\limits_{i=1}^{I} \widehat{x}_{i0} \leq \sum\limits_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} y_{j0},\ \widehat{\lambda}_0 \cdot \left(\sum\limits_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} y_{j0} - \sum\limits_{i=1}^{I} \widehat{x}_{i0} \right) = 0 \\
\sum\limits_{i=1}^{I} \widehat{x}_{is} \leq \sum\limits_{i=1}^{I} w_{is} + \sum_{j=1}^{J} y_{js},\ \forall s = 1,...,S, \\
\widehat{\lambda}_s \cdot \left(\sum\limits_{i=1}^{I} w_{is} + \sum_{j=1}^{J} y_{js} - \sum\limits_{i=1}^{I} \widehat{x}_{is} \right) = 0,\ \forall s = 1,...,S, \\
\sum\limits_{i=1}^{I} \widehat{\theta}_i = \sum_{j=1}^{J} \xi_j, \\
p_s \cdot \left(w_{is} + A_s(p,q)\cdot\widehat{\theta}_i - \widehat{x}_{is} \right) \geq 0,\ \forall s = 1,...,S,\ \forall i \\
\widehat{\gamma}_{is} p_s \cdot \left(w_{is} + A_s(p,q)\cdot\widehat{\theta}_i - \widehat{x}_{is} \right) = 0,\ \forall s = 1,...,S,\ \forall i.
\end{array}} \tag{5}$$

[1] See Avriel (1976), Th. 4.41, pp 99-100, Bertsekas (1995), Prop. 5.1.5, 5.1.6, pp. 427-428 and Bazaraa et al (1979), Th. 6.2.5, pp 209-210.

Note that in view of the Assumption $(H.5)$, it follows that $\widehat{\lambda} \neq 0$, hence $\left(\widehat{\lambda}, \widehat{\phi}\right) \neq 0$, and $\widehat{\gamma} \neq 0$. Let us denote by $\widehat{\lambda}(\delta, p, q, y, \xi)$, $\widehat{\phi}(\delta, p, q, y, \xi)$, $\widehat{\gamma}(\delta, p, q, y, \xi)$ the nonempty convex set of all admissible $\widehat{\lambda}$, $\widehat{\phi}$, $\widehat{\gamma}$ respectively.

Then, by these optimality conditions, we can show that $\left(\left(\widehat{x}, \widehat{\theta}\right), (y, \xi), \widehat{\lambda}, \widehat{\phi}\right)$ is a candidate to be a GEI equilibrium:

a) By Condition 2 in Lagrange theorem (eq. (5)), $\left(\left(\widehat{x}, \widehat{\theta}\right), (y, \xi), \widehat{\lambda}, \widehat{\phi}\right)$ satisfies the feasibility condition of a GEI equilibrium with free disposability:

$$\begin{cases} \sum\limits_{i=1}^{I} \widehat{x}_{i0} \leq \sum\limits_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} y_{j0}, \; \widehat{\lambda}_0 \cdot \left(\sum\limits_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} y_{j0} - \sum\limits_{i=1}^{I} \widehat{x}_{i0} \right) = 0 \\ \sum\limits_{i=1}^{I} \widehat{x}_{is} \leq \sum\limits_{i=1}^{I} w_{is} + \sum_{j=1}^{J} y_{js}, \qquad \forall s = 1, ..., S, \\ \widehat{\lambda}_s \cdot \left(\sum\limits_{i=1}^{I} w_{is} + \sum_{j=1}^{J} y_{js} - \sum\limits_{i=1}^{I} \widehat{x}_{is} \right) = 0, \qquad \forall s = 1, ..., S, \\ \sum\limits_{i=1}^{I} \widehat{\theta}_i = \sum_{j=1}^{J} \xi_j. \end{cases}$$

b)Let now study $i' - th$ consumer's problem. Taking $\forall i \neq i'$, $x_i = \widehat{x}_i$, $\theta_i = \widehat{\theta}_i$ from (4), it follows that

$$\begin{aligned} & \delta_{i'} u_{i'} \left(\widehat{x}_{i'0}, ..., \widehat{x}_{i'S}\right) + \widehat{\lambda}_0 \cdot \left(w_{i'0} - \widehat{x}_{i'0}\right) - \widehat{\phi} \cdot \widehat{\theta}_{i'} + \sum_{s=1}^{S} \widehat{\lambda}_s \cdot \left(w_{i's} + A_s(p,q) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \\ & + \sum_{s=1}^{S} \widehat{\gamma}_{i's} \, p_s \cdot \left(w_{i's} + A_s(p,q) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \\ \geq \; & \delta_{i'} u_{i'} \left(x_{i'0}, ..., x_{i'S}\right) + \widehat{\lambda}_0 \cdot \left(w_{i'0} - x_{i'0}\right) - \widehat{\phi} \cdot \theta_{i'} + \sum_{s=1}^{S} \widehat{\lambda}_s \cdot \left(w_{i's} + A_s(p,q) \cdot \theta_{i'} - x_{i's}\right) \\ & + \sum_{s=1}^{S} \widehat{\gamma}_{i's} \, p_s \cdot \left(w_{i's} + A_s(p,q) \cdot \theta_{i'} - x_{i's}\right), \end{aligned}$$

that, equivalently,

$$\begin{aligned} & \delta_{i'} u_{i'} \left(\widehat{x}_{i'0}, ..., \widehat{x}_{i'S}\right) + \widehat{\lambda}_0 \cdot \left(w_{i'0} - \widehat{x}_{i'0}\right) - \widehat{\phi} \cdot \widehat{\theta}_{i'} \\ & + \sum_{s=1}^{S} \left(\widehat{\lambda}_s + \widehat{\gamma}_{i's} p_s\right) \cdot \left(w_{i's} + A_s(p,q) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \\ \geq \; & \delta_{i'} u_{i'} \left(x_{i'0}, ..., x_{i'S}\right) + \widehat{\lambda}_0 \cdot \left(w_{i'0} - x_{i'0}\right) - \widehat{\phi} \cdot \theta_{i'} \\ & + \sum_{s=1}^{S} \left(\widehat{\lambda}_s + \widehat{\gamma}_{i's} p_s\right) \cdot \left(w_{i's} + A_s(p,q) \cdot \theta_{i'} - x_{i's}\right). \end{aligned}$$

We will distinguish two cases according to the positivity of $\delta_{i'}$.

-Whenever $\delta_{i'} = 0$,

$$\widehat{\lambda}_0 \cdot (w_{i'0} - \widehat{x}_{i'0}) - \widehat{\phi} \cdot \widehat{\theta}_{i'} + \sum_{s=1}^{S} \left(\widehat{\lambda}_s + \widehat{\gamma}_{i's} p_s\right) \cdot \left(w_{i's} + A_s(p,q) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \geq$$
$$\widehat{\lambda}_0 \cdot (w_{i'0} - x_{i'0}) - \widehat{\phi} \cdot \theta_{i'} + \sum_{s=1}^{S} \left(\widehat{\lambda}_s + \widehat{\gamma}_{i's} p_s\right) \cdot (w_{i's} + A_s(p,q) \cdot \theta_{i'} - x_{i's}), \ \forall (x_{i'}, \theta_{i'}).$$

By Condition 2 in Lagrange theorem (eq. (5)), we have

$$\widehat{\lambda}_0 \cdot (w_{i'0} - \widehat{x}_{i'0}) - \widehat{\phi} \cdot \widehat{\theta}_{i'} + \sum_{s=1}^{S} \widehat{\lambda}_s \cdot \left(w_{i's} + A_s(p,q) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \geq$$
$$\widehat{\lambda}_0 \cdot (w_{i'0} - x_{i'0}) - \widehat{\phi} \cdot \theta_{i'} + \sum_{s=1}^{S} \left(\widehat{\lambda}_s + \widehat{\gamma}_{i's} p_s\right) \cdot (w_{i's} + A_s(p,q) \cdot \theta_{i'} - x_{i's}), \ \forall (x_{i'}, \theta_{i'}).$$

Let define the transfers of the $i-th$ consumers as

$$t_i = \widehat{\lambda}_0 \cdot (\widehat{x}_{i0} - w_{i0}) + \widehat{\phi} \cdot \widehat{\theta}_i + \sum_{s=1}^{S} \widehat{\lambda}_s \cdot \left(\widehat{x}_{is} - A_s(p,q) \cdot \widehat{\theta}_i - w_{is}\right).$$

Taking $\theta_{i'} = 0$, we have

$$t_{i'} \leq \widehat{\lambda}_0 \cdot (x_{i'0} - w_{i'0}) + \sum_{s=1}^{S} \left(\widehat{\lambda}_s + \widehat{\gamma}_{i's} p_s\right) \cdot (x_{i's} - w_{i's}), \ \forall x_{i'}.$$

Whenever it is satisfied

$$(p,q) \in \left\{ \left(\frac{\lambda}{\|(\lambda,\phi)\|}, \frac{\phi}{\|(\lambda,\phi)\|} \right) : (\lambda,\phi) \in \left(\widehat{\lambda}(\delta^*, y^*, \xi^*), \widehat{\phi}(\delta^*, y^*, \xi^*)\right) \right\}, \tag{6}$$

we have

$$t_{i'} \leq \widehat{\lambda}_0 \cdot (x_{i'0} - w_{i'0}) + \sum_{s=1}^{S} \left(1 + \widehat{\gamma}_{is} / \left(\left\|\left(\widehat{\lambda}, \widehat{\phi}\right)\right\|\right)\right) \widehat{\lambda}_s \cdot (x_{i's} - w_{i's}), \ \forall x_{i'}.$$

with $\left(1 + \widehat{\gamma}_{is} / \left(\left\|\left(\widehat{\lambda}, \widehat{\phi}\right)\right\|\right)\right) \geq 0$.

By Assumption $(H.6)$ ($\exists \underline{x}_{i'} \in \mathbb{X}_{i'}$ such that $\underline{x}_{i'} \ll w_{i'}$), we have

$$t_{i'} \leq \widehat{\lambda}_0 \cdot (\underline{x}_{i'0} - w_{i'0}) + \sum_{s=1}^{S} \left(1 + \widehat{\gamma}_{is} / \left(\left\|\left(\widehat{\lambda}, \widehat{\phi}\right)\right\|\right)\right) \widehat{\lambda}_s \cdot (\underline{x}_{i's} - w_{i's}) < 0.$$

whenever $\widehat{\lambda} \neq 0$.

We will see in step IV, we will show that $\delta_i^* \gg 0$, by contradiction with this part. In step V, we also will prove that in equilibrium $\widehat{\lambda} \neq 0$.

-Whenever $\delta_{i'} \neq 0$, we have

$$\begin{aligned}
& u_{i'}\left(\widehat{x}_{i'0},...,\widehat{x}_{i'S}\right) + \frac{1}{\delta_{i'}}\widehat{\lambda}_0 \cdot \left(w_{i'0} - \widehat{x}_{i'0}\right) - \\
& \frac{1}{\delta_{i'}}\widehat{\phi} \cdot \widehat{\theta}_{i'} + \sum_{s=1}^{S} \frac{\left(\widehat{\lambda}_s + \widehat{\gamma}_{i's}p_s\right)}{\delta_{i'}} \cdot \left(w_{i's} + A_s(p,q) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \quad - \\
\geq \; & u_{i'}\left(x_{i'0},...,x_{i'S}\right) + \frac{1}{\delta_{i'}}\widehat{\lambda}_0 \cdot \left(w_{i'0} - x_{i'0}\right) - \\
& \frac{1}{\delta_{i'}}\widehat{\phi} \cdot \theta_{i'} + \sum_{s=1}^{S} \frac{\left(\widehat{\lambda}_s + \widehat{\gamma}_{i's}p_s\right)}{\delta_{i'}} \cdot \left(w_{i's} + A_s(p,q) \cdot \theta_{i'} - x_{i's}\right), \; \forall \left(x_{i'}, \theta_{i'}\right).
\end{aligned}$$

Whenever the condition (6) is satisfied, we have

$$\frac{\left(\widehat{\lambda}_s + \widehat{\gamma}_{i's}p_s\right)}{\delta_{i'}} = \frac{\left(1 + \widehat{\gamma}_{i's} \Big/ \left(\left\|\left(\widehat{\lambda}, \widehat{\phi}\right)\right\|\right)\right)}{\delta_{i'}}\widehat{\lambda}_s, \text{ with } \frac{\left(1 + \widehat{\gamma}_{i's} \Big/ \left(\left\|\left(\widehat{\lambda}, \widehat{\phi}\right)\right\|\right)\right)}{\delta_{i'}} \neq 0,$$

for all $s = 1,...,S$. Let define

$$\nu_{i0} = \frac{1}{\delta_i} \text{ and } \nu_{is} = \frac{\left(1 + \widehat{\gamma}_{is} \Big/ \left(\left\|\left(\widehat{\lambda}, \widehat{\phi}\right)\right\|\right)\right)}{\delta_i}, \quad \forall s = 1,...,S.$$

Note that $\nu_{is} \geq 0$, $\forall s$.

