FRACTIONAL COINTEGRATING REGRESSIONS IN THE PRESENCE OF LINEAR TIME TRENDS

Uwe Hassler and Francesc Marmol

98-12

WORKING PAPERS

Universidad Carlos III de Madrid

Working Paper 98-12
Statistics and Econometrics Series 06
January 1998

Departamento de Estadística y Econometría
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-9849

FRACTIONAL COINTEGRATING REGRESSIONS IN THE PRESENCE OF LINEAR TIME TRENDS

Uwe Hassler and Francesc Marmol*

Abstract

We consider regressions of nonstationary fractionally integrated variables dominated by linear time trends. The regression errors are short memory, long memory or even nonstationary, and hence allow for a very flexible cointegration model. In case of simple regressions, least squares estimation gives rise to limiting normal distribucions independently of the order of integration of the regressor, whereas the customary t-statistics diverge. We also investigate the possibility of testing for mean reverting equilibrium deviations by means of a residual-based log-periodogram regression. Asymptotic results become more complicated in the multivariate case.

Key Words

Nonstationary regressors; stationary or nonstationary errors; limiting normality; residual-based cointegration testing.

*Hassler, Institute of Statistics and Econometrics, Free University of Berlin; Mármol, Departamento de Estadística y Econometría, Universidad Carlos III de Madrid, e-mail: fmarmol@est-econ.es. J.E.L. Classification: C15, C22, C32

FRACTIONAL COINTEGRATING REGRESSIONS IN THE PRESENCE OF LINEAR TIME TRENDS

Uwe Hassler

Institute of Statistics and Econometrics, Free University of Berlin

and

Francesc Marmol

Department of Statistics and Econometrics, Universidad Carlos III *of Madrid*

This Version: January 1998

Abstract

We consider regressions of nonstationary fractionally integrated variables dominated by linear time trends. The regression errors are short memory, long memory or even nonstationary, and hence allow for a very flexible cointegration model. In case of simple regressions, least squares estimation gives rise to limiting normal distributions independently of the order of integration of the regressor, whereas the customary t-statistics diverge. We also investigate the possibility of testing for mean reverting equilibrium deviations by means of a residual-based log-periodogram regression. Asymptotic results become more complicated in the multivariate case.

Keywords: Nonstationary regressors; stationary or nonstationary errors; limiting normality; residual-based cointegration testing.

J.E.L. Classification: C15, C22, C32.

I. INTRODUCTION

In his seminal paper, Granger (1981) introduced both the concept of cointegration and of fractional integration to econometrics. Since then, cointegration has revolutionized time series econometrics, as is well documented through textbooks such as e.g., Banerjee et al. (1993). Fractionally integrated time series models have become popular with economic data, too. Successful application include Diebold and Rudebusch (1991), Sowell (1992), Cheung and Lai (1993), Crato and Rothman (1994) and Hassler and Wolters (1995). See Baillie (1996) for a recent overview. Empirical evidence hence ranges from consumption over *GNP* to prize indexes. These three examples are not only driven by a possibly fractional stochastic trend, but certainly also by a deterministic trend often approximated as linear. Therefore, the title of the present paper suggests itself.

We consider the regression model $y_t = \alpha + \beta' x_t + z_t$. To our knowledge, not many related results have been published in the fractional context assuming stochastic regressors. An early exception is the work by Cheung and Lai (1993) with an application and some heuristic theory. Very recently, Robinson and Hidalgo (1997) covered the case where x_t and z_t are both stationary fractionally integrated and independent of each other. Our assumptions are very different. The error term is fractionally integrated of order d_z and may be correlated with the regressor The latter is nonstationary fractionally integrated and moreover driven by a linear time trend. The assumptions allow for very flexible cointegration models where the equilibrium error z_t is *i*) short memory $(d_z = 0)$, *ii*) long memory but stationary $(0 < d_z < 0.5)$, *iii*) nonstationary but mean reverting $(0.5 < d_z < 1)$, *c.f.*, Cheung and Lai (1993) for a discussion, or *iv*) nonstationary and not mean reverting $(d_z \geq 1)$. Our results even hold if d_z equals or exceeds the order of integration of the regressors, which gives rise to nonsense or spurious regression.

This paper does not consider detrended regressions because it is often believed in econometrics that "*the cointegrating vector will annihilate both the stochastic trend and the deterministic trend*" (Watson, 1994, p. 2895). In case of simple regressions we establish limiting normality of the appropriately normalized least squares estimator, thus generalizing the prominent result by West (1988). The limits as well as the rate of convergence are independent of the degree of integration of the regressor, which is dominated by its linear trend. On the other hand, *t*-statistics diverge with the rate depending on the (non)stationarity of the error terms. We propose a residual-based test applying the log-periodogram regression in order to decide whether the error term is mean reverting or not. In multivariate regressions, things get more complicated because the asymptotic results are influenced by the order of integration of both the regressors and the error In particular, asymptotic normality no longer holds.

This paper is organized as follows. The next Section becomes precise on the model and the underlying assumptions. In Section III asymptotic results are presented for simple regressions. Section IV treats the multivariate case. Section V provides Monte Carlo evidence on the possibility of residual-based cointegration testing. Concluding remarks are collected in Section VI Proofs are mainly relegated to the Appendix.

II. THE MODEL AND UNDERLYING ASSUMPTIONS

Let us start by considering a simple regression model

(1) $y_t = \alpha + \beta x_t + z_t$, $\beta \neq 0$, $t = 1,\ldots,T$,

where

(2) $x_t = \gamma + \mu t + x_t^0$, $\mu \neq 0$, $\Delta^{d_x} x_t^0 = u_{xt}$, $0.5 < d_x < 1.5$,

(3) $\Delta^{d_z} z_t = u_{zt}$, $0 \leq d_z < 1.5$, $d_z \neq 0.5$,

(4) $d_x > d_z$.

Without loss of generality, we shall assume that $u_{it} = 0$, $i \in \{x, z\}$, for $t \leq 0$. Given the data generating process (henceforth denoted *DGP*) (1)-(3), condition (4) implies that $(y_t, x_t)'$ are (fractionally) cointegrated, allowing for stationary as well as nonstationary fractionally integrated innovation terms. This condition, in turn, justifies the cointegration title of the paper, but it is not needed in the derivations of the results in Sections 3 and 4. The provided distributional theory also covers the case of nonsense regressions in the presence of linear trends. Effectively, assume that

$$y_t = a + bt + y_t^0,$$

and

$$x_t = \gamma + \mu t + x_t^0,$$

have independent stochastic components, where d_y from $y_t^0 \propto I(d_y)$ must not be smaller than d_x. In this case, although the series are stochastically independent, model (1) shows up if $b = \beta\mu$ and $z_t = y_t^0 - \beta x_t^0$, $d_z = max\{d_x, d_y\}$.

With respect to the innovation vector $u_t = (u_{xt}, u_{zt})'$ driving the stochastic processes, we shall assume that it satisfies the following general characterization.

