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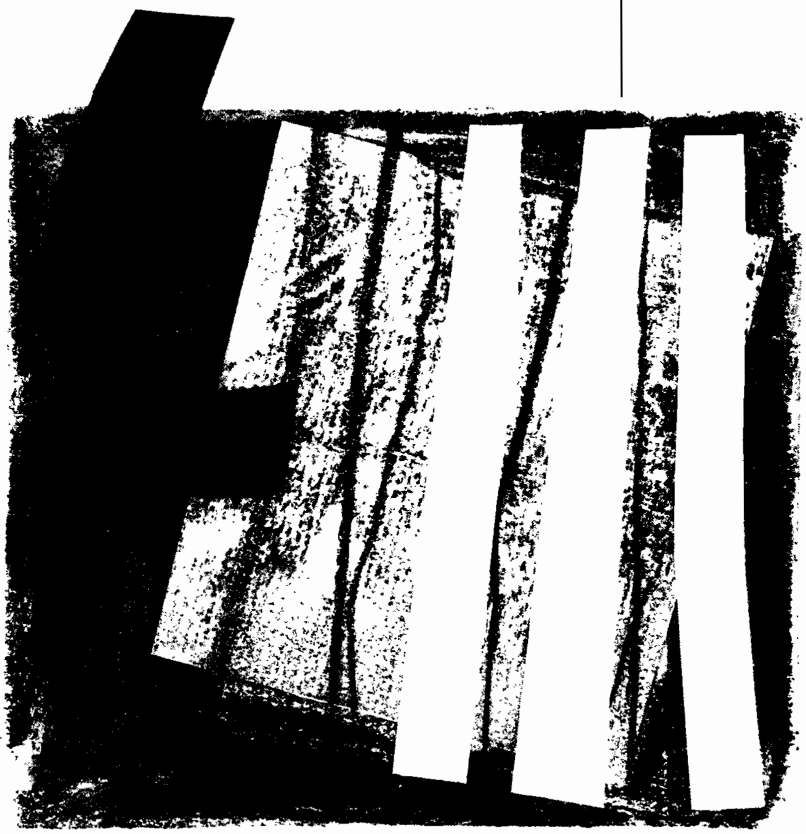

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PRESERVATION OF SOME STOCHASTIC ORDERS BY ORDER STATISTICS.

Rosa E. Lillo, Asok K. Nanda and Moshe Shaked*

Abstract

We show that the order statistics, in a sample from a distribution that has a logconcave density function, are ordered in the up shifted likelihood ratio order. We also show that the order statistics from two different collections of random variables are ordered in the up shifted likelihood ratio order or in the regular likelihood ratio order, if the underlying random variables are so ordered. Some results about the down shifted likelihood ratio order are also included in this paper. Finally it is indicated how the results can be applied in reliability theory.

Keywords: Likelihood ratio order; DFR; equilibrium rate; logconcave and logconvex densities; k-out-of-n reliability systems.

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1 Introduction

Let X be a random variable with an interval support, having the density, distribution and survival functions f, F and $\overline{F}$, respectively, and let $X_{(1:m)} \leq X_{(2:m)} \leq \cdots \leq X_{(m:m)}$ denote the order statistics from a sample of m independent random variables, all identically distributed as X. Raqab and Amin (1996) and Khaledi and Kochar (1999) proved that

$$X_{(i:m)} \leq_{\rm lr} X_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j, \tag{1.1}$$

where $\leq_{\rm lr}$ denotes the likelihood ratio order (the formal definitions of the stochastic orders that are mentioned in this section can be found in the sequel or in Shaked and Shanthikumar (1994)). Since the likelihood ratio order is stronger than the usual stochastic order $\leq_{\rm st}$, and than the hazard rate order $\leq_{\rm hr}$, it follows from (1.1) that

$$X_{(i:m)} \leq_{\rm st} X_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j,$$

and that

$$X_{(i:m)} \leq_{\rm hr} X_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j.$$