Then, we have that it is satisfied

$$\begin{aligned}
& u_{i'}\left(\widehat{x}_{i'0},...,\widehat{x}_{i'S}\right) + \nu_{i'0}\widehat{\lambda}_0 \cdot \left(w_{i'0} - \widehat{x}_{i'0}\right) - \\
& \nu_{i'0}\widehat{\phi} \cdot \widehat{\theta}_{i'} + \sum_{s=1}^{S} \nu_{i's}\widehat{\lambda}_s \cdot \left(w_{i's} + A_s(p,q) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \\
\geq \; & u_{i'}\left(x_{i'0},...,x_{i'S}\right) + \nu_{i'0}\widehat{\lambda}_0 \cdot \left(w_{i'0} - x_{i'0}\right) - \\
& \nu_{i'0}\widehat{\phi} \cdot \theta_{i'} + \sum_{s=1}^{S} \nu_{i's}\widehat{\lambda}_s \cdot \left(w_{i's} + A_s(p,q) \cdot \theta_{i'} - x_{i's}\right), \; \forall \left(x_{i'}, \theta_{i'}\right).
\end{aligned}$$

By the nonsatiation assumption $(H.5)$, we have that $\widehat{\gamma}_{is} > 0$; hence,

$$p_s \cdot \left(w_{is} + A_s(p,q) \cdot \widehat{\theta}_i - \widehat{x}_{is}\right) = 0, \; \forall s = 1,...,S, \; \forall i.$$

Moreover, whenever $\widehat{\lambda} \neq 0$ and $\delta_i > 0$, it is satisfied that

$$\nu_{is} > 0, \; \forall s = 0,1,...,S, \; \forall i = 1,...,I.$$

Then, $\left(\widehat{x}_{i'}, \widehat{\theta}_{i'}\right)$ is a saddle point of the following Lagrangian function

$$\begin{aligned}\Phi_{i'}\left(x_{i'}, \theta_{i'}, \nu_{i'}\right) &= u_{i'}\left(x_{i'0}, ..., x_{i'S}\right) + \nu_{i'0}\left[\widehat{\lambda}_0 \cdot \left(w_{i'0} - x_{i'0}\right) - \widehat{\phi} \cdot \theta_{i'}\right] + \\ &+ \sum_{s=1}^{S} \nu_{i's}\left[\widehat{\lambda}_s \cdot \left(w_{i's} + A_s\left(p, q\right) \cdot \theta_{i'} - x_{i's}\right)\right]\end{aligned}$$

with $x_{i'} \in \mathbb{X}_{i'}$, $\theta_{i'} \in \mathbb{Z}$ and $\nu_{i'} \geq 0$, iff

$$\widehat{\lambda}_0 \cdot \left(w_{i'0} - \widehat{x}_{i'0}\right) - \widehat{\phi} \cdot \widehat{\theta}_{i'} \geq 0, \ \nu_{i'0}\left(\widehat{\lambda}_0 \cdot \left(w_{i'0} - \widehat{x}_{i'0}\right) - \widehat{\phi} \cdot \widehat{\theta}_{i'}\right) = 0,$$

and, $\forall s = 1, ..., S$

$$\widehat{\lambda}_s \cdot \left(w_{i's} + A_s \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right) \geq 0, \ \nu_{i's}\left[\widehat{\lambda}_s \cdot \left(w_{i's} + A_s\left(p, q\right) \cdot \widehat{\theta}_{i'} - \widehat{x}_{i's}\right)\right] = 0.$$

And if $\left(\widehat{x}_i, \widehat{\theta}_i\right)$ is a saddle point, then $\left(\widehat{x}_i, \widehat{\theta}_i\right)$ is one of the decision plans of the $i - th$ consumer; i.e.

$$\boxed{\left(\widehat{x}_i, \widehat{\theta}_i\right) = \arg \max_{x_i \in \mathbb{X}_i,\ \theta_i \in \mathbb{Z}} \left\{u_i\left(x_{i0}, ..., x_{iS}\right) : \widehat{\lambda}_0 \cdot x_{i0} \leq \widehat{\lambda}_0 \cdot w_{i0} - \widehat{\phi} \cdot \theta_i, \ \widehat{\lambda}_s \cdot x_{is} \leq \widehat{\lambda}_s \cdot w_{is} + \widehat{\lambda}_s \cdot A_s\left(p, q\right) \cdot \theta_i, \ \forall s = 1, ..., S\right\}.}$$

In other words, by homogeneity of matrix $A_s\left(p, q\right)$, $\left(\widehat{x}_i, \widehat{\theta}_i\right)$ is one of the decision plans of the $i - th$ consumer whenever it is satisfied (6) and

$$\begin{aligned}\widehat{\lambda}_0 \cdot \left(w_{i0} - \widehat{x}_{i0}\right) - \widehat{\phi} \cdot \widehat{\theta}_i &= 0, \text{ with } \nu_{i0} > 0 \\ \widehat{\lambda}_s \cdot \left(w_{is} + A_s\left(p, q\right) \cdot \widehat{\theta}_i - \widehat{x}_{is}\right) &= 0, \text{ with } \nu_{is} > 0, \ \forall s = 1, ..., S.\end{aligned}$$

Note that, for all $i - th$ consumer, the equalities

$$\widehat{\lambda}_s \cdot \left(w_{is} + A_s\left(p, q\right) \cdot \widehat{\theta}_i - \widehat{x}_{is}\right) = 0, \ \forall s = 1, ..., S, \tag{7}$$

are satisfied trivially, whenever it is satisfied (6).

In step III, we will see that, in equilibrium, there exists a vector $(\delta^*, p^*, q^*, y^*, \xi^*)$ that satisfies the condition (6) with $(p^*, q^*) \neq 0$, $\delta^* \gg 0$ and $0 \in t_i$, where

$$t_i = \widehat{\lambda}_0 \cdot \left(\widehat{x}_{i0} - w_{i0}\right) - \widehat{\phi} \cdot \widehat{\theta}_i,$$

by condition (7). In step IV, we also will show that $\delta_i^* \gg 0$. Furthermore, in step V we will show that $(p^*, q^*) \neq 0$.

Step III:

Let define

$$t_i(\delta,p,q,x,\theta,y,\xi) = \widehat{\lambda}_0(\delta,p,q,x,\theta,y,\xi)\cdot(x_{i0}-w_{i0})+\widehat{\phi}(\delta,p,q,x,\theta,y,\xi)\cdot\theta_i + \sum_{s=1}^{S}\widehat{\lambda}_s(\delta,p,q,x,\theta,y,\xi)\cdot(x_{is}-A_s(p,q)\cdot\theta_i-w_{is}).$$

Since the set of all feasible consumption and production sets are bounded, we may construct a compact and convex set K which contains all feasible consumption and assets of agent i in its interior, respectively all feasible production plan and assets for firm j.

In view of Assumptions $(H.2)$ and $(H.6)$, we can construct some compact and convex sets K_1, K_2, K_3 and K_4 such that for all $(\delta,p,q)\in\Lambda\times\Delta_+$ and all allocation $(x,\theta,y,\xi)\in K^I\times K^J$, the set

$$\left(\widehat{\lambda}(\delta,p,q,x,\theta,y,\xi),\widehat{\phi}(\delta,p,q,x,\theta,y,\xi),\widehat{\gamma}(\delta,p,q,x,\theta,y,\xi),t(\delta,p,q,x,\theta,y,\xi)\right)$$

is in $K_1\times K_2\times K_3\times K_4$ and $K_1\subset\mathbb{R}_+^{D(S+1)}\setminus\{0\}$, $K_2\subset\mathbb{R}^C\setminus\{0\}$, $K_3\subset\mathbb{R}_+^{IS}\setminus\{0\}$, $K_4\subset\left\{t\in\mathbb{R}^I:\ \sum_{i=1}^I t_i=0\right\}$.

Note that these sets

$$\left(\widehat{\lambda}(\delta,p,q,x,\theta,y,\xi),\widehat{\phi}(\delta,p,q,x,\theta,y,\xi),\widehat{\gamma}(\delta,p,q,x,\theta,y,\xi),t(\delta,p,q,x,\theta,y,\xi)\right)$$

are possibly empty-valued since (x,θ) is not necessarily a solution of the Problem (3), in particular (x,θ,y,ξ) is not necessarily feasible.

We can now define the correspondence ψ from the convex and compact set

$$\Lambda\times\Delta_+\times K_1\times K_2\times K_3\times K_4 \times\prod_{i=1}^{I}(\mathbb{X}_i\cap K)\times\prod_{i=1}^{I}(\mathbb{Z}\cap K)\times\prod_{j=1}^{J}(\mathbb{Y}_j\cap K)\times\prod_{j=1}^{J}(\mathbb{Z}\cap K)$$

to itself,

$$\psi:(\delta,p,q,\lambda,\phi,\gamma,t,x,\theta,y,\xi)\longmapsto$$

$$\Big(r(\delta-t),\tfrac{\lambda}{\|(\lambda,\phi)\|},\tfrac{\phi}{\|(\lambda,\phi)\|},\widehat{\lambda}(\delta,p,q,x,\theta,y,\xi),\widehat{\phi}(\delta,p,q,x,\theta,y,\xi),\widehat{\gamma}(\delta,p,q,x,\theta,y,\xi), t(\delta,p,q,x,\theta,y,\xi),\widehat{x}(\delta,p,q,y,\xi),\widehat{\theta}(\delta,p,q,y,\xi),\widehat{y}(p,q),\widehat{\xi}(p,q)\Big)$$

where r is a continuous retraction from $\{u\in\mathbb{R}^I:\ \sum_{i=1}^I u_i=1\}$ to Λ that satisfies[2]: (i) $r(u)\in\partial\Lambda$ implies that there exists some i such that $r(u)_i=0$ and $u_i\leq 0$ and (ii) $u\notin\Lambda$ implies that $r(u)\in\partial\Lambda$.

[2]Note that these conditions are satisfied by the two following cases :
$r(u)=proj_\Lambda(u)$ the orthogonal projection over Λ
$r(u)=\left(\frac{u_1^+}{u_1^+ +\dots u_I^+},\dots,\frac{u_I^+}{u_1^+ +\dots u_I^+}\right)$ (approach used by Neghishi)

By applying the Maximum Theorem under convexity[3] to Problem (3), we have

$$\left(\widehat{x}\left(\delta,p,q,y,\xi\right),\widehat{\theta}\left(\delta,p,q,y,\xi\right),\widehat{y}\left(p,q\right),\widehat{\xi}\left(p,q\right),\widehat{\lambda}\left(\delta,p,q,x,\theta,y,\xi\right),\widehat{\phi}\left(\delta,p,q,x,\theta,y,\xi\right)\right)$$

are compact-valued, convex-valued, usc correspondences. Since by construction, $t\left(\delta,p,q,x,\theta,y,\xi\right)$, $r\left(\delta-t\right)$, $\frac{\lambda}{\|(\lambda,\phi)\|}$ and $\frac{\phi}{\|(\lambda,\phi)\|}$ are also compact-valued, convex-valued, usc correspondences. Finally, applying Gourdel Fixed-Point theorem[4] (see Gourdel (1995)) on the correspondence ψ, we get the existence of some

$$\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$$

such that:

$\delta^*=r\left(\delta^*-t^*\right)$,

$p^*=\frac{\lambda^*}{\|(\lambda^*,\phi^*)\|}$,

$q^*=\frac{\phi^*}{\|(\lambda^*,\phi^*)\|}$,

$\lambda^*\in\widehat{\lambda}\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ or $\widehat{\lambda}\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ is empty,

$\phi^*\in\widehat{\phi}\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ or $\widehat{\phi}\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ is empty,

$\gamma^*\in\widehat{\phi}\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ or $\widehat{\gamma}\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ is empty,

$t^*\in t\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ or $t\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ is empty,

$x^*\in\widehat{x}\left(\delta^*,p^*,q^*,y^*,\xi^*\right)$,

$\theta^*\in\widehat{\theta}\left(\delta^*,p^*,q^*,y^*,\xi^*\right)$,

$y^*\in\widehat{y}\left(p^*,q^*\right)$,

$\xi^*\in\widehat{\xi}\left(p^*,q^*\right)$.