<u>Assumption 1.</u> *Let* $u_t = (u_{xt}, u_{zt})'$ *be generated by the linear process*

$$(5) \qquad u_t = \sum_{j=0}^{\infty} C_j v_{t-j} \, ,$$

where the sequence of random vectors $v_t = (v_{xt}, v_{zt})'$ *is i.i.d.* $(0, \Sigma)$ *with* $\Sigma > 0$ *and the sequence of matrix coefficients* $\left\{C_j\right\}_{j=0}^{\infty}$ *is* 1-*summable in the sense of Brillinger* (1981). *Further, assume that* $max_{i \; \{x,z\}} \sup_t E|v_{it}|^{\tau} < \infty$, *for* $\tau \geq max\left\{4, (8-8d)/(2d-1)\right\}$, *with* $d = \max\left\{d_x, d_z\right\} < 1.5$. [1]

Hence, throughout this paper, we shall allow u_t be generated by the linear process (5). This general class of stationary $I(0)$ processes includes all stationary and invertible *ARMA* processes and is therefore of wide applicability. On the other hand, under this assumption, the process u_t is strictly stationary and ergodic with continuous spectral density given by

$$f_{uu}(\lambda) = \frac{1}{2\pi}\left(\sum_{j=0}^{\infty} C_j \exp(ij\lambda)\right)\Sigma\left(\sum_{j=0}^{\infty} C_j \exp(ij\lambda)\right)^*$$

Let us first be concerned with the case where $0.5 < d_z < 1.5$ such that $W_t = (x_t^0, z_t)'$ is a nonstationary fractionally integrated vector process with component Wold representations given by (2) and (3) and define the vector random element

(6) $\quad K_T(r) = \Upsilon_T^{-1} W_{[Tr]}$,

for $r \in [0,1]$, with $\Upsilon_T = diag\{T^{d_x - 1/2}, T^{d_z - 1/2}\}$ and where $[\cdot]$ denotes the integer part. Note that $K_T(r) \in D[0,1]^2 = D[0,1] \times D[0,1]$, the product metric space of all real valued vector functions on $[0,1]$ that are right continuous at each element of $[0,1]$ and possess finite left limits. Endow each component space $D[0,1]$ with the Skorohod metric. Then, under Assumption 1, the following functional central limit theorem for nonstationary fractionally integrated processes holds.

Theorem 1. Under Assumption 1, *with* $d_z > 0.5$,

(7) $\quad K_T(r) \Rightarrow B_d(r)$,

where $\Rightarrow$ *denotes weak convergence of the associated probability measures in the sense of Billingsley* (1968) *and* $B_d(r)$ *is a two-dimensional fractional Brownian motion with long-run covariance matrix* Ω, *denoted* $B_d(r) \equiv FBM(\Omega)$, *with*

(8) $$\Omega = 2\pi f_{uu}(0) = \left(\sum_{j=0}^{\infty} C_j\right)\Sigma\left(\sum_{j=0}^{\infty} C_j\right)' > 0,$$

and given by

(9) $$B_d(r) = \int_0^r \Phi(r-s)dB(s),$$

where $\Phi(r-s) = diag\{(r-s)^{d_x - 1}/\Gamma(d_x), (r-s)^{d_z - 1}/\Gamma(d_z)\}$, $\Gamma(\cdot)$ *is the gamma or generalized factorial function and* $B(r)$ *is a two-dimensional Brownian motion with covariance matrix* Ω.

[1] Note that under condition (4), $max\{d_x, d_z\} = d_x$ so that $\tau \geq max\{4, (8 - 8d_x)/(2d_x - 1)\}$. In spite of this fact, however, we have written Assumption 1 in the most general manner in order to also encompass the spurious case.

Both random vector processes $B_d(r)$ and $B(r)$ belong to $C[0,1]^2 = C[0,1] \times C[0,1]$ almost surely, where $C[0,1]$ is the space of continuous functions defined on the unit interval. This functional central limit theorem has been recently proved by Marmol and Dolado (1998a). Let $B_d(r) = \left(B_{d_x}(r), B_{d_z}(r)\right)'$ and $B(r) = \left(B_x(r), B_z(r)\right)'$ be divided conformably with W_t. Then, using (7) jointly with the Continuous Mapping Theorem (Billingsley, 1968), the following lemma can be deduced. As a matter of notation, for the rest of this paper all sums run from 1 to T, and all integrals are from 0 to 1 if not indicated otherwise. We write integrals with respect to Lebesgue measure such as $\int B(r)dr$ as $\int B$ in order to save space. Similarly, stochastic integrals such as $\int B(r)dW(r)$ are written simply as $\int BdW$ for similar reasons. Lastly, all limits given in the paper are as the sample size $T \to \infty$.

<u>*Lemma* 1.</u> *Under Assumption* 1, *with* $d_z > 0.5$,

(10) $T^{-0.5-d_x} \sum x_t^0 \Rightarrow \int B_{d_x} \quad [\equiv \Theta_1(d_x)]$, $\qquad T^{-0.5-d_z} \sum z_t \Rightarrow \int B_{d_z} \quad [\equiv \Theta_1(d_z)]$,

(11) $T^{-1.5-d_x} \sum t x_t^0 \Rightarrow \int r B_{d_x} \quad [\equiv \Theta_2(d_x)]$, $\qquad T^{-1.5-d_z} \sum t z_t \Rightarrow \int r B_{d_z} \quad [\equiv \Theta_2(d_z)]$,

(12) $T^{-2d_x} \sum \left(x_t^0 - \bar{x}^0\right)^2 \Rightarrow \int B_{d_x}^2 - \left(\int B_{d_x}\right)^2 \quad [\equiv \Theta_3(d_x)]$,

$T^{-2d_z} \sum \left(z_t - \bar{z}\right)^2 \Rightarrow \int B_{d_z}^2 - \left(\int B_{d_z}\right)^2 \quad [\equiv \Theta_3(d_z)]$,

(13) $T^{-d_x-d_z} \sum x_t^0 z_t \Rightarrow \int B_{d_x} B_{d_z} \quad [\equiv \Theta_4(d_x, d_z)]$,

and joint convergence of (10) *through* (13) *also applies, where* $\bar{x}^0 = T^{-1}\sum x_t^0$ *and* $\bar{z} = T^{-1}\sum z_t$.

Assume now that $d_z < 0.5$, i.e., that z_t is an essentially stationary stochastic process with uniformly bounded second moments (*c.f.*, Wooldridge, 1994) and define the partial sum process $S_{z,t} = \sum_{j=1}^{t} z_j$, $S_{z,0} = 0$. Note that $S_{z,t}$ is a nonstationary fractionally integrated process of order $1 \leq s_z = 1 + d_z < 1.5$ for which Assumption 1 and hence Theorem 1 apply, yielding

(14) $$T^{1/2-s_z} S_{z,[Tr]} \Rightarrow B_{s_z}(r) = \frac{1}{\Gamma(1+d_z)} \int_0^r (r-e)^{d_z} dB_z(e),$$

which is a fractional Brownian motion in the sense of Mandelbrot and Van Ness (1968). Now, using expression (14) and the Continuous Mapping Theorem, it is straightforward to derive the following results.

Lemma 2. Under Assumption 1, *with* $d_z < 0.5$,

(15) $$T^{-0.5-d_z} \sum z_t \Rightarrow B_{s_z}(1) \quad [\equiv \Theta_5(d_z)]$$

and

(16) $$T^{-1.5-d_z} \sum t z_t \Rightarrow B_{s_z}(1) - \int B_{s_z} \quad [\equiv \Theta_6(d_z)].$$

On the other hand, the weak convergence of the sample moment $\sum x_t^0 z_t$ when $d_z < 0.5$ is further more complicated, as we cannot appeal to the invariance principle (7) and the Continuous

Mapping Theorem, since in this case the latter result does not longer apply. Nonetheless, in the case where z_t is a short-memory process, i.e., in the case where $d_z = 0$, Marmol and Dolado (1998a) prove the following lemma.