Khaledi and Kochar (2000) proved that if the survival function $\overline{F}$ is logconvex (that is, DFR (decreasing failure rate)) then

$$X_{(i:m)} \leq_{\rm disp} X_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j,$$

where $\leq_{\rm disp}$ denotes the dispersive order. In this paper we prove, among other things, that if f is logconcave then

$$X_{(i:m)} \leq_{\rm lr\uparrow} X_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j, \tag{1.2}$$

where $\leq_{\rm lr\uparrow}$ denotes the up shifted likelihood ratio order, formally defined in Section 2 below.

Now consider another random variable Y with an interval support, having the density, distribution and survival functions g, G and $\overline{G}$, respectively, and let $Y_{(1:n)} \leq Y_{(2:n)} \leq \cdots \leq Y_{(n:n)}$ denote the order statistics from a sample of n independent random variables, all identically distributed as Y. Khaledi and Kochar (1999) proved that

$$X \leq_{\rm st} Y \Longrightarrow X_{(i:m)} \leq_{\rm st} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j.$$

Khaledi and Kochar (2000) proved that if the survival functions $\overline{F}$ or $\overline{G}$ are logconvex (that is, DFR) then

$$X \leq_{\rm disp} Y \Longrightarrow X_{(i:m)} \leq_{\rm disp} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j.$$

As a corollary they obtained that if the survival functions $\overline{F}$ or $\overline{G}$ are logconvex then

$$X \leq_{\rm hr} Y \Longrightarrow X_{(i:m)} \leq_{\rm disp} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j.$$

In this paper we show, among other things, that

$$X \leq_{\rm lr\uparrow} Y \Longrightarrow X_{(i:m)} \leq_{\rm lr\uparrow} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j. \tag{1.3}$$

As a corollary we also obtain

$$X \leq_{\rm lr} Y \Longrightarrow X_{(i:m)} \leq_{\rm lr} Y_{(j:n)} \qquad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j. \tag{1.4}$$

The results (1.3) and (1.4) are also extended in this paper to the case in which the X_i's and the Y_j's are not identically distributed.

Below, 'increasing' and 'decreasing' mean 'nondecreasing' and 'nonincreasing,' respectively.

2 The Shifted Likelihood Ratio Orders

First let us recall the definition of the likelihood ratio order when the compared random variables have interval supports (possibly infinite) that need not be identical. Let X and Y be two absolutely continuous random variables, each with an interval support. Let l_X and u_X be the left and the right endpoints of the support of X. Similarly define l_Y and u_Y. The values l_X, u_X, l_Y and u_Y may be infinite. Let f and g denote the density functions of X and Y, respectively.

Definition 2.1. Let X and Y be two absolutely continuous random variables as above. We say that X is smaller than Y in the *likelihood ratio order*, denoted as $X \leq_{\rm lr} Y$, if

$$\frac{g(t)}{f(t)} \quad \text{is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y). \tag{2.1}$$

Note that in (2.1), when $u_X < u_Y$, we use the convention $a/0 = \infty$ when $a > 0$. In particular, it is seen that if $u_X < l_Y$ then $X \leq_{\rm lr} Y$.

Shanthikumar and Yao (1986a) have introduced and studied an order which they called the shifted likelihood ratio order. The definition below is slightly more general than the definition of Shanthikumar and Yao (1986a) who considered only nonnegative random variables.

Definition 2.2. Let X and Y be two absolutely continuous random variables as above. We say that X is smaller than Y in the *up shifted likelihood ratio order*, denoted as $X \leq_{\rm lr\uparrow} Y$, if

$$X - x \leq_{\rm lr} Y \quad \text{for each } x \geq 0. \tag{2.2}$$

Rewriting (2.2) using (2.1) we obtain the following result. Below f and g denote the density functions of X and Y, respectively.

