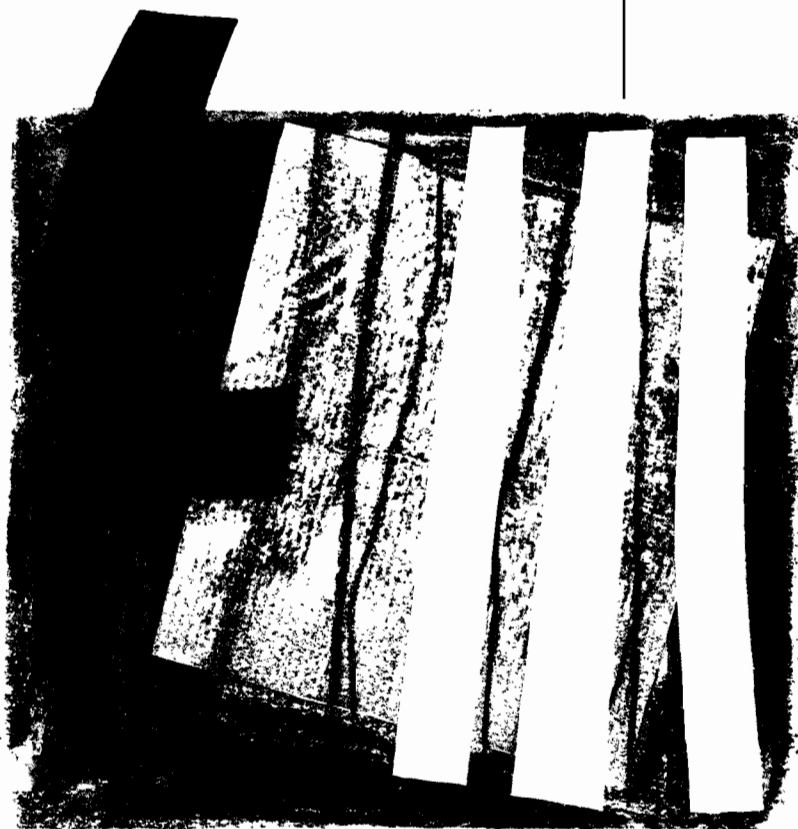


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Departamento de Economía de la Empresa
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 6249608

A TEST OF THE MIXTURE OF DISTRIBUTIONS MODEL

Ainhoa Zárraga Alonso*

Abstract

In this paper a direct test of the mixture of distributions model is conducted using daily stock return and trading volume of the Spanish continuous stock market for the period April 1990 to January 1996. Both the standard mixture of distributions model of Tauchen and Pitts (1983) and the modified version proposed by Andersen (1996) are estimated by the Generalized Method of Moments and tested using the overidentified restrictions. The results of the tests show the rejection of the restrictions that the standard and modified models impose on the data, that is, the dynamics of the Spanish returns and volume are not directed by a common factor, namely the flow of information, according to the specifications of the mixture models considered.

*Universidad Carlos III de Madrid. Dept. Economía de la Empresa, C/ Madrid, 126, 28903 Getafe (Madrid), Spain. E-mail: azarraga@teleline.es

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1 Introduction

The study of the factors determining the distribution of the stock returns has been an important topic in financial literature. Firstly, univariate models with ARCH specifications (see Engle (1982)) or its extension into GARCH (see Bollerslev (1986)) were studied but, more recently, multivariate models that capture the interaction between the return volatility and other economic variables have been considered.

The empirical evidence shows the existence of a positive contemporaneous correlation between the trading volume and the price changes (see Stickel and Verrecchia (1994) for a summary). Clark (1973) offers a theoretical explanation for the relationship between the price changes and the volume through the mixture of distributions model (MODM henceforth), in which price changes and trading volume are related due to their common dependence on a mixing variable, the information flow. Similarly, Epps and Epps (1976) use a mixture model complementary to that of Clark (1973) in order to explain such a relationship between the variables. However, Clark (1973) as well as Epps and Epps (1976) consider volume as a weakly exogenous variable and this assumption is not adequate if price changes and volume are determined jointly. Tauchen and Pitts (1983) modify the univariate model of Clark (1973) by including trading volume as an endogenous variable. They develop a bivariate mixture model in which the joint distribution of daily price changes and daily volume is modeled as a mixture of bivariate normal distributions. In the model, the process of information arrival to the market directs the dynamics of price changes and volume.

Clark (1973), Epps and Epps (1976), Tauchen and Pitts (1983) and Harris (1987), among others, find evidence in favour of the MODM since the distributional patterns expected from

the mixture model appear consistent with those of the data they use. However, they do not conduct a direct test of the mixture model but of its implications. On the contrary, Richardson and Smith (1994) and Andersen (1996) estimate the bivariate mixture model using the generalized method of moments (GMM) and conduct a direct test of the MODM. The former finds evidence in favour of the MODM less strong than implied by existing studies and the latter rejects the MODM for the data series he uses.

However, Andersen (1996) modifies the standard MODM by integrating the microstructure framework of Glosten and Milgrom (1985) and finds that the modified version of the mixture model captures the characteristics of the data series. The dynamic features are governed by the information flow, modeled as a stochastic volatility process.

In this paper I carry out a direct test of both the standard bivariate mixture of distributions model developed by Tauchen and Pitts (1983) and the modified version of Andersen (1996). Since the models impose restrictions on the joint moments of stock returns and trading volume as a function of a few parameters, it is possible to test a number of restrictions on the data. Using average daily stock returns and trading volume for the Spanish continuous stock market the models are estimated by the GMM procedure of Hansen (1982) and tested using the overidentified restrictions. Also, characteristics of the distribution of the random rate of information flow, not observable, can be estimated.

