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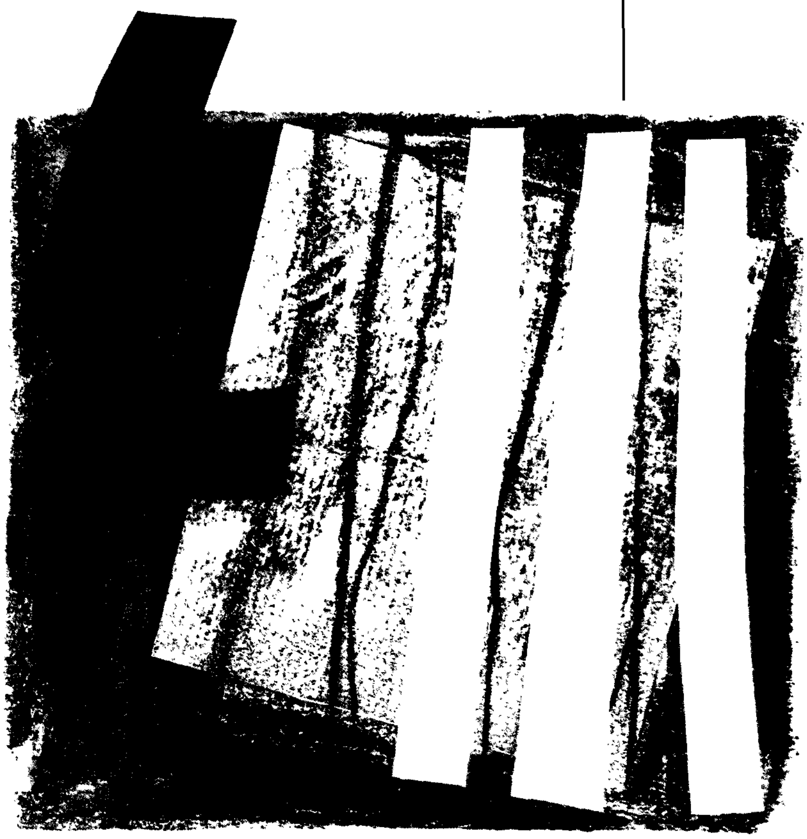

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M. Esteban-Bravo *

Abstract

In this paper we present a general approach to existence problems in Cournot-Walras (CW) economies, based on mathematical programming theory. We propose a definition of the decision problem of firms which avoids the profit maximization rule as the only rational criterion for the firms and uses the excess demand function instead of the inverse demand function. We prove the existence of a CW equilibrium and we state practical conditions to characterize a CW equilibrium. We also propose efficient algorithms for computing CW equilibria. Finally, we consider some extensions such as externalities, Stackelberg, collusive and Nash equilibrium models.

Keywords: General equilibrium; Cournot-Walras equilibrium; computation of equilibria; Newton-type algorithms.

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1 Introduction

In this paper we provide a general framework to deal with Cournot-Walras (CW) equilibrium models. Such a concept is the natural extension of Cournot's ideas into a general equilibrium framework. This model assumes that firms are the only noncompetitive agents while consumers are considered to be competitive. In other words, CW models take into account the situation where each firm maximizes its objectives given the production decisions of the remaining firms, exactly as in the Cournot model.

There exists a large literature on partial noncompetitive markets that are characterized by the behavior of firms which do not treat prices as exogenous parameters. Most of these models derive from the work of Cournot (1838) and the notion of Nash equilibrium introduced later (Nash, 1950, 1951). Subsequent contributions have refined the Nash equilibrium concept and developed a theory of noncompetitive equilibrium that seems to work well in partial analysis. Today, partial equilibrium theories of imperfect competition embody a great variety of models which are characterized by different assumptions on firms' behavior, on the nature of products, on market mechanisms and so on. See, for example, Friedman (1977, 1982), Tirole (1988), Fudenberg and Tirole (1992), among others. But, unfortunately, there is not satisfactory theory of general equilibrium with imperfect competition yet. Two main approaches have been considered to date.

The first one is called the *subjective demand approach* and was proposed by Negishi (1961, 1972, 1989, 1994). He studied the existence of a general equilibrium for a monopolistic competition economy. He assumed that each monopolistic firm has a subjective inverse demand function for its outputs and a supply curve for its inputs. Then, given their conjectures, each firm chooses production plans that maximize profits. The main criticism is that there is an element of arbitrariness in the conjectures of monopolistic firms. Furthermore, Negishi's approach rules out bilateral monopoly and oligopoly.

The second approach was studied originally by Gabszewicz and Vial (1972) (see also Fitzroy (1974), Roberts (1980), Mas Colell (1982), Hart (1985) and Gary-Bobo (1989), among others). Given a production allocation, they defined a price vector corresponding to a Walrasian equilibrium of the pure exchange economy. This is the objective inverse demand function facing each firm. Thus, firms choose production plans that maximize their profits, taking the production decision of the remaining firms as given. This leads to a CW equilibrium. These authors were already aware of the main difficulties raised by their approach: for example, their concept of CW equilibrium depends on the rule chosen to normalize prices, and hence, the results would be dependent on this rule.

Surveys in this area are due to Bonano (1990), Hart (1985), Gabszewicz and Michel (1992), Gabszewicz and Thisse (1999). Codognato (1994) also discusses the main problems raised by these approaches.

The purpose of this paper is to develop a general approach to existence problems in the CW framework, and appropriate algorithms for the computation of equilibria in large-scale CW models.

We establish the existence of a CW equilibrium assuming that, instead of just maximizing profits, firms have a general objective to maximize. In the literature of competitive general equilibrium it is shown that, under some assumptions, profit maximization is the reasonable goal that all firms agree upon. However, in the case of imperfect competition, profit maximization may not be a rational objective for firms and there is no objective of the firm which shareholders unanimously agree to, as it is pointed out in Gabszewicz and Vial (1972), Dierker and Grodal (1986, 1988) and Grodal (1992), and is discussed in Codognato (1994). We also assume that firms have perfect knowledge of their demand conditions, but we avoid the use of inverse demand functions.

Although the computation of CW equilibria plays an essential role in the applied general equilibrium theory, as far as we are aware the CW equilibrium literature provides approaches to compute such equilibria that may be difficult or inefficient to apply to large problems. In this paper we state practical conditions that characterize CW equilibria and suggest algorithms for finding these points. The chief idea is that an equilibrium should be defined in terms of local optimizers. Therefore, we introduce a definition for what we call local CW equilibria.

Finally, we prove the existence of equilibria under the main imperfect competition assumptions that the traditional partial equilibrium theory takes into account, such as externalities, Stackelberg, collusive and Nash equilibrium models.

The rest of the paper is organized as follows. In Section 2 we describe the basic model and indicate its main properties. Section 3 is devoted to the existence of a CW equilibrium. In Section 4, we characterize the local CW equilibrium as the solution of a nonlinear system of equations under convexity assumptions and we provide sufficient conditions for local CW equilibria relaxing the standard convexity assumptions. In section 5, we prove the uniqueness of such equilibria. In section 6, we propose algorithms to compute such a equilibrium. Section 7 extends all previous results to others noncompetitive models. For instance, we consider models where each firm possesses market power for some commodities, or models that present externalities. Finally, we also consider Stackelberg, collusive and Nash equilibrium models.

2 The model

We consider an economy with D perfectly divisible commodities, I consumers and J firms. We take $\mathbb{R}^D$ as the commodity space. For any $x \in \mathbb{R}^D$, x^T denotes the transpose of x, which is an D-dimensional row vector. For any $x,\ y \in \mathbb{R}^D$, $x \cdot y = x^T y$ denotes the inner product of vectors x and y. We assume that consumers are price-taking utility maximizers and firms behave in the Cournot way. We also assume that all relevant information is available when agents take decisions (i.e. perfect information).

In this setting, prices are decision variables for firms. We define prices on the nonnegative simplex,

$$\Delta_p = \left\{ p \in \mathbb{R}^D : \sum_{d=1}^{D} p_d = 1, \quad p_d \geq 0, \quad d = 1, ..., D \right\}.$$

2.1 The consumers

We assume that the choice set for the ith consumer is given by a subset $\mathbb{X}_i \subset \mathbb{R}^D$ which describes feasible consumption vectors. Each consumer has preferences given by a utility function $u_i : \mathbb{X}_i \longrightarrow \mathbb{R}$, and is endowed with a vector $w_i \in \mathbb{X}_i$. Assuming market economies with private ownership (the consumers own firms), $r_{ij} \in \mathbb{R}_+$ denotes the ith consumer's participation in the jth firm, with $\sum_{i=1}^{I} r_{ij} = 1, \forall j = 1, ..., J$. Thus, the ith consumer's demand can be obtained as the solution to the following program:

$$\boxed{\underset{x_i \in \mathbb{X}_i}{Max} \quad \{u_i(x_i) : p \cdot x_i \leq R_i\},} \tag{1}$$

where

$$R_i = R_i(p, y^*) = p \cdot \left(w_i + \sum_{j=1}^{J} r_{ij} y_j^*\right), \; \forall i = 1, \ldots, I,$$

for $y^* = (y_1^*, ..., y_J^*) \in \prod_{j=1}^{J} \mathbb{Y}_j$ the optimal production decisions of the firms and $p \in \Delta_p$ a price vector.

Under the following assumptions:

C.1. The ith consumer's consumption set $\mathbb{X}_i \subset \mathbb{R}^D$ is closed, bounded below, convex and $w_i \in \mathbb{X}_i$,

C.2. The utility function $u_i : \mathbb{X}_i \rightarrow \mathbb{R}$ that represents the ith consumer's preference relation $\succeq_i$ is continuous and concave,

the existence of a solution for this problem is guaranteed. This solution is known as the *demand correspondence* and denoted by $\widehat{x}_i(p, R_i(p, y^*))$. By the Maximum Theorem under convexity[1], this correspondence is non-empty, compact-valued, convex-valued and usc.

Furthermore, if we also assume that the utility function is strictly concave, such a solution $\widehat{x}_i(p, R_i(p, y^*))$ is a continuous function. In this case, let

$$\widehat{x}(p, y^*) = \sum_{i=1}^{I} \widehat{x}_i(p, R_i(p, y^*)) \subset \sum_{i=1}^{I} \mathbb{X}_i$$

be the *market demand function*, defined for all $p \in \Delta_p$ and for all $y^* \in \prod_{j=1}^{J} \mathbb{Y}_j$.

2.2 The firms

In this paper, we assume that the jth firm maximizes a general objective function $o_j(y_j, p_j)$ on its production set $\mathbb{Y}_j \subset \mathbb{R}^D$ and the price set $\Delta_p \subset \mathbb{R}^D$, given the production decisions of the remaining firms, exactly as in the Cournot model. In this context, firms believe that they may affect market prices and therefore, the decision problem of firms is to choose a production plan and a vector price that maximize a general objective and satisfy the market clearing condition given the expected behavior of the remaining firms and the market demand function. Usually, it is assumed that the firm's objective is to maximize

[1]The Maximum Theorem under convexity restrictions is a consequence of the Maximum Theorem given by Berge (see Berge (1963), pp. 115-116). The Maximum Theorem under convexity is presented in Sundaram (1996, pp. 237-239) and Ginsburg and Keyzer (1997, pp. 472-476).

its profit, i.e. $o_j(y_j, p) = p \cdot y_j$. Our proposed setting allows the consideration of other alternative objectives.