Proposition 2.3. *Let X and Y be two absolutely continuous random variables as above. Then $X \leq_{\rm lr\uparrow} Y$ if, and only if, for each $x \geq 0$ we have*

$$\frac{g(t)}{f(t+x)} \quad \textit{is increasing in } t \in (l_X - x, u_X - x) \cup (l_Y, u_Y). \tag{2.3}$$

Recall that a density function f is said to be *logconcave* if its support $\{t \mid f(t) > 0\}$ is an interval, with endpoints $l < u$, say, and if $\log f$ is concave on (l, u); that is, if for any $x \in (0, u-l)$ we have that

$$\frac{f(t)}{f(t+x)} \quad \text{is increasing in } t \in (l, u-x). \tag{2.4}$$

Logconcavity, as defined above, can be interpreted as logconcavity over the whole real line with the convention $\log 0 = -\infty$. When f is a density function of a nonnegative random variable, then logconcavity can be interpreted as a positive aging notion (see, for example, Shaked and Shanthikumar (1987)).

Proposition 2.4. *Let X be an absolutely continuous random variable as above. Then $X \le_{\rm lr\uparrow} X$ if, and only if, f is logconcave on $(-\infty, \infty)$.*

Proof. If $X \le_{\rm lr\uparrow} X$ then by (2.3), for any $x \ge 0$ we have that

$$\frac{f(t)}{f(t+x)} \quad \text{is increasing in } t \in (l_X - x, u_X - x) \cup (l_X, u_X).$$

In particular, for any $x \in (0, u_X - l_X)$ we get (2.4).

Conversely, suppose that f is logconcave. If $x \in (0, u_X - l_X]$ then, by (2.4), we have that (2.3) holds (because then $(l_X - x, u_X - x) \cup (l_X, u_X)$ is the union of the three disjoint intervals $(l_X - x, l_X) \cup [l_X, u_X - x] \cup (u_X - x, u_X)$, and the ratio $f(t)/f(t+x)$ is equal 0 on the first interval, is increasing in t on the second interval, and is equal ∞ on the third interval). If $x > u_X - l_X$ then $(l_X - x, u_X - x)$ and (l_X, u_X) are disjoint, and the ratio $f(t)/f(t+x)$ is equal 0 on the first interval, and is equal ∞ on the second interval; so (2.3) holds in this case too. □

An important property of the up shifted likelihood ratio order is stated next. Shanthikumar and Yao (1986a) proved it for nonnegative random variables, but it is true in general as follows.

Proposition 2.5. *Let (X_1, X_2) and (Y_1, Y_2) be two pairs of independent absolutely continuous random variables. If $X_i \le_{\rm lr\uparrow} Y_i$, $i = 1, 2$, then $X_1 + X_2 \le_{\rm lr\uparrow} Y_1 + Y_2$.*

In the sequel we will also touch upon another stochastic order which is given in the following definition.

Definition 2.6. Let X and Y be two absolutely continuous random variables with support $[0, \infty)$. We say that X is smaller than Y in the *down shifted likelihood ratio order*, denoted as $X \le_{\rm lr\downarrow} Y$, if

$$X \le_{\rm lr} [Y - x \,|\, Y > x] \quad \text{for all } x \ge 0. \tag{2.5}$$

Note that in the above definition we compare only nonnegative random variables. This is because for the down shifted likelihood ratio order we cannot take an analog of (2.2), such as, $X \le_{\rm lr} Y - x$, as a definition. The reason is that here, by taking x very large, it is seen that practically there are no random variables that satisfy such an order relation. Note that in the definition above, the right hand side $[Y - x \,|\, Y > x]$ can take on (when x varies) any value in the right neighborhood of 0. Therefore we restricted the support of the compared random variables to $[0, \infty)$.