The importance of this analysis is that it allows to know whether the MODM captures the features of the Spanish data series. If that is the case, the theoretical model should be taken into account for future research on financial markets and what is more, investors should take their decisions on the choice of the optimal portfolio in terms of a mixture of

distributions instead of the portfolio returns distribution, which provides a basis for the derivation of the known capital asset pricing model (CAPM). Although empirical evidence against the traditional CAPM has been found for the Spanish market (see Sánchez-Torres and Sentana (1998)), more recently asset pricing models that consider higher order moments as the co-skewness and co-kurtosis of an asset with the market portfolio have been considered to represent an empirical approximation to the stock markets (see Sánchez-Torres and Sentana (1998) and Fang and Lai (1997), among others).

The paper is organized as follows. Section 2 presents the theoretical basis of the mixture models. Next, section 3 explains the methodology used to test the bivariate mixture models. The estimation and tests results are shown in Section 4, and the last section concludes.

2 Mixture of distributions model

This section reviews the mixture of distributions theory and presents the models to be estimated in the next section.

Clark (1973) introduces the mixture of distributions model in order to explain why the probability distribution of the daily price change is leptokurtotic. He considers that the daily price change is the sum of a random number of within-day price changes. He presents and tests the hypothesis that the distribution of daily price change is subordinated to a normal distribution, which comes from a generalization of the Central Limit Theorem¹. Next, subordination term is defined.

A discrete stochastic process is usually indexed as $X_0, X_1, \dots, X_t, \dots$ or, equally, $X(0)$,

¹See Clark (1973) for the theorem and its proof.

$X(1), \dots, X(t), \dots$, where $X(s)$ is the value that a particular realization of the stochastic process takes at time s . But the process could be indexed by a set of numbers t_1, t_2, \dots , where these numbers are a realization of a stochastic process with positive increments, that is, $t_1 \leq t_2 \leq \dots$. Therefore, if $T(t)$ is a positive stochastic process, a new process $X(T(t))$ may be formed. This process is said to be subordinated to $X(t)$, being $T(t)$ the directing process and the distribution of $\Delta X(T(t))$ is said to be subordinated to the distribution of $\Delta X(t)$.

Clark (1973) considers that the distribution of daily price changes, $\Delta X(T(t))$, is subordinated to the distribution of within-day price changes, $\Delta X(t)$, and it is directed by the distribution of trading volume, which assumes lognormal. Therefore, trading volume is the mixing variable, $T(t)$, which measures the speed of evolution in the price process. The different evolution of price series on different days is because the information is available to traders at a varying rate.

According to the model of Clark (1973), trading volume is related positively to the number of within-day transactions and therefore, trading volume is related positively to the variability of price changes.

Clark (1973) finds evidence that finite-variance distributions subordinate to the normal distribution appears to be more consistent with cotton futures price data for the period 1945-1958 than members of the stable family.

Epps and Epps (1976) support the theoretical framework of the model of Clark (1973) and provide further empirical evidence on the thesis that the conditional variance of price changes is a function of trading volume. In their model, the change in the market price on

each within-day transaction is the average of the changes in all traders' price expectations. They assume a positive relationship between the extent to which traders disagree when they revise their reservation prices and the absolute value of the change in the market price. The relationship between the price variability and volume appears because the latter is related to the extent to which traders disagree when they revise their reservation prices. Using price changes and transaction volume data from NYSE on a sequence of individual transactions during the month of January, 1971, they obtain evidence in favour of MODM.

The models of Clark (1973) and Epps and Epps (1976) consider volume as a weakly exogenous variable, which is not adequate if both price changes and trading volume are jointly determined. On the contrary, Tauchen and Pitts (1983) present a model in which the joint distribution of price changes and volume can be modeled as a mixture of bivariate normal distributions. The sum of the random number of the within-day price changes and volume gives the daily value of each variable, resulting a bivariate mixture model. Also, Tauchen and Pitts (1983) consider that the models of Clark (1973) and Epps and Epps (1976) are incomplete and can be extended in two directions.

First, both models use the conditional distribution of the square of the price change over an interval of time, ΔP^2 , given the trading volume, V , for the same interval of time. Therefore, it is necessary to know the functional form of the conditional expectation $E[\Delta P^2|V]$. The model Tauchen and Pitts (1983) derive does not require this calculation and provides an explicit expression for the joint probability distribution of the price change and the trading volume over any interval of time. Second, neither model considers growth in the size of speculative markets.

Tauchen and Pitts (1983) take into account these two possible extensions and develop a bivariate mixture model for the joint distribution of price changes and trading volume. They use a Walrasian sequential equilibrium in which the flow of information is the only factor that determines the trading of volume. The market consists of J traders who have different reservation prices. P_{ij}^* is the j th trader's reservation price. The movement from one to another within-day equilibrium is initiated when traders change their reservation prices as new information arrives at the market. The resulting price change, ΔP_i , is the average change in reservation prices. The associated trading volume, V_i , is a proportion of the sum of the absolute values of the changes between reservation and market prices.

The specification of the joint probability distribution of the increments ΔP_{ij}^* induces a joint probability distribution for the change in the market price and trading volume. Tauchen and Pitts (1983) assume the following model for the increments in the reservation price:

$$\Delta P_{ij}^* = \phi_i + \psi_{ij}, \quad E[\phi_i] = E[\psi_{ij}] = 0, \quad \text{var}[\phi_i] = \sigma_\phi^2, \quad \text{var}[\psi_{ij}] = \sigma_\psi^2, \quad (1)$$

where ϕ and ψ are mutually independent both across traders and through time. The component ϕ_i is common to all traders, while the component ψ_{ij} is specific to each one.