Lemma 3. *Under Assumption* 1, *with* $d_z = 0$,

(17) $T^{-d_x} \sum x_t^0 z_t \Rightarrow \int B_{d_x} dB_z \quad [\equiv \Theta_7(d_x)]$ *if* $d_x > 1$,

(18) $T^{-1} \sum x_t^0 z_t \Rightarrow \int B_x dB_z + \Delta_{xz} \quad [\equiv \Theta_8(d_x)]$ *if* $d_x = 1$,

and

(19) $T^{-1} \sum x_t^0 z_t \xrightarrow{p} \Delta_{xz}^{\xi} \quad [\equiv \Theta_9(d_x)]$ *if* $0.5 < d_x < 1$,

where $\Delta_{xz} = \sum_{j=0}^{\infty} E(u_{x,0} u_{z,j})$ *and* $\Delta_{xz}^{\xi} = \sum_{j=0}^{\infty} E(\Delta x_0^0 u_{z,j})$.

Finally, when z_t is a long memory process, with $0 < d_z < 0.5$, it can be proved (*c.f.*, Cheung and Lai, 1993, Chan and Terrin, 1995 and Marmol and Dolado 1998b) the following result.

Lemma 4. *Under Assumption* 1, *with* $0 < d_z < 0.5$,

(20) $T^{-d_x - d_z} \sum x_t^0 z_t \Rightarrow \Psi_{xz} \quad [\equiv \Theta_{10}(d_x, d_z)]$,

where Ψ_{xz} *is a function of the fractional Brownian motions* B_{d_x} *and* B_{d_z}.

Lastly, using the above results, it is rather direct to prove the next result.

Lemma 5. Under DGP (1)-(3) *and Assumption* 1,

(21) $T^{-3}\sum\left(x_t-\bar{x}\right)\xrightarrow{p}\frac{\mu^2}{12}$

and

(22) $T^{-3}\sum\left(y_t-\bar{y}\right)\xrightarrow{p}\frac{\mu^2}{12}\beta^2$.

III. REGRESSION RESULTS

Consider now the estimation by ordinary least squares of the β coefficient in the cointegrating regression (1)

(23) $\left(\hat{\beta}-\beta\right)=\dfrac{\sum\left(x_t-\bar{x}\right)z_t}{\sum\left(x_t-\bar{x}\right)^2}$.

Theorem 2. Under DGP (1)-(3) *and Assumption* 1,

(24) $\hat{\beta}\xrightarrow{p}\beta$ *for all* $0\le d_z<1.5$ *and* $0.5<d_x<1.5$.

(25) *If* $0.5<d_z<1.5$, $T^{1.5-d_z}\left(\hat{\beta}-\beta\right)\Rightarrow\dfrac{12}{\mu}\left\{\Theta_2(d_z)-\dfrac{1}{2}\Theta_1(d_z)\right\}\equiv N\left\{0,\dfrac{144}{\mu^2}V_1^z\right\}$.

(26) *If* $0\le d_z<0.5$, $T^{1.5-d_z}\left(\hat{\beta}-\beta\right)\Rightarrow\dfrac{12}{\mu}\left\{\Theta_6(d_z)-\dfrac{1}{2}\Theta_5(d_z)\right\}\equiv N\left\{0,\dfrac{144}{\mu^2}V_2^z\right\}$,

where

(27) $V_1^z=2\omega_z^2\left(\dfrac{1}{(d_z+2)(2d_z+3)}+\dfrac{1}{4(d_z+1)(2d_z+1)}-\dfrac{(2d_z+3)}{2(d_z+1)(d_z+2)(2d_z+2)}\right)$

and

(28) $$V_2^z = \omega_z^2\left(\frac{2+d_z(2d_z-1)}{4(2+d_z)(3+2d_z)}\right),$$

with ω_z^2 *denoting the long-run variance of* u_{zt}, *i.e.,* $\omega_z^2 = (0,1)\Omega(0,1)'$.

Consequently, the least squares estimator $\hat{\beta}$ is consistent for all $0 \le d_z < 1.5$ and $0.5 < d_x < 1.5$. There is no bias resulting from the correlation between regressors and regression errors. Note, however, that the convergence of $\hat{\beta}$ to its theoretical counterpart is slower as $d_z \to 1.5$ and larger samples are needed in order to improve the reliability of any finite sample analysis. Converse comments apply for $d_z \to 0$. On the other hand, upon appropriate normalization, we obtain well defined limiting distributions given by expressions (25) and (26). These limits, in turn, do not depend on the order of integration of the stochastic regressor component x_t^0. In particular, in case of simply detrending, $x_t = t$, expression (25) has been recently provided by Hassler (1997, expression (19)).

Moreover, Theorem 2 shows also that, upon appropiate normalization, $\hat{\beta}$ has limiting Gaussian distributions for all $0 \le d_z < 1.5$. In particular, when $d_z = 0$, from expressions (26) and (28), West's (1988) classic result follows

(29) $$T^{1.5}\left(\hat{\beta}-\beta\right) \equiv N\left(0, \frac{12}{\mu^2}\omega_z^2\right).$$

Let now R^2 and t_β denote the standard coefficient of determination and t-Student statistic testing for the true parameter values, respectively. Their asymptotic behavior is characterized in the next theorem.

Theorem 3. Under DGP (1)-(3) *and Assumption* 1, *then*

(30) *If* $0.5 < d_z < 1.5$, $T^{3-2d_z}(R^2 - 1) \Rightarrow -\frac{12}{\beta^2\mu^2}\{\Theta_3(d_z) - 12(\Theta_2(d_z) - 0.5\Theta_1(d_z))\}^2$.

(31) *If* $0 \le d_z < 0.5$, $T^2(R^2 - 1) \xrightarrow{p} -\frac{12\,\mathrm{var}(z_t)}{\beta^2\mu^2}$.

(32) *If* $0.5 < d_z < 1.5$, $T^{-1/2}t_\beta \Rightarrow \frac{12^{1/2}\{(\Theta_2(d_z) - 0.5\Theta_1(d_z))\}}{\left[\{\Theta_3(d_z) - 12(\Theta_2(d_z) - 0.5\Theta_1(d_z))\}^2\right]^{1/2}}$.

(33) *If* $0 \le d_z < 0.5$, $T^{-d_z}t_\beta \Rightarrow \left(\frac{12}{\mathrm{var}(z_t)}\right)^{1/2}\left\{\Theta_6(d_z) - \frac{1}{2}\Theta_5(d_z)\right\}$.

From the above theorem, the following comments are in order. Firstly, note from expressions (30)-(33) that both the limiting distributions and the rates of convergence of the coefficient of determinations and the *t*-Student statistic only depend on the stochastic behavior of the perturbation term, z_t. Secondly, the coefficient of determination tends to one in probability whether $0.5 < d_z < 1.5$ or $0 \le d_z < 0.5$, even that at different rates. Indeed, note that the rate is higher in the stationary case than in the nonstationary case, since $3 - 2d_z < 2$. The behavior of the coefficient of determination, in turn, can be explained under classical arguments. Since the behavior of the regressor x_t is asymptotically dominated by the assumed deterministic trend, then, asymptotically, regression (1) becomes equivalent to a regression among trending variables and, consequently, the coefficient of determination tends to one, independently of the goodness of fit of the proposed regression.