Given the expected production decision of the firms $y^e \in \mathbb{C}$, where

$$\mathbb{C} = \left\{ y \in \prod_{j=1}^{J} \mathbb{Y}_j : \sum_{j=1}^{J} y_j + \sum_{i=1}^{I} w_i \in \sum_{i=1}^{I} \mathbb{X}_i \right\}$$

denotes the set of attainable productions, the jth firm faces the problem

$$\boxed{\begin{array}{ll} \underset{(y_j, p_j) \in \mathbb{Y}_j \times \Delta_p}{Max} & o_j(y_j, p_j) \\ s.t. & \\ & y_j = \sum_{i=1}^{I} \widehat{x}_i \left(p_j, R_i \left(p_j, y_j, y_{-j}^e \right) \right) - \sum_{j' \neq j} y_{j'}^e - \sum_{i=1}^{I} w_i, \\ & p_j \cdot y_j \geq 0, \end{array}} \tag{2}$$

where $y_{-j}^e = \left(y_1^e, ..., y_{j-1}^e, y_{j+1}^e, ..., y_J^e \right)$ is the vector of expected production decisions of the remaining firms $j' \neq j$.

Note that if we remove the no-loss constraints ($p_j \cdot y_j \geq 0$, for all $j = 1, .., J$), firms could have negative profits. Since the wealth of consumers depends on the profits or losses of firms, the total income of consumers could be negative and the market clearing condition might not be satisfied. As a consequence, it could lead to the nonexistence of equilibria. When firms are profit maximizers it is not necessary to take into account the no-loss constraint if we assume $0 \in \mathbb{Y}_j$.

Under the following assumptions:

P.1. The objective function for the jth firm $o_j : \mathbb{Y}_j \times \Delta_p \to \mathbb{R}$ is continuous in $\mathbb{Y}_j$ and Δ_p,

P.2. The set of feasible points of Problem (2) is nonempty and compact,

the existence of a solution for this problem can be proved from standard results. Let $\widehat{y}_j \left(y_{-j}^e \right) \subset \mathbb{Y}_j$ be the set of optimal quantity choices for the jth firm and $\widehat{p}_j \left(y_{-j}^e \right) \subset \mathbb{R}^D$ be the optimal price vector. By the Maximum Theorem, these solutions are non-empty, compact-valued, convex-valued and usc correspondences.

2.3 The economy

An economy can be described by a set

$$E = \left(\{ \mathbb{X}_i, u_i \}_{i=1}^{I}, \{ \mathbb{Y}_j, o_j \}_{j=1}^{J}, \{ w_i \}_{i=1}^{I}, \{ r_{ij} \}_{i,j} \right),$$

whose elements satisfy the following conditions:

H.1. The $i-th$ consumer's consumption set $\mathbb{X}_i \subset \mathbb{R}^D$ is closed, bounded below and $w_i \in \mathbb{X}_i$.

H.2. The utility function $u_i : \mathbb{X}_i \to \mathbb{R}$ that represents the $i-th$ consumer's preference relation $\succeq_i$ is continuous.

H.3. $\mathbb{X}_i$ is convex, $\forall i = 1, ..., I$.

H.4. $u_i(\cdot)$ is strictly concave in $\mathbb{X}_i$, $\forall i = 1, ..., I$.

H.5. $u_i(\cdot)$ is a strictly monotonous increasing function, $\forall i = 1, ..., I$.

H.6. The market demand (by consumers) is one-to-one in prices (and consequently, it is invertible in prices); i.e. $\forall y \in \prod_{j=1}^{J} \mathbb{Y}_j$, the function $\widehat{x}(\cdot, y)$ defined on Δ_p is one-to-one.

H.7. The production set for the $j-th$ firm, $\mathbb{Y}_j \subset \mathbb{R}^D$, is closed, bounded and $0 \in \mathbb{Y}_j$.

H.8. The objective function for the $j-th$ firm $o_j : \mathbb{Y}_j \times \mathbb{R}^D \to \mathbb{R}$ is continuous, and $\sum_{i=1}^{I} r_{ij} = 1$.

H.9. $\mathbb{Y}_j$ is convex, $\forall j = 1, ..., J$.

H.10. $o_j(y_j, p)$ is quasiconcave in $\mathbb{Y}_j$, $\forall p \in \mathbb{R}^D$, $\forall j = 1, ..., J$.

H.11. $o_j(y_j, p)$ is homogeneous of degree α in p, for some $\alpha > 0$, $\forall j = 1, ..., J$.

H.12. For each firm $j \in \{1, \ldots, J\}$ and for each $y^e \in \mathbb{C}$, the set of feasible productions and prices

$$\left\{ (y_j, p_j) \in \mathbb{Y}_j \times \Delta_p : p_j \cdot y_j \geq 0, \quad y_j = \widehat{x}\left(p_j, y_j, y_{-j}^e\right) - \sum_{j' \neq j} y_{j'}^e - \sum_{i=1}^{I} w_i \right\} \tag{3}$$

is non-empty, closed, bounded and convex. Suppose further that there exists a point $(\overline{y}_j, \overline{p}_j) \in \mathbb{Y}_j \times \Delta_p$ such that $\overline{p}_j \cdot \overline{y}_j > 0$ (Slater's condition).

These assumptions are very similar to the standard ones used in the context of CW models. In particular, the assumptions made on the consumers ($(H.1)$ to $(H.5)$) and the technology of the firms ($(H.7)$, $(H.9)$) are standard in this literature.

Assumption $(H.6)$ cannot be relaxed, in general. The *parametric monotonicity theory* identifies sufficient conditions on parametric families of optimization problems under which optima vary monotonically with the parameter. Unfortunately, these conditions are not satisfied by the utility maximization problem (1). See Sundaram (1996), pp. 253-267.

Assumptions $(H.8)$ and $(H.11)$ are satisfied when firms are profit maximizers.

The quasiconcavity of the objective functions for the firms, Assumption $(H.10)$, could be a very restrictive assumption. For example, $o_j(y_j, p_j) = p_j \cdot y_j$ is only quasiconcave when $p_j \geq 0$ and $y_j \geq 0$. Almost every CW example in the literature avoids this problem assuming there are not inputs in the economy ($\mathbb{Y}_j \subset \mathbb{R}_+^D$, $\forall j$) or assuming that the allocation input is fixed a priori (as a technological parameter). Both assumptions are hardly realistic. See, for instance Gabszewicz and Vial (1972).

Finally, note that Assumption $(H.12)$ is quite restrictive. The set

$$\left\{(y_j,p_j)\in\mathbb{Y}_j\times\Delta_p:\quad y_j=\widehat{x}\left(p_j,y_j,y^e_{-j}\right)-\sum_{j'\neq j}y^e_{j'}-\sum_{i=1}^{I}w_i\right\}$$

is convex when the demand market function is quasimonotone[2] in $\mathbb{Y}_j$ and Δ_p. On the other hand, the functions $p_j\cdot y_j$ are only quasiconcave when $p_j\geq 0$ and $y_j\geq 0$. Hence, the set

$$\{(y_j,p_j)\in\mathbb{Y}_j\times\Delta_p:\quad p_j\cdot y_j\geq 0\}$$

is convex when $p_j\geq 0$ and $y_j\geq 0$. Consequently, we can guarantee that the set (3) is convex if the demand function is quasimonotone in $\mathbb{Y}_j$ and Δ_p and the functions $p_j\cdot y_j$ are quasiconcave. However, these conditions are not necessary.

2.4 The concept of a CW equilibrium

We now introduce the concept of a CW equilibrium.

Definition 1 ***CW Equilibrium***. *The allocation*

$$(x^*,y^*,p^*)\in\left(\prod_{i=1}^{I}\mathbb{X}_i\times\prod_{j=1}^{J}\mathbb{Y}_j\times\Delta_p\right),$$

with $p^*\neq 0$, *is a CW equilibrium for an economy* E *if:*

CP *Each consumer solves its Problem* (1).

FP *Each firm solves its Problem* (2).

FPC *Fixed Point conditions:* $p^*_j=p^*$ *and in Problem* (2), $y^e_j=y^*_j$, $\forall j=1,\ldots,J$.

In other words, a CW equilibrium is a vector y^* such that $y^*_j=\widehat{y}_j\left(y^*_{-j}\right)$, for all $j=1,\ldots,J$. As a consequence, the equilibrium price p^* is defined biunivocally by the equation

$$\widehat{x}(p,y^*)=\sum_{j=1}^{J}y^*_j+\sum_{i=1}^{I}w_i,$$

where $\widehat{x}(p,y^*)=\sum_{i=1}^{I}\widehat{x}_i(p,R_i(p,y^*))$.

Note that a CW equilibrium also satisfies the traditional market clearing condition

$$\sum_{i=1}^{I}x^*_i=\sum_{j=1}^{J}y^*_j+\sum_{i=1}^{I}w_i.$$

The traditional Cournot model avoids the use of prices as decision variables by means of the inverse demand correspondence. We will see that both methodologies are analogous in the next section.

[2] A function f defined on $X\subset\mathbb{R}^D$, nonempty and convex set, is called quasimonotonote if f is both quasiconvex and quasiconcave. If f is continuous and $\{x\in X: f(x)=\alpha\}$ is a convex set for every $\alpha\in\mathbb{R}$, then f is quasimonotone.

2.5 The inverse demand approach

In this section, we consider an alternative way to define the decision problem of firms by means of the inverse demand function.

Each firm uses quantities as strategic variables. This corresponds to the traditional Cournot approach. As discussed in the introduction, the literature on Cournot-Walras equilibria uses the inverse demand correspondence, expressing the price of each commodity as a function of the quantity demanded. The inverse demand correspondence is defined as

$$p\left(x,y\right)=\left\{p\in\Delta_{p}:\sum_{i=1}^{I}x_{i}^{*}\left(p,R_{i}\left(p,y\right)\right)=x\right\}.$$

Assumption $(H.6)$ enssures that $p\left(x,y\right)$ is a one-to-one function.

First, note that the inverse demand takes into account prices and decisions of firms. In the context of general equilibrium theory, the consumers wealth depends on the distribution of profits and, of course, on prices.

In order to allow the existence of a one-to-one demand function, it is necessary to consider a normalization of prices. This is due to the homogeneity of degree zero of the demand function in the prices. On the other hand, in order to assure the existence of optimal prices, we must assume that prices are defined on a compact set.

In this setting, we define prices on the simplex Δ_p without loss of generality, since the demand functions are homogeneous of degree zero in prices (i.e. $\widehat{x}_i\left(p,R_i\left(p,y^*\right)\right)=\widehat{x}_i\left(cp,R_i\left(cp,y^*\right)\right)$ for all $c>0$) and the firms' objectives are homogeneous of degree $\alpha>0$ in prices.

The Walrasian convention is to normalize the price vector by setting a particular commodity as the numéraire. In other words, the numéraire price is chosen to be equal to one (for example). But this normalization does not meet the required assumption on the compactness of the price set. Nevertheless, at equilibrium it is possible to rescale such prices taking one of the commodities as numéraire to measure the relative prices of the other commodities.