We deduce from the last two conditions that $\widehat{\lambda}\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ is non-empty (and consequently $t\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)$ is also non-empty) which implies that the point is a fixed-point for ψ

$$\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right)\in\psi\left(\delta^*,p^*,q^*,\lambda^*,\phi^*,\gamma^*,t^*,x^*,\theta^*,y^*,\xi^*\right).$$

In Step V, we will prove that $\lambda^*\neq0$, and consequently, $p^*\neq0$ since $p^*=\frac{\lambda^*}{\|(\lambda^*,\phi^*)\|}$. Hence, $(p^*,q^*)\neq0$.

Step IV:

In view of step II, it only remains to show that $\delta_i^*\gg0$, in order to prove that $\left(\left(x^*,\theta^*\right),\left(y^*,\xi^*\right),\left(p^*,q^*\right)\right)$ is a GEI equilibrium.

[3] The Maximum Theorem under convexity restrictions is a consequence of the Maximum Theorem given by Berge (see Berge (1963), pp 115-116). The Maximum Theorem under convexity is presented in Sundaram (1996, pp 237-239) and Ginsburg and Keyzer (1997, pp 472-476).

[4] An alternative redaction may use the more sophisticated fixed-point theorem of Eilemberg-Montgomery (1946) to some acyclic correspondences.

We will prove by contradiction. Suppose that it is not the case, then since δ^* is in the (relative) bordary of Λ, it follows from the property of the retraction r that for some i', $\delta^*_{i'} = 0$ and $\delta^*_{i'} - t^*_{i'} \leq 0$.

By the colinearity of p^*, q^* and λ^*, ϕ^*, respectively, it follows from the part c) of step II that $t^*_{i'} < 0$. This is a contradiction and therefore, wee have proved that δ is in the (relative) interior of Λ, and consequently by condition (ii) on the retraction r, we deduce that $\delta^* - t^* \in \Lambda$, which implies that $t^* = 0$.

Step V:

Finally, note that $\lambda^* \gg 0$ will be implied by the nonsatiation assumption. We will use a proof by contradiction. Suppose that $\lambda^*_{s'k} = 0$. Let us consider i' such that $\delta^*_{i'} > 0$. Taking $\forall i \neq i'$, $x_i = \widehat{x}_i$, $\theta_i = \widehat{\theta}_i$, $\theta_{i'} = \widehat{\theta}_{i'}$ and $\forall s \neq s'$, $x_{i's} = \widehat{x}_{i's}$ from (4) , it follows that

$$\begin{aligned} & u_{i'}\left(x^*_{i'0}, ..., x^*_{i'S}\right) + \nu_{i's}\lambda^*_s \cdot \left(w_{i's'} - x^*_{i's'}\right) \\ \geq \quad & u_{i'}\left(x^*_{i'0}, .., x_{i's'}, .., x^*_{i'S}\right) + \nu_{i's'}\lambda^*_{s'} \cdot \left(w_{i's'} - x_{i's'}\right) \end{aligned}$$

since $\delta^*_{i'} > 0$, in particular $u_{i'}(x^*_{i'}) \geq u_{i'}(x^*_{i'} + z_k)$, for $z_k = (0, \dots, 0, 1, 0, \dots, 0)$, that contradicts the monotonicity assumption.

Moreover, since $\lambda^* \gg 0$, we have $\sum\limits_{i=1}^{I} x^*_i = \sum\limits_{i=1}^{I} w_i + \sum_{j=1}^{J} y^*_j$; i.e. the market clearing condition is verified.

■

We can also prove the existence of a GEI equilibrium $((x^*, \theta^*), (y^*, \xi^*), (p^*, q^*))$, where $(p^*, q^*) \neq 0$. Our approach will follow the plan already established in Theorem 3. In this context, we assume that $(p, q) \in \Delta$.

Theorem 4 ***Existence of a GEI equilibrium.*** *Let* E *be an economy satisfying conditions* $(H.1)$ *to* $(H.4)$, $(H.7)$ *to* $(H.15)$ *and* $(H.17)$ *and the following:*

H.5'. *Locally nonsatiated: There exists at least one consumer* i *such that* $\forall x_i \in \mathbb{X}_i$ *and* $\varepsilon > 0$, *there exists* $z_i \in \mathbb{X}_i \cap B(x_i, \varepsilon)$ *that satisfied* $u_i(z_i) > u_i(x_i)$.

H.6'. *Cheaper Consumption Assumption : For every consumer, given a price vector* $(p, q) \in \Delta$ *with* $p \neq 0$, *there exists* $\underline{x}_i \in \mathbb{X}_i$ *such that* $\sum\limits_{s=0}^{S} (p_s \cdot \underline{x}_{is}) < \sum\limits_{s=0}^{S} (p_s \cdot w_{is})$.

H.16'. *The set of allocations*

$$\mathbb{A} = \left\{ ((x, \theta), (y, \xi)) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) \times \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z}) : \sum_{i=1}^{I} \theta_i = \sum_{j=1}^{J} \xi_i, \right.$$
$$\left. \sum_{i=1}^{I} x_{i0} = \sum_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} y_{j0}, \ \sum_{i=1}^{I} x_{is} = \sum_{i=1}^{I} w_{is} + \sum_{j=1}^{J} y_{js}, \ \forall s = 1, ..., S \right\}$$

is nonempty, closed, and bounded.

H.17'. *For each* $(p,q) \in \Delta$, *the set*

$$\mathbb{F}(p,q) = \left\{ (x,\theta) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) : \exists (y,\xi) \in \left(\widehat{y}(p,q), \widehat{\xi}(p,q)\right), \ ((x,\theta),(y,\xi)) \in \mathbb{A}, \right.$$
$$\left. p_s \cdot x_{is} \leq p_s \cdot w_{is} + p_s \cdot A_s(p,q) \cdot \theta_i, \ \forall s = 1,...,S, \forall i = 1,...,I \right\}$$

is nonempty.

Then there exists a GEI equilibrium $((x^*,\theta^*),(y^*,\xi^*),(p^*,q^*))$, *with* $(p^*,q^*) \neq 0$.

Proof.

The Theorem 4 states sufficient conditions under which an GEI equilibrium exists. The proof is similar to that of Theorem 3 considering the following mathematical programming problem, instead of Problem (3) of Theorem 3:

Let $\left(\widehat{y}_j(p,q), \widehat{\xi}_j(p,q)\right)$ be the optimal decisions of firms for all $(p,q) \in \Delta$. These correspondences are compact-valued, convex-valued and usc. Note that these correspondences are well-defined at $(p,q) = 0$.

Then, consider $(p,q) \in \Delta$, $(y,\xi) \in \widehat{\mathbb{Y}}$ and $\delta \in \Lambda$ such that define the following problem,

$$\boxed{\begin{array}{l}
\underset{s.t.}{Max} \sum_{i=1}^{I} \delta_i u_i(x_{i0},...,x_{iS}) \\
\sum_{i=1}^{I} x_{i0} = \sum_{i=1}^{I} w_{i0} + \sum_{j=1}^{J} \widehat{y}_{j0}(p,q), \\
\sum_{i=1}^{I} x_{is} + \sum_{j=1}^{J} A_s(p,q) \cdot \widehat{\xi}_j(p,q) = \sum_{i=1}^{I} w_{is} + \sum_{j=1}^{J} \widehat{y}_{js}(p,q) + \sum_{i=1}^{I} A_s(p,q) \cdot \theta_i, \ \forall s = 1,...,S, \\
\sum_{i=1}^{I} \theta_i = \sum_{j=1}^{J} \widehat{\xi}_j(p,q), \\
p_s \cdot x_{is} \leq p_s \cdot w_{is} + p_s \cdot A_s \cdot \theta_i, \ \forall s = 1,...,S, \ \forall i = 1,...,I, \\
x_i \in \mathbb{X}_i, \ \theta_i \in \mathbb{Z}.
\end{array}} \tag{8}$$

The solution of this problem

$$\left(\widehat{x}(\delta,p,q,y,\xi), \widehat{\theta}(\delta,p,q,y,\xi), \widehat{\lambda}(\delta,p,q,y,\xi), \widehat{\phi}(\delta,p,q,y,\xi), \widehat{\gamma}(\delta,p,q,y,\xi)\right)$$

are compact-valued, convex-valued, usc correspondences. Note that by Assumption $(H.5')$, the multipliers are nonzero for all $(p,q) \in \Delta$, $(y,\xi) \in \widehat{\mathbb{Y}}$ and $\delta \in \Lambda$.

Using a similar fixed point argument to that of Theorem 3, there exists a fixed point

$$(\delta^*, p^*, q^*, \lambda^*, \phi^*, \gamma^*, t^*, x^*, \theta^*, y^*, \xi^*)$$

of correspondence ψ. Since

$$(p^*,q^*) \in \left(\frac{\lambda^*}{\|(\lambda^*,\phi^*)\|}, \frac{\phi^*}{\|(\lambda^*,\phi^*)\|}\right),$$

we can assure $(p^*, q^*) \neq 0$.

The rest of proof is identical to that of Theorem 3 without considering step V.

■

We can also prove the existence of a GEI equilibrium $((x^*, \theta^*), (y^*, \xi^*), (p^*, q^*))$, with asset prices nonnegative $q^* \geq 0$. The proof is similar to that of Theorem 3 considering the following inequality constraints

$$\sum_{i=1}^{I} \theta_i \leq \sum_{j=1}^{J} \widehat{\xi}_j (p, q)$$

instead of

$$\sum_{i=1}^{I} \theta_i = \sum_{j=1}^{J} \widehat{\xi}_j (p, q)$$

in Problem (3) of Theorem 3. This result allows the excess supply of assets, in equilibrium, is free since the utility functions do not depend on the financial assets and hence, the nonsatiation assumption does not imply that asset prices are nonzero in equilibrium; i.e. $q^* \neq 0$.

4 Characterization of a two-period GEI equilibrium

In this section we provide a characterization of an equilibrium relaxing the standard convexity assumptions and the limited short-selling assumption. We also consider a two period production model to be as simple as possible

We first introduce the concept of *local equilibria*. Then, we characterize a local equilibrium as a solution of a system of nonlinear equations.

Definition 5 ***Local GEI Equilibrium.*** *The vector prices $(p^*, q^*) \in \mathbb{R}^{D(S+1)} \times \mathbb{R}^C$, with $(p^*, q^*) \neq 0$, and the allocation*

$$((x^*, \theta^*), (y^*, \xi^*)) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) \times \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z}),$$

is a local is a GEI equilibrium for the economy E, *if there exists $\varepsilon > 0$ such that:*

CPL *Each consumer solves its local problem; i.e.*

$$\boxed{\begin{array}{ll} \underset{s.t.}{Max} & u_i (x_{i0}, ..., x_{iS}) \\ & p_0 \cdot x_{i0} \leq p_0 \cdot w_{i0} - q \cdot \theta_i, \\ & p_s \cdot x_{is} \leq p_s \cdot w_{is} + p_s \cdot A_s \cdot \theta_i, \qquad \forall s = 1, ..., S, \\ & x_i \in \mathbb{X}_i \cap \overline{B}(x_i^*, \varepsilon), \quad \theta_i \in \mathbb{Z} \cap \overline{B}(\theta_i^*, \varepsilon). \end{array}}$$

FPL *Each firm solves its local problem; i.e.*

$$\boxed{\begin{array}{ll} \underset{s.t.}{Max} & o_j\left(y_j,\xi_j,p,q\right) \\ & p_0\cdot y_{j0}+q\cdot\xi_j=0 \\ & p_s\cdot y_{js}-p_s\cdot A_s\cdot\xi_j=0, \qquad \forall s=1,...,S, \\ & y_j\in\mathbb{Y}_j\cap\overline{B}\left(y_j^*,\varepsilon\right), \quad \xi_j\in\mathbb{Z}\cap\overline{B}\left(\xi_j^*,\varepsilon\right). \end{array}}$$

MC *Market clearing:*

$$\sum_{i=1}^{I}x_i^*=\sum_{i=1}^{I}w_i+\sum_{j=1}^{J}y_j^*, \quad \sum_{i=1}^{I}\theta_i^*=\sum_{j=1}^{J}\xi_i^*.$$

The local GEI equilibrium concept fails to satisfy the completeness property of consumers' preferences, see Mas Colell et al. (1995) p. 6. But note that in applied models we may not be able to know a full specification of the market. Modelers may address these problems by using local estimations of production functions, preferences and consumption sets. A further discussion of this issue can be found in Esteban, Gourdel and Prieto (2000).