With respect to the inferential results, we can observe from expressions (32) and (33) that the standard t-Student statistic testing for the true slope parameter in regression (1) diverges except in the case where $d_z = 0$. This result has also been noted by Marmol (1998) for the case where no deterministic terms in expression (2) is assumed. Therefore, with long memory or nonstationary fractionally integrated error terms, standard inference is not valid in our context, since the t-Student statistic will reject with probability one any null hypothesis. Moreover, note that when $0.5 < d_z < 1.5$, the t-Student statistic does not have an asymptotically Gaussian distribution, in spite of the fact that the corresponding $\hat{\beta}$ estimator does have an asymptotically Gaussian distribution given by expression (25). In contrast, when $0 \leq d_z < 0.5$, it follows from expression (28) that the standardized t-Student statistic $t_\beta^* = T^{-d_z} t_\beta$ has an asymptotically Gaussian distribution given by

(34) $$t_\beta^* \equiv N\left(0, \frac{12}{\text{var}(z_t)} V_2^z\right),$$

which, in the particular case where $d_z = 0$ and u_{zt} an i.i.d. process, allow us to obtain the classical inferential result that

(35) $$t_\beta^* = t_\beta \equiv N(0,1).$$

IV. THE MULTIVARIATE CASE

In this section, we shall consider the multivariate extension of the cointegrating regression (1)

(1') $$y_t = \alpha + \beta' x_t + z_t,$$

where now x_t is an m-dimensional stochastic vector process generated as

(2') $x_t = \Pi t + x_t^0, \ \Pi \neq 0,$

with $x_t^0 = \left(x_{1,t}^0, x_{2,t}^0, \ldots, x_{m,t}^0\right)'$ being an m-dimensional stochastic vector process of nonstationary fractionally integrated processes with memory parameter d_x and Π an m-dimensional vector of constants acting as nuisance parameters in our set-up.

Under the obvious generalization of Assumption 1 from the case $m = 1$ to this multivariate framework, where now $u_{xt} = \left(u_{1,xt}, \ldots, u_{m,xt}\right)'$, the long-run covariance matrix of x_t, denoted Ω_{xx}, is positive definite, which in turn implies that the components of x_t are not allowed to be cointegrated among themselves.

Moreover, following the analysis of Section 2, it is straightforward to prove a multivariate version of Lemma 2,

(36) $T^{-3}\sum\left(x_t - \bar{x}\right)\left(x_t - \bar{x}\right)' \xrightarrow{p} \frac{1}{12}\Pi\Pi',$

that is no longer invertible for $m > 1$. This fact complicates the limiting distribution theory of the least squares estimator of β in regression (1'). In order to develop a complete asymptotic theory in this case, we can follow the treatment suggested by Park and Phillips (1988, p. 477). For this, let us define an orthogonal matrix (ξ, Ξ) of order m with $\xi = \left(\Pi'\Pi\right)^{-1/2}\Pi$ so that Ξ expands the null space of Π, and transform the regression equation (1') as

(37) $y_t = \alpha + \beta'(\xi, \Xi)(\xi, \Xi)' x_t + z_t = \alpha + \gamma_1 f_{1t} + \gamma_2' f_{2t} + z_t,$

where $\gamma_1 = \beta'\xi$, $f_{1t} = \xi' x_t$, $\gamma_2' = \beta'\Xi$ and $f_{2t} = \Xi' x_t$, with dimensions (1×1), (1×1), $(1 \times m - 1)$ and $(m - 1 \times 1)$, respectively. With this transformation, the deterministic trends of x_t are now concentrated in f_{1t} and the stochastic trends in f_{2t}. Specifically,

(38) $f_{1t} = (\Pi'\Pi)^{1/2} t + \xi' x_t^0$

and

(39) $f_{2t} = \Xi' x_t^0$.

Now, writing the transformed regression (37) in a more compact way as

(40) $y_t = \rho' F_t + z_t$,

where $\rho' = (\alpha, \gamma_1, \gamma_2')$ and $F_t' = (1, f_{1t}, f_{2t}')$, the asymptotic results for the least squares estimators of the parameters in (37) are given in the following theorem.

<u>*Theorem 4.*</u> *Under DGP* (1'), (2') *and* (3) *and Assumption* 1, *when* $d_z > 0.5$,

(41) $T^{-d_z} \aleph_T (\hat{\rho} - \rho) \Rightarrow \Lambda^{-1} \wp_1$.

When $0 < d_z < 0.5$, $0.5 < d_x < 1.5$ *or* $d_z = 0$, $d_x \geq 1$,

(42) $T^{-d_z} \aleph_T (\hat{\rho} - \rho) \Rightarrow \Lambda^{-1} \wp_2$.

When $d_z = 0$ *and* $d_x < 1$,

(43) $T^{d_x - 1} \aleph_T (\hat{\rho} - \rho) \Rightarrow \Lambda^{-1} \wp_3$,

where $\Lambda, \wp_1, \wp_2, \wp_3$ *are defined in the Appendix, expressions* (A21), (A14), (A15) *and* (A26), *respectively, and where* $\aleph_T = diag\{T^{0.5}, T^{1.5}, T^{d_x} I_{m-1}\}$.

Now, denoting

$$\Lambda^{-1} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{pmatrix}, \quad \wp_k = \begin{pmatrix} \wp_1^k \\ \wp_2^k \\ \wp_3^k \end{pmatrix}, \ k = 1,2,3,$$

and

$$\Re_k = \Lambda_{31}\wp_1^k + \Lambda_{32}\wp_2^k + \Lambda_{33}\wp_3^k,$$

the limiting distributions of β in regression (1') can be characterized from Theorem 4 in a straightforward manner using the expression $\hat{\beta} - \beta = \xi(\hat{\gamma}_1 - \gamma_1) + \Xi(\hat{\gamma}_2 - \gamma_2)$.

<u>Corollary 1.</u> *When* $d_z > 0.5$,

(44) $$T^{d_x - d_z}\left(\hat{\beta} - \beta\right) \Rightarrow \Xi\Re_1.$$

When $0 < d_z < 0.5$, $0.5 < d_x < 1.5$ *or* $d_z = 0$, $d_x \geq 1$,

(45) $$T^{d_x - d_z}\left(\hat{\beta} - \beta\right) \Rightarrow \Xi\Re_2.$$

When $d_z = 0$ *and* $d_x < 1$,

(46) $$T^{2d_x - 1}\left(\hat{\beta} - \beta\right) \Rightarrow \Xi\Re_3.$$

Therefore, from Corollary 1 we deduce that the limiting distribution of the least squares estimator $\hat{\beta}$ of the slope coefficient in the cointegrating relationship (1') is consistent but not normal if $m > 1$ and invariant to Π for all $0 \leq d_z < 1.5$ and $0.5 < d_x < 1.5$. In the particular case where $d_x = 1$ and $d_z = 0$, this result was proved by Park and Phillips (1988, Theorem 3.6, part *d*). It is also worth noting that the distributions and rates of convergence of $\hat{\beta}$ depend now on d_x, in contrast with the results obtained for the simple regression model (1) in Theorem 2. Notice in particular that the rates of convergence are slower in the multivariate case than in Theorem 2, because $d_x < 1.5$.

In order to obtain Gaussian limiting distributions in the multivariate case under our set-up, when $d_x = 1$ and $d_z = 0$, Hansen (1992) proposes the use of Fully-Modified tests statistics (*cf.*, Phillips and Hansen, 1990). His claim has recently been extended to the $0.5 < d_x < 1.5$ and $d_z = 0$ case by Dolado and Marmol (1998) with the same conclusions. However, from the results obtained by Marmol and Dolado (1998b), the extension of these claims to the $0 < d_z < 0.5$ case seems not to be so clear, and more research in this direction is clearly needed.

V. RESIDUAL-BASED COINTEGRATION TESTS

Cheung and Lai (1993) investigate the purchasing power parity (PPP) hypothesis in fractional context. They do so by analyzing the residuals from a regression of the logarithms of the foreign price index in domestic currency on the logarithm of the domestic price index. If we follow Hassler and Wolters (1995), Baillie et al. (1996), or more recently Ooms and Hassler (1997), then logs of consumer price indexes of several industrial countries can be considered as fractionally integrated of order d_x with $1 < d_x < 1.5$. At the same time those series display approximately linear time trends (which has not been taken into account by Cheung and Lai, 1993), so that the present framework seems to be adequate to test, e.g., the PPP hypothesis.