Accounting for this specification, i th price change and trading volume are given by:

$$\Delta P_i = \phi_i + \bar{\psi}_i, \quad \bar{\psi}_i \equiv \frac{1}{J} \sum_{j=1}^J \psi_{ij} \quad (2)$$

$$V_i = \frac{\alpha}{2} \sum_{j=1}^J |\psi_{ij} - \bar{\psi}_i| \quad (3)$$

Tauchen and Pitts (1983) assume that the components ϕ_i and ψ_{ij} follow a normal distribution. Hence, the distribution of ΔP_i is normal and, for large J , V_i is approximately

normally distributed. Also, ΔP_i and V_i are stochastically independent but their common dependence on the components ϕ_i and ψ_i makes functional dependencies among their moments exist².

Fixing the number of traders, J , and allowing the information flow to vary every day, daily price changes and trading volume are obtained as the sum, with respect to the information flow, of the within-day price changes and trading volume. The resulting distribution conditional on the information flow, K , is as follows.

$$\Delta P|K \sim \mathcal{N}(0, \sigma_1^2 K) \quad (4)$$

$$V|K \sim \mathcal{N}(\mu_2 K, \sigma_2^2 K) \quad (5)$$

As the information flow is a random variable, the unconditional joint distribution of ΔP and V is a mixture of independent normals with the same mixing variable, K . Therefore, the information flow represents a stochastic volatility process that directs both price changes and volume and each series will give information on that volatility process, which is not observable.

The relationship between the variability of price changes and trading volume appears because both variables are positively related to the mixing variable in the following way: $Cov(\Delta P^2, V) = \sigma_1^2 \mu_2 Var[K] > 0$. However, conditional on the information flow the relation is null: $Cov(\Delta P^2, V|K) = 0$.³

Tauchen and Pitts (1983) estimate the bivariate mixture model by maximum likelihood using daily price change and volume of trading on the 90-day T-bills futures market for the

²See Tauchen and Pitts (1983) for a detailed explanation.

³See appendix A for the proof.

period 1976-1979. They find that the parameter estimates have the correct signs and are of reasonable magnitude, supporting the model.

The bivariate mixture model (equations (4) and (5)) explains several characteristics on the shape of the distribution of the price changes, ΔP_t , and the trading volume, V_t , such as the kurtosis of the unconditional distribution of ΔP_t and the skewness of the unconditional distribution of V_t . Also, the model predicts a positive correlation between ΔP_t^2 and V_t .⁴

In the literature presented so far, the empirical evidence is in favour of the mixture of distributions model. However, this result is obtained from the fact that the distributional patterns generated from the data appear consistent with those derived from the mixture model (see Clark (1973), Epps and Epps (1976), Tauchen and Pitts (1983), Harris (1987) or Lamoureux and Lastrapes (1990), among others), but they do not conform a direct test of the model.

Richardson and Smith (1994) consider the bivariate mixture model of Tauchen and Pitts (1983) as the theoretical framework to conduct a direct test of the model using the generalized method of moments. The model imposes restrictions on the joint unconditional moments of ΔP and V , and on their crossed moments. The test of those restrictions is a direct test of the bivariate mixture model.

Richardson and Smith (1994) perform the test of the MODM using daily prices and volume for the Dow Jones 30 firms over the sample period 1982-1986. The evidence found supporting the model is not as strong as previous studies find. On the other hand, they test the restrictions that several distributions of the information flow place on the parameters

⁴The proof can be found in Harris (1987).

and obtain that the lognormal distribution provides better results than other distributions.

More recently, Andersen (1996) develops a bivariate mixture model from the microstructure framework of Glosten and Milgrom (1985), where the trading volume is determined by both information asymmetries and liquidity needs. The market has a unique asset with a random liquidation value of V at a point in the future. The risk-neutral traders are of three types: a specialist, informed and uninformed investors. Investors arrive sequentially to the market and decide whether to trade one unit of the asset or not. The informed investors receive private information inducing a dynamic learning process through the sequence of trades and transaction prices. When all agents agree on the price, the market achieves a new temporal equilibrium until new information arrives. Therefore, every day can be decomposed into a random number of small intervals with varying length, each of them consisting of a learning process and an equilibrium phase.

Using the conclusion of Glosten and Milgrom (1985) that the sequence of transaction prices follows a martingale and denoting the transaction price recorded during the j th temporary equilibrium of day t by $P_{j,t}$, $j = 1, \dots, J_t - 1$, where the number of information arrivals on day t is random but large, Andersen (1996) determines the return on day t by:

$$1 + R_t = \prod_{j=1}^{J_t} \frac{P_{j,t}}{P_{j-1,t}} = \frac{P_{J_t,t}}{P_{0,t}} \quad (6)$$

where $P_{0,t}$ and $P_{J_t,t}$ denote the first and last transaction price of the day, respectively.

Andersen (1996) takes into account that J is large but displays a significant variation across the trading days. In order to capture this variation he considers that $J_t = K_t J$, where J is a benchmark day with a fixed large number of arrivals and K_t denotes the intensity of information arrivals relative to the benchmark. Writing $R_t = \ln(P_{J_t,t}/P_{0,t})$, the return

specification is as follows⁵:

$$R_t = \sigma K_t^{1/2} \frac{1}{(JK_t)^{1/2}} \sum_{j=1}^{JK_t} \epsilon_{j,t} \quad (7)$$

For large J and under weak regularity conditions, the distribution of returns conditional on the information flow is normal⁶:

$$R_t|K_t \sim \mathcal{N}(0, \sigma^2 K_t) \quad (8)$$

Hence, the systematic return volatility dynamics is governed by the time series properties of the information flow. Taking into account that the scale of K_t itself is arbitrary, Andersen (1996) normalizes the equation (8) by choosing $\sigma = 1$.