Let f and g denote the density functions of X and Y, respectively. Also, let $\overline{G}$ denote the survival function of Y. For $x \ge 0$, the density function of $[Y - x \,|\, Y > x]$ is $g(\cdot + x)/\overline{G}(x)$

on $[0, \infty)$, and is 0 otherwise. Thus, for nonnegative random variables X and Y as above, the analog of (2.3) is

$$X \leq_{\mathrm{lr}\downarrow} Y \iff \frac{g(t+x)}{f(t)} \text{ is increasing in } t \geq 0 \text{ for all } x \geq 0. \tag{2.6}$$

Recall that a density function f with support $\{t | f(t) > 0\} = [0, \infty)$ is said to be *logconvex* if $\log f$ is convex on $[0, \infty)$; that is, if for any $x > 0$ we have that

$$\frac{f(t)}{f(t+x)} \quad \text{is decreasing in } t \geq 0. \tag{2.7}$$

Using (2.6) and (2.7) we obtain at once the following result.

Proposition 2.7. *Let X be a nonnegative absolutely continuous random variable as above. Then $X \leq_{\mathrm{lr}\downarrow} X$ if, and only if, f is logconvex on $[0, \infty)$.*

3 Comparisons of Order Statistics

In this section we first obtain some results that compare, in the likelihood ratio and in the shifted likelihood ratio orders, order statistics from two different samples.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_m$ be m independent random variables, and let $Y_1, Y_2, \ldots, Y_n$ be other n independent random variables, all having absolutely continuous distributions.*
(a) *If $X_i \leq_{\mathrm{lr}\uparrow} Y_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, then*

$$X_{(i:m)} \leq_{\mathrm{lr}\uparrow} Y_{(j:n)} \qquad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j.$$

(b) *If $X_i \leq_{\mathrm{lr}} Y_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, then*

$$X_{(i:m)} \leq_{\mathrm{lr}} Y_{(j:n)} \qquad \text{whenever} \quad i \leq j \quad \text{and} \quad m-i \geq n-j.$$

Proof. In this proof we use an idea of Chan, Proschan and Sethuraman (1990) which is also described in the proof of Theorem 1.C.9 in Shaked and Shanthikumar (1994).

Let f_i, F_i and $\overline{F}_i$ denote the density, distribution, and survival functions of X_i. Similarly, let g_j, G_j and $\overline{G}_j$ denote the density, distribution, and survival functions of Y_j. The density functions of $X_{(i:m)}$ and $Y_{(j:n)}$ are given by

$$f_{X_{(i:m)}}(t) = \sum_{\boldsymbol{\pi}} f_{\pi_1}(t) F_{\pi_2}(t) \cdots F_{\pi_i}(t) \overline{F}_{\pi_{i+1}}(t) \cdots \overline{F}_{\pi_m}(t),$$

and

$$g_{Y_{(j:n)}}(t) = \sum_{\boldsymbol{\sigma}} g_{\sigma_1}(t) G_{\sigma_2}(t) \cdots G_{\sigma_j}(t) \overline{G}_{\sigma_{j+1}}(t) \cdots \overline{G}_{\sigma_n}(t),$$

where $\sum_{\boldsymbol{\pi}}$ signifies the sum over all permutations $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ of $(1, 2, \ldots, m)$, and $\sum_{\boldsymbol{\sigma}}$ similarly denotes the sum over all permutations $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$.

In order to prove (a), fix an $x \geq 0$, and write

$$\frac{g_{Y_{(j:n)}}(t)}{f_{X_{(i:m)}}(t+x)} = \frac{\sum_\sigma g_{\sigma_1}(t)G_{\sigma_2}(t)\cdots G_{\sigma_j}(t)\overline{G}_{\sigma_{j+1}}(t)\cdots\overline{G}_{\sigma_n}(t)}{\sum_\pi f_{\pi_1}(t+x)F_{\pi_2}(t+x)\cdots F_{\pi_i}(t+x)\overline{F}_{\pi_{i+1}}(t+x)\cdots\overline{F}_{\pi_m}(t+x)}. \tag{3.1}$$