Generally in case of cointegrating regressions, we consider the OLS residuals from (1) or (1)' and want to find out, whether the equilibrium deviations z_t are mean-reverting or not. This amounts to a one-sided test for

$(H1)\quad H_0\colon\ d_z = 1 \quad versus \quad H_1\colon\ d_z < 1.$

In a first step, we could consider the differences of the OLS residuals (substituting the true models (1) and (1)'):

$$\Delta\hat{z}_t = \Delta z_t - (\hat{\beta} - \beta)' \Delta x_t .$$

By assumption, Δx_t is stationary, so that Theorem 2 and Corollary 1 yield under H_0

(47) $$\Delta\hat{z}_t = \Delta z_t - \begin{cases} O_p(T^{-0.5}) & m = 1 \\ O_p(T^{1-d_x}) & m > 1 \end{cases} .$$

This provides the following intuition: In bivariate regressions $(m = 1)$ $\Delta\hat{z}_t$ equals Δz_t up to $O_p(T^{-0.5})$. Moreover, the asymptotic distribution of the estimator from Theorem 2 is independent of d_x and correlation between the regressor and error term. Hence we may expect that residual-based tests behave asymptotically like tests based on the unobserved series Δz_t directly. If $m > 1$, however, the difference between $\Delta\hat{z}_t$ and Δz_t is less negligible because $d_x < 1.5$. Plus, the asymptotic distribution from Corollary 1 depends on the order of integration of the regressors. Thus, residual-based tests are likely to behave differently from tests based on Δz_t. This intuition will be confronted with Monte Carlo evidence for the log-periodogram regression first proposed by Geweke and Porter-Hudak (1983).

For this. notice that, in terms of differences $(H1)$ becomes

$(H2)$ H_0: $\delta_z := d_z - 1 = 0$ *versus* H_1: $\delta_z < 0$.

Assume for the moment that z_t is observable. Then we could compute the periodogram of the differences,

$$I(\lambda_j) = T^{-1} \left| \sum_{t=1}^{T} \Delta z_t \exp(i\lambda_j t) \right|^2 ,$$

at the harmonic frequencies $\lambda_j = 2\pi j / T$, $j = 1, 2, \ldots, n$. Following Geweke and Porter-Hudak (1983) the log-periodogram regression amounts to

(48) $\ln I(\lambda_j) = \hat{c} + \hat{\delta}_x R_j + res._j$,

where $R_j = -\ln\{4sin^2(\pi j / T)\}, j = 1,2,\ldots,n$.

A test for ($H2$) could be based on the t-statistic where we make use of the asymptotic variance of the error term in (48):

$$(49) \quad t_\delta = \frac{\hat{\delta}_z \sqrt{\sum_{j=1}^{n} (R_j - \bar{R})^2}}{\pi / \sqrt{6}} \xrightarrow{d} N(0,1) \text{ as } n \to \infty .$$

Normality of the t-statistic testing for the true value δ_z has recently be established rigorously by Robinson (1995). If δ_z differs from zero, this requires that the first harmonic frequencies are neglected in (48), so that $j = n_1 + 1, n_1 + 2, \ldots, n$, where n_1 as well as n has to grow with T. In nonfractional context, however, i.e., under ($H2$), asymptotic normality as in (49) arises for $n_1 + 1 = 0$ already. The number of harmonic frequencies may not grow too fast in order to avoid a bias of $\hat{\delta}_z$ due to eventual short memory ($ARMA$) parameters of Δz_t. Following the suggestion by Geweke and Porter-Hudak (1983), we choose $n = [T^{0.5}]$.

Our proposal is to replace the unobserved Δz_t by the differences of the OLS residuals in order to compute the periodogram and run regression (48). It will now be analyzed whether the approximation (49) still holds in this situation.

For this, in Table 1 we investigate the level of a one-sided cointegration test in case of bivariate regressions by simulation.[2] If residuals and regressors are independent the experimental levels are close to the nominal ones. This holds, as expected, irrespective of d_x. In case of (strong) positive

[2] All computations were made by means of GAUSS386. Stationary fractionally integrated series were generated without approximation using the algorithm by Hosking (1984). Nonstationary series were generated according to

correlation between errors and regressors and $d_x > 1$, the empirical levels may be very far below the nominal ones. This contrasts the intuition provided below (47) and therefore deserves further consideration.

The correlation allowed for in Table 1 is

$$corr\left(x_t^0, z_t\right) = \frac{\varphi\sqrt{\operatorname{var}\left(x_t^0\right)}}{\sqrt{\varphi^2 \operatorname{var}\left(x_t^0\right) + \operatorname{var}\left(z_t^0\right)}} = \frac{1}{\sqrt{1 + \operatorname{var}\left(z_t^0\right) / \left(\varphi^2 \operatorname{var}\left(x_t^0\right)\right)}} \underset{\substack{(\varphi=1) \\ (d_x = d_z)}}{=} 0.707,$$

where $z_t = \varphi x_t^0 + z_t^0$, $\operatorname{cov}\left(x_t^0, z_t^0\right) = 0$.

Hence, the correlation for $\varphi = 1$ is rather strong and growing with d_x because the variance of x_t^0 is positively related to d_x,

$$\frac{\partial corr\left(x_t^0, z_t\right)}{\partial d_x} > 0.$$

This is one explanation for the growing size distortion with increasing d_x. Another reason arises from the proof of Theorem 2:

$$T^{1.5-d_z}\left(\hat{\beta} - \beta\right) = T^{-1.5-d_z} \frac{\mu\sum t z_t - \mu T^{-1}\sum t \sum z_t + \sum x_t^0 z_t - T^{-1}\sum x_t^0 \sum z_t}{T^{-3}\sum\left(x_t - \bar{x}\right)^2}$$

$$= T^{-1.5-d_z} \frac{\mu\sum t z_t - \mu T^{-1}\sum t \sum z_t}{T^{-3}\sum\left(x_t - \bar{x}\right)^2} + O_p\left(T^{d_x - 1.5}\right).$$

In other words: the largest d_x, the more slowly vanishes the influence of the correlation on the distribution. Please notice that in case of $I(1)$ regressors the correlation does not affect the experimental levels being again close to the nominal ones.

---

an autoregressive scheme with 50 additional valued discarded for the log-periodogram regression in order to get rid of the starting value 0.

On the other hand, in Table 2 the power of the proposed test is studied (in case of exogenous regressors) for $m=1$. As expected, it does not depend on d_x. The power increases reasonably fast with d_z becoming smaller than one.

Lastly, Table 3 turns to the multivariate case, $m=2,3$. As suspected in our heuristic consideration following (47), the log-periodogram regression does not provide a valid test based on (49). Hence, more refined tools seem to be required to test cointegration in multivariate regressions.

VI. CONCLUDING REMARKS

In this paper we considered the limiting behavior of the slope coefficient as well as the customary least squares statistics in the regression model $y_t = \alpha + \beta' x_t + z_t$, where the regressors are assumed to be nonstationary fractionally integrated driven by a linear time trend and where the purely stochastic process z_t can be either short memory, long memory, nonstationary but mean reverting or nonstationary and not mean reverting.

In the case of simple linear regressions, the least squares estimator converges to a normal distribution after adequate normalization. Neither the limit nor the rate of convergence depend on the order of integration of the regressors. The t-statistics diverge and the coefficient of determination approaches one. The latter does not necessarily mean that a reasonable model is estimated. Even in case of nonsense regressions where the memory parameter of z_t may exceed the order of integration of the regressor, the estimator eliminates the linear trend by converging to the ratio of the trend coefficients of y_t and x_t.