Whenever the demand is one-to-one, we can rewrite the jth firm problem, Problem (2), as follows

$$\begin{array}{ll}\underset{(y_j,p_j)\in\mathbb{Y}_j\times\Delta_p}{Max} & o_j\left(y_j,p_j\right)\\ s.t. & \\ & p_j=p\left(\left(y_j+\sum\limits_{j'\neq j}y_{j'}^{e}+\sum\limits_{i=1}^{I}w_i\right),\left(y_j,y_{-j}^{e}\right)\right),\\ & p_j\cdot y_j\geq 0.\end{array}$$

This problem is equivalent to

$$\begin{array}{ll}\underset{y_j\in\mathbb{Y}_j}{Max} & o_j\left(y_j,p\left(\left(y_j+\sum\limits_{j'\neq j}y_{j'}^{e}+\sum\limits_{i=1}^{I}w_i\right),\left(y_j,y_{-j}^{e}\right)\right)\right)\\ s.t. & \\ & p\left(\left(y_j+\sum\limits_{j'\neq j}y_{j'}^{e}+\sum\limits_{i=1}^{I}w_i\right),\left(y_j,y_{-j}^{e}\right)\right)\cdot y_j\geq 0.\end{array}$$

Many theoretical models of imperfect competition assume that firms are profit maximizers despite the limitations of this criterion. Although it is in accordance with the partial equilibrium approach used in industrial organization, as pointed out by Gabszewicz and Vial (1972), the owner of a firm may prefer to have lower profits distributed to him, in exchange for a much lower price in purchasing some desired commodity.

In particular, if we assume firms are profit maximizers and $0 \in \mathbb{Y}_j$, the problem that the jth firm faces is

$$\boxed{\underset{y_j \in \mathbb{Y}_j}{Max} \quad o_j \left(y_j, p \left(\left(y_j + \sum_{j' \neq j} y^e_{j'} + \sum_{i=1}^{I} w_i \right), \left(y_j, y^e_{-j} \right) \right) \right),} \tag{4}$$

where $o_j(y_j, p) = p \cdot y_j$. The solution of this problem is a correspondence denoted by $\widehat{y}_j \left(y^e_{-j} \right)$.

In this setting, an equilibrium is a vector y^* such that $y^*_j = \widehat{y}_j \left(y^*_{-j} \right)$ for all $j = 1, ...J$. Then, the price and the consumption in equilibrium are unique and are defined as

$$\begin{aligned} p^* &= p \left(\left(\sum_{j=1}^{J} y^*_j + \sum_{i=1}^{I} w_i \right), y^* \right), \\ x^*_i &= \widehat{x}_i \left(p^*, R_i \left(p^*, y^* \right) \right), i = 1, ..., I. \end{aligned}$$

Therefore, in the context of general equilibrium, an equivalent formulation in terms of the inverse demand function can be considered. Using the inverse demand function, the firm's problem is to decide on the level of production allocation, and the price at which it can sell this production is given by the inverse demand function.

However, in practice it might be difficult to get the inverse demand function $p(x, y)$ explicitly. Furthermore, in order to compute a CW equilibrium it is convenient to work with the excess demand function instead of the inverse demand function. The use of the direct functions is computationally more efficient than the use of the inverse functions. In the practical optimization literature, it is widely known that computing a solution using inverse functions is more likely to give rise to ill-conditioned problems. See e.g. Wright, M. H. (1997) and its references.

As discussed in the introduction, the literature on CW equilibria uses the inverse demand function. For example, in order to guarantee the existence of a solution for this problem, Negishi (1961) assumes that the inverse demand functions are linear and decreasing with respect to outputs. Gabszewicz and Vial (1972) assume that the profit functions are strictly quasiconcave with respect to outputs. In this setting, almost every model of imperfect competition assumes that the profit functions, defined by the inverse demand function as in (4), are quasiconcave in the outputs. But the quasiconcavity of these functions does not follows from simple assumptions on preferences and technologies. Bonanno (1990, pp. 311-315) presents a simple model where the profit functions are not quasiconcave. For further discussion, see Codognato (1994).

On the other hand, note that, in a partial equilibrium setting, only a subset of equilibrium relations is specified. The unspecified part is covered by the assumption that "other things are assumed constant". In this setting, consumers' wealth is exogenously determined and consequently, the inverse demand correspondence depends only on prices

$$p(x) = \left\{ p \in \Delta_p : \sum_{i=1}^{I} x_i^* (p, R_i) = x \right\}.$$

Finally, observe that these simplifications may introduce significant distortions on the equilibrium values.

3 Existence of a CW equilibrium

Under mild assumptions, it is possible to prove the existence of a solution for the consumers and firms problems in a CW economy. However, we require additional and stronger assumptions to show the existence of a CW equilibrium (x^*, y^*, p^*). This is the purpose of this section.

Theorem 2 ***Existence of a CW equilibrium.*** *Let* E *be an economy satisfying conditions* $(H.1)$ *to* $(H.12)$. *Then there exists a CW equilibrium*

$$(x^*, y^*, p^*) \in \prod_{i=1}^{I} \mathbb{X}_i \times \prod_{j=1}^{J} \mathbb{Y}_j \times \Delta_p.$$

Proof.

Let $y^e \in \mathbb{C}$ and consider Problem (2), for all $j = 1, ..., J$. To prove the existence of a CW equilibrium, we first prove that there exists a solution of Problem (2). In step II, we show that this solution could be a CW equilibrium, under certain conditions that depend on the parameters y^e. In step III we verify that such conditions are satisfied. Finally, we prove that this solution satisfies the fixed point conditions and solves the consumers' problems.

Step I:

By Weierstrass' theorem, for each $y^e \in \mathbb{C}$ we can guarantee the existence of a solution for Problem (2). Denote this solution as

$$(\widehat{y}(y^e), \widehat{p}(y^e)) \in \prod_{j=1}^{J} (\mathbb{Y}_j \times \Delta_p).$$

Observe that, for example, $\widehat{y}_j(y^e)$ only depends on y^e_{-j} and not on the $j-th$ coordinate of y^e. Notationally, however, it is much easier to write $\widehat{y}_j$ as depending on the entire vector y^e.

Step II:

We now show that there exists $y^* \in \mathbb{C}$ such that $y_j^* \in \widehat{y}_j(y^*)$, $\forall j$.

Consider the correspondence g from $\mathbb{C}$ to itself defined by $g(y^e) = (\widehat{y}_1(y^e), ..., \widehat{y}_J(y^e))$. By applying the Maximum Theorem under convexity to Problem (2), $\widehat{y}_j(y^e)$, $\forall j$ are nonempty, compact-valued, convex-valued, usc correspondences on $\mathbb{C}$, a nonempty, convex and compact set. Hence, by Kakutani's theorem, this correspondence admits a fixed point $y^* \in \mathbb{C}$ such that:

$$y^* \in \left(\widehat{y}_1(y^*), ..., \widehat{y}_J(y^*)\right).$$

Step III:

We now prove that the uniqueness of equilibrium prices, i.e. $\widehat{p}_1(y^*) = ... = \widehat{p}_J(y^*)$ holds. Since $y^* \in \widehat{y}(y^*)$, we have that

$$\sum_{i=1}^{I} \widehat{x}_i\left(\widehat{p}_j(y^*), R_i\left(\widehat{p}_j(y^*), y^*\right)\right) = \sum_{j=1}^{J} y_j^* + \sum_{i=1}^{I} w_i, \quad \forall j = 1, ..., J.$$

But then, by Assumption $(H.6)$, we also have $\widehat{p}_1(y^*) = ... = \widehat{p}_J(y^*)$. Let us denote this price vector as p^*.

Step IV:

Finally, note that for all i,

$$x_i^* = \widehat{x}_i\left(p^*, R_i(p^*, y^*)\right) = \arg\max \left\{u_i(x_i) : \ p^* \cdot x_i \leq R_i(p^*, y^*)\right\}$$

■

Assumption $(H.6)$ is necessary to avoid that individual prices $\widehat{p}_1(y^*), ..., \widehat{p}_J(y^*)$ could be different. This assumption presupposes the existence of equilibrium prices, and it is quite strong. But, as discussed in the introduction, other approaches considered to date assume even stronger hypothesis.

As discussed in the definition of the economy, Assumptions $(H.10)$ and $(H.12)$ could be very restrictive. If we relax these assumptions, Problem (2) is not convex and the CW equilibrium may fail to exist. Hence, the preceding approach cannot be applied to that case.

4 Characterization of a local CW equilibrium

Having established in Section 3 conditions under which a CW equilibrium (x^*, y^*, p^*) is guaranteed to exist, in this section we provide a characterization of a CW equilibrium as the solution of a system of nonlinear equations.

We first introduce the concept of *local equilibria*. Then, we characterize a local equilibrium as a solution of a system of nonlinear equations.

Definition 3 ***Local CW Equilibrium.*** *The allocation*

$$(x^*, y^*, p^*) \in \left(\prod_{i=1}^{I} \mathbb{X}_i \times \prod_{j=1}^{J} \mathbb{Y}_j \times \mathbb{R}^D\right),$$

with $p^* \neq 0$ is a local CW equilibrium for an economy E *if there exists $\varepsilon > 0$ such that:*

CP *Each consumer solves its Problem* (1) *in $\mathbb{X}_i \cap \overline{B}(x_i^*, \varepsilon)$.*

FP *Each firm solves its Problem* (2) *in $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ and $\Delta_p \cap \overline{B}\left(p_j^*, \varepsilon\right)$.*

FPC *Fixed Point conditions:* $p_j^* = p^*$ *and in Problem* (2), $y_j^e = y_j^*$, $\forall j = 1, \dots, J$.

The local CW equilibrium concept fails to satisfy the completeness property, see Mas Colell et al. (1995) p. 6. But note that in applied models we may not be able to know a full specification of the market. Modelers may address these problems by using local estimations of production functions, preferences and consumption sets. A further discussion of this issue can be found in Esteban, Gourdel and Prieto (2000).

The local equilibrium concept is more realistic than the traditional equilibrium definition. It is important to emphasize that in this model, the local equilibria with nonnegative prices are defined on the sets $\left\{\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)\right\}_{j=1}^J$, $\{\mathbb{X}_i \cap \overline{B}(x_i^*, \varepsilon)\}_{i=1}^I$ and $\Delta_p \cap \overline{B}(p^*, \varepsilon)$. These sets can be interpreted as information sets for the agents regarding their technologies and their preferences. We can assume that a local equilibrium will change when the agents get new information. This is what we observe in the real world.

On the other hand, the market demand function may not be one-to-one for any prices on the nonnegative simplex. Note that the concept of local CW equilibria only requires that the market demand function is locally one-to-one in a neighborhood of the equilibrium's prices.

If Definition 3 is satisfied $\forall \varepsilon > 0$, then (x^*, y^*, p^*) is a CW equilibrium for a economy E.

In order to compute a local equilibrium, it is necessary to provide practical conditions that characterize local CW equilibria and suggest algorithms for finding these points. This characterization requires strong assumptions such as pseudoconcavity of the firms' objective functions, since quasiconcave functions may have stationary points where the gradient vanishes that are not local maxima.

The next theorem states conditions to characterize a local equilibrium for the economy E. Without loss of generality, we assume that the technology of the $j-th$ firm is described by inequality constraints

$$\mathbb{Y}_j = \left\{y_j \in \mathbb{R}^D : F_j(y_j) \leq 0\right\}.$$

An inequality constraint is said to be *active* at a given point if it is satisfied with equality at this point.