With respect to the trading volume, both informational asymmetries and liquidity needs motivate trade. Andersen (1996) assumes that the distribution of each component of the volume is Poisson and specifies the following conditional distribution of the trading volume:

$$V_t|K_t \sim \mathcal{Po}(m_0 + IK_t\mu) \quad (9)$$

where I denotes the maximum number of insiders that might obtain a private signal associated with the event. The constant term m_0 is the component of the volume generated by liquidity needs, while the component generated by informed traders is proportional to the information flow. $m_1 = I \cdot \mu$ is the factor of proportionality, which determines how strongly volume fluctuates in response to the information arrivals.

⁵See Andersen (1996) for more details.

⁶The distribution is obtained from a generalization of the Central Limit Theorem proposed by Clark (1973).

Andersen (1996) proposes to detrend the volume series to get stationarity. After removing the trend, the new volume series is $\hat{V}_t = cV_t$, which has the following conditional distribution:

$$\hat{V}_t|K_t \sim c \cdot \mathcal{P}o(m_0 + m_1 K_t) \quad (10)$$

Finally, Andersen (1996) allows for a nonzero mean in the return equation, so the bivariate distribution of return and trading volume conditional on the information flow is:

$$R_t|K_t \sim \mathcal{N}(\bar{r}, K_t) \quad (11)$$

$$\hat{V}_t|K_t \sim c \cdot \mathcal{P}o(m_0 + m_1 K_t) \quad (12)$$

The main difference between the so-called modified mixture of distributions model (equations (11) and (12)) and the standard mixture model (equations (4) and (5)) is the specification of the equation of volume. On the one hand, the term m_0 , that measures the part of daily volume generated by the demand of liquidity and, hence, it is independent from the information flow, and, on the other hand, the imposition of a conditional Poisson rather than normal distribution. The latter respects the nonnegativity constraint on trading volume, while estimates of the parameters implying negative observations of volume could be obtained with the standard model. In the modified model there exists a contemporaneous relationship between returns and volume because both series depend on the flow of information arriving at the market, K_t .

Andersen (1996) uses daily NYSE return and trading volume on IBM common stock over the period 1973-1991 and performs a direct test for both the standard and modified mixture of distributions models. He finds evidence against the standard MODM, but in favour of the modified version of the model.

3 Econometric methodology

This section describes the methodology used to estimate the parameters and test the restrictions on the standard and modified bivariate MODM.

The MODM is estimated by GMM, proposed by Hansen (1982), based on the convergence of selected sample moments to their unconditionally expected values. Let $X_t = (R_t, V_t)$ be the vector of observables. If X_t conforms to MODM, then its unconditional moments should also conform to those of the MODM,

$$E[m(X_t, \theta)] = 0, \quad t = 1, \dots, T \quad (13)$$

where θ is a vector of M parameters of MODM, which could be the means and variances of X_t and the central moments of the information flow arriving at the market, K_t , and where $m(\cdot)$ is a J -vector of functional forms implied by MODM.

In large samples, under the null hypothesis that X_t is distributed as MODM, the sample moments of equation (13) should tend to zero:

$$\bar{m}(\theta) = \frac{1}{T} \sum_{t=1}^T m(X_t, \theta) \rightarrow 0 \quad (14)$$

The estimator of θ must satisfy the J -equation system and M unknown parameters:

$$\bar{m}(\theta) = \frac{1}{T} \sum_{t=1}^T m(X_t, \theta) = 0 \quad (15)$$

If the number of independent equations is greater than the number of unknown parameters ($J > M$), the system is overidentified and the estimator of θ is obtained by minimizing:

$$q = T \bar{m}(\theta)' (S_0)^{-1} \bar{m}(\theta), \quad (16)$$

where $S_0 = \sum_{t=-\infty}^{+\infty} E[m(X_t, \theta)m(X_t, \theta)']$.

Hansen (1982) provides the following results for the distribution of the estimator $\hat{\theta}$ and for the overidentifying test statistic q :

$$\sqrt{T}(\hat{\theta} - \theta) \stackrel{a}{\sim} \mathcal{N}\left(0, [D_0' S_0^{-1} D_0]^{-1}\right) \quad (17)$$

$$q = T\bar{m}(\theta)'(S_0)^{-1}\bar{m}(\theta) \stackrel{a}{\sim} \chi_{J-M}^2, \quad (18)$$

where $D_0 = E[\partial m(X_t, \theta)/\partial \theta]$.

These results hold for any consistent estimators of S_0 and D_0 . In this paper, the matrix S_0 used in the objective function is obtained according to the procedure of Newey and West (1987) with 4 lags.

4 Estimation and test of the mixture models

This section presents the results of the estimation of the bivariate mixture models as well as the results of the specification tests carried out to check whether the data conform to the standard and modified MODM, presented in section 2.

The data used in this work are daily stock returns and the logarithm of trading volume from 100 financial stocks of the Spanish continuous stock market for the period 19-04-1990 to 29-01-1996. The trading volume is the number of shares traded daily and the returns are obtained from the series of closing price, P_t , accounting for dividends, DI_t , and subscription rights, SU_t , according to the following expression: $R_t = \ln(P_t + DI_t + SU_t) - \ln(P_{t-1})$. The data series used in the empirical analysis are the sample average of the returns and trading volume of the stocks⁷. Both series are stationary and significant daily and monthly seasonal

⁷The analysis was also carried out using data of some individual stocks but the results did not change.

effects are removed using the two step procedure of Gallant, Rossi and Tauchen (1992).