Hence, we require tests that allow to discriminate between cointegration an such nonsense regressions. Practitioners might appreciate for instance a test under the null hypothesis $d_z = 1$ with power against the alternative $d_z < 1$. Under the alternative, the equilibrium error may be only mean reverting but not necessarily stationary, which seems to be too strong an assumption with some applications. In this sense, we provide Monte Carlo evidence supporting the intuition that such a test is available by applying the log-periodogram regression to bivariate OLS residuals.

We also tackled multivariate regressions under the simplifying assumption that all regressors are integrated of the same order. Here this order affects the limiting distributions that are no longer normal. Moreover, the rates of convergence depend on the order of integration of the regressors, and convergence is more slowly than with simple regressions. This is likely to complicate the development of eventual residual-based cointegration tests. Experimentally, we found that the residual-based log-periodogram regression does not provide a valid cointegration test in multivariate regressions.

Table 1: Level of bivariate tests

T	$d_x = 1.0$	1.1	1.2	1.3	1.4	φ
	1.2	2.2	1.3	1.9	1.0	
250	4.6	5.7	4.4	6.1	4.8	0
	8.9	10.1	8.7	9.8	8.3	
	1.2	0.7	0.7	0.3	0.2	
250	5.1	3.2	2.0	0.8	0.6	1
	10.0	6.2	3.9	1.4	1.1	
	1.5	2.0	1.9	1.2	1.3	
500	5.8	5.0	6.0	5.2	5.6	0
	9.5	10.3	11.0	9.0	10.3	
	1.5	1.1	0.4	0.1	0.0	
500	4.8	3.4	1.2	0.5	0.1	1
	10.2	6.0	1.9	0.8	0.1	

The true model is $y_t = x_t + z_t, t = 1,2,\ldots,T$, $x_t = t + x_t^0, z_t = \varphi x_t^0 + z_t^0$, $\Delta^{d_x} x_t^0 = u_{xt}$, $\Delta z_t = u_{zt}$, where u_{xt}, u_{zt} are standard normal white noise sequences independent of each other. We present the percentage of rejections of one-sided tests at the 1%, 5% and 10% level from 1000 replications.

Table 2: Power of bivariate tests

	$d_z = 1.0$	0.9	0.8	0.7	0.6
$d_x = 1.4$					
$T = 250$	4.8	13.8	24.2	38.9	54.4
$T = 500$	5.6	15.7	33.2	53.5	70.1
$d_x = 1.0$					
$T = 250$	4.6	14.4	26.0	36.8	52.6
$T = 500$	5.8	15.8	33.3	52.3	69.7

The true model is $y_t = x_t + z_t$, $t = 1,2,\ldots,T$, $x_t = t + x_t^0$, $z_t = \varphi x_t^0 + z_t^0$, $\Delta^{d_x} x_t^0 = u_{xt}$, $\Delta^{d_z} z_t = u_{zt}$, where u_{xt}, u_{zt} are standard normal white noise sequences independent of each other. We present the percentage of rejections of one-sided tests at the 5% level from 1000 replications.

Table 3: Level of multivariate tests

	$d_x = 1.0$	1.1	1.2	1.3	1.4
	3.3	2.5	4.8	3.9	4.1
$m = 2$	12.1	10.9	12.5	10.9	10.5
	20.1	16.5	21.6	17.8	16.7
	8.3	9.9	7.2	9.1	7.8
$m = 3$	19.5	22.7	17.8	21.4	18.5
	30.0	29.4	27.3	31.1	25.9

The true model is $y_t = x_{1t} + \cdots + x_{mt} + z_t$, $t = 1,2,\ldots,500$, $x_{it} = t + x_{it}^0$, $i = 1,2,\ldots,m$, $\Delta^{d_x} x_{it}^0 = u_{x_i t}$, $\Delta z_t = u_{zt}$, where $u_{x_i t}, u_{zt}$ are standard normal white noise sequences independent of each other. We present the percentage of rejections of one-sided tests at the 1%, 5% and 10% level from 1000 replications.

APPENDIX

Proof of Lemma 5. The lemma follows in a rather direct manner using Lemmas 1-4 and the Continuous Mapping Theorem (henceforth denoted *CMT*).

Consider first the case where $d_z > 0.5$. From the manipulation of $\sum(x_t - \bar{x})^2$ we get

$$\text{(A1)} \quad \sum(x_t - \bar{x})^2 = \mu^2 \sum(t - \bar{t})^2 + \sum(x_t^0 - \bar{x}^0)^2 + 2\mu\sum t x_t^0 + 2\mu\bar{t}\sum x_t^0 ,$$

where $\bar{t} = T^{-1}\sum t$. Now, as $T^{-2}\sum t \to 1/2$, it follows from (10), (11) and the *CMT* that $\sum t x_t^0 - \bar{t}\sum x_t^0 = O_p\left(T^{1.5+d_x}\right)$. On the other hand, given that $T^{-3}\sum(t-\bar{t})^2 \to 1/12$, then expression (21) follows from (12) and the *CMT*, as $2d_x < 3$. Moreover, since (A1) does not depend on z_t, then (21) holds also for $d_z < 0.5$.

In the same manner, from *DGP* (1)-(3), it can be deduced that

$$\text{(A2)} \quad \sum(y_t - \bar{y})^2 = \beta^2 \sum(x_t - \bar{x})^2 + \sum(z_t - \bar{z})^2 + 2\beta\sum x_t(z_t - \bar{z})$$

$$= \beta^2 \sum(x_t - \bar{x})^2 + \sum(z_t - \bar{z})^2 + 2\beta\mu\sum t z_t - 2\beta\mu T^{-1}\sum t\sum z_t + 2\beta\sum x_t^0 z_t - 2\beta T^{-1}\sum z_t \sum x_t^0$$

so that, using Lemmas 1-4 and the *CMT*, we have that the first term in the right side of (A2) is the leading term of order $O_p(T^3)$ whether $d_z > 0.5$ or $d_z < 0.5$, giving rise to expression (22). ■

Proof of Theorem 2. Manipulating the numerator of $(\hat{\beta} - \beta)$ in (23) yields

$$\text{(A3)} \quad \sum(x_t - \bar{x})z_t = \mu\sum t z_t - \mu T^{-1}\sum t \sum z_t + \sum x_t^0 z_t - T^{-1}\sum x_t^0 \sum z_t .$$

Using Lemmas 1-4, it can be proved that $\mu\sum t z_t - \mu T^{-1}\sum t\sum z_t = O_p\left(T^{1.5+d_z}\right)$ and $\sum x_t^0 z_t - T^{-1}\sum x_t^0 \sum z_t = O_p\left(T^{d_x+d_z}\right)$ whether $d_z > 0.5$, $0 < d_z < 0.5$, or $d_z = 0$ and $d_x \geq 1$, and $O_p(T)$ for $d_z = 0$ and $d_x < 1$. Consequently, (A3) becomes

(A4) $$T^{-1.5-d_z}\sum(x_t-\bar{x})z_t = T^{-1.5-d_z}\mu\sum tz_t - T^{-1.5-d_z}\mu T^{-1}\sum t\sum z_t + o_p(1).$$