Theorem 4 ***Characterization of local CW equilibria.*** *Let* E *be an economy satisfying conditions* $(H.1)$ *to* $(H.7)$, $(H.8)$ *and* $(H.11)$. *If there exists a vector*

$$z^* = (y^*, p^*, \alpha^*, \beta^*, \mu^*) \in \prod_{j=1}^{J} \mathbb{Y}_j \times \Delta_p \times \mathbb{R}^{DJ} \times \mathbb{R}_+^J \times \mathbb{R}_+^J,$$

that satisfies the following conditions:

H.6'. *The market demand is (locally) one-to-one in the price set* $\Delta_p \cap \overline{B}(p^*, \varepsilon)$.

H.9'. $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ *is convex and* $F_j(\cdot)$ *is continuously differentiable in* $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$, $\forall j$,

H.10'. *o_j is continuously differentiable and pseudoconcave*[3] *in* $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ *and* $\Delta_p \cap \overline{B}(p^*, \varepsilon)$, $\forall j$.

H.12'. *For each firm* $j \in \{1, \ldots, J\}$ *and for each* $y^e \in \mathbb{C}$, *the set of feasible productions and prices*

$$\left\{(y_j, p_j) \in \mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right) \times \Delta_p \cap \overline{B}(p^*, \varepsilon): \quad p_j \cdot y_j \geq 0, \quad y_j = \widehat{x}\left(p_j, y_j, y_{-j}^e\right) - \sum_{j' \neq j} \overline{y}_{j'}^e - \sum_{i=1}^{I} w_i \right\}$$

is nonempty, closed, bounded and convex.

H.13. *The market demand (by consumers) is continuously differentiable*[4] *in* $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ *and* $\Delta_p \cap \overline{B}(p^*, \varepsilon)$, $\forall j$.

H.14. *The allocation* (y^*, p^*) *is a regular point*[5] *of Problems* (2), *for all* $j = 1, ..., J$,

and the following holds:

$$\begin{array}{|l|}\hline \nabla_{y_j} o_j\left(y_j^*, p^*\right) + \beta_j^* p^* - \alpha_j^* \nabla_{y_j} \widehat{x}(p^*, y^*) + \alpha_j^* e_j - \mu_j^* \nabla_{y_j} F_j\left(y_j^*\right) = 0, \ \forall j = 1, ..., J, \\ \nabla_p o_j\left(y_j^*, p^*\right) + \beta_j^* y_j^* - \alpha_j^* \nabla_p \widehat{x}(p^*, y^*) = 0, \ \forall j = 1, ..., J, \\ \mu_j^* F_j\left(y_j^*\right) = 0, \quad F_j\left(y_j^*\right) \leq 0, \ \forall j = 1, ..., J, \\ \beta_j^*\left(p^{*T} y_j^*\right) = 0, \quad p^{*T} y_j^* \geq 0, \ \forall j = 1, ..., J, \\ \sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} y_j^* + \sum_{i=1}^{I} w_i, \ \text{with } x_i^* = \widehat{x}_i(p^*, y^*), \\ \hline \end{array} \tag{5}$$

where $e_j = \left(0, ..., 1^{(j)}, ...0\right)^T$, *then* (x^*, y^*, p^*) *is a local CW equilibrium for the economy* $\mathbb{E}$.

Proof.

Consider Problems (2) for all $j = 1, ..., J$. Note that (y^*, p^*) is a feasible point of these problems. Furthermore, the vector z^* defines a local optimizer for Problems (2), for all $j = 1, ..., J$. This is an immediate consequence of the Karush-Kuhn-Tucker optimality (sufficient and necessary) conditions for pseudoconcave functions; see Avriel (1976), Th. 6.7, pp. 152-153 and Bazaraa et al. (1979), Th. 4.3.8, pp. 164-165.

■

[3] If o_j is quasiconcave, twice continuously differentiable and its gradient does not vanish in $\mathbb{Y}_j \times \Delta_p$, then o_j is pseudoconcave. See Arrow and Enthoven (1961), p. 783.

[4] See Mas-Colell (1985), pp. 84-89, for sufficient conditions. The market demand function $x(p, w)$ is differentiable if and only if the determinant of the bordered Hessian of $u(\cdot)$ is nonzero at $x(p, w)$.

[5] A feasible vector (x^*, y^*) for which the active constraint gradients are linearly independent is called regular. Equivalently, a feasible vector (x^*, y^*) for which the matrix of active constraint gradients has full row rank is called regular.

Note that Assumption $(H.14)$ is a constraint qualification under differentiability. Note also that a sufficient condition for Assumption $(H.12')$ is that the no loss constraints are quasiconcave in $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ and $\Delta_p \cap \overline{B}\,(p^*, \varepsilon)$, $\forall j$, and the residual demand functions

$$y_j - \widehat{x}\left(p_j, (y_j, y_{-j}^e)\right) + \sum_{j' \neq j} y_{j'}^e + \sum_{i=1}^{I} w_i$$

are quasimonotone in $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ and $\Delta_p \cap \overline{B}\,(p^*, \varepsilon)$, $\forall j$.

The characterization of a CW equilibrium is described by a set of equality and inequality constraints. The inequalities can be transformed into equations and simple bounds by adding nonnegative *slack variables*. For example, let us consider a production set defined as

$$\mathbb{Y}_j = \left\{ y_j \in \mathbb{R}^D : F_j(y_j) \leq 0 \right\},$$

then we may consider an equivalent technology set

$$\mathbb{Y}_j = \left\{ y_j \in \mathbb{R}^D : F_j\,(y_j) + f_j = 0, \;\; f_j \geq 0 \right\},$$

where f_j are nonnegative slack variables. In an optimization context, the simple bounds can be treated as part of the objective functions using barrier terms. See Fiacco and McCormick (1968). As a consequence, a CW equilibrium can be characterized by a system of nonlinear equations together with bound constraints. Let $H\,(z) = 0$ denote the system (5) of nonlinear equations that characterize a CW equilibrium, where z now contains the variables and slacks; and $l \leq z \leq u$ denote the bound constraints, where l and u are vectors of lower and upper bounds on the components of z. Some components of z may lack a lower or an upper bound, in these cases we set the appropriate components of l and u to $-\infty$ and $+\infty$, respectively.

Another important issue is the difficulty of meeting the required assumptions on pseudoconcavity. For example, the profit functions $o_j\,(y_j, p_j) = p_j \cdot y_j$ are only quasiconcave when $p_j \geq 0$ and $y_j \geq 0$ and pseudoconcave when $p_j > 0$ and $y_j > 0$. However, almost every model of imperfect competition imposes this condition to characterize a CW equilibrium from the solution of a system of equations $H\,(z) = 0$. See for example, the productive economy defined by Gabszewicz and Vial (1972).

A similar difficulty arises in satisfying the quasimonotonicity of the residual demand. For example, a linear fractional function defined on a convex set where the denominator never vanishes is quasimonotone. In particular, the demand functions are quasimonotone when they are linear and decreasing with respect to outputs (as Negishi (1961) assumes).

In spite of the difficulties to meet the required convexity assumptions of Theorem 4, under differentiability assumptions, the system of nonlinear equations (5) gives a necessary condition for a CW equilibrium. In other words, a CW equilibrium satisfies the system of nonlinear equations (5) given by Theorem 4.

The following result provides additional sufficient conditions for a vector (x^*, y^*, p^*) to be a local CW equilibrium in a general setting. This result suggests an efficient computational method to find local CW equilibria, if they exist.

Consider Problems (2), for all $j = 1, ..., J$. Let

$$\begin{array}{l} g_j^1(y_j, p_j) = p_j \cdot y_j \geq 0, \\ g_j^2(y_j, p_j) = -F_j(y_j) \geq 0, \text{ and} \\ h_j(y_j, p_j) = y_j - \widehat{x}\left(p_j, \left(y_j, y_{-j}^e\right)\right) + \sum_{j' \neq j} y_{j'}^e + \sum_{i=1}^{I} w_i; \end{array}$$

denote the constraints of Problem (2) and

$$L_j\left(y_j, p_j, \alpha_j, \beta_j, \mu_j\right) = o_j(y_j, p_j) + \alpha_j h_j(y_j, p_j) + \beta_j g_j^1(y_j, p_j) + \mu_j g_j^2(y_j, p_j),$$

denotes the Lagrangian function of Problem (2), where $\beta_j, \mu_j \geq 0$, for all $j = 1, ..., J$.

Theorem 5 ***Sufficient conditions for local CW equilibria.*** *Let* E *be an economy. If there exists a vector*

$$z^* = (y^*, p^*, \alpha^*, \beta^*, \mu^*) \in \prod_{j=1}^{J} \mathbb{Y}_j \times \Delta_p \times \mathbb{R}^{DJ} \times \mathbb{R}_+^J \times \mathbb{R}_+^J,$$

that satisfies (5) *and the following conditions for some* $\varepsilon > 0$,

H.9". $F_j(\cdot)$ *is twice continuously differentiable in* $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$, $\forall j$,

H.10". o_j *is twice continuously differentiable in* $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ *and* $\Delta_p \cap \overline{B}(p^*, \varepsilon)$, $\forall j$.

H.13. *The market demand function is twice continuously differentiable in* $\mathbb{Y}_j \cap \overline{B}\left(y_j^*, \varepsilon\right)$ *and* $\Delta_p \cap \overline{B}(p^*, \varepsilon)$, $\forall j$

H.14. *For all* $j = 1, ..., J$, *it holds that*

$$v_j^T\left(\nabla^2 L_j\left(y_j^*, p^*, \alpha_j^*, \beta_j^*, \mu_j^*\right)\right) v_j < 0, \ \forall v_j \neq 0, \ \text{with } v_j \in V_j\left(y_j^*, p^*\right), \qquad (6)$$

where

$$V_j\left(y_j^*, p^*\right) = \left\{v : \begin{array}{l} \nabla g_j^1\left(y_j^*, p^*\right)^T v = 0, \ si\ g_j^1\left(y_j^*, p^*\right) = 0 \ y\ \beta_j^* > 0, \\ \nabla g_j^1\left(y_j^*, p^*\right)^T v \geq 0, \ si\ g_j^1\left(y_j^*, p^*\right) = 0 \ y\ \beta_j^* = 0, \\ \nabla g_j^2\left(y_j^*, p^*\right)^T v = 0, \ si\ g_j^2\left(y_j^*, p^*\right) = 0 \ y\ \mu_j^* > 0, \\ \nabla g_j^2\left(y_j^*, p^*\right)^T v \geq 0, \ si\ g_j^2\left(y_j^*, p^*\right) = 0 \ y\ \mu_j^* = 0, \\ \nabla h_j\left(y_j^*, p^*\right)^T v = 0 \end{array}\right\}$$

then (y^*, p^*) *is a local CW equilibrium.*

Proof.

Consider Problems (2), for all $j = 1, ..., J$. The vector z^* defines a local optimizer for these problems. This is an immediate consequence of the Karush-Kuhn-Tucker optimality (sufficient) conditions for pseudoconcave functions. See Avriel (1976), Th. 3.11, pp. 48-51 and Bazaraa et al. (1979), Th. 4.4.2, pp. 169-170.

■

These necessary and sufficient conditions for a CW equilibrium are very useful when the conditions in Theorem 4 do not hold.