Now, for any choice of a permutation π of $(1,2,\ldots,m)$ and a permutation σ of $(1,2,\ldots,n)$ we have

$$\frac{g_{\sigma_1}(t)G_{\sigma_2}(t)\cdots G_{\sigma_j}(t)\overline{G}_{\sigma_{j+1}}(t)\cdots\overline{G}_{\sigma_n}(t)}{f_{\pi_1}(t+x)F_{\pi_2}(t+x)\cdots F_{\pi_i}(t+x)\overline{F}_{\pi_{i+1}}(t+x)\cdots\overline{F}_{\pi_m}(t+x)}$$
$$= \frac{g_{\sigma_1}(t)}{f_{\pi_1}(t+x)}\cdot\frac{G_{\sigma_2}(t)\cdots G_{\sigma_i}(t)}{F_{\pi_2}(t+x)\cdots F_{\pi_i}(t+x)}\cdot\frac{\overline{G}_{\sigma_{j+1}}(t)\cdots\overline{G}_{\sigma_n}(t)}{\overline{F}_{\pi_{m-n+j+1}}(t+x)\cdots\overline{F}_{\pi_m}(t+x)}\cdot\frac{G_{\sigma_{i+1}}(t)\cdots G_{\sigma_j}(t)}{\overline{F}_{\pi_{i+1}}(t+x)\cdots\overline{F}_{\pi_{m-n+j}}(t+x)}.$$

Since $X_{\pi_1} \leq_{\text{lr}\uparrow} Y_{\sigma_1}$ we see from (2.3) that the first fraction above is increasing in t. From $X_{\pi_k} \leq_{\text{lr}\uparrow} Y_{\sigma_k}$ and (2.2) it follows that $X_{\pi_k} - x \leq_{\text{rh}} Y_{\sigma_k}$, where $\leq_{\text{rh}}$ denotes the reversed hazard rate order; but that means that $G_{\sigma_k}(t)/F_{\pi_k}(t+x)$ is increasing in t, $k = 2,\ldots,i$, and therefore the second fraction above is increasing in t. From $X_{\pi_{k+m-n}} \leq_{\text{lr}\uparrow} Y_{\sigma_k}$ and (2.2) it also follows that $X_{\pi_{k+m-n}} - x \leq_{\text{hr}} Y_{\sigma_k}$, where $\leq_{\text{hr}}$ denotes the hazard rate order; but that means that $\overline{G}_{\sigma_k}(t)/\overline{F}_{\pi_{k+m-n}}(t+x)$ is increasing in t, $k = j+1,\ldots,n$, and therefore the third fraction above is increasing in t. The fourth fraction above obviously increases in t too, and thus the whole product increases in t.

Note that if $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ are all nonnegative univariate functions, such that $a_i(t)/b_j(t)$ is increasing in t for all $1 \leq i \leq m$ and $1 \leq j \leq n$, then $\sum_{i=1}^m a_i(t)/\sum_{j=1}^n b_j(t)$ is also increasing in t. It follows from this fact, and from (3.1), that $g_{Y_{(j:n)}}(t)/f_{X_{(i:m)}}(t+x)$ is increasing in t, and from (2.3) we obtain (a).

The proof of (b) is similar — just take $x = 0$ in the above argument. □

One may wonder whether an analog of Theorem 3.1(a) exists for the down likelihood ratio order; that is, whether $X_i \leq_{\text{lr}\downarrow} Y_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ imply that $X_{(i:m)} \leq_{\text{lr}\downarrow} Y_{(j:n)}$ for i, j, n and m as in Theorem 3.1(a). It will be shown after Theorem 3.5 below that this is not the case. However, the next result shows that the first order statistics can be compared in the order $\leq_{\text{lr}\downarrow}$ under the above conditions.