Now, using Lemma 1 and the *CMT*, we have that, when $d_z > 0.5$,

(A5) $$T^{-1.5-d_z}\sum(x_t-\bar{x})z_t \Rightarrow \mu\Theta_2(d_z) - \frac{\mu}{2}\Theta_1(d_z),$$

whereas that, when $d_z < 0.5$, from Lemma 2 and the *CMT*,

(A6) $$T^{-1.5-d_z}\sum(x_t-\bar{x})z_t \Rightarrow \mu\Theta_6(d_z) - \frac{\mu}{2}\Theta_5(d_z),$$

obtaining in this way expressions (25) and (26) using (A5), (A6), (21) and the *CMT*. Since $d_z < 1.5$, consistency of $\hat{\beta}$ to its theoretical counterpart follows for all $d_z \geq 0$. Finally, normality of the limiting distributions of $T^{1.5-d_z}(\hat{\beta}-\beta)$ in (25) and (26) follows from Marmol (1997) and Haldrup and Marmol (1997), respectively, noting that $\Theta_2(d_z) - 0.5\Theta_1(d_z) = \int(r-0.5)B_{d_z}$ and $\Theta_6(d_z) - 0.5\Theta_5(d_z) = \int(r-0.5)dB_{s_z}$. ■

Proof of Theorem 3. Consider first the asymptotic behavior of the least squares residuals, $\hat{z}_t$, in regression (1):

(A7) $$\sum\hat{z}_t^2 = \sum(z_t-\bar{z})^2 + (\hat{\beta}-\beta)^2\sum(x_t-\bar{x})^2 - 2(\hat{\beta}-\beta)\sum(x_t-\bar{x})z_t$$

$$= \sum(z_t-\bar{z})^2 - (\hat{\beta}-\beta)^2\sum(x_t-\bar{x})^2.$$

When $d_z > 0.5$, we have that $\sum\hat{z}_t^2 = O_p(T^{2d_z}) + O_p(T^{2d_z}) + O_p(1)$, respectively, using Lemma 1, (21) and (25), obtaining

(A8) $$T^{-2d_z}\sum\hat{z}_t^2 \Rightarrow \Theta_3(d_z) - 12\left\{\Theta_2(d_z) - \frac{1}{2}\Theta_1(d_z)\right\}^2.$$

On the other hand, when $d_z < 0.5$, $\sum \hat{z}_t^2 = O_p(T) + O_p\left(T^{2d_z}\right) + O_p\left(T^{2d_z}\right)$, respectively, using Lemma 2, (21) and (26). Thus, in the stationary case and by ergodicity,

(A9) $\quad T^{-1}\sum \hat{z}_t^2 \xrightarrow{\ p\ } \operatorname{var}(z_t)$.

Finally, using Lemma 2, Theorem 2, (A8), (A9) and the *CMT*, expressions (30)-(33) follow from the definitions of the coefficient of determination and the t-Student statistic

$$R^2 - 1 = -\frac{\sum \hat{z}_t^2}{\sum (y_t - \bar{y})^2},\ t_\beta = \frac{\left(\hat{\beta} - \beta\right)\left(\sum (x_t - \bar{x})^2\right)^{1/2}}{\left(T^{-1}\sum \hat{z}_t^2\right)^{1/2}}. \ \blacksquare$$

Proof of Theorem 4. The least squares estimators in model (40) turn out to be

(A10) $\quad \left(\hat{\rho} - \rho\right) = \left(\sum F_t F_t'\right)^{-1}\left(\sum F_t z_t\right)$.

Let us first be concerned with the sample vector

(A11) $\quad \sum F_t z_t$,

and for this, notice that, under Assumption 1, the multivariate version of the functional central limit theorem (7) jointly with the *CMT* imply that

(A12) $\quad T^{1/2-d_x}\Xi' x_{[Tr]}^0 \Rightarrow \Xi' B_{d_x} = \widetilde{B}_{d_x}$,

where along the proof of this theorem, B_{d_x} will stand for an m-dimensional fractional Brownian motion associated with the x_t^0 stochastic vector sequence so that $\widetilde{B}_{d_x}(r)$ will be an $(m-1)$-dimensional fractional Brownian motion with associated (positive definite) covariance matrix given by $\widetilde{\Omega}_{xx} = \Xi'\Omega_{xx}\Xi$.

Now, when $d_z > 0.5$, using (10), (11), (A12) and the *CMT*, we have that

$$T^{-0.5-d_z}\sum z_t \Rightarrow \Theta_1(d_z),$$

$$T^{-1.5-d_z}\sum f_{1t}z_t = T^{-1.5-d_z}(\Pi'\Pi)^{1/2}\sum tz_t + T^{-1.5-d_z}\sum \xi' x_t^0 z_t$$

$$= T^{-1.5-d_z}(\Pi'\Pi)^{1/2}\sum tz_t + o_p(1) \Rightarrow (\Pi'\Pi)^{1/2}\Theta_2(d_z),$$

and

$$T^{-d_x-d_z}\sum f_{2t}z_t = T^{-d_x-d_z}\sum \Xi' x_t^0 z_t \Rightarrow \int \widetilde{B}_{d_x} B_{d_z} = \widetilde{\Theta}_4(d_x, d_z),$$

so that, by defining the diagonal matrix of order $m+1$

(A13) $\Im_T = diag\{T^{0.5+d_z}, T^{1.5+d_z}, T^{d_x+d_z} I_{m-1}\}$,

where I_{m-1} denotes the identity matrix of order $m-1$, it follows that

(A14) $$\Im_T^{-1}\sum F_t z_t \Rightarrow \begin{pmatrix} \Theta_1(d_z) \\ (\Pi'\Pi)^{1/2}\Theta_2(d_z) \\ \widetilde{\Theta}_4(d_x, d_z) \end{pmatrix} \equiv \wp_1.$$

Equally, using the corresponding multivariate extensions of Lemmas 2-4 with $\widetilde{B}_{d_x}(r)$ replacing $B_{d_x}(r)$, jointly with the *CMT*, it is not difficult to prove that when $d_z < 0.5$,

(A15) $$\Im_T^{-1}\sum F_t z_t \Rightarrow \begin{pmatrix} \Theta_5(d_z) \\ (\Pi'\Pi)^{1/2}\Theta_6(d_z) \\ \widetilde{\Theta}_*(d_x, d_z) \end{pmatrix} \equiv \wp_2,$$

where

(A16) $\widetilde{\Theta}_*(d_x, d_z) = \widetilde{\Theta}_7(d_x, d_z) = \int \widetilde{B}_{d_x} dB_z$,

if $d_x > 1, d_z = 0$,

(A17) $\widetilde{\Theta}_*(d_x, d_z) = \widetilde{\Theta}_8(d_x, d_z) = \int \widetilde{B}_x dB_z + \Xi'\Delta_{xz}$,

if $d_x = 1, d_z = 0$, and

(A18) $\widetilde{\Theta}_*(d_x, d_z) = \widetilde{\Theta}_{10}(d_x, d_z) = \Xi'\Psi_{xz}$

if $0 < d_z < 0.5$. On the other hand, when $d_x < 1, d_z = 0$, we must replace the weight matrix $\Im_T$ for the following one

(A19) $\Im_T^* = diag\{T^{0.5}, T^{1.5}, TI_{m-1}\}$,

obtaining

(A20) $$\left(\Im_T^*\right)^{-1}\sum F_t z_t \Rightarrow \begin{pmatrix} \Theta_5(d_z) \\ (\Pi'\Pi)^{1/2}\Theta_6(d_z) \\ \widetilde{\Theta}_9(d_x, d_z) \end{pmatrix},$$

where $\widetilde{\Theta}_9(d_x, d_z) = \Xi' \Delta_{xz}^{\xi}$.