Estimation procedure requires nonlinear optimization and thus, an iterative procedure is needed. In this paper the BHHH algorithm (Berndt, Hall, Hall and Hausman (1974)) is used considering that convergence exists when the value of the function to minimize does not vary more than 0.001 from one iteration to the next one. The moment conditions involved in the GMM procedure must be selected arbitrary⁸.

For the standard MODM, the following system involving the daily returns, R_t , the volume, V_t , and the flow of information, K_t , is estimated by GMM (Hansen (1982)):

$$R_t|K_t \sim \mathcal{N}(\bar{r}, K_t) \quad (19)$$

$$V_t|K_t \sim \mathcal{N}(\mu_v K_t, \sigma_v K_t) \quad (20)$$

To estimate the model, the following set of unconditional moment conditions is used⁹:

$$E[R_t] - \bar{r} = 0 \quad (21)$$

$$E|R_t - \bar{r}| - (2/\pi)^{1/2} E[K_t^{1/2}] = 0 \quad (22)$$

$$E[(R_t - \bar{r})^2] - E[K_t] = 0 \iff E[(R_t - \bar{r})^2] - \bar{K} = 0, \quad (23)$$

where $E[K_t] = \bar{K}$

$$E[|R_t - \bar{r}|^3] - 2(2/\pi)^{1/2} E[K_t^{3/2}] = 0 \quad (24)$$

⁸Estimation has been performed using different moment conditions finding that in some cases, the results do not vary with respect to those presented in this paper and, in other cases, the estimation of the models is not possible because the covariance matrix S_0 is not regular.

⁹See appendix B.1 for the proof.

$$E \left[(R_t - \bar{r})^4 \right] - 3E \left[\bar{K}^2 + var(K_t) \right] = 0 \quad (25)$$

$$E[V_t] - \mu_v \bar{K} = 0 \quad (26)$$

$$E \left[(V_t - \mu_v \bar{K})^2 \right] - \sigma_v \bar{K} - \mu_v^2 var(K_t) = 0 \quad (27)$$

$$E[R_t V_t] - \bar{r} \mu_v \bar{K} = 0 \quad (28)$$

$$E \left[|R_t - \bar{r}| (V_t - \mu_v \bar{K}) \right] = 0 \quad (29)$$

$$E \left[R_t (V_t - \mu_v \bar{K})^2 \right] - \bar{r} \left[\sigma_v \bar{K} + \mu_v^2 var(K_t) \right] = 0 \quad (30)$$

The parameter vector is: $\theta = (\bar{r}, E(K_t^{1/2}), \bar{K}, E(K_t^{3/2}), \mu_v, \sigma_v, E(K_t^2))$. Thus, there are 10 moment conditions and 7 free parameters and then, 3 overidentifying restrictions. The system is estimated by minimizing with respect to θ , the distance between the sample and theoretical moments in the quadratic form (16), where S_0 is estimated in accordance with Newey and West (1987) procedure taking 4 lags.

Table 1 presents the results of the estimation of the standard MODM. It shows the estimated parameters with the associated estimated standard errors in parenthesis as well as the value of the specification test statistic with the p-value. The estimated parameters are not of the expected sign since the estimated average returns are negative and the estimated variance of the information flow arriving at the market is also negative. On the other hand, the value of the specification test is large and hence the null hypothesis that the data conform the standard bivariate MODM is rejected. This result is similar to that obtained by Andersen (1996) and also by Richardson and Smith (1994) for some of the stocks they consider.

Table 1: Estimation results for the standard MODM

\bar{r}	$E[K_t^{1/2}]$	\bar{K}	$E[K_t^{3/2}]$	μ_v	σ_v	$E[K_t^2]$	χ^2_3 (p-val.)
-0.031	0.0389	2.32e-04	-5.41e-05	-0.237	-126.33	-4.01e-06	14519.1
(1.2e-04)	(2.3e-05)	(8.0e-06)	(5.8e-07)	(84.09)	(733.36)	(2.8e-08)	(.000)

One possible explanation of the rejection of MODM is that linking returns and volume to the information flow through a bivariate conditional normal distribution may not be the correct specification even though a MODM is appropriate. In fact, Andersen (1996) modifies the standard MODM assuming that volume follows a conditional Poisson distribution instead of a normal and finds evidence in favour of the modified MODM.

Next, the modified MODM of Andersen (1996) is estimated to check whether a simple modification of the standard model fits the Spanish data.

For the modified MODM, the following system involving the daily returns, R_t , the volume, V_t , and the flow of information, K_t , is estimated by GMM (Hansen (1982)):

$$R_t|K_t \sim \mathcal{N}(\bar{r}, K_t) \quad (31)$$

$$V_t|K_t \sim \mathcal{P}o(m_0 + m_1 K_t) \quad (32)$$

Andersen (1996) detrends the volume series to get stationarity. However, the volume series used in this paper is stationary and hence, the conditional distribution of volume does not have the parameter c of equation (12).