The local equilibrium concept is more realistic than the traditional equilibrium definition. It is important to emphasize that in this model, the local equilibria with nonnegative prices are defined on the sets $\left\{\mathbb{X}_i\cap\overline{B}\left(x_i^*,\varepsilon\right)\right\}_{i=1}^{I}$, $\left\{\mathbb{Z}\cap\overline{B}\left(\theta_i^*,\varepsilon\right)\right\}_{i=1}^{I}$, $\left\{\mathbb{Y}_j\cap\overline{B}\left(y_j^*,\varepsilon\right)\right\}_{j=1}^{J}$ and $\left\{\mathbb{Z}\cap\overline{B}\left(\xi_j^*,\varepsilon\right)\right\}_{j=1}^{J}$. These sets can be interpreted as information sets of the agents regarding their technologies and their preferences. We can assume that a local equilibrium will change when the agents get new information. This is actually what we observe in the real world.

As discussed in the introduction, the most of the literature on computation of general equilibria uses the excess demand function using the concept of no-arbitrage equilibrium. But, note that in applied models it is really difficult to specify the functional form of the demand and supply functions. In order to compute a local equilibrium, it is necessary to state practical conditions that characterize local GEI equilibria and suggest algorithms for finding these points. We will assume some smoothness properties to characterize a local GEI equilibrium as the solution of a system of non-linear equations.

Another important issue is the difficulty of meeting the required assumptions of existence's theorems given in Section 3. The next theorem states sufficient conditions to characterize a local equilibrium for the economy E, relaxing the limited short-selling and the standard convexity assumptions. Furthermore, the characterization of a GEI equilibrium only require the continuity of the matrix asset $A\left(p,q\right)$ in the neighborhood of the solution point $\left(p^*,q^*\right)$.

Without loss of generality, we assume that the consumption set of the $i-th$ consumer is described by inequality constraints

$$\mathbb{X}_i=\left\{x_i\in\mathbb{R}^{D(S+1)}:G_i\left(x_i\right)\leq 0\right\},$$

the technology of the $j-th$ firm is described by

$$\mathbb{Y}_j = \left\{ y_j \in \mathbb{R}^{D(S+1)} : F_j(y_j) \leq 0 \right\},$$

and the asset space $\mathbb{Z} = \mathbb{R}^C$. An inequality constraint is said to be *active* at a given point if it is satisfied with equality at this point.

Consider the product spaces

$$\Gamma_+ = \mathbb{R}^{I(S+1)}_{++} \times \prod_{i=1}^{I} \left(\mathbb{X}_i \times \overset{o}{\mathbb{Z}} \right) \times \prod_{j=1}^{J} \left(\mathbb{Y}_j \times \overset{o}{\mathbb{Z}} \right) \times \Delta_+ \times \mathbb{R}^I_+ \times \mathbb{R}^J_+ \times \overline{\mathbb{R}^{J(S+1)}}$$

and

$$\Gamma = \mathbb{R}^{I(S+1)}_{++} \times \prod_{i=1}^{I} \left(\mathbb{X}_i \times \overset{o}{\mathbb{Z}} \right) \times \prod_{j=1}^{J} \left(\mathbb{Y}_j \times \overset{o}{\mathbb{Z}} \right) \times \Delta \times \mathbb{R}^I_+ \times \mathbb{R}^J_+ \times \mathbb{R}^{J(S+1)}.$$

The next theorem provides a characterization of local GEI equilibria.

Theorem 6 ***Characterization of local GEI equilibria.*** *Let* E *be an economy satisfying conditions* $(H.1)$, $(H.2)$, $(H.5)$, $(H.7)$, $(H.8)$, $(H.11)$, $(H.12)$ *and* $(H.15)$. *If there exists a point*

$$z^* = (\delta^*, x^*, \theta^*, y^*, \xi^*, p^*, q^*, \omega^*, \mu^*, \gamma^*) \in \Gamma_+,$$

that satisfies the following conditions:

H.3'. $\mathbb{X}_i \cap \overline{B}(x_i^*, \varepsilon)$ *is convex* $\forall i$,

H.4'. $u_i(\cdot)$ *is concave in* $\mathbb{X}_i \cap \overline{B}(x_i^*, \varepsilon)$, $\forall i$,

H.9'. $\mathbb{Y}_j \cap \overline{B}(y_j^*, \varepsilon) \neq \emptyset$ *is convex,* $\forall j$,

H.10'. $o_j(y_j, \xi_j, p, q)$ *is concave in* $\mathbb{Y}_j \cap \overline{B}(y_j^*, \varepsilon)$ *and* $\mathbb{Z} \cap \overline{B}(\xi_j^*, \varepsilon)$, $\forall (p,q) \in \mathbb{R}^{D(S+1)+C}$, $\forall j$,

H.13'. $\mathbb{Z} \cap \overline{B}(\theta_i^*, \varepsilon)$, $\forall i$, *and* $\mathbb{Z} \cap \overline{B}(\xi_j^*, \varepsilon)$, $\forall j$, *are convex.*

H.14'. *The return functions* $A_s(p,q)$ *are nonnegative, continuous, and homogeneous of degree zero in* $\overline{B}((p^*, q^*), \varepsilon)$.

H.18. $u_i(\cdot)$ *is continuously differentiable in* $\mathbb{X}_i \cap \overline{B}(x_i^*, \varepsilon)$ *and* $G_i(\cdot)$ *is continuously differentiable in* $\mathbb{X}_i \cap \overline{B}(x_i^*, \varepsilon)$, $\forall i$,

H.19. $o_j(y_j, \xi_j, p, q)$ *is continuously differentiable in* $\mathbb{Y}_j \cap \overline{B}(y_j^*, \varepsilon)$ *and* $\mathbb{Z} \cap \overline{B}(\xi_j^*, \varepsilon)$, $\forall (p,q) \in \mathbb{R}^{D(S+1)+C}$, *and* $F_j(\cdot)$ *is continuously differentiable in* $\mathbb{Y}_j \cap \overline{B}(y_j^*, \varepsilon)$, $\forall j$,

H.20. *rank* $(A_s(p^*, q^*)) = D$, $\forall s$,

and the following holds:

$$\boxed{\begin{array}{l}
\delta^*_{is} \cdot \nabla_{x_{is}} u_i(x^*_i) - \lambda^{*T}_s - \omega^*_{is} \nabla_{x_{is}} G_{is}(x^*_{is}) = 0, \text{ con } \delta^*_{is} > 0, \forall i, \forall s, \\
\nabla_{y_{js}} o_j\left(y^*_j, \xi^*_j, \lambda^*, \phi^*\right) - \gamma^*_{js} \lambda^{*T}_s - \mu^*_{js} \nabla_{y_{js}} F_{js}(y^*_j) = 0, \forall j, \forall s, \\
\sum\limits_{s=1}^{S} \frac{1}{\delta^*_{is}} \lambda^*_s A_s - \frac{1}{\delta^*_{i0}} \phi^* = 0, \ \forall i, \\
\sum\limits_{s=1}^{S} \gamma^*_{js} \lambda^*_s A_s - \gamma^*_{j0} \phi^* = 0, \ \forall j, \\
\omega^*_{is} G_{is}(x^*_{is}) = 0, \quad G_{is}(x^*_{is}) \leq 0, \ \forall i, \forall s, \\
\mu^*_{js} F_{js}\left(y^*_j\right) = 0, \quad F_{js}\left(y^*_j\right) \leq 0, \ \forall j, \forall s, \\
\lambda^{*T}_0 (x^*_{i0} - w_{i0}) + \phi^{*T} \theta^*_i = 0, \ \forall i, \\
\lambda^{*T}_s (x^*_{is} - w_{is} - A_s \cdot \theta^*_i) = 0, \ \forall i, \forall s, \\
\lambda^{*T}_0 y^*_{j0} - \phi^{*T} \xi^*_j = 0, \ \forall j, \\
\lambda^{*T}_s y^*_{js} + \lambda^{*T}_s A_s \xi^*_j = 0, \ \forall j, \ \forall s, \\
\sum\limits_{i=1}^{I} x^*_i = \sum\limits_{i=1}^{I} w_i + \sum_{j=1}^{J} y^*_j, \\
\sum\limits_{i=1}^{I} \theta^*_i = \sum_{j=1}^{J} \xi^*_i,
\end{array}} \tag{9}$$

then $((x^*, \theta^*), (y^*, \xi^*), (p^*, q^*))$ *is a local GEI equilibrium for the economy* E.

Proof.

First, note that it is assumed that $(x^*, \theta^*, y^*, \xi^*)$ is a feasible point of Problem (3), i.e. the allocation $(x^*, \theta^*, y^*, \xi^*) \in \mathbb{A}$.

On the other hand, z^* satisfies the first-order condition of Problem (3) and Problem (2), by Assumption $(H.5)$, $(H.11)$ and taking $\delta^*_{i0} = 1/\nu_{i0}$, $\delta^*_{is} = 1/\nu_{is}$, $\forall s = 1, ..., S$ and $(p^*, q^*) = \left(\frac{\lambda^*}{\|(\lambda^*, \phi^*)\|}, \frac{\phi^*}{\|(\lambda^*, \phi^*)\|}\right)$. Then, by $(H.3')$, $(H.4')$, $(H.9')$, $(H.10')$, $(H.13')$ and $(H.14')$, we have that z^* is a local optimum of Problem (3). See Avriel (1976), Th. 4.38, p. 96, Bertsekas (1995), Prop. 3.1., pp 254-281 and Bazaraa et al (1979), Th. 6.2.5, pp. 209-210.

On the other hand, z^* also satisfies the fixed point conditions required by Theorem

$$(p^*, q^*) = \left(\frac{\lambda^*}{\|(\lambda^*, \phi^*)\|}, \frac{\phi^*}{\|(\lambda^*, \phi^*)\|}\right),$$
$$\lambda^{*T}_0 (x^*_{i0} - w_{i0}) + \phi^{*T} \theta^*_i = 0, \forall i,$$
$$\lambda^{*T}_s (x^*_{is} - w_{is} - A_s \cdot \theta^*_i) = 0, \forall i, \forall s = 1, ..., S,$$

and $\delta^*_i > 0$, we have that $((x^*, \theta^*), (y^*, \xi^*), (p^*, q^*))$ is a local GEI equilibrium.

■

Assumption $(H.16)$ is the constraint qualification under differentiability. Assuming $rank\left(A_s\left(p^*, q^*\right)\right) = D$, $\forall s$, the constraint gradients associated to conditions defined by $\mathbb{A}$ and balance equations are always linear independent. Then, $(x^*, \theta^*, y^*, \xi^*) \in \mathbb{A}$ is a regular point[5] of Problem (3).

Note that the Assumption $rank\left(A_s\left(p^*, q^*\right)\right) = D$, $\forall s$, does not imply that the financial markets are complete. For example, if $A_s\left(p^*, q^*\right) = A_{s'}\left(p^*, q^*\right)$, $\forall s \neq s'$, such that $rank\left(A_s\left(p^*, q^*\right)\right) = D$, we have that $rank\left(p^* A\left(p^*, q^*\right)\right) = 1 \neq S$.

Analogously, we can establish other sufficient conditions for the characterization of a local GEI equilibrium allowing for the existence of negative spot prices.