Theorem 3.2. *Let $X_1, X_2, \ldots, X_m$ be m independent random variables, and let $Y_1, Y_2, \ldots,$ Y_n be other n independent random variables, all having absolutely continuous distributions with support $[0,\infty)$. If $X_i \leq_{\text{lr}\downarrow} Y_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$, then*

$$X_{(1:m)} \leq_{\text{lr}\downarrow} Y_{(1:n)} \qquad \textit{whenever} \quad m \geq n.$$

Proof. Fix an $x \geq 0$. Using the notation of the proof of Theorem 3.1 we write

$$\frac{g_{Y_{(1:n)}}(t+x)}{f_{X_{(1:m)}}(t)} = \frac{\sum_\sigma g_{\sigma_1}(t+x)\overline{G}_{\sigma_2}(t+x)\cdots\overline{G}_{\sigma_n}(t+x)}{\sum_\pi f_{\pi_1}(t)\overline{F}_{\pi_2}(t)\cdots\overline{F}_{\pi_m}(t)}.$$

Now, for any choice of a permutation π of $(1,2,\ldots,m)$ and a permutation σ of $(1,2,\ldots,n)$ we have

$$\frac{g_{\sigma_1}(t+x)\overline{G}_{\sigma_2}(t+x)\cdots\overline{G}_{\sigma_n}(t+x)}{f_{\pi_1}(t)\overline{F}_{\pi_2}(t)\cdots\overline{F}_{\pi_m}(t)}$$
$$= \frac{g_{\sigma_1}(t+x)}{f_{\pi_1}(t)}\cdot\frac{\overline{G}_{\sigma_2}(t+x)\cdots\overline{G}_{\sigma_n}(t+x)}{\overline{F}_{\pi_2}(t)\cdots\overline{F}_{\pi_n}(t)}\cdot\frac{1}{\overline{F}_{\pi_{n+1}}(t)\cdots\overline{F}_{\pi_m}(t)}.$$

Since $X_{\pi_1} \leq_{\rm lr\downarrow} Y_{\sigma_1}$ we see from (2.6) that the first fraction above is increasing in t. From $X_{\pi_k} \leq_{\rm lr\downarrow} Y_{\sigma_k}$ and (2.5) it follows that $X \leq_{\rm hr} [Y - x|Y > x]$; but that means that $\overline{G}_{\sigma_k}(t + x)/\overline{F}_{\pi_k}(t)$ is increasing in t, $k = 2, \ldots, n$, and therefore the second fraction above is increasing in t. The third fraction above obviously increases in t too, and thus the whole product increases in t. To end the proof use the final argument in the proof of Theorem 3.1 to conclude that $g_{Y_{(1:n)}}(t + x)/f_{X_{(1:m)}}(t)$ is increasing in t and use (2.6). □

As a corollary of Theorem 3.1 we will now obtain the results stated in (1.3) and (1.4) as described below. Recall the notation used in Section 1; that is, let below $X_{(1:m)} \leq X_{(2:m)} \leq \cdots \leq X_{(m:m)}$ be the order statistics from a sample of m independent random variables, all identically distributed as some X, and let $Y_{(1:n)} \leq Y_{(2:n)} \leq \cdots \leq Y_{(n:n)}$ be the order statistics from a sample of n independent random variables, all identically distributed as some Y. By letting all the X_i's in Theorem 3.1 be distributed as X, and all the Y_j's be distributed as Y, we get the following results.

Corollary 3.3. *Let X and Y be two absolutely continuous random variables with interval supports. Then*

$$X \leq_{\rm lr\uparrow} Y \Longrightarrow X_{(i:m)} \leq_{\rm lr\uparrow} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m - i \geq n - j, \tag{3.2}$$

and

$$X \leq_{\rm lr} Y \Longrightarrow X_{(i:m)} \leq_{\rm lr} Y_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m - i \geq n - j. \tag{3.3}$$

Similarly we have the following corollary of Theorem 3.2.