To estimate the modified model the 10 moment conditions of the standard model (equations (21) to (30)) are used. However, as the conditional distribution of volume is Poisson

instead of normal, moment conditions involving volume vary in the following way¹⁰:

$$E[V_t] - (m_0 + m_1\bar{K}) = 0 \iff E[V_t] - \bar{V} = 0, \quad (33)$$

where $\bar{V} = (m_0 + m_1\bar{K})$

$$E[(V_t - \bar{V})^2] - \bar{V} - m_1^2 var(K_t) = 0 \quad (34)$$

$$E[R_t V_t] - \bar{r}\bar{V} = 0 \quad (35)$$

$$E[|R_t - \bar{r}|(V_t - \bar{V})] - (2/\pi)^{1/2} m_1 [E(K_t^{3/2}) - \bar{K}E(K_t^{1/2})] = 0 \quad (36)$$

$$E[R_t(V_t - \bar{V})^2] - \bar{r}[\bar{V} + m_1^2 var(K_t)] = 0 \quad (37)$$

Also, another moment condition is considered for estimation:

$$E[(R_t - \bar{r})^2 V_t] - \bar{K}\bar{V} - m_1 var(K_t) = 0 \quad (38)$$

The parameter vector is: $\theta = (\bar{r}, E(K_t^{1/2}), \bar{K}, E(K_t^{3/2}), var(K_t), m_0, m_1)$. There are 11 equations and 7 free parameters and thus, 4 overidentifying restrictions. As it was the case in the standard model, the covariance matrix, S_0 , is estimated according to Newey and West (1987) taking 4 lags.

Table 2 presents both the estimated parameters of the modified model with their corresponding standard errors and the value of the overidentifying statistic with the associated p-value. Although the estimated average returns are positive, the variance of the information flow is negative. Also, the value of the χ^2 statistic for the specification test is very large and hence, the data do not conform the modified MODM, result similar to that of the standard model.

¹⁰See appendix B.2 for the proof.

As Andersen (1996) explains, it is possible to assess the standard MODM within the modified MODM framework by testing the restriction $m_0 = 0$. This test has been conducted obtaining a test statistic of 48831903.48, which is asymptotically distributed as a chi-squared with 1 degree of freedom. Therefore, the null hypothesis of $m_0 = 0$ is soundly rejected for a 5% significance level and hence, the standard version of the MODM is rejected, as it was obtained previously.

Table 2: Estimation results for the modified MODM

\bar{r}	$E[K_t^{1/2}]$	\bar{K}	$E[K_t^{3/2}]$	$var(K_t)$	m_0	m_1	χ_4^2 (p-val.)	$m_0 = 0$ (p-val.)
0.035	0.428	3.068	2.635	-2.688	4.712	4.119	15970.6	48.8e+06
(5.2e-06)	(6.5e-05)	(.0002)	(.0002)	(.001)	(.0006)	(.0003)	(.000)	(.000)

Therefore, according to the evidence obtained in this paper, it can be concluded that although most of the parameters estimated in both the standard and modified MODM are significant, the null hypothesis that daily average returns and trading volume from the Spanish continuous stock market conform to the models is rejected.

5 Conclusions

This paper studies the joint distribution of daily returns and trading volume from the Spanish continuous stock market using the theoretical framework of the mixture of distributions model. According to this model, the dynamic features of the joint system are determined by a mixing random variable, that represents the information flow arriving at the market.

The model is estimated by GMM using daily average returns and volume from the Spanish continuous stock market for the period 19-04-1990 to 29-01-1996. The estimation method allows to test the model using the overidentified restrictions. Also, in spite of the fact that the information flow is an unobservable variable, the parameters estimated can provide information about it.

Apart from the test of the standard model of Tauchen and Pitts (1983), this paper considers the modified version of Andersen (1996), in which volume is generated by liquidity needs and informational asymmetries, and the conditional distribution of volume is Poisson instead of Normal.

As Richardson and Smith (1994) and Andersen (1996) find, the results of the tests show that the data do not conform to the standard MODM. However, in contrast to the results of Andersen (1996), there is also evidence against the modified version of the model. Thus, there is no evidence that the dynamics of Spanish returns and volume are directed by a common factor, namely the flow of information, according to the specifications of the standard and modified MODM.

Since the MODM is rejected, investors should not take their decisions about the choice of the optimal portfolio by considering a mixture of distributions model (in which case investors should consider moments and co-moments of two distributions), at least in the way it has been specified in this paper. Unfortunately, the rejection of the MODM does not imply the validity of the CAPM for the optimal decision of the investors (focused on the moments of the distribution of expected returns), and therefore, an analysis of the model is required to determine its validity for the Spanish market.

A Covariance between price change variability and trading volume

This appendix provides formal derivations of the expressions $Cov(\Delta P^2, V)$ and $Cov(\Delta P^2, V|K)$ of the bivariate mixture model of Tauchen and Pitts (1983).

The mixture model of Tauchen and Pitts (1983) considers the following distributions conditional on the flow of information, K :

$$\Delta P|K \sim \mathcal{N}(0, \sigma_1^2 K) \tag{39}$$

$$V|K \sim \mathcal{N}(\mu_2 K, \sigma_2^2 K) \tag{40}$$

The mixture model can also be written as:

$$\Delta P = \sigma_1 \sqrt{K} Z_1 \tag{41}$$

$$V = \mu_2 K + \sigma_2 \sqrt{K} Z_2 \tag{42}$$

where Z_1 and Z_2 are $\mathcal{N}(0, 1)$ random variables, and Z_1, Z_2 and K are mutually independent.