Consider a private ownership economy E in which firms are profit maximizers and the inverse demand functions are linear and decreasing in $\mathbb{Y}_j \subset \mathbb{R}^D$, as in Negishi's approach. Then, the firms' problem is as follows: given $y^e \in \mathbb{C}$,

$$\boxed{\begin{array}{ll} \underset{(y_j,p_j)\in\mathbb{Y}_j\times\Delta_p}{Max} & p_j \cdot y_j \\ s.t & \\ & y_j = \widehat{x}\left(p_j, y_j, y^e_{-j}\right) - \sum_{j'\neq j} y^e_{j'} - \sum_{i=1}^{I} w_i; \end{array}} \qquad (7)$$

where $\widehat{x}(p,y) = (p-a)/b$ for some $a > 0$ and $b < 0$, for all $j = 1, ..., J$. In this example, the firms objective functions are not quasiconcave and hence Theorem 2 cannot be applied. However, the sufficient conditions (6) are trivially satisfied for the solution of the system of nonlinear equations (5). Therefore, from Theorem 5 such a solution is a CW equilibrium.

In certain situations, however, it could be difficult to get a closed-form expression for the demand market functions. In this case, we should replace the demand function by other practical conditions. Under conditions $(H.1)$ to $(H.5)$ and

H.6'. The utility function $u_i(\cdot)$ is continuously differentiable,

the solution of each consumer Problem (1) can be characterized by its first-order conditions[6]. Given $y^e \in \mathbb{C}$, the jth firm then faces the problem

$$\boxed{\begin{array}{ll} \underset{(y_j,p_j)\in\mathbb{Y}_j\times\Delta_p}{Max} & o_j(y_j, p_j) \\ s.t. & \\ & y_j = \sum_{i=1}^{I} x_i - \sum_{j'\neq j} y^e_{j'} - \sum_{i=1}^{I} w_i, \\ & p_j \cdot y_j \geq 0, \\ & \nabla_{x_i} u_i(x_i) - \lambda_i p_j = 0, \forall i, \\ & p_j \cdot x_i - p_j \cdot w_i - p_j \cdot \left(r_{ij} y_j + \sum_{j'\neq j} r_{ij'} y^e_{j'} \right) = 0, \forall i, \end{array}} \qquad (8)$$

where λ_i is the Lagrange's multiplier of Problem (1) for the $i-th$ consumer.

In this case, conditions that characterize a CW equilibrium can be established in a similar way, except that we have to take into account the first-order conditions of Problem (1) for each consumer. As discussed in the definition of the economy, in this context, it

[6]To be rigorous, the Kuhn-Tucker necessary conditions are valid only if the constraint qualification condition holds. In this context, this requirement is always met.

could be difficult to check the invertibility requirement of the demand market function. Nevertheless, the concept of local CW equilibria requires only that the market demand function is locally one-to-one in a neighborhood of equilibrium's prices. And in practice, the utility functions commonly used in applied economics are such that they define locally one-to-one demand functions.

5 Global uniqueness of equilibria

Having established in Section 3 conditions under which a CW equilibrium $(\overline{x^*, y^*}, p^*)$ is guaranteed to exist, we study its uniqueness.

Proposition 6 *Under the assumptions in Theorem 4 and*

U.M. *H is uniformly monotone on D_0*[7]*, where H is the system of nonlinear equations*

$$H : D_0 \longrightarrow \prod_{j=1}^{J} \mathbb{Y}_j \times \Delta_p \times \mathbb{R}^{DJ} \times \mathbb{R}_+^J \times \mathbb{R}_-^J,$$

defined by Theorem 4, and $D_0 \subset \prod_{j=1}^{J} \mathbb{Y}_j \times \Delta_p \times \mathbb{R}^{DJ} \times \mathbb{R}_+^J \times \mathbb{R}_-^J$ is an open and convex set,

there exists a unique $z^* \in D_0$ such that $H(z^*) = 0$. Hence, there exists a unique CW equilibrium (y^*, p^*) for the economy E.

Proof.

The result is an immediate consequence of the monotonicity of operator H. See Ortega and Rheinboldt (1970), pp. 141-145. ■

Ortega and Rheinboldt (1970, pp. 142) provide sufficient conditions for Assumption (*U.M.*).

6 Computing local CW equilibria

The traditional literature on general CW equilibria constructs such equilibria in three stages. In a first stage, all noncompetitive agents (e.g. firms) anticipate the consequences of their actions (construct their inverse demand function), then they make their optimal decisions and finally the market clears.

In particular, the approach called the *objective demand approach,* proposed by Gabszewicz and Vial (1972), is a good way to prove the existence of CW equilibrium when firms are profit maximizers. This approach can be summarized as follows:

Step I: Get the inverse demand function $\pi(y)$: given y^e, they solve

$$z(p) = \sum_{j=1}^{J} \widehat{x}_i(p, R_i(p, y^e)) - \sum_{j=1}^{J} y_j^e - \sum_{i=1}^{I} w_i = 0.$$

[7] A mapping $H : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is uniformly monotone on $D_0 \subset D$, an open convex set, iff there exists $\gamma > 0$ such that $(H(x) - H(y))(x - y) \geq \gamma (x - y)^T (x - y)$, $\forall x, y \in D_0$.

Let

$$\pi\left(y^{e}\right)=p^{-1}\left(\sum_{j=1}^{J} y_{j}^{e}+\sum_{i=1}^{I} w_{i}\right)$$

Step II: Solve the firms problems:

$$\widehat{y}_{j}=\arg\max_{y_{j}\in\mathbb{Y}_{j}} \pi\left(y_{j},\left\{y_{j'}^{e}\right\}_{j'\neq j}\right)\cdot y_{j}=\arg\max_{y_{j}\in\mathbb{Y}_{j}} p^{-1}\left(y_{j}+\sum_{j'\neq j} y_{j'}^{e}+\sum_{i=1}^{I} w_{i}\right)\cdot y_{j},$$

where the profit functions of the firms are assumed to be quasiconcave in outputs (the inverse demand functions are assumed to be such that the profit functions are quasiconcave in outputs).

Step III: Apply Fixed Point arguments:

$$y^{*}=\widehat{y}_{j}\left(y^{*}\right),\ \forall j.$$

The main problem of this approach is that, in practice, the inverse demand function is difficult to obtain explicitly. In this section, we propose algorithms that avoid the use of the inverse demand function.

First, we present an algorithm (Algorithm 1) based on the characterization given by Theorem 4. But, as we mentioned before, it is quite difficult to meet these convexity assumptions. An alternative algorithm is proposed later; this second approach solves the firms problems iteratively (Algorithm 2).

6.1 Algorithm 1

In this section we outline how a local equilibrium can be computed using traditional methods (Newton's method). We consider the system of nonlinear equations given by Section 4, $H(z)=0$, where $z\in D_0$ and the bounds constraints of the form $l\leq z\leq u$. As we have shown, the solutions of this system are equilibria for a given economy E under certain convexity assumptions. Note that we cannot solve this system using traditional methods (Newton's method) due to the simple bounds. We will solve an alternative inequality-constrained optimization problem:

$$\boxed{\begin{array}{ll} \underset{z}{Min} & \frac{1}{2}\left\|H(z)\right\|_2^2 \\ s.t. & \\ & l\leq z\leq u. \end{array}} \tag{9}$$

using *Interior Point Methods* for non-linear programming. In Appendix A, we present a scheme of the interior point algorithm for a general constrained problem. For additional details on interior point methods applied to Problem (9), see Esteban, Gourdel and Prieto (2000).

To illustrate this approach, we present some examples.

Example 7 *Consider a two-commodity static exchange economy with two consumers and two goods. The excess demand function is defined as*

$$\widehat{x} = [a_1 - (1 - a_1)\, p_1, a_2 - (1 - a_2)\, p_2]$$

with a_1, $a_2 > 0$. *Firms are profit maximizers and their production sets are* $\mathbb{R}^2$. *Taking as an initial point* $z_0 = e^T$ *and the parameters* $a_1 = \frac{2}{3}$, $a_2 = \frac{2}{3}$, *the interior-point algorithm converges to*

$$z^* = [.11, .11, .11, .11, 1.333, 1.333, -0.11, -0.11, -0.11, -0.11]$$

in 6 iterations.

We can be certain that $y_1^* = (.11, .11)$, $y_2^* = (.11, .11)$ *and* $p^* = (1.333, 1.333)$ *is a CW equilibrium since the excess demand function is linear and decreasing in outputs as in Negishi's framework and, consequently, it satisfies the assumptions given in Theorem 4.*

Example 8 *Consider a two-commodity static exchange economy with two consumers and two goods. The excess demand function is defined as*

$$\widehat{x} = [a_1 - (1 - a_1)\, p_1, a_2 - (1 - a_2)\, p_2]$$

with a_1, $a_2 > 0$. *Firms are profit maximizers and their production sets are, respectively,*

$$G_1 = \left\{(y_1, y_2) \in \mathbb{R}^2 : 2y_1 + y_2 \leq 10\right\},$$
$$G_2 = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_1 + 2y_2 \leq 10\right\}.$$

Taking as an initial point $z_0 = e^T$ *and the parameters* $a_1 = \frac{1}{4}$, $a_2 = \frac{3}{4}$, *the interior-point algorithm converges to*

$$z^* = [4, 0, 0.09, 3.68, 0, 0.5, 2.5, 2.88, -5.18, -2.9, -1.89, -6.8]$$

in 7 iterations.

We can be certain that $y_1^* = (4, 0)$, $y_2^* = (0.09, 3)$ *and* $p^* = (0, 0.5)$ *is a CW equilibrium since the excess demand function is linear and decreasing in outputs as in Negishi's framework and, consequently, it satisfies assumptions given in Theorem 4.*

Example 9 *This is a slightly modified version of the example given by Gabszewicz and Vial (1972). Consider a two-commodity static exchange economy with two consumers and two goods. The utility functions of consumers* 1 *and* 2 *are, respectively,* $u_1(x_1, x_2) = (x_1)^{\frac{1}{4}} (x_2)^{\frac{3}{4}}$, *and* $u_2(x_1, x_2) = (x_1)^{\frac{3}{4}} (x_2)^{\frac{1}{4}}$, *the participations of the consumers in the firms are*

r_{ij}	1	2
1	1	0
2	0	1

without initial endowments. Firms are profit maximizers and their production sets are, respectively,

$$G_1 = \left\{(y_1, y_2) \in \mathbb{R}^2 : 2y_1 + y_2 \leq 10\right\},$$
$$G_2 = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_1 + 2y_2 \leq 10\right\}.$$

Taking as an initial point $z_0 = e^T$, *the interior-point algorithm converges to*

$$z^* = [4.4, 0, 0, 4.4, .25, .25, 0, 0, .083, .25, .25, .083]$$

in 6 iterations.

However $y_1^* = (4.4, 0)$, $y_2^* = (0, 4.4)$ *and* $p^* = (.25, .25)$ *is not a CW equilibrium since Theorem 4 cannot be applied (Cobb-Douglas demand functions are not quasimonotone in prices) and the second-order conditions* (6) *are not satisfied.*

Example 10 *This is the example given by Gabszewicz and Vial (1972) in which productions are bounded*

$$G_1 = \{(y_1, y_2) \in \mathbb{R}^2 : 2y_1 + y_2 \leq 10, 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 8\},$$
$$G_2 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + 2y_2 \leq 10, 0 \leq y_1 \leq 8, 0 \leq y_2 \leq 2\}.$$

Taking as an initial point $z_0 = e^T$, *the interior-point algorithm converges to*

$$z^* = [2, 3.3, 3.3, 2, .25, .25, 0, 0, -.0357, -.1072, , -.1072, -.0357]$$

in 6 iterations. As before, $y_1^* = (2, 3.3)$, $y_2^* = (3.3, 2)$ *and* $p^* = (.25, .25)$ *is not a CW equilibrium since Theorem 4 cannot be applied (Cobb-Douglas demand functions are not quasimonotone in prices) and the second-order conditions* (6) *are not satisfied.*

6.2 Algorithm 2

Let us consider an economy E under the assumptions in Theorem 2 and assume that $o_j(\cdot)$ and $\widehat{x}(\cdot)$ are differentiable. An algorithm to get CW equilibrium is as follows:

Step 0. Let $y^e \in \mathbb{C}$.