In the same manner, from Lemma 1, (38), (39) and the *CMT*, it can be proved that

(A21) $$\aleph_T^{-1}\sum F_t F_t' \aleph_T^{-1} \Rightarrow \begin{pmatrix} 1 & \frac{1}{2}(\Pi'\Pi)^{1/2} & \int \widetilde{B}_{d_x}' \\ \frac{1}{2}(\Pi'\Pi)^{1/2} & \frac{1}{3}(\Pi'\Pi) & (\Pi'\Pi)^{1/2}\int r\widetilde{B}_{d_x}' \\ \int \widetilde{B}_{d_x} & (\Pi'\Pi)^{1/2}\int r\widetilde{B}_{d_x} & \int \widetilde{B}_{d_x}\widetilde{B}_{d_x}' \end{pmatrix} \equiv \Lambda ,$$

where

(A22) $\aleph_T = diag\{T^{0.5}, T^{1.5}, T^{d_x} I_{m-1}\}$.

Now, noting from (A13) and (A22) that $\Im_T = T^{d_z}\aleph_T$ and that the matrix Λ in (A21) is invertible (a.s.), from the manipulation of (A10) it follows that, when $d_z > 0.5$, then

(A23) $$T^{-d_z}\aleph_T(\hat{\rho} - \rho) = \left(\aleph_T^{-1}\sum F_t F_t' \aleph_T^{-1}\right)^{-1} \Im_T^{-1}\left(\sum F_t z_t\right) \Rightarrow \Lambda^{-1}\wp_1,$$

when $0 < d_z < 0.5$ or $d_z = 0$ and $d_x \geq 1$,

(A24) $$T^{-d_z}\aleph_T(\hat{\rho} - \rho) = \left(\aleph_T^{-1}\sum F_t F_t' \aleph_T^{-1}\right)^{-1} \Im_T^{-1}\left(\sum F_t z_t\right) \Rightarrow \Lambda^{-1}\wp_2$$

and when $d_z = 0$ and $d_x < 1$, (A19)-(A22) and the *CMT* yields

(A25) $T^{d_x-1}\aleph_T(\hat{\rho}-\rho)\Rightarrow\Lambda^{-1}\wp_3$

where

(A26) $$\wp_3=\begin{pmatrix}0\\0\\\tilde{\Theta}_9(d_x,d_z)\end{pmatrix}.$$

This last result proves the theorem. ■

REFERENCES

Baillie, R.T., 1996, "Long Memory Processes and Fractional Integration in Econometrics", *Journal of Econometrics* 73, 5-59.

Baillie, R.T., C.F. Chung and M.A. Tieslau, 1996, "Analyzing Inflation by the Fractionally Integrated *ARFIMA-GARCH* Model", *Journal of Applied Econometrics* 11, 23-40.

Banerjee, A., J.J. Dolado, J.W. Galbraith and D.F. Hendry, 1993, *Co-Integration, Error Correction and the Econometric Analysis of Non-Stationary Data*, Oxford University Press, Oxford.

Billingsley, P., 1968, *Convergence of Probability Measures*, Wiley, New York.

Brillinger, D.R., 1981, *Time Series: Data Analysis and Theory*, Holden Day, San Francisco.

Chan, N.H. and N. Terrin, 1995, "Inference for Unstable Long-Memory Processes with Applications to Fractional Unit Root Autoregressions", *Annals of Statistics* 23, 1662-1683.

Cheung, Y.-W. and K.S. Lai, 1993, "A Fractional Cointegration Analysis of Purchasing Power Parity, *Journal of Business and Economic Statistics* 11, 103-122.

Crato, N and Ph. Rothman, 1994, "Fractional Integration Analysis of Long-run Behavior for *US* Macroeconomic Time Series", *Economics Letters* 45, 287-291.

Diebold, F.X. and G.D. Rudebusch, 1991, "Is Consumption too Smooth? Long Memory and the Deaton Paradox", *The Review of Economics and Statistics* 73, 1-9.

Dolado, J.J. and F. Marmol, 1998, "Efficient Estimation of Cointegrating Relationships in the Presence of Nonstationary Fractionally Integrated Processes", *unpublished manuscript*, Universidad Carlos III de Madrid.

Geweke, J. And S. Porter-Hudak, 1983, "The Estimation and Application of Long Memory Time Series Models", *Journal of Time Series Analysis*4, 221-238.

Granger, C.W.J., 1981, "Some Properties of Time Series Data and their Use in Econometric Model Specification", *Journal of Econometrics* 16, 121-130.

Haldrup, N. and F. Marmol, 1997, "On the Interactions between Deterministics and Stochastics Trends in Fractionally Integrated Processes", *unpublished manuscript*, Aarhus University.

Hansen, B., 1992, "Efficient Estimation and Testing of Cointegrating Vectors in the Presence of Deterministic Trends", *Journal of Econometrics* 53, 87-121.

Hassler, U., 1997, "Sample Autocorrelation of Nonstationary Fractionally Integrated Series", *Statistical Papers* 38, 43-62.

Hassler, U and J. Wolters, 1995, "Long Memory in Inflation Rates: International Evidence", *Journal of Business and Economic Statistics* 13, 37-45.

Hosking, J.R.M., 1984, "Modeling Persistence in Hydrological Time Series Using Fractional Differencing", *Water Resources Research* 20, 1898—1908.

Mandelbrot, B.B. and J.W. Van Ness, 1968, "Fractional Brownian Motions, Fractional Brownian Noises and Applications", *SIAM Review* 10, 422-437.

Marmol, F., 1997, "Fractional Integration versus Trend Stationarity in Time Series Analysis", *Working Paper* 97-02, Universidad Carlos III de Madrid.

Marmol, F., 1988, "Can We Distinguish between Spurious and Cointegrating Relationships on the Basis of the Standard *OLS* Statistics?", *unpublished manuscript*, Universidad Carlos III de Madrid.

Marmol, F. and J.J. Dolado, 1998a, "Asymptotic Inference for Nonstationary Fractionally Integrated Processes", *unpublished manuscript*, Universidad Carlos III de Madrid.

Marmol, F. and J.J. Dolado, 1998b, "Some Results on the Weak Convergence of Strong Dependence Stochastic Processes", *unpublished manuscript*, Universidad Carlos III de Madrid.

Ooms, M. and U. Hassler, 1997, "On the Effect of Seasonal Adjustment on the Log-Periodogram Regression", *Economics Letters* 56, 135-141.

Park, J.Y. and P.C.B. Phillips, 1988, "Statistical Inference in Regressions with Integrated Processes. Part 1", *Econometric Theory* 4, 468-497.

Phillips, P.C.B. and B. Hansen, 1990, "Statistical Inference in Intrumental Variables Regression with $I(1)$ Processes", *Review of Economic Studies* 57, 99-125.

Robinson, P. 1995, "Log-Periodogram Regression of Time Series with Long Range Dependence", *The Annals of Statistics* 23, 1048-1072.

Robinson, P. and J. Hidalgo, 1997, "Time Series Regression with Long Range Dependence", *The Annals of Statistics* 25, 77-104.

Sowell, F., 1992, "Modeling Long-Run Behavior with the Fractional *ARIMA* Model", *Journal of Monetary Economics* 29, 277-302.

Watson, M.W., 1994, "Vector Autoregressions and Cointegration", in *Handbook of Econometrics* Vol. IV, 2843-2915, ed. by R.F. Engle and D.L. McFadden, Elsevier Science B.V., North Holland.

West, K. D., 1988, "Asymptotic Normality, when Regressors Have a Unit Root", *Econometrica* 56, 1397-1418.

Wooldridge, J.M., 1994, "Estimation and Inference for Dependent Processes", in *Handbook of Econometrics* Vol. IV, 2639-2738, ed. by R.F. Engle and D.L. McFadden, Elsevier Science B.V., North Holland.