Proposition 7 ***Characterization of local equilibria.*** *Let* E *be an economy satisfying conditions* $(H.1)$, $(H.2)$, $(H.3')$, $(H.4')$, $(H.5')$, $(H.7)$, $(H.8)$, $(H.9')$, $(H.10')$, $(H.11)$, $(H.12)$, $(H.13')$, $(H.14')$, $(H.15)$, $(H.18)$, $(H.19)$ *and* $(H.20)$. *If there exists a point*

$$z^* = (\delta^*, x^*, \theta^*, y^*, \xi^*, p^*, q^*, \omega^*, \mu^*, \gamma^*) \in \Gamma,$$

that satisfies the conditions (9), *then* $\left((x^*, \theta^*), (y^*, \xi^*), (p^*, q^*)\right)$ *is a local GEI equilibrium for the economy* E.

The characterization a GEI equilibrium is described by inequality constraints that can be transformed them into equations by adding nonnegative *slack variables.* For example, let consider

$$\mathbb{Y}_j = \left\{y_j \in \mathbb{R}^{D(S+1)} : F_j\left(y_j\right) \leq 0\right\},$$

then we may consider an equivalent technology set $f_j \geq 0$

$$\mathbb{Y}_j = \left\{y_j \in \mathbb{R}^{D(S+1)} : F_j\left(y_j\right) + f_j = 0,\ f_j \geq 0\right\},$$

where f_j are nonnegative slack variables. Therefore, a GEI equilibrium is characterized by equalities constraints together with bound constraints. Let $H(z) = 0$ denote the system (9) of nonlinear equations that characterize a GEI equilibrium, where z now contains the variables and slacks; and $l \leq z \leq u$ denote the bound constraints, where l and u are vectors of lower and upper bounds on the components of z. Some components of z may lack a lower or an upper bound, in these cases we set the appropriate components of l and u to $-\infty$ and $+\infty$, respectively.

Note that we assume that the portfolios θ^*, ξ^* are interior points of $\mathbb{Z}$. We can extend all previous results allowing that $\theta^*, \xi^* \in \partial\mathbb{Z}$. In such a case, let consider $\mathbb{Z} = \{\theta : H(\theta) \leq 0\}$. Then, the characterization a GEI equilibrium is given by the following additional conditions: $\eta H(\theta) = 0$, $H(\theta) \leq 0$, where η denote the Lagrange multiplier associated to $H(\theta) \leq 0$, and the conditions

[5] A feasible vector (x^*, y^*) for which the active constraint gradients are linearly independent is called regular. Equivalently, a feasible vector (x^*, y^*) for which the matrix of active constraint gradients has full row rank is called regular.

$$\boxed{\begin{array}{l}\sum\limits_{s=1}^{S} \frac{1}{\delta_{is}^*}\lambda_s^* A_s - \frac{1}{\delta_{i0}^*}\phi^* = 0, \ \forall i, \\ \sum\limits_{s=1}^{S} \gamma_{js}^*\lambda_s^* A_s - \gamma_{j0}^*\phi^* = 0, \ \forall j,\end{array}}$$

should be modified adequately.

Finally, note that when the functions u_i, o_j, G_i and $\bar{F_j}$ are not differentiable (but still convex and finite everywhere) the conditions to characterize an equilibrium are similar, except that we have to take into account the subdifferentials of such functions instead of their gradients. For an introduction to nonsmooth optimization see, for example, Rockafellar (1970, 1981), Clarke (1990). In practice most economic models consider differentiability assumptions.

4.1 No arbitrage condition

As discussed in the introduction, the idea of arbitrage and absence of arbitrage opportunities is a basic concept of finance. It is well-known that under Assumption (H.5), if there exists an equilibrium, then the financial market must not offer arbitrage opportunities.

Consider the system (9) of nonlinear equations that characterize a GEI equilibrium given in Section 4. Note that a local GEI equilibrium satisfies the following conditions:

$$\boxed{\begin{array}{l}\sum\limits_{s=1}^{S} \frac{1}{\delta_{is}^*}\lambda_s^* A_s - \frac{1}{\delta_{i0}^*}\phi^* = 0, \ \forall i, \\ \sum\limits_{s=1}^{S} \gamma_{js}^*\lambda_s^* A_s - \gamma_{j0}^*\phi^* = 0, \ \forall j,\end{array}}$$

whenever the asset's portfolio in equilibrium is an interior point of $\mathbb{Z}$.

In Finance Theory, this is known as *no-arbitrage condition*. If this condition is satisfied, the market is said to not offer arbitrage opportunities. This condition is equivalent to the no-free-lunch property (it is not possible to produce any good in positive amount without using some other good as a input).

Under *no-arbitrage condition*, we can consider an alternative concept of equilibrium known as *normalized no-arbitrage equilibrium*. See Magill and Shafer (1991). The importance of this equilibrium concept is that its allocations coincide with those of the GEI equilibrium and the proofs of existence of equilibria based on Grassmanians are simplified. See Duffie and Shafer (1985).

5 Local Uniqueness of two-period GEI equilibria

Having established in Section 3 conditions under which a GEI equilibrium

$$((x^*, \theta^*), (y^*, \xi^*), p^*, q^*) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) \times \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z}) \times \Delta$$

is guaranteed to exist, we now study its local uniqueness.

Firstly, we establish sufficient conditions under a GEI equilibrium are functions (instead of correspondences) in $\delta^* \in \Lambda$ so that $0 \in t(\delta^*)$. This is an immediate consequence of Maximum Theorem for a convex program (see Sundaram (1996), pp. 237-239).

Then, we prove that, under certain conditions, there exist a unique δ^* that defines an unique GEI equilibrium for the economy E.

Proposition 8 *Let* $((x^*, \theta^*), (y^*, \xi^*), p^*, q^*)$ *be a local GEI equilibrium. For each* $\delta^* \in \Lambda$ *such that* $0 \in t(\delta^*)$, *we have:*

1. *Under strictly local concavity, for each vector price* $(p^*, q^*) \in \Delta$ *the allocations in equilibrium* $((x^*, \theta^*), (y^*, \xi^*))$ *is locally unique.*

2. *Under the assumptions* $(H.4')$, $(H.10')$, $(H.18)$, $(H.19)$ *and* $(H.20)$, *the prices* $(p^*, q^*) \in \Delta$ *associated to* $((x^*, \theta^*), (y^*, \xi^*))$ *are locally unique.*

Proof.

1. The result is an immediate consequence of the optimality properties for a convex constrained problem. See Avriel (1976), Th. 4.31, 4.32, pp. 92-93, Bertsekas (1995), Prop. 1.1.2, p. 12 and Bazaraa et al (1979), Th. 3.4.2, p. 101.

2. The result is an immediate consequence of the Lagrange Multiplier Theorem for a convex constrained problem. See Avriel (1976), Th. 3.8, 3.9 pp. 41-45, Bertsekas (1995), Prop. 3.1, 3.2, pp. 254-282 and Bazaraa et al (1979), Th. 4.3.7, 4.3.8, pp. 162-165. ■

But, note that the previous proposition does not guarantee the local uniqueness of an equilibrium due to the parameter $\delta^* \in \Lambda$. We now prove, under stronger conditions, the uniqueness of such an equilibrium.

Proposition 9 *Under assumptions given by Theorem 6 and the following one:*

U.M. *H is uniformly monotone on ${D_0}$*[6]*, where H is the system of non-linear equations $H : D_0 \longrightarrow \Gamma$, defined by Section 4, such that $D_0 \subset \Gamma$ is an opened and convex set,*

[6] A mapping $H : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is uniformly monotone on $D_0 \subset D$, an opened convex set, iff there exists $\gamma > 0$ such that $(H(x) - H(y))(x - y) \geq \gamma (x - y)^T (x - y)$, $\forall x, y \in D_0$.

there exists a unique $z^* \in D_0$ *such that* $H(z^*) = 0$. *Hence, there exists a unique GEI equilibrium* $((x^*, \theta^*), (y^*, \xi^*), p^*, q^*)$ *for the economy* E.

Proof.

It is an immediate consequence of the monotonicity of operator H. See Ortega and Rheinboldt (1970), pp. 141-145.

■

Ortega and Rheinboldt (1970, pp. 142) provide sufficient conditions for this assumption $(U.M.)$

6 Computing local two-period GEI equilibria

In this section we outline how a local equilibrium can be computed using *Interior-Point Methods*. We consider the system of nonlinear equations given by Section 4, $H(z) = 0$, where $z \in D_0$ and the bounds constraints of the form $l \leq z \leq u$. As we have shown, the solutions of this system are equilibria for a given economy E. Note that we cannot solve this system using traditional methods (Newton's method) due to the simple bounds. We will solve an alternative inequality-constrained optimization problem:

$$\boxed{\begin{array}{ll} \underset{z}{Min} & \frac{1}{2}\|H(z)\|_2^2 \\ s.t. & \\ & l \leq z \leq u. \end{array}} \tag{10}$$

using *Interior Point Methods* for non-linear programming.

During the 1960s, many techniques were derived for unconstrained optimization. It was standard practice to convert a constrained problem into a sequence of unconstrained problems, by incorporating to the objective function additional terms that would add arbitrarily-high costs either for infeasibility or for approaching the boundary of the feasible region. The most popular of these approaches for inequality constrained problems was the use of barrier methods. Interior point methods are closely related to the classical logarithmic barrier methods. The barrier method is defined by introducing a parameter μ, called the *barrier parameter*, and a logarithmic barrier function that is defined in the interior of the feasible set of the original problem.

Interior point methods transform this inequality-constrained optimization problem into a sequence of equality-constrained optimization subproblems defined as:

$$\boxed{Min \quad \frac{1}{2}\|H(z)\|_2^2 - \mu \sum_{i=1}^{I} \log(z_i - l_i) - \mu \sum_{i=1}^{I} \log(u_i - z_i).} \tag{11}$$

Under mild conditions, every limit point of a sequence $\{z^*(\mu)\}$ of local minimizers of these problems is a local minimum of the original constrained problem; i.e. $z^*(\mu) \longrightarrow z^*$ as $\mu \rightarrow 0$. This method was studied by Fiacco and McCormick (1968).

In spite of the good properties of this method, it became unpopular because of the numerical ill-conditioning of the barrier Hessian. Recently, it was proved that under conditions that normally hold in practice, this ill-conditioning does not degrade the accuracy of the computed solution. See Wright, M. (1997) and Wright, S. (1998).

In 1984, Karmarkar presented a polynomial-time linear programming method. In 1986, Gill et al. showed that there is an equivalence between Karmarkar's method and logarithmic barrier methods. Since then, interior-points have become very popular. For an introduction to interior point methods see, e.g. Wright, M. (1998) and its references, and for details, see Nesterov and Nemirovskii (1994).

When interior point methods are applied to Problem (11), the corresponding first-order conditions have the form:

$$J(z_k)^T H(z_k) - \mu (Z_k - L)^{-1} + \mu (U - Z_k)^{-1} = 0,$$

where $Z_k = diag(z_k)$, $L = diag\,(l)$, $U = diag\,(u)$ and $J(z_k)$ denote the Jacobian matrix of H.

Let $W_k^1 = \mu (Z_k - L)^{-1}$ and $W_k^2 = \mu (U - Z_k)^{-1}$, then we can rewrite the first-order conditions as

$$\begin{aligned}
&J(z_k)^T H(z_k) - w_k^1 + w_k^2 = 0, \\
&(Z_k - L)\, W_k^1 - \mu = 0, \\
&(U - Z_k)\, W_k^2 - \mu = 0, \\
&w_k^1, w_k^2 > 0,
\end{aligned}$$

that we will denote as by $F\,(z_k, w_k^1, w_k^2) = 0$. This is the standard primal-dual system that we will solve using Newton algorithm (See e.g. Dennis and Schnabel (1996, pp 86-154)):

Step 1. Let z_0, w_0^1, w_0^2 and $\varepsilon > 0$. Set $k = 1$, $z_k \leftarrow z_0$, $w_k^1 \leftarrow w_0^1$, and $w_k^2 \leftarrow w_0^2$.