Step 1. Solve the of Problem (2) for all $j = 1, ..., J$, and obtains $\widehat{p}_j(y^e)$ and $\widehat{y}_j(y^e)$ for all $j = 1, ..., J$.

Step 2. Update $y_j^e \longleftarrow \widehat{y}_j$. The algorithm stops if the convergence criteria are satisfied; otherwise, go to Step 1.

This iterative method is known as the *successive substitution method.* This method uses

$$y_{k+1}^e = \widehat{y}(y_k^e) \tag{10}$$

to generate the sequence $\{y_k^e\}$ from a given initial point y_1^e. If this method converges then it converges to the solution. In other words, if there exists a limit point y^* for this sequence $\{y_k^e\}$, then y^* is the vector of optimal productions. The most popular of these approaches are the Jacobi and Gauss-Seidel iterative methods to solve systems of linear equations. For details to these nonlinear iterative methods see Ortega and Rheinboldt (1970) and Rheinboldt (1998).

Unfortunately, the resulting iteration may converge quite slowly, or not at all. The rate of convergence depends upon how small is the spectral radius of matrix $\nabla\widehat{y}$. In other words, it depends on the condition $\|\nabla\widehat{y}(y^*)\| < 1$ being satisfied. The smaller (in modulus) the eigenvalues of $\nabla\widehat{y}$, the more rapid is the rate of convergence. See Golub and Van Loan (1996).

This result is well-known in the literature on partial noncompetitive markets. See, for example, Varian (1992, ch. 16). He presents examples of Cournot oligopoly models in which this type of algorithm diverges. In this context, $\widehat{y}(\cdot)$ is a so-called *reaction function.* Note that the equations of the form (10) are difference equations. The solutions or *stationary states* of the system (10) are CW equilibria. In this context, y^* is called a *steady* or *stationary state* that is *asymptotically stable* if $\|\nabla\widehat{y}(y^*)\| < 1$ and *unstable* if $\|\nabla\widehat{y}(y^*)\| > 1$. If y^* is asymptotically stable, there exits a neighbourhood of y^* in which iteration (10) converges, and the order is at worst linear.

The main advantage of the successive substitution methods is that they do not require computing the first derivatives $\nabla\widehat{y}$. However, under differentiability, there exits alternative approaches that ensure the convergence, such as variants of Newton method.

Using a Newton-like method is reasonable since it has the potential for a much more rapid rate of convergence. Note that the updating of parameter y^e is given by the system of nonlinear equations

$$\widehat{y}(y^*) - y^* = 0, \tag{11}$$

and $\widehat{y}$ is assumed to be continuously differentiable[8].

The *kth* iteration of a Newton-like method can be written as

$$y_k^e = y_{k-1}^e + \alpha p, \tag{12}$$

where $\alpha = 1$ for the standard Newton method and p is given by

$$\left(\nabla\widehat{y}_j\left(y_{k-1}^e\right) - I\right)p = -\left(\widehat{y}_j\left(y_{k-1}^e\right) - y_{k-1}^e\right). \tag{13}$$

The convergence of this method at a rapid rate happens when the initial point y_1^e is close to the fixed point y^*. In other words, local convergence is assured.

Since the computation of $\nabla\widehat{y}_j$ could be a very expensive task, we consider an approximate matrix B_k instead of $\left(\nabla\widehat{y}_j\left(y_{k-1}^e\right) - I\right)$, updated using Broyden's method (a quasi-Newton method). The matrices are updated to take account of the new information in each iteration. The update formula is defined as

$$B_{k+1} = B_k + \frac{(s_k - B_k p_k)\,p_k^T}{p_k^T p_k},$$

where $s_k = \left(\widehat{y}\left(y_k^e\right) - y_k^e\right) - \left(\widehat{y}\left(y_{k-1}^e\right) - y_{k-1}^e\right)$. The initial approximation B_0 is usually taken as the identity matrix if no additional information is available. However, we use the finite-difference approximation to $\nabla\widehat{y}_j\left(y_1^e\right)$ minus the identity matrix. For an introduction to quasi-Newton methods and details about the local convergence behavior of these methods see Dennis and Schnabel (1996).

However the above approach is only appropriate for providing local convergence. An algorithm that is locally convergent offers no guarantee of returning any meaningful answer (since these methods converge when started close enough to a local optimum). In general, we can obtain algorithms with **global convergence** properties (these algorithms converge

[8] Necessary conditions are given by the Implicit Function Theorem. See Ortega and Rheinboldt (1970), pp. 128-129.

to a local optimum when started far from all local minima). An iterative algorithm is globally convergent if any limit point of the sequence generated by the algorithm is a solution of the problem. Generally, algorithms with global convergence properties may be slower than those with local convergence properties.

There are several alternatives to ensure global convergence: line search methods, trust-region methods and filter-type methods. We briefly present the first two (the third one is very promising but very recent):

A) LINE SEARCH METHODS

The algorithm mentioned above can ensure global properties by enforcing that a certain function, known as *merit function,* decreases sufficiently at each iteration. The merit function should be constructed so that an optimum of this function is an optimum of the original problem.

To ensure sufficient descent, each iteration of the line search approach decides how far to move along the direction p_{k-1}. In other words, each iteration of this algorithm has the form:

$$y_k^e = y_{k-1}^e + \alpha_{k-1}\, p_{k-1},$$

where α_{k-1} is a scalar obtained by the merit function such that in each iteration the value of merit function decreases. In this context, an adequate merit function could be

$$\|\widehat{y}_j\left(y^e\right) - y^e\|_2^2.$$

To ensure that α_{k-1} is an adequate value we require that p_{k-1} is a descent direction of the merit function (to assure this property, it could be necessary to modify the lineal system of equations that defines p_{k-1}). For details, see Dennis and Schnabel (1996).

However, in this context this result cannot be used. Note that we cannot assure that the Newton step (13) is a descent direction for (11), since B_k is not equal to the true Jacobian $\left(\nabla\widehat{y}_j\left(y_{k-1}^e\right) - I\right)$. As a consequence, it might be impossible to get an appropriate value of α_{k-1} in some iterations. The descent property ensures that the sequence $\{y_k^e\}$ generated by Newton's method converges to a limit y^*, when the initial point y_1^e is far away from the limit point y^*.

B) TRUST-REGION METHODS

In this setting, the iterates are defined by the formula

$$y_k^e = y_{k-1}^e + \; p_{k-1}.$$

In order to get global convergence, p_{k-1} is defined as the solution of a certain optimization problem whose behavior near y_{k-1}^e is similar to that of the original problem. Since the approximation may not be good when the optimum is far from y_{k-1}^e, we restict the length of the direction p_{k-1}, i.e. $\|p_{k-1}\| \leq \Delta_k$. In other words, we obtain the search direction p_{k-1} by solving the following subproblem:

$$\boxed{\begin{array}{ll} \min\limits_{s.t} & \left\|\left(\nabla\widehat{y}_j\left(y_{k-1}^e\right) - I\right)p + \widehat{y}_j\left(y_{k-1}^e\right) - y_{k-1}^e\right\|_2^2 \\ & \|p\| \leq \Delta_k. \end{array}}$$

Whenever we use a quasi-Newton approximation B_k instead of the true Jacobian $\nabla\widehat{y}_j\left(y_{k-1}^e\right) - I$, the search direction p is given by

$$\left(B_k^T B_k + \mu_k\; I\right)p = -B_k^T\left(\widehat{y}_j\left(y_{k-1}^e\right) - y_{k-1}^e\right),$$

for a certain scalar $\mu_k \geq 0$.

This approach is known as the Levenberg-Marquardt method. By choosing μ_k adequately, the global convergence property is guaranteed. See Dennis and Schnabel (1996).

We present a scheme of the proposed algorithm, summarizing the aspects described previously:

Step 0. Let $y^e \in \mathbb{C}$ and $\varepsilon > 0$.

Step 1. Solve Problem (2) for all $j = 1, ..., J$, and obtain $\widehat{p}_j(y^e)$ and $\widehat{y}_j(y^e)$ ~~for~~ all $j = 1, ..., J$.

Step 2. Update $y_j^e \longleftarrow \widehat{y}_j + p$, where p is given by $\left(B_k^T B_k + \mu_k I\right) p = -B_k^T\left(\widehat{y}_j\left(y_{k-1}^e\right) - y_{k-1}^e\right)$. The algorithm stops if the convergence criteria $\left\|y_k^e - y_{k-1}^e\right\| < \varepsilon$ are satisfied; otherwise, go to Step 1.

Finally, we present a methodology to compute the reaction functions $\widehat{y}(\cdot)$ numerically. In other words, we propose an approach to solve Problems (2) for all $j = 1, ..., J$ at each step. The literature on partial noncompetitive markets always assumes that the functional form of $\widehat{y}_j(\cdot)$ is known. But note that in applied models it is difficult to specify in advance the functional form of $\widehat{y}_j(\cdot)$.

Clearly, the approach used to solve Problems (2) for all $j = 1, ..., J$ at each step plays a crucial role in this algorithm. We solve these problems using interior-point methods. For details, see Appendix A.

To illustrate this approach, we present some examples.

Example 11 *Consider a two-commodity static exchange economy with two consumers and two goods. The excess demand function is defined as*

$$\widehat{x} = [a_1 - (1 - a_1) p_1, a_2 - (1 - a_2) p_2]$$

with a_1, $a_2 > 0$. *Firms are profit maximizers and their production sets are* $\mathbb{R}^2$. *Taking as an initial point* $z_0 = e^T$ *and the parameters* $a_1 = \frac{1}{4}$, $a_2 = \frac{3}{4}$, *the Algorithm 2 converges to* $y_1^* = (.11, .11)$, $y_2^* = (.11, .11)$ *and* $p^* = (1.333, 1.333)$ *in 3 steps.*

Example 12 *This is the example given by Gabszewicz and Vial (1972) in which productions are bounded*

$$G_1 = \left\{(y_1, y_2) \in \mathbb{R}^2 : 2y_1 + y_2 \leq 10, 0 \leq y_1 \leq 2, 0 \leq y_2 \leq 8\right\},$$
$$G_2 = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_1 + 2y_2 \leq 10, 0 \leq y_1 \leq 8, 0 \leq y_2 \leq 2\right\}.$$

Taking as an initial point $z_0 = e^T$, *the Algorithm 2 converges to* $y_1^* = (1.76, 6.47)$, $y_2^* = (6.47, 1.76)$ *and* $p^* = (.5, .5)$ *in 7 steps.*

Example 13 *This is the example given by Gabszewicz and Vial (1972) in which firm 1 maximize its expected profits. Taking as an initial point* $z_0 = e^T$, *the Algorithm 2 converges to* $y_1^* = (1.36, 7.27)$, $y_2^* = (6.84, 1.57)$ *and* $p^* = (.52, .47)$ *in 14 steps.*

Finally, note that if the functions $\widehat{x}$, o_j and F_j are not differentiable (but still convex and finite everywhere) the conditions to characterize an equilibrium are similar, except that we have to take into account the subdifferentials of $\widehat{x}$, o_j and F_j instead of their gradients. For an introduction to nonsmooth optimization see, for example, Rockafellar (1970, 1981), Clarke (1990). In practice most economic models satisfy differentiability assumptions.