Denoting E the expectations operator, the following expressions are derived as:

$$\begin{aligned} Cov(\Delta P^2, V) &= E[\Delta P^2 V] - E[\Delta P^2]E[V] \\ &= \sigma_1^2 \mu_2 E[K^2] - \sigma_1^2 E[K]^2 \mu_2 \\ &= \sigma_1^2 \mu_2 [E[K^2] - E[K]^2] \\ &= \sigma_1^2 \mu_2 Var(K) \end{aligned}$$

On the other hand, conditioning on the flow of information, K :

$$\begin{aligned}
Cov(\Delta P^2, V|K) &= E[\Delta P^2 V|K] - E[\Delta P^2|K]E[V|K] \\
&= E\left[\sigma_1^2 K Z_1^2 (\mu_2 K + \sigma_2 \sqrt{K} Z_2)|K\right] - (\mu_2 K)(\sigma_1^2 K) \\
&= (\sigma_1^2 \mu_2 K^2 + \sigma_1^2 K \sigma_2 \sqrt{K} 0) - \sigma_1^2 \mu_2 K^2 \\
&= \sigma_1^2 \mu_2 K^2 - \sigma_1^2 \mu_2 K^2 = 0
\end{aligned}$$

B Moment conditions

This appendix provides formal derivations of the moment conditions derived from the standard and modified MODM.

B.1 Standard mixture of distributions model

Let E_K be the expectations operator over the distribution of K , and let $E_{R|K}$ and $E_{V|K}$ be the expectations operator over the conditional distributions of R and V , respectively, given K . The following moments are derived using the law of iterated expectations.

$$\begin{aligned}
E[R_t] &= E_K E_{R|K}[R_t] \\
&= E_K[\bar{r}] \\
&= \bar{r}
\end{aligned}$$

$$\begin{aligned}
E|R_t - \bar{r}| &= E_K E_{R|K}|R_t - \bar{r}| \\
&= E_K \int_{-\infty}^{\infty} |R_t - \bar{r}| \frac{1}{\sqrt{2\pi\sqrt{K_t}}} e^{-\frac{(R_t - \bar{r})^2}{2K_t}} dR_t \\
&= E_K \left[2 \frac{K_t}{\sqrt{2\pi\sqrt{K_t}}} \right]
\end{aligned}$$

$$\begin{aligned}
&= E_K \left[\frac{\sqrt{2}\sqrt{K_t}}{\sqrt{\pi}} \right] \\
&= (2/\pi)^{1/2} E[K_t^{1/2}]
\end{aligned}$$

$$\begin{aligned}
E[(R_t - \bar{r})^2] &= E_K E_{R|K} [(R_t - \bar{r})^2] \\
&= E_K [\text{var}(R_t)] = E_K[K_t] \\
&= \bar{K}
\end{aligned}$$

$$\begin{aligned}
E[|R_t - \bar{r}|^3] &= E_K E_{R|K} [|R_t - \bar{r}|^3] \\
&= E_K \int_{-\infty}^{\infty} |R_t - \bar{r}|^3 \frac{1}{\sqrt{2\pi}\sqrt{K_t}} e^{-\frac{(R_t - \bar{r})^2}{2K_t}} dR_t \\
&= E_K \left[2 \left(\frac{2K_t^2}{\sqrt{2\pi}\sqrt{K_t}} \right) \right] \\
&= 2(2/\pi)^{1/2} E[K_t^{3/2}]
\end{aligned}$$

$$\begin{aligned}
E[(R_t - \bar{r})^4] &= E_K E_{R|K} [(R_t - \bar{r})^4] \\
&= E_K \left[\frac{K^2 4!}{2^2 2!} \right] \\
&= 3E_K[K_t^2] \\
&= 3E_K [(K_t - \bar{K} + \bar{K})^2] \\
&= 3E_K [(K_t - \bar{K})^2 + \bar{K}^2 + 2(K_t - \bar{K})\bar{K}] \\
&= 3E_K [(K_t - \bar{K})^2 + \bar{K}^2] \\
&= 3E [\bar{K}^2 + \text{var}(K_t)]
\end{aligned}$$

$$\begin{aligned}
E[V_t] &= E_K E_{V|K}[V_t] \\
&= E_K[\mu_v K_t]
\end{aligned}$$

$$= \mu_v E_K[K_t]$$

$$= \mu_v \bar{K}$$

$$\begin{aligned} E[(V_t - \mu_v \bar{K})^2] &= E_K E_{V|K}[(V_t - \mu_v \bar{K})^2] \\ &= E_K E_{V|K}[(V_t - \mu_v K_t + \mu_v K_t - \mu_v \bar{K})^2] \\ &= E_K E_{V|K}(V_t - \mu_v K_t)^2 + E_K E_{V|K}(\mu_v K_t - \mu_v \bar{K})^2 \\ &\quad + 2E_K E_{V|K}(V_t - \mu_v K_t)(\mu_v K_t - \mu_v \bar{K}) \\ &= E_K [\text{var}(K_t)] + \mu_v^2 E_K(K_t - \bar{K})^2 \\ &= \sigma_v \bar{K} + \mu_v^2 \text{var}(K_t) \end{aligned}$$

$$\begin{aligned} E[R_t V_t] &= E_K [E_{R|K}(R_t) E_{V|K}(V_t)] \\ &= E_K [\bar{r} \mu_v K_t] \\ &= \bar{r} \mu_v \bar{K} \end{aligned}$$

$$\begin{aligned} E[|R_t - \bar{r}|(V_t - \mu_v \bar{K})] &= E_K [E_{R|K}|R_t - \bar{r}| E_{V|K}(V_t - \mu_v \bar{K})] \\ &= 0 \end{aligned}$$

$$\begin{aligned} E[R_t(V_t - \mu_v \bar{K})^2] &= E_K [E_{R|K}(R_t) E_{V|K}(V_t - \mu_v \bar{K})^2] \\ &= \bar{r} [E_K(\sigma_v K_t) + \mu_v^2 \text{var}(K_t)] \\ &= \bar{r} [\sigma_v \bar{K} + \mu_v^2 \text{var}(K_t)] \end{aligned}$$