Corollary 3.4. *Let X and Y be two absolutely continuous random variables with support $[0, \infty)$. Then*

$$X \leq_{\rm lr\downarrow} Y \Longrightarrow X_{(1:m)} \leq_{\rm lr\downarrow} Y_{(1:n)} \quad \text{whenever} \quad m \geq n.$$

Taking $X =_{\rm st} Y$ in Corollaries 3.3 and 3.4 we obtain the following results (in particular, (1.2)) from Propositions 2.4 and 2.7.

Theorem 3.5. (a) *Let X be an absolutely continuous random variable with an interval support. If X has a logconcave density function then*

$$X_{(i:m)} \leq_{\rm lr\uparrow} X_{(j:n)} \quad \text{whenever} \quad i \leq j \quad \text{and} \quad m - i \geq n - j. \tag{3.4}$$

(b) *Let X be an absolutely continuous random variable with support $[0, \infty)$. If X has a logconvex density function then*

$$X_{(1:m)} \leq_{\rm lr\downarrow} X_{(1:n)} \quad \text{whenever} \quad m \geq n. \tag{3.5}$$

It is not true that if a nonnegative random variable X has a logconvex density on $[0, \infty)$ then $X_{(i:m)} \leq_{\rm lr\downarrow} X_{(j:n)}$ for i, j, n and m as in (3.4). For example, let X be an exponential random variable. Then X has a logconvex density on $[0, \infty)$. If the inequality above were true, then we would have obtained, for instance, that $X_{(2:2)} \leq_{\rm lr\downarrow} X_{(2:2)}$; which, by Proposition 2.7, would have implied that $X_{(2:2)}$ had a logconvex density — a contradiction to the fact that $X_{(2:2)}$ is strictly IFR (increasing failure rate). This discussion also shows that it is not possible to replace $\leq_{\mathrm{lr}\uparrow}$ by $\leq_{\mathrm{lr}\downarrow}$ in (3.2).

One may wonder whether the conditions that X has a logconcave or logconvex density are necessary for the conclusions of Theorem 3.5. The following counterexample shows that this is indeed the case — it shows that for any fixed i, j, n and m such that

$$i \leq j \quad \text{and} \quad m - i \geq n - j, \tag{3.6}$$

there exists a random variable X whose density is not logconcave such that the inequality in (3.4) does not hold. The following counterexample also shows that for any fixed m and n such that $m \geq n$, there exists a random variable whose density is not logconvex such the inequality in (3.5) does not hold.

Counterexample 3.6. Consider the random variable X with density function

$$f(x) = \begin{cases} \frac{2}{3}x, & 0 \leq x \leq 1; \\ \frac{2}{3x^2}, & x > 1. \end{cases}$$

It is easy to verify that f is neither logconcave nor logconvex. Let i, j, n and m satisfy (3.6). For $x > 0$ and $t > 1$ note that

$$h(t) \equiv \frac{f_{X_{(j:n)}}(t)}{f_{X_{(i:m)}}(t+x)} = C\frac{(t+x)^{m+1}(3t-2)^{j-1}}{t^{n+1}(3t+3x-2)^{i-1}},$$

where C is a positive constant. Thus

$$\frac{d}{dt}\log h(t) = \frac{m+1}{t+x} + \frac{3j-3}{3t-2} - \frac{n+1}{t} - \frac{3i-3}{3t+3x-2}.$$

If x is very large, then the first and the fourth fractions above are negligible. The remaining difference is negative when $t > \frac{2n+2}{3(n+2-j)}$. Thus h decreases on some interval of $[0, \infty)$, and therefore, by (2.3), the inequality in (3.4) does not hold.