7 Extensions

Throughout this paper, we have assumed that firms behave in the Cournot way for every commodity. In this section we present an existence proof of equilibrium, assuming that firms posses market power for some goods. We extend all previous results in a simple manner to Stackelberg, collusive and Nash equilibrium models. Finally, we describe the existence of CW equilibria in the presence of externalities.

7.1 General Equilibrium in the CW model with "competitive" and "noncompetitive" commodities

In practice, it is unrealistic to assume that firms are sufficiently powerful to affect all market prices. In this section, we consider that firms act as price takers for some goods and as noncompetitive for the rest.

Let E be an economy in which there are D_1 competitive commodities $D_c = \{1, ..., D_1\}$ and $D - D_1$ noncompetitive commodities $D_{nc} = \{D_1 + 1, ..., D\}$.

The jth firm faces the problem

$$\begin{array}{ll} \underset{(y_j, p_j^{nc})}{Max} & o_j\left(y_j, p^c, p_j^{nc}\right) \\ s.t. & \\ & y_j = \left(y_j^c, y_j^{nc}\right) \in \mathbb{Y}_j,\ \left(p^c, p_j^{nc}\right) \in \Delta_p, \\ & \left(p^c, p_j^{nc}\right) \cdot y_j \geq 0, \\ & y_j^{nc} = \sum_{i=1}^{I} \widehat{x}_i^{nc}\left(p^c, p_j^{nc}, R_i\left(p^c, p_j^{nc}, y_j^{nc}, \left(y_{-j}^{nc}\right)^e\right)\right) - \sum_{j' \neq j} \left(y_{j'}^{nc}\right)^e - \sum_{i=1}^{I} w_i^{nc}, \end{array}$$

where $\left(y_{-j}^{nc}\right)^e$ is the vector of optimal noncompetitive production decisions of the remaining firms $j' \neq j$ and p^c is the price of the competitive commodities.

Proposition 14 ***Existence of a CW equilibrium with "competitive" and "noncompetitive" commodities.*** *Under the assumptions in Theorem 2, there exists a CW equilibrium.*

Proof.

Let $y^e \in \mathbb{C}$ and $p^e = (p_c^e, p_{nc}^e) \in \Delta_p$, and define the following problems for all $j = 1, ..., J$,

$$
\begin{array}{|l|}
\hline
\underset{\left(y_j, p_j^{nc}\right)}{Max} \quad o_j\left(y_j, p_c^e, p_j^{nc}\right) \\
s.t. \\
\qquad y_j = \left(y_j^c, y_j^{nc}\right) \in \mathbb{Y}_j, \left(p_c^e, p_j^{nc}\right) \in \Delta_p, \\
\qquad \left(p_c^e, p_j^{nc}\right) \cdot y_j \geq 0, \\
\qquad y_j^{nc} = \sum_{i=1}^{I} \widehat{x}_i^{nc}\left(p_c^e, p_j^{nc}, R_i\left(p_c^e, p_j^{nc}, y_j^{nc}, \left(y_{-j}^{nc}\right)^e\right)\right) - \sum_{j' \neq j} \left(y_{j'}^{nc}\right)^e - \sum_{i=1}^{I} \overline{w}_i^{nc}, \\
\hline
\end{array} \qquad (14)
$$

where $\widehat{x}(\cdot)$ is the market demand. Let $z(p^e, y^e)$ be the excess demand function, that should satisfy Walras' law

$$p^e z(p^e, y^e) = 0.$$

Note that in equilibrium $z^{nc}(p^*, y^*) = 0$ by definition, and $z^c(p^*, y^*) = 0$ by monotonicity of utility functions.

The proof is very similar to that of Theorem 2.

Step I:

By Weierstrass' theorem, for each $y^e \in \mathbb{C}$, there exists a solution for the above Problem, denoted as

$$\left(\widehat{y}(y^e, p^c), \left(p^c, \widehat{p}^{nc}(y^e, p^c)\right)\right) \in \prod_{j=1}^{J} \left(\mathbb{Y}_j \times \Delta_p\right).$$

Let $\widehat{x}_i\left(p^c, \widehat{p}^{nc}(y^e, p^c), R_i\left(p^c, \widehat{p}^{nc}(y^e, p^c), y^e\right)\right)$ denote the consumption decisions of consumers.

Step II:

We now show that there exists $(y^*, (p^{*c}, p^{*nc})) \in \mathbb{C} \times \Delta_p$ such that

$$
\begin{aligned}
y^* &= \widehat{y}(y^*, p^{*c}, p^{*nc}), \\
p^{*nc} &= \widehat{p}_j^{nc}(y^*, p^{*c}, p^{*nc}), \quad \forall j = 1, ..., J, \\
0 &= z(y^*, p^{*c}, p^{*nc}).
\end{aligned}
$$

We can define the correspondence ψ from the convex and compact set $\mathbb{C} \times \Delta_p$ to itself,

$$\psi : (y, p) \longmapsto \left(\widehat{y}(y, p), \frac{p + \max\{0, z(y, p)\}}{\sum_{d=1}^{D} p_d + \sum_{d=1}^{D} \max\{0, z_d(y, p)\}}\right).$$

Following the lines of Arrow and Hahn (1971, ch. 2), by Brouwer's theorem, there exists a fixed point (y^*, p^*) such that:

$$y^* = \widehat{y}(y^*, p^*), \; p^* = \frac{p^* + \max\{0, z(y^*, p^*)\}}{\sum_{d=1}^{D} p_d^* + \sum_{d=1}^{D} \max\{0, z_d(y^*, p^*)\}}.$$

Then, $z_d(y^*, p^*) \leq 0$, $\forall d = 1, ..., D$. Note that the monotonicity of utility functions ensures that there are nonzero prices in equilibrium and, as a consequence, $z_d(y^*, p^*) = 0$,

$\forall d$. This property is known as *desirability* and it is satisfied when utility functions are increasing in all commodities.

As a consequence, we have proved the existence of an optimal production allocation y^* and a optimal price p^{*c} of the competitive commodities. It only remains to prove that p^{*nc} is the vector of optimal prices for noncompetitive commodities.

From the residual demand constraint of Problem (14) for all $j = 1, ..J$, we have that

$$\widehat{x}\left(p^{*c}, \widehat{p}_j^{nc}\left(y^*, p^*\right), y^*\right) = \sum_{j=1}^{J} y_j^* + \sum_{i=1}^{I} w_i, \quad \forall j = 1, ..., J.$$

implying $\widehat{p}_1^{nc}\left(y^*, p^*\right) = ... = \widehat{p}_J^{nc}\left(y^*, p^*\right)$. On the other hand, by assumption $(H.6)$ and

$$\widehat{x}\left(p^{*c}, \widehat{p}_j^{nc}\left(y^*, p^*\right), y^*\right) = \widehat{x}\left(p^{*c}, p^{*nc}, y^*\right), \quad \forall j,$$

we have $\widehat{p}_j\left(y^*, p^*\right) = p^{*nc}, \quad \forall j$.

■

The quasiconcavity of the objective functions of firms is still a very restrictive assumption. In this setting, the decision variables of firms are the production plans and the price of noncompetitive commodities, that are often assumed to be defined on $\mathbb{R}^D_{++}$.

When firms are profit maximizers and have market power in outputs, given the price of competitive commodities p^c, the jth firm's objective is as follows:

$$o_j\left(y_j, p^c, p_j^{nc}\right) = p^c \cdot y_j^c + p_j^{nc} \cdot y_j^{nc};$$

with $y_j^{nc} \gg 0$ and $p_j^{nc} \gg 0$. But the quasiconcavity assumption of o_j is not satisfied since the sum of a linear function and a quasiconcave function is not a quasiconcave function. As a consequence, the CW equilibrium may fail to exist.

An example of a firm objective function that satisfies the quasiconcavity property is the following: consider an economy in which the set of goods that can be outputs, $y_j^O = (y_1, ..., y_{D_1})$, is distinct from the set that can be inputs, $y_j^I = (y_{D_1+1}, ..., y_D)$. In such a case, the technology set is given by

$$\mathbb{Y}_j = \left\{\left(y_j^I, y_j^O\right) \in \mathbb{R}^D : y_j^O \leq f\left(y_j^I\right), y_j^I \leq 0, y_j^O \geq 0\right\}.$$

For some appropriate parameters α and β, the firms' objective can be defined as follows:

$$\begin{aligned} o_j\left(y_j, p_j\right) &= \sum_{d=1}^{D_1}\left[\alpha_d \log\left(p_d\right) + \beta_d \log\left(y_d\right)\right] \\ &\quad - \sum_{d=D_1+1}^{D}\left[\alpha_d \log\left(p_d\right) + \beta_d \log\left(-y_d\right)\right]. \end{aligned}$$

In this setting, firms are maximizing revenues and minimizing costs.

Analogously, we can also characterize such a equilibrium as a solution of a set of equations under some appropriate conditions.

7.2 General Equilibrium in the Stackelberg Model

Let E be an economy in which $L = \{1, ..., J_1\}$ firms are the so-called leaders and $S = \{J_1 + 1, ..., J\}$ firms are the followers. Given a price vector $p \in \mathbb{R}^D$, the firm $j \in S$ faces the problem

$$\boxed{\underset{y_j \in \mathbb{Y}_j}{Max} \quad \{o_j(y_j, p) : p \cdot y_j \geq 0\}.}$$

Under convexity and compactness assumptions, the existence of solution for this problem can be proved. This solution is a correspondence denoted by $\widehat{y}_j(p)$. Furthermore, under strong convexity, such a solution is a function.

Let define the residual demand

$$\widehat{r}(p) = \sum_{i=1}^{I} \widehat{x}_i(p) - \sum_{j' \in S} \widehat{y}_{j'}(p)$$

where $\widehat{x}_i(p)$ are consumer's demand for all $i = 1, ..., I$. We assume that $\widehat{r}(p)$ is a one-to-one function. Then, the leaders face the following problem

$$\boxed{\begin{array}{ll} \underset{(y_j, p_j) \in \mathbb{Y}_j \times \Delta_p}{Max} & o_j(y_j, p_j) \\ s.t. & \\ & y_j = \widehat{r}(p_j) - \sum\limits_{j' \in L \backslash \{j\}} y_{j'}^e - \sum\limits_{i=1}^{I} w_i, \\ & p_j \cdot y_j \geq 0. \end{array}}$$

Under convexity and compactness assumptions, the existence of solution for this problem can be proved. This solution is a correspondence denoted by $\widehat{y}_j\left(y_{-j}^e\right) \subset \mathbb{Y}_j$ and $\widehat{p}_j\left(y_{-j}^e\right) \subset \mathbb{R}^D$.

In equilibrium it is satisfied

$$y_j^* = \widehat{y}_j\left(y_{-j}^*\right), \quad \forall j = 1, \ldots, J,$$

i.e. $y_j^e = y_j^*$, $\forall j = 1, \ldots, J$. Moreover, $p_j^* = \widehat{p}_j\left(y_{-j}^*\right) = p$, $\forall j = 1, \ldots, J$ since $\widehat{r}(p)$ is one-to-one.