Step 2. If $\|F\,(z_k, w_k^1, w_k^2)\|_2 < \varepsilon$, stop (the problem is solved); else, solve the system

$$\begin{pmatrix} J(z_k)^T H(z_k) & I & -I \\ W_k^1 & (Z_k - L) & 0 \\ -W_k^2 & 0 & (U - Z_k) \end{pmatrix} \begin{pmatrix} \Delta z \\ \Delta w^1 \\ \Delta w^2 \end{pmatrix} = -F\left(z_k, w_k^1, w_k^2\right).$$

Step 3. Compute α_z, α_{w^1}, $\alpha_{w^2} \in (0,1)$ such that $z_{k+1} = z_k + \alpha_z \Delta z$, $w_{k+1}^1 = w_k^1 + \alpha_{w^1} \Delta w^1$ and $w_{k+1}^2 = w_k^2 + \alpha_{w^2} \Delta w^2$ are feasible.

Step 4. Consider the merit function

$$M\,(z;\mu) = \frac{1}{2}\,\|H(z)\|_2^2 - \mu \sum_{i=1}^{I} \log\,(z_i - l_i) - \mu \sum_{i=1}^{I} \log\,(u_i - z_i)\,,$$

and let $m\,(\alpha) = M\,(z + \alpha \Delta z; \mu)$.

While $m\,(0) - m\,(\alpha_z) < -\rho\,\alpha_z\, \nabla m\,(0)^T\, \Delta z$, where $0 < \rho < 1$, set $\alpha_z \leftarrow \alpha_z / 2$ and $z_{k+1} = z_k + \alpha_z \Delta z$.

Step 5. Update

$$\mu \leftarrow \gamma \frac{(z_k - l)^T w_k^1 + (u - z_k)^T w_k^2}{2D},$$

where $0 \leq \gamma < 1$, and $k \leftarrow k+1$ and go back to step 2.

To apply the interior-point method, we have to ensure that $J(z)$ exists at the solution z^*. This is not a restrictive requirement. Hens (1998, p. 143) pointed out that the point where demand is discontinuous is no longer a candidate equilibrium point. Therefore, these methods are applicable in a neighbourhood of the solution z^* and as a consequence, the counterexamples like that of Hart (1975) are exceptional.

The parameter μ measures the average value of the pairwise products $(z_k - l)^T w_k^1$ and $(u - z_k)^T w_k^2$. The success of this algorithm depends critically on the choice of the parameters μ and γ. Unfortunately, difficulties can arise if unsuitable values of these parameters are used. See e.g. Wright, M. (1998).

6.1 Some examples

To illustrate this approach, we present some examples.

Example 10 ***Two period exchange economy. DeMarzo and Eaves (1996).***

Consider a GEI exchange economy with three consumers, three states in the second period, two assets and two goods. The consumer $i-th$ has an utility function of the form $u_i(x) = \sum_{s=1}^{3} \pi_s \left(B - x_{s1}^{\alpha_i} x_{s2}^{1-\alpha_i}\right)$, where $B = 57$, $\pi = \left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\alpha_1 = \alpha_2 = \frac{3}{4}$ and $\alpha_3 = \frac{1}{4}$ and initial endowments

$$\begin{aligned} w_1 &= w_2 = (10, 10; 25, 20; 20, 20; 15, 20)^T, \\ w_3 &= (20, 20; 5, 10; 10, 10; 15, 20)^T. \end{aligned}$$

The return matrix A is given by

$$A^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 2 & -1 \end{bmatrix}.$$

Taking as an initial point $z_0 = [1, w_1, w_2, w_3, 1]^T$, the interior-point algorithm converges to the equilibrium

$$\begin{aligned} x_1^* &= x_2^* = [17.01, 7.76; 24.39, 11.66; 21.61, 10.37; 18.13, 7.9]^T, \\ x_3^* &= [5.96, 24.47; 6.2, 26.67; 6.77, 29.25; 8.72, 34.19]^T, \\ \theta_1^* &= \theta_2^* = [-0.6340, -4.4395]^T, \ \theta_3^* = [1.2681, 8.879]^T, \end{aligned}$$

in 17 steps. The following table provides some information on the implementation of the algorithm for this example:

k	$\left\Vert \frac{z_{k+1}-z_k}{z_k} \right\Vert_2$	$\frac{1}{2}\Vert H(z_k)\Vert_2^2$	$\nabla H(z_k)^T H(z_k)$	μ	*alpha*
1	1.16e-1	6.47e+6	2.58e+10	6.98e-2	0.99
2	1.09e-1	3.1e+5	1e+9	1.12e-2	1
3	2.05e-1	7.43e+03	2.5e+7	2.56e-3	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
17	7e-5	3.74e-23	2.34e-7	2.05e-15	1

Example 11 ***Two period exchange economy. Schemedders (1998).***

Now, we consider a slight variation of Example 1. We assume that the initial endowments of agent 3 is given by

$$w_3 = (20, 20; 8, 24; 10, 30; 6, 18)^T.$$

Taking as an initial point $z_0 = [\mathbf{1}, w_1, w_2, w_3, \mathbf{1}]^T$, the interior-point algorithm converges to the equilibrium

$$\begin{aligned} x_1^* &= x_2^* = [16.70, 7.2; 24.77, 12.62; 21.04, 13; 14.59, 9.34]^T, \\ x_3^* &= [6.58, 25.58; 8.45, 38.75; 7.9, 43.98; 6.81, 39.3]^T, \\ \theta_1^* &= \theta_2^* = [4.0106, -6.7346]^T, \ \theta_3^* = [-8.0211, 13.4692]^T, \end{aligned}$$

in 17 steps. The following table provides some information on the implementation of the algorithm for this example:

k	$\left\Vert \frac{z_{k+1}-z_k}{z_k} \right\Vert_2$	$\frac{1}{2}\Vert H(z_k)\Vert_2^2$	$\nabla H(z_k)^T H(z_k)$	μ	*alpha*
1	1.4e-1	6.33e+6	2.53e+10	7.49e-2	0.98
2	8.42e-2	2.86e+3	2.5e+5	2.25e-2	1
3	4.5e-1	1.98e+02	9.05e+4	7.66e-3	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
17	2.26e-5	1.4e-24	4.98e-8	1.53e-15	1

Example 12 ***Two period production economy.***

Consider a production economy with three consumers, one producer, three states in the second period, two assets and two goods. We assume that the consumers are characterized as Example 1. The firm is profit-maximizers and has a production set $Y = \{(y_1, y_2) : y_2 = 16 - (y_1 + 4)^2\}$. Given an initial point $z_0 = \mathbf{1}^T$, the interior-point algorithm converges to

$$\begin{aligned} x_1^* &= x_2^* = [13.82, 8.56; 23.16, 15.45; 20.41, 13.75; 17.6, 9.57]^T, \\ x_3^* &= [5.19, 28.95; 5.67, 34.08; 6.18, 37.47; 8.29, 40.6]^T, \\ y^* &= [-7.15, 6.07; -2.99, 14.99; -2.98, 14.97; -1.501, 9.75]^T, \\ \theta_1^* &= \theta_2^* = [0.1773, -2.8564]^T, \ \theta_3^* = [3.9281, 5.8534]^T, \\ \xi^* &= [4.2828, 0.1405]^T, \end{aligned}$$

in 25 steps.

It is important to clarify that interior point methods converge to a stationary point. Such a point is a global minimum if H is convex, but this need not be so for nonconvex problems. Thus, it is recommended to run this method from multiple starting points.

On the other hand, if a interior point method starts at any stationary point, including a local minimum, it may not stop at that point. This depends critically on the choice of the parameter μ.

As discussed in the introduction, Schmedders (1998) compute equilibria with homotopy techniques using the first-order conditions of the nonarbitrage agents' problem. In order to avoid discontinuities in the excess demand correspondence, he consider one agent with penalties for transactions on the asset markets instead of assuming lower bounds on short sales. By making these penalties larger and larger, the solutions of the homotopy function are closer and closer to the GEI equilibrium.

The main inconvenience of homotopy methods is that these methods may fail to produce a solution even to a fairly simple system of nonlinear equations. Furthermore, these methods typically require significantly more function and derivative evaluations and linear algebra operations than the methods presented in this section.

7 General GEI model

Having studied the standard two periods GEI model, in this section we extend all previous results of the two periods GEI model to multiple periods in a simple manner.

We also consider a private ownership market economy with I consumers and J firms and a finite number of perfectly divisible commodities and financial assets. We also assume that there exist L goods, that can be placed at M states of the world in which it is available, E locations in space and T dates of availability. Each financial asset is a promise to deliver an amount of good in a certain state of world. The markets on which the commodities and the financial assets are traded are assumed to be competitive, so that agents believe that they can buy and sell as many commodities or assets as they want without affecting their prices. As before, all relevant information is symmetric across economic agents. Hence, the first extension is of the information structure.

7.1 Information structure

Now we consider a model that is characterized by the information available at each period of time. We also assume that this information is the same for all agents. In this setting we need to consider a formal representation of information that is given by the concept of *information structure.*

Definition 13 ***Information structure.*** *Given a finite sample space* $\Omega = \{\omega_1, ..., \omega_M\}$ *that represents the states of world, an information structure is a sequence of* σ*-algebra* $\{\mathcal{F}_t\}_{t=1}^{T}$ *such that:*

1. $\mathcal{F}_1 = \{\Omega, \emptyset\}$,
2. $\mathcal{F}_T = \mathcal{P}(\Omega)$,
3. $\mathcal{F}_{t+1}$ *is finer than* $\mathcal{F}_t$, $\forall t = 1, ..., T-1$.

In other words, each σ-algebra $\mathcal{F}_t$ is a collection of events that are known at time t. The condition (3) excludes the loss of information between $t+1$ and t, that is, as time passes their knowledge does not decrease.

The information structure can be specified endogenous or exogenously. The information is exogenous when it is revealed by variables that agents do not control. On the other hand, the information is endogenous when the decision of agents depends on decision taking until the current state. Note that if all agents know the decision rules of the rest of agents and the information structure is symmetric, then the endogenous information coincides with the exogenous one.

With this temporal setting, we now need to be explicit about the commodity space. We assume that the commodity space of the model is vector space $\prod_{t=1}^{T} \mathbb{R}^{D \times \mathcal{F}_t}$, where $D = L\ E$ and

$$\mathbb{R}^{D \times \mathcal{F}_t} = \left\{h_t : (\Omega, \mathcal{F}_t) \longrightarrow \mathbb{R}^D,\ \mathcal{F}_t - measurable\right\}.$$

In the context of stochastic programming, the constraints

$$h_t \in \mathbb{R}^{D \times \mathcal{F}_t},\ \forall t = 1, ..., T$$

are known as *non-anticipativity constraints* since these constraints guarantee that decisions made today cannot depend on information received tomorrow (or any day thereafter).

Note that a model defined by these constraints quickly becomes large as the number of decision dates and the cardinality of the sample space increase. And so, these models may be difficult to compute. Fortunately, we can consider an equivalent and simple way to consider these constraints that reduces the dimension of the model. To this end, we present the concepts of determining class and finitely generated σ-algebra.

Definition 14 *The partition* $\mathbf{F}_t = \{F_t^s\}_{s=1}^{S_t} \subset \Omega$ *is said to be a determining class of the* σ*-algebra* $\mathcal{F}_t$*, if it is the smallest* σ*-algebra generated by the class* $\mathbf{F}_t$*. That is denoted by* $\sigma(\mathbf{F}_t) = \mathcal{F}_t$*. (See Billingsley (1968), p. 15). The sets* $F_t^s \in \mathbf{F}_t$ *are known as scenarios.*

If $\{\mathcal{F}_t\}_{t=1}^{T}$ is a filtration, then their determining classes $\{\mathbf{F}_t\}_{t=1}^{T}$ are nested.

Definition 15 *The σ-algebra $\mathcal{F}_t$ is said to be finitely generated if there exist a finite determining class $\mathbf{F}_t$. If $\mathcal{F}_t$ is finite, then it is finitely generated.*

If $\Omega = \{\omega_1, ..., \omega_M\}$ is finite, every filtration $\{\mathcal{F}_t\}_{t=1}^T$ has finitely generated σ-algebras. Note that $\mathbf{F}_T = \Omega$ and $S_T = M$, since $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$.

For example, the information structure of the two-periods GEI model is given by the σ-algebras $\{\mathcal{F}_0, \mathcal{F}_1\}$ where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_1 = \Omega = \{\omega_1, ..., \omega_S\}$ and the finite determining classes $\mathbf{F}_0 = \Omega$ and $\mathbf{F}_1 = \{\{\omega_1\}, ..., \{\omega_S\}\}$, respectively.