On the other hand, for $0 < x < 1$ and $0 \leq t \leq 1 - x$ we have

$$\tilde{h}(t) \equiv \frac{f_{X_{(1:n)}}(t+x)}{f_{X_{(1:m)}}(t)} = C'\frac{(t+x)(3-(x+t)^2)^{n-1}}{t(3-t^2)^{m-1}},$$

where C' is a positive constant. Thus

$$\frac{d}{dt}\log \tilde{h}(t) = \frac{1}{t+x} - \frac{2(n-1)(x+t)}{3-(t+x)^2} - \frac{1}{t} + \frac{2(m-1)t}{3-t^2}.$$

Let $t \to 0$. Then the third fraction tends to ∞, and therefore it dominates the sign of the above expression which is negative in some right neighborhood of 0. Thus $\tilde{h}$ decreases there, and therefore, by (2.6), the inequality in (3.5) does not hold. ◀

The shifted likelihood ratio orders can often be easily identified or derived. Suppose that the random variables X and Y have differentiable densities f and g, respectively. Define the transforms $k_X \equiv f'/f$ and $k_Y \equiv g'/g$; these are continuous analogs of the discrete equilibrium rates studied in Shanthikumar and Yao (1986a, 1986b). From (2.1) it is seen that

$$X \leq_{\text{lr}} Y \Longleftrightarrow k_X(t) \leq k_Y(t) \quad \text{for all } t \in (l_Y, u_X).$$

Similarly,

$$X \leq_{\text{lr}\uparrow} Y \Longleftrightarrow k_X(t') \leq k_Y(t) \quad \text{whenever } l_Y \leq t \leq t' \leq u_X; \tag{3.7}$$

and for random variables with supports $[0, \infty)$ we have

$$X \leq_{\text{lr}\downarrow} Y \Longleftrightarrow k_X(t) \leq k_Y(t') \quad \text{whenever } t' \geq t \geq 0.$$

The easy identification of the shifted likelihood ratio orders, combined with the fact that order statistics are lifetimes of k-out-of-n systems, yield interesting bounds in reliability theory. As an example, consider a k-out-of-n reliability system with independent identically distributed component lifetimes $X_1, X_2, \ldots, X_n$. The lifetime of the system then is $X_{(n-k+1:n)}$. Suppose that another such system, with component lifetimes $X_1', X_2', \ldots, X_n'$ that are distributed as the X_i's, is used as a cold standby; that is, it replaces the original system when it fails. Thus the total lifetime of the combined system is $X_{(n-k+1:n)} + X'_{(n-k+1:n)}$, where $X_{(n-k+1:n)}$ and $X'_{(n-k+1:n)}$ are identically distributed and independent. Denote by X the lifetime of a generic component; that is, $X =_{\text{st}} X_i =_{\text{st}} X_i'$, $i = 1, 2, \ldots, n$. Suppose that the density f of X is not completely known or that it is complicated, but that it can be bounded from above, in the sense of $\leq_{\text{lr}\uparrow}$, by a lifetime Y that has a simple distribution. For example, if the distribution of X is only known to have a transform $k \equiv f'/f$ such that $\sup\{k(t), t \geq 0\} \leq -\lambda$ for some known $\lambda > 0$, then Y can be taken to be an exponential random variable with hazard rate λ. From Corollary 3.3 and (3.7) we then get that $X_{(n-k+1:n)} \leq_{\text{lr}\uparrow} Y_{(n-k+1:n)}$, and Proposition 2.5 then yields

$$X_{(n-k+1:n)} + X'_{(n-k+1:n)} \leq_{\text{lr}\uparrow} Y_{(n-k+1:n)} + Y'_{(n-k+1:n)},$$

where the notation $Y'_{(n-k+1:n)}$ is self-explanatory. Various probabilistic quantities of interest in reliability theory (such as the survival function) can be computed, at least numerically, for $Y_{(n-k+1:n)} + Y'_{(n-k+1:n)}$, and we thus obtain upper bounds on the respective quantities that are associated with the lifetime $X_{(n-k+1:n)} + X'_{(n-k+1:n)}$.

Of course, bounds as above can also be obtained when the lifetimes X_i are not identically distributed (use Theorem 3.1 with identically distributed Y_j's), or when more information about the transform k of X is known (so that the transform of Y can be taken to be, say, some step function, rather than a constant).

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