Proposition 15 ***Existence of a Stackelberg equilibrium.*** *Under the assumptions in Theorem 2 (assuming that $\widehat{r}(p)$ is one-to-one instead of demand), there exists a Stackelberg equilibrium.*

7.3 General Equilibrium in the Collusive Model

Let E be an economy in which firms collude and form a cartel, that is, firms make optimal decisions by maximizing a joint objective function. Then, the problem to be solved is

$$\begin{array}{ll} \underset{(\{y_j\},p)}{Max} & o_j\left(\sum_{j=1}^{J} y_j, p\right) \\ s.t & \\ & (\{y_j\},p) \in \prod_{j=1}^{J} \mathbb{Y}_j \times \Delta_p, \\ & y_j = \sum_{i=1}^{I} \widehat{x}_i\left(p, R_i\left(p, y_j, y^e_{-j}\right)\right) - \sum_{j' \neq j} y^e_{j'} - \sum_{i=1}^{I} w_i, \ \forall j = 1, ..., J. \end{array}$$

Proposition 16 ***Existence of a collusive equilibrium.*** *Under the assumptions in Theorem 2, there exists a collusive equilibrium.*

7.4 Nash General Equilibrium in the non-cooperative static game

In a Nash equilibrium every agent faces given values of variables $x^e_{-i} = \{x^e_{i'}\}_{i' \neq i}$ set by other agents and makes his optimal choice $\widehat{x}_i$, choosing from elements belonging to a set $\mathbb{X}_i\left(x^e_{-i}\right)$. In this case, every agent i faces the following problem

$$\widehat{x}_i\left(x^e_{-i}\right) = \underset{x_i \in \mathbb{X}_i\left(x^e_{-i}\right)}{\arg\max}\ u_i\left(x_i, x^e_{-i}\right), \tag{15}$$

where the function $u_i\left(x_i, x^e_{-i}\right)$ is the agent's objective function, for all $i = 1, ..., I$.

Then, $x^* = (x^*_1, ..., x^*_I)$ is a Nash equilibrium if every agent solves Problem (15) and there is a coordination mechanism such that

$$x^*_i = \widehat{x}_i\left(x^*_{-i}\right), \quad \forall i = 1, ..., I.$$

By applying the Maximum theorem to Problem (15), under continuity and strict concavity, there exists a continuous function $\widehat{x}_i$ of the other agents' decisions that satisfies the condition for Kakutani's theorem. Then, there exists a fixed point x^* of $\widehat{x}_i$. As a consequence, the Nash equilibrium is a fixed point of $\widehat{x}_i$. The function $\widehat{x}_i\left(x^e_{-i}\right)$ which associates the best response of the agent i to the decisions of other agents x^e_{-i} is known as the *reaction function* of i.

Note that the system of equations $x^*_i = \widehat{x}_i\left(x^*_{-i}\right), \forall i$ characterizes a Nash equilibrium. This setting is an appropriate representation of a Nash equilibrium to describe the relation between agents, but in applied models it is difficult to specify their functional forms.

Under convexity and smoothness conditions, a Nash equilibrium is characterized as a solution of the first order conditions of Problem (15). See e.g. Ho (1970) and Basar and Olsder (1982). However these convexity assumptions are very difficult to satisfy in practice. In order to overcome these requirements, alternative approaches had been developed based on Jacobi and Gauss-Seidel methods.

Following a Jacobi approach, each agent optimizes its decision given the previous iteration solution for other agents. Following a Gauss-Seidel approach, each agent optimizes its decision given the current iteration solution for other (preceding) agents. However, both algorithms may converge slowly and convergence cannot be always assured. See Ortega and Rheinboldt (1970).

To overcome these problems, overrelaxation procedures have been proposed. The main disadvantage of these methods is that they depend critically on the choice of some parameters. See Ortega and Rheinboldt (1970).

Finally, note that Algorithm 2 proposed in this paper is also appropriate to solve this type of problems.

7.5 Externalities

In this section, we outline how externalities can be incorporated into noncompetitive economy.

Assume that firms are directly affected by the actions of the remaining firms. Then, we can describe externalities in the $j-th$ firm's production set as follows

$$\mathbb{Y}_j\left(y_{-j}\right)=\left\{y_j\in\mathbb{R}^D : F_j\left(y_j,\left\{y_{j'}\right\}_{j'\neq j}\right)\leq 0\right\},$$

where $y_{-j}=\left\{y_{j'}\right\}_{j'\neq j}$. These externalities can also be incorporated to the $j-th$ firm's objective as $o_j\left(y_j,\left\{y_{j'}\right\}_{j'\neq j},p\right)$.

All results derived in the previous sections are valid for this section. Further discussions on this extension can be found in Esteban, Gourdel, Prieto (2000).

8 Appendix A: Interior-Point Methods

In this section we describe the implementation of Interior-Point Methods. We consider the following nonlinear, and possibly non-convex problem:

$$\boxed{\begin{array}{ll}\underset{z\in\mathbb{R}^D}{Max} & f(z)\\ s.t. & \\ & c(z)=0,\\ & l\leq z\leq u,\end{array}}$$

where $f:\mathbb{R}^D\longrightarrow\mathbb{R}$ and $c:\mathbb{R}^D\longrightarrow\mathbb{R}^m$. Note that an inequality constraint $c(z)\geq 0$ can be rewritten as equality by adding a slack variable, $c(z)+s=0$ with $s\geq 0$.

During the 1960s, many techniques were derived for unconstrained optimization. It was standard practice to convert a constrained problem into a sequence of unconstrained problems, by incorporating to the objective function additional terms that would add arbitrarily high costs either for infeasibility or for approaching the boundary of the feasible region. The most popular of these approaches for inequality constrained problems was the use of barrier methods. Interior point methods are closely related to the classical logarithmic barrier methods. The barrier method is defined by introducing a parameter μ, called the *barrier parameter*, and a logarithmic barrier function that is defined in the interior of the feasible set of the original problem.

Interior point methods transform this inequality-constrained optimization problem into a sequence of equality-constrained optimization subproblems defined as:

$$\boxed{\begin{array}{ll} \underset{z\in\mathbb{R}^D}{Max} & f(z)+\mu\sum_{i=1}^{I}\log(z_i-l_i)+\mu\sum_{i=1}^{I}\log(u_i-z_i) \\ s.t. & \\ & c(z)=0. \end{array}} \qquad (16)$$

Under mild conditions, every limit point of a sequence $\{z^*(\mu)\}$ of local minimizers of these problems is a local minimum of the original constrained problem; i.e. $z^*(\mu)\longrightarrow z^*$ as $\mu\rightarrow 0$. This method was studied by Fiacco and McCormick (1968). In spite of the good properties of this method, it became unpopular because of the numerical ill-conditioning of the barrier Hessian. Recently, it was proved that under conditions that normally hold in practice, this ill-conditioning does not degrade the accuracy of the computed solution. See Wright, M. (1997) and Wright, S. (1998).

In 1984, Karmarkar presented a polynomial-time linear programming method. In 1986, Gill et al. showed that there is an equivalence between Karmarkar's method and logarithmic barrier methods. Since then, interior points have become very popular. For an introduction to interior-point methods see, e.g. Wright, M. (1998) and its references, and for details, see Nesterov and Nemirovskii (1994).

When interior point methods are applied to Problem (16), the corresponding first-order conditions have the form:

$$\begin{array}{l} \nabla f(z_k)+\nabla c(z_k)^T\lambda_k+\mu(Z_k-L)^{-1}-\mu(U-Z_k)^{-1}=0, \\ c(z_k)=0, \end{array}$$

where $Z_k=diag(z_k)$, $L=diag(l)$ and $U=diag(u)$.

Let $W_k^1=\mu(Z_k-L)^{-1}$ and $W_k^2=\mu(U-Z_k)^{-1}$, then we can rewrite the first-order conditions as

$$\begin{array}{l} \nabla f(z_k)+\nabla c(z_k)^T\lambda_k+w_k^1-w_k^2=0, \\ (Z_k-L)W_k^1-\mu=0, \\ (U-Z_k)W_k^2-\mu=0, \\ c(z_k)=0, \\ w_k^1,w_k^2>0, \end{array}$$

that we will denote as by $F(z_k,\lambda_k,w_k^1,w_k^2)=0$, where $W_k^1=diag(w_k^1)$ and $W_k^2=diag(w_k^2)$. This is the standard primal-dual system that we will solve using Newton algorithm (See e.g. Dennis and Schnabel (1996, pp 86-154)):

Step 1. Let $z_0,w_0^1,w_0^2\in\mathbb{R}^D$, $\lambda_0\in\mathbb{R}^m$ and $\varepsilon>0$. Set $k=1$, $z_k\leftarrow z_0$, $w_k^1\leftarrow w_0^1$, $w_k^2\leftarrow w_0^2$ and $\lambda_k\leftarrow\lambda_0$.

Step 2. If $\|F(z_k,\lambda_k,w_k^1,w_k^2)\|_2<\varepsilon$, stop (the problem is solved); else, solve the system

$$\begin{pmatrix} \nabla^2 f(z_k)+\lambda_k\nabla^2 c(z_k) & I & -I & \nabla c(z_k)^T \\ W_k^1 & (Z_k-L) & 0 & 0 \\ -W_k^2 & 0 & (U-Z_k) & 0 \\ \nabla c(z_k) & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \Delta z \\ \Delta w^1 \\ \Delta w^2 \\ \Delta\lambda \end{pmatrix}=-F(z_k,\lambda_k,w_k^1,w_k^2).$$

Step 3. Compute α_z, α_λ, α_{w^1}, $\alpha_{w^2} \in (0,1)$ such that $z_{k+1} = z_k + \alpha_z \Delta z$, $w_{k+1}^2 = w_k^2 + \alpha_{w^2} \Delta w^2$, $w_{k+1}^1 = w_k^1 + \alpha_{w^1} \Delta w^1$ and $\lambda_{k+1} + \alpha_\lambda \, \Delta\lambda$ are feasible.

Step 4. Consider the merit function

$$L(z, \lambda; \rho) = f(z) + \mu \sum_{i=1}^{I} \log(z_i - l_i) + \mu \sum_{i=1}^{I} \log(u_i - z_i) - \lambda^T c(z) + \frac{1}{2} \sum_{j=1}^{m} \rho_j c_j(z)^2.$$

and let $m(\alpha) = L(z + \alpha \, \Delta z, \lambda; \rho)$.

While $m(0) - m(\alpha_z) < -\rho \, \alpha_z \, \nabla m(0)^T \, \Delta z$, where $0 < \rho < 1$; set $\alpha_z \leftarrow \alpha_z / 2$ and $z_{k+1} = z_k + \alpha_z \Delta z$.

Step 5. Update

$$\mu \leftarrow \gamma \frac{(z_k - l)^T w_k^1 + (u - z_k)^T w_k^2}{2D},$$

where $0 \leq \gamma < 1$, and $k \leftarrow k + 1$ and go back to step 2.

The parameter μ measures the average value of the pairwise products $(z_k - l)^T w_k^1$ and $(u - z_k)^T w_k^2$. The success of this algorithm depends critically on the choice of the parameters μ and γ. Unfortunately, difficulties can arise if unsuitable values of these parameters are used. See e.g. Wright, M. (1998).

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