If the σ-algebra $\mathcal{F}_t$ is finitely generated by $\mathbf{F}_t$, then h_t is $\mathcal{F}_t - measurable$ iff $h_t(\omega)$ is constant in $\omega \in F_t^s$, for each $s = 1, ..., S_t$. Let

$$h_{F_t^s} = h_t(\omega), \quad \forall \omega \in F_t^s$$

denote such a value. Hence, the non-anticipativity constraints $h_t \in \mathbb{R}^{D \times \mathcal{F}_t}$ is equivalent to rewrite $h_t(\omega)$ as

$$h_{\mathbf{F}_t} = \left(h_{F_t^1}, ..., h_{F_t^{S_t}}\right) \text{ with } h_{F_t^s} \in \mathbb{R}^D, \ \forall s = 1, ..., S_t.$$

Note that $h_{\mathbf{F}_t} \in \mathbb{R}^{DS_t}$ whereas we obtain $h_t(\cdot) \in \mathbb{R}^{DM}$ by dropping the non-anticipativity constraints. Hence, these constraints reduce the dimension of the problem.

For example, in the two-periods GEI model, $x_0 = x(\mathbf{F}_0) = x(\Omega)$ denotes the consumption plan at $t = 0$ and $x_1 = (x(\omega_1), ..., x(\omega_S)) = x(\mathbf{F}_1)$ denotes the consumption plan at $t = 1$.

Summarizing, the commodity space is $\prod_{t=1}^{T} \mathbb{R}^{DS_t} = \mathbb{R}^{L^*}$ where $L^* = D \cdot \left(\sum_{t=1}^{T} S_t\right)$ and the vectors $h_{\mathbf{F}_t}$ are defined as above. Each commodity has a market price that is defined in the vector space $\mathbb{R}^{L^*}$.

Note that for each $h \in \mathbb{R}^{L^*}$, a unique sequence of decisions corresponding to $\omega \in \Omega$ is realized. This idea can be expressed by means of the concept of *scenario tree*, that is defined as

$$\Im = \bigcup_{t \in \mathbb{T},\, s \in \mathbb{S}_t} F_t^s,$$

where $\mathbb{T} = \{1, ..., T\}$ and $\mathbb{S}_t = \{1, ..., S_t\}$. Each F_t^s is called *tree node* or *scenario.* Each scenario tree is entitled to the preorder relation $F \succ F'$ iff the node F' succeeded to F if $F' \subset F$. In this case, it is obvious that $\exists\ t,\ t'$ such that $t' > t$ and $F \subset \mathbf{F}_t$, $F' \subset \mathbf{F}_{t'}$.

In general, we can consider that each $\omega \in \Omega$ is identified by a terminal node $F_T^{s_T} \in \mathbf{F}_T$. On the other hand, note that each branch of the tree (the path from the root F_0 of the tree to a leaf) corresponds to a event $\omega \in \Omega$. Hence, without lack of generality, we can identify each ω with the associated vector of nodes of the brand tree[7]. Analogously, an non-terminal node $F_t^{s_t} \in \mathbf{F}_t$ can be identified by the subpath from the root of the tree to that node.

[7] In practice, the notation

$$\omega = (F_1^{s_1}, ..., F_T^{s_T}), \quad F_{t+1}^{s_{t+1}} \succ F_t^{s_t}.$$

The main issue of this approach in practical is dimensionality. If we consider T time periods, the number of scenarios needed to model the informational structure is $\sum_{t=1}^{T} S_t$. Moreover, S_t often grows exponentially with t. Hence, the scenario tree can be expanded to arbitrarily large sizes as the temporal horizon T increases. Then great difficulties arise from the computational point of view. However, in the literature on stochastic programming, there exist computational methods that allows to handle large models. Surveys in this area are due to Ermoliev and Wets (1988) ~~and~~ Wets (1989).

7.2 Financial system

Trading occurs at each information set. In order to provide instruments that enable each agent to trade among the different markets, we must extend the economy by the addition of *financial assets.*

For each scenario F, we assume that there exist C_F financial assets. Let C denote the number of all financial assets. Let $\theta_F^c \in \mathbb{R}$ denote the number of units of the asset c that is held by some agent at period t. When $\theta^c(t) > 0$, the agent is worthy of the asset c, and otherwise, the agent is debtor. A portfolio of assets at period t is defined as

$$\theta_F = \left(\theta_F^1, ..., \theta_F^{C_F}\right)^T \in \mathbb{R}^{C_F}.$$

Each portfolio θ_F has a vector of market price $q_F \in \mathbb{R}^{C_F}$ and deliver a future dividends that are defined by means of a family of matrices

$$\underset{D\times C_F}{A_{(G,F)}}\colon\ \forall G \succ F,$$

where each $A_{(G,F)}$ represents the dividends of the portfolio θ_F at scenario $G \succ F$. The short-lived dividends are delivered at the immediate successors of its node of issue F, and the long-lived dividends are deliver after the immediate successors of F. Let p_G denote the prices of real goods at scenario G, then the dividends' value of the portfolio θ_F at scenario $G \succ F$ is

$$\underset{1\times D}{p_G} \cdot \underset{D\times C_F}{A_{(G,F)}} \underset{C_F\times 1}{\theta_F} .$$

On the other hand, the dividends' value at scenario G of a collection of portfolios purchased at the predecessor's scenarios of G is

$$p_G \cdot \left(\sum_{F \prec G} A_{(G,F)} \theta_F \right).$$

can be useful to design the set Ω.

If we consider all scenarios at period t, instead of a scenario F, then for each period t, it can be possible to define the financial assets

$$\theta_{\mathbf{F}_t} = \left(\theta^T_{F^1_t}, ..., \theta^T_{F^{S_t}_t}\right)^T \in \mathbb{R}^{\sum_{s=1}^{S_t} C_{F^s_t}}$$

with a price

$$q = \left(q^T_{F^1_t}, ..., q^T_{F^{S_t}_t}\right)^T \in \mathbb{R}^{\sum_{s=1}^{S_t} C_{F^s_t}}.$$

Obviously, the dividends of $\theta_{\mathbf{F}_t}$ at period $\tau > t$ can be written as

$$\underset{DS_\tau \times \left(\sum_{s=1}^{S_t} C_{F^s_t}\right)}{A_{(\mathbf{F}_\tau,\mathbf{F}_t)}} \quad \underset{\left(\sum_{s=1}^{S_t} C_{F^s_t}\right)\times 1}{\theta_{\mathbf{F}_t}} = \begin{pmatrix} A_{(F^1_\tau,F^1_t)} & \cdots & A_{(F^1_\tau,F^{S_t}_t)} \\ \vdots & \cdots & \vdots \\ A_{(F^{S_\tau}_\tau,F^1_t)} & \cdots & A_{(F^{S_\tau}_\tau,F^{S_t}_t)} \end{pmatrix} \begin{pmatrix} \theta_{F^1_t} \\ \vdots \\ \theta_{F^{S_t}_t} \end{pmatrix}$$

where we assume that the matrix $A_{(G,F)}$ is zero if it is not satisfied that $G \succ F$.

Hence, the set of all feasible financial assets is described by the vectors

$$\theta = \begin{pmatrix} \theta_{\mathbf{F}_1} \\ \vdots \\ \theta_{\mathbf{F}_{T-1}} \end{pmatrix} \in \mathbb{Z} \subset \mathbb{R}^{\sum_{t=1}^{T-1}\sum_{s=1}^{S_t} C_{F^s_t}}.$$

with a market price

$$q = \begin{pmatrix} q_{\mathbf{F}_1} \\ \vdots \\ q_{\mathbf{F}_{T-1}} \end{pmatrix} \in \mathbb{R}^{\sum_{t=1}^{T-1}\sum_{s=1}^{S_t} C_{F^s_t}}$$

If we summarize the dividends of all assets θ for every period of time, we have the following dividends matrix

$$\underset{D\sum_{t=1}^{T} S_t \times 1}{\mathbf{A}\boldsymbol{\theta}} = \underset{D\sum_{t=1}^{T} S_t \times \sum_{t=1}^{T-1}\sum_{s=1}^{S_t} C_{F^s_t}}{\begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ A_{(\mathbf{F}_2,\mathbf{F}_1)} & 0 & \cdots & \cdots & 0 \\ A_{(\mathbf{F}_3,\mathbf{F}_1)} & A_{(\mathbf{F}_3,\mathbf{F}_2)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{(\mathbf{F}_T,\mathbf{F}_1)} & A_{(\mathbf{F}_T,\mathbf{F}_2)} & \cdots & \cdots & A_{(\mathbf{F}_T,\mathbf{F}_{T-1})} \end{pmatrix}} \cdot \underset{\sum_{t=1}^{T-1}\sum_{s=1}^{S_t} C_{F^s_t} \times 1}{\begin{pmatrix} \theta_{\mathbf{F}_1} \\ \cdots \\ \theta_{\mathbf{F}_{T-1}} \end{pmatrix}}.$$

The matrix $\mathbf{A}$ is known as *dividends or returns matrix.*

7.3 The firms

We assume that the jth firm maximizes an objective function (e.g. its profit) on its production set $\mathbb{Y}_j \subset \mathbb{R}^{L^*}$. We also allow that firm issues or purchases real financial assets. In such a case, firms are covered against a risk of future production. For each scenario F, let ξ_{jF} denote the portfolio of real assets of the jth firm, characterized by the price market q_F and the dividend's matrix $A_{(G,F)}$, with $G \succ F$. In such a case, firm is under an obligation to deliver the production promised at previous scenarios.

A production set $\mathbb{Y}_j$ is described by means of a set of functions $F_j(\cdot) : \mathbb{R}^{L^*} \longrightarrow \mathbb{R}^{K_j}$, called transformation functions,

$$\mathbb{Y}_j = \left\{ y_j \in \mathbb{R}^{L^*} : F_j(y_j) \leq 0 \right\}.$$

These dynamic constraints play a crucial role in the production decision process, since the firm is directly affected by its past decision.

Without loss of generality, we assume that the jth firm's objective is given by the function

$$o_j : \mathbb{Y}_j \times \mathbb{Z} \times \mathbb{R}^{L^*} \times \mathbb{R}^C \longrightarrow \mathbb{R}.$$

To model the behavior of the firm in this context, it is a difficult task. We can consider a large number of different objectives as many as the firms' behavior. Usually, firms are profit maximizers over production and financial transactions

$$o_j\left(y_j, \xi_j, p, q\right) = E\left[\sum_{F \in \Im} p_F \cdot y_{jF} - \sum_{F \in \Im} q_F \cdot \xi_{jF}\right].$$

Then, given the price vectors $(p,q) \in \mathbb{R}^{L^*} \times \mathbb{R}^C$, the jth firm faces the problem

$$\boxed{\begin{array}{ll} \underset{s.t.}{Max} & o_j\left(y_j, \xi_j, p, q\right) \\ & p_{F_0} \cdot y_{jF_0} + q_{F_0} \cdot \xi_{jF_0} = 0, \\ & p_F \cdot y_{jF} - p_F \cdot \sum_{G \prec F} A_{(F,G)} \xi_{jG} + q_F \cdot \xi_{jF} = 0, \forall F \in \Im \setminus \{F_0 \cup \mathbf{F}_T\}, \\ & p_F \cdot y_{jF} - p_F \cdot \sum_{G \prec F} A_{(F,G)} \xi_{jG} = 0, \forall F \in \mathbf{F}_T \\ & y_j \in \mathbb{Y}_j, \quad \xi_j \in \mathbb{Z}. \end{array}} \tag{12}$$

Under convexity and compactness assumptions, the existence of solutions for this problem can be proved by the Maximum Theorem. These solutions are correspondences denoted by $y_j^* = y_j(p,q)$and $\xi_j^* = \xi_j(p,q)$. Furthermore, under strong convexity, such solutions are functions.

7.4 The consumers

A consumer is an individual agent (a single household or a family) who takes sequentially decisions regarding its demand for goods and services and the supply of different types of labor.

The choice set for the ith consumer is given by a subset $\mathbb{X}_i \subset \mathbb{R}^{L^*}$ which describes feasible consumption vectors. Each consumer has preferences given by a utility function $u_i : \mathbb{X}_i \longrightarrow \mathbb{R}$ and is endowed with a vector $w_i \in \mathbb{X}_i$. In order to transfer income between scenarios, consumer have to hold assets. For each scenario F, let

$\theta_{iF} \subset \mathbb{R}^{C_F}$ denote the portfolio of real assets traded at market price $q_F \in \mathbb{R}^{C_F}$. The payoff of an asset is given by $A_{(G,F)}$ at scenario $G \succ F$.

Thus, given the price vectors $(p,q) \in \mathbb{R}^{L^*} \times \mathbb{R}^C$, the ith consumer faces the problem,

$$
\boxed{
\begin{array}{ll}
\underset{s.t..}{Max} & u_i(x_i) \\
 & p_{F_0} \cdot x_{iF_0} + q_{F_0} \cdot \theta_{iF_0} \leq p_{F_0} \cdot w_{iF_0}, \\
 & p_F \cdot x_{iF} + q_F \cdot \theta_{iF} \leq p_F \cdot w_{iF} + p_F \cdot \sum\limits_{G \prec F} A_{(F,G)} \cdot \theta_{iG},\ \forall F \in \Im \setminus \{F_0 \cup \mathbf{F}_T\}, \\
 & p_F \cdot x_{iF} \leq p_F \cdot w_{iF} + p_F \cdot \sum\limits_{G \prec F} A_{(F,G)} \cdot \theta_{iG},\ \forall F \in \mathbf{F}_T, \\
 & x_i \in \mathbb{X}_i, \quad \theta_i \in \mathbb{Z}.
\end{array}
}
\tag{13}
$$

Under convexity and compactness assumptions, the existence of solutions for this problem can be proved by the Maximum Theorem. These solutions are correspondences denoted by $(x_i^*,\ \theta_i^*)$ that depends on the market prices (p,q). Furthermore, under strong convexity, such solutions are functions.

7.5 The economy

In this section we define a multiperiod GEI economy and extend the assumptions on the characteristics of this economy under which the GEI model works.

Consider the product space of economy's commodities

$$\Gamma^b = \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) \times \prod_{j=1}^{J} (\mathbb{Y}_j \times \mathbb{Z})$$

and the sets

$$
\begin{aligned}
\Delta_+ &= \left\{ (p,q) \in \mathbb{R}_+^{L^*} \times \mathbb{R}^C : \sum_{d=1}^{L^*} p_d + \sum_{d=1}^{C} |q_d| \leq 1 \right\}, \\
\Delta &= \left\{ (p,q) \in \mathbb{R}^{L^*} \times \mathbb{R}^C : \sum_{d=1}^{L^*} |p_d| + \sum_{d=1}^{C} |q_d| \leq 1 \right\}.
\end{aligned}
$$

An economy can thus be described by a set

$$E = \left(\{\mathbb{X}_i, u_i, w_i\}_{i=1}^{I}, \{\mathbb{Y}_j, o_j\}_{j=1}^{J}, \Im, (\mathbb{Z}, R) \right),$$

whose elements satisfy the conditions:

H.M1. The ith consumer's consumption set $\mathbb{X}_i \subset \mathbb{R}^{L^*}$ is closed and $w_i \in \mathbb{X}_i$.

H.M2. The utility function $u_i(\cdot)$ that represents the ith consumer's preference relation $\succeq_i$ is continuous in $\mathbb{X}_i$.

H.M3. $\mathbb{X}_i$ is convex, $\forall i = 1, ..., I$.

H.M4. $u_i(\cdot)$ is concave in $\mathbb{X}_i$, $\forall i = 1, ..., I$.

H.M5. The utility function $u_i(\cdot)$ is monotonous in $\mathbb{X}_i$.

H.M6. Survival assumption: For every consumer, there exists $\underline{x}_i \in \mathbb{X}_i$ such that $\underline{x}_i \ll w_i$.

H.M7. The production set for the $j-th$ firm, $\mathbb{Y}_j \subset \mathbb{R}^{L^*}$, is closed, bounded and $0 \in \mathbb{Y}_j$.

H.M8. The objective function for the $j-th$ firm $o_j : \mathbb{Y}_j \times \mathbb{Z} \times \mathbb{R}^{L^*} \times \mathbb{R}^C \to \mathbb{R}$ is continuous.

H.M9. $\mathbb{Y}_j$ is convex, $\forall j = 1, ..., J$.

H.M10. For each firm $j \in \{1, \ldots, J\}$, the function $o_j(\cdot)$ is concave in $\mathbb{Y}_j \times \mathbb{Z}$, $\forall (p, q) \in \mathbb{R}^{L^*} \times \mathbb{R}^C$.

H.M11. For each firm $j \in \{1, \ldots, J\}$, the function $o_j(\cdot)$ is homogeneous of degree α in p and q, for some $\alpha > 0$.

H.M12. The set of feasible financial assets $\mathbb{Z} \subset \mathbb{R}^C$ is closed and $0 \in \mathbb{Z}$.

H.M13. $\mathbb{Z} \subset \mathbb{R}^C$ is convex.

H.M14. The return functions $A_{(G,F)}(p, q) : \forall G \succ F$ are nonnegative, continuous, and homogeneous of degree zero in (p, q).

H.M15. Financial asset survival assumption: $\exists z \in \mathbb{Z} : A_{(G,F)}(p, q)\, z \gg 0$.

H.M16. The set of allocations $\mathbb{A}$ is nonempty, closed, and bounded.

H.M17. For each $(p, q) \in \Delta_+$, the set

$$\begin{aligned}\mathbb{F}(p,q) = \; & \Big\{ (x,\theta) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}) : \; \exists (y,\xi) \in \left(\widehat{y}(p,q), \widehat{\xi}(p,q)\right), \; ((x,\theta),(y,\xi)) \in \mathbb{A}, \\ & p_F \cdot x_{iF} + q_F \cdot \theta_{iF} \leq p_F \cdot w_{iF} + p_F \cdot \sum_{G \prec F} A_{(F,G)} \cdot \theta_{iG}, \; \forall F \in \Im \setminus \{F_0 \cup \mathbf{F}_T\}, \; \forall i \\ & p_F \cdot x_{iF} \leq p_F \cdot w_{iF} + p_F \cdot \sum_{G \prec F} A_{(F,G)} \cdot \theta_{iG}, \; \forall F \in \mathbf{F}_T, \; \forall i \Big\}\end{aligned}$$

is nonempty, where $\widehat{y}(p,q) = \sum_{j=1}^{J} \widehat{y}_j(p,q)$ and $\widehat{\xi}(p,q) = \sum_{j=1}^{J} \widehat{\xi}_j(p,q)$ defined as

$$\boxed{\begin{aligned}\left(\widehat{y}_j(p,q), \widehat{\xi}_j(p,q)\right) = \; & \arg\max \Big\{ o_j(y_j, \xi_j, p, q) : \quad p_{F_0} \cdot y_{jF_0} + q_{F_0} \cdot \xi_{jF_0} = 0, \\ & p_F \cdot y_{jF} - p_F \cdot \sum_{G \prec F} A_{(F,G)} \xi_{jG} + q_F \cdot \xi_{jF} = 0, \forall F \in \Im \setminus \{F_0 \cup \mathbf{F}_T\}, \\ & p_F \cdot y_{jF} - p_F \cdot \textstyle\sum_{G \prec F} A_{(F,G)} \xi_{jG} = 0, \forall F \in \mathbf{F}_T, \\ & y_j \in \mathbb{Y}_j, \quad \xi_j \in \mathbb{Z} \Big\}.\end{aligned}}$$

H.M18. All relevant information is symmetric across economic agents, given by the informational structure $\{\mathcal{F}_t\}_{t=0}^{T}$, finitely generated by $\mathbf{F}_t = \{F_t^s\}_{s=1}^{S_t}$, $\forall t = 0, ..., T$, and summarized by $\Im$.

7.6 The concept of a sequential equilibrium

We now introduce the concept of a sequential equilibrium.

Definition 16 ***Sequential Equilibrium.*** *The vector prices* $(p^*, q^*) \in \overline{\mathbb{R}^{L^*}} \times \mathbb{R}^C$, *with* $(p^*, q^*) \neq 0$, *and the consumers' and producers' allocations* $((x^*, \theta^*), (y^*, \xi^*)) \in \Gamma^b$ *is a sequential equilibrium for an economy* E, *if*

FP *Each firm solves its decision Problem* (12)*;*

CP *Each consumer solves its decision Problem* (13)*;*

MC *Market clearing:*

$$\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} w_i + \sum_{j=1}^{J} y_j^*, \quad \sum_{i=1}^{I} \theta_i^* = \sum_{j=1}^{J} \xi_j^*. \tag{14}$$

7.7 Existence, characterization and local uniqueness of sequential equilibria

The previous result on the existence, characterization and local uniqueness of a GEI equilibrium can be extended accordingly to the multiperiods model.

Theorem 17 ***Existence of a sequential equilibrium.*** *Let* E *be an economy satisfying conditions* $(H.M1)$ *to* $(H.M18)$*. Then there exists a sequential equilibrium*

$$((x^*, \theta^*), (y^*, \xi^*), p^*, q^*) \in \Gamma^b \times \mathbb{R}_+^{L^*} \times \mathbb{R}^C.$$

Proof.

The proof is similar to that of Theorem considering the following mathematical programming problem, instead of Problem (3) of Theorem 3:

Let $\left(\widehat{y}_j(p, q), \widehat{\xi}_j(p, q)\right)$ the solutions of the $j - th$ firms' problem for all $j = 1, ..., J$.

Now, consider $(p,q) \in \Delta_+$, $(y,\xi) \in \widehat{\mathbb{Y}}$ and $\delta \in \Lambda$ such that define the following problem,

$$
\boxed{
\begin{array}{l}
\underset{s.a}{Max} \sum_{i=1}^{I} \delta_i \, u_i\,(x_i) \\
\sum_{i=1}^{I} x_{iF_0} \leq \sum_{i=1}^{I} w_{iF_0} + \sum_{j=1}^{J} y_{jF_0}, \\
\sum_{i=1}^{I} x_{iF} + \sum_{j=1}^{J} A_{(F,G)} \xi_{jF} \leq \sum_{i=1}^{I} w_{iF} + \sum_{j=1}^{J} y_{jF} + \sum_{i=1}^{I} A_{(F,G)} \theta_{iF}, \quad \forall F \in \Im \setminus \{F_0\}, \\
\sum_{i=1}^{I} \theta_{iF} = \sum_{j=1}^{J} \xi_{jF}, \quad \forall F \in \Im, \\
p_F \cdot x_{iF} + q_F \cdot \theta_{iF} \leq p_F \cdot w_{iF} + p_F \cdot \sum_{G \prec F} A_{(F,G)} \cdot \theta_{iG}, \forall F \in \Im \setminus \{F_0 \cup \mathbf{F}_T\}, \\
p_F \cdot x_{iF} \leq p_F \cdot w_{iF} + p_F \cdot \sum_{G \prec F} A_{(F,G)} \cdot \theta_{iG}, \forall F \in \mathbf{F}_T, \\
(x,\, z) \in \prod_{i=1}^{I} (\mathbb{X}_i \times \mathbb{Z}).
\end{array}
}
\tag{15}
$$

The rest of proof is identical to that of Theorem 3.

■

We can also prove the existence of a GEI equilibrium with spot prices $p^* \neq 0$. The proof is similar to that of Theorem 17 considering the following equality constraints

$$
\left|
\begin{array}{l}
\sum_{i=1}^{I} x_{iF_0} = \sum_{i=1}^{I} w_{iF_0} + \sum_{j=1}^{J} y_{jF_0}, \\
\sum_{i=1}^{I} x_{iF} + \sum_{j=1}^{J} A_{(F,G)} \xi_{jF} = \sum_{i=1}^{I} w_{iF} + \sum_{j=1}^{J} y_{jF} + \sum_{i=1}^{I} A_{(F,G)} \theta_{iF}, \quad \forall F \in \Im \setminus \{F_0\},
\end{array}
\right|
$$

instead of inequalities in Problem (15) of Theorem 17.

Analogously, we can establish sufficient conditions for the characterization of a local sequential equilibrium under differentiability assumptions. These conditions are the first order conditions of Problem (15) and the fixed-point conditions required by Theorem 17.

Finally, the local uniqueness of a local sequential equilibrium can be proved using these sufficient conditions.

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