B.2 Modified mixture of distributions model

The first 5 moment conditions used for the estimation of the modified MODM (equations (21) to (25)) refer to returns and have the same expressions to those used to estimate the

standard model because the conditional distribution of returns is the same in both models. Next, the formal derivations of the rest of the equations used to estimate the modified MODM are presented¹¹.

$$\begin{aligned}
E[V_t] &= E_K E_{V|K}(V_t) \\
&= E_K [m_0 + m_1 K_t] \\
&= m_0 + m_1 E_K(K_t) \\
&= m_0 + m_1 \bar{K} \\
&= \bar{V}
\end{aligned}$$

$$\begin{aligned}
E[(V_t - \bar{V})^2] &= E_K [E_{V|K}(V_t - \bar{V})^2] \\
&= E_K \left\{ E_{V|K} [V_t - (m_0 + m_1 K_t) + (m_0 + m_1 K_t) - \bar{V}]^2 \right\} \\
&= E_K \left\{ E_{V|K} [V_t - (m_0 + m_1 K_t) + (m_0 + m_1 K_t) - (m_0 + m_1 \bar{K})]^2 \right\} \\
&= E_K \left\{ E_{V|K} [V_t - (m_0 + m_1 K_t)]^2 + (m_1 K_t - m_1 \bar{K})^2 \right\} \\
&\quad + 2E_K \left\{ [V_t - (m_0 + m_1 K_t)] [(m_0 + m_1 K_t) - (m_0 + m_1 \bar{K})] \right\} \\
&= E_K \left\{ var(V_t|K_t) + E_{V|K} [m_1^2 (K_t - \bar{K})^2] \right\} \\
&= E_K [(m_0 + m_1 K_t) + m_1^2 (K_t - \bar{K})^2] \\
&= \bar{V} + m_1^2 var(K_t)
\end{aligned}$$

$$E[R_t V_t] = E_K [E_{R|K}(R_t) E_{V|K}(V_t)]$$

¹¹One of the equations used by Andersen (1996) for the estimation of the modified model is $E[|R_t - \bar{r}|(V_t - \bar{V})]$. However, there is an errata in his paper, since he derives $E[|R_t - \bar{r}|(V_t - \bar{V})] = c(2/\pi)^{1/2} m_1 [E(K_t^{3/2}) - E(K_t^{1/2})]$, which is not correct as it is proofed in this appendix.

$$\begin{aligned}
&= E_K [E_{R|K}(R_t) \cdot (m_0 + m_1 K_t)] \\
&= E_K [\bar{r}(m_0 + m_1 K_t)] \\
&= \bar{r}(m_0 + m_1 \bar{K}) \\
&= \bar{r}\bar{V}
\end{aligned}$$

$$\begin{aligned}
E [|R_t - \bar{r}|(V_t - \bar{V})] &= E_K [E_{R|K}|R_t - \bar{r}|E_{V|K}(V_t - \bar{V})] \\
&= E_K \{E_{R|K}|R_t - \bar{r}| \cdot \\
&\quad E_{V|K}[V_t - (m_0 + m_1 K_t) + (m_0 + m_1 K_t) - (m_0 + m_1 \bar{K})]\} \\
&= E_K \{E_{R|K}|R_t - \bar{r}|E_{V|K} [(V_t - (m_0 + m_1 K_t)) + m_1(K_t - \bar{K})]\} \\
&= E_K \{E_{R|K}|R_t - \bar{r}|E_{V|K}[m_1(K_t - \bar{K})]\} \\
&= E_K \{m_1(K_t - \bar{K})(2/\pi)^{1/2} K_t^{1/2}\} \\
&= (2/\pi)^{1/2} m_1 E_K [(K_t - \bar{K}) K_t^{1/2}] \\
&= (2/\pi)^{1/2} m_1 E_K (K_t^{3/2} - \bar{K} K_t^{1/2}) \\
&= (2/\pi)^{1/2} m_1 [E(K_t^{3/2}) - \bar{K} E(K_t^{1/2})]
\end{aligned}$$

$$\begin{aligned}
E [(R_t - \bar{r})^2 V_t] &= E_K [E_{R|K}(R_t - \bar{r})^2 E_{V|K}(V_t)] \\
&= E_K [(m_0 + m_1 K_t) E_{R|K}(R_t - \bar{r})^2] \\
&= E_K [(m_0 + m_1 K_t) K_t] \\
&= E_K (m_0 K_t + m_1 K_t^2) \\
&= m_0 \bar{K} + m_1 E_K (K_t^2) \\
&= m_0 \bar{K} + m_1 E_K [(K_t - \bar{K} + \bar{K})^2] \\
&= m_0 \bar{K} + m_1 E_K [(K_t - \bar{K})^2 + \bar{K}^2 + 2(K_t - \bar{K})\bar{K}]
\end{aligned}$$

$$\begin{aligned}
&= m_0 \bar{K} + m_1 [\text{var}(K_t) + \bar{K}^2] \\
&= \bar{K}(m_0 + m_1 \bar{K}) + m_1 \text{var}(K_t) \\
&= \bar{K}\bar{V} + m_1 \text{var}(K_t)
\end{aligned}$$

$$\begin{aligned}
E [R_t(V_t - \bar{V})^2] &= E_K [E_{R|K}(R_t)E_{V|K}(V_t - \bar{V})^2] \\
&= E_K \{ \bar{r} [(m_0 + m_1 K_t) + m_1^2 (K_t - \bar{K})^2] \} \\
&= \bar{r} [\bar{V} + m_1^2 \text{var}(K_t)]
\end{aligned}